Resolution and Reconstruction for a Helical CT-Scanner

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A helical CT-scanner is a fan-beam scanner in which the patient is moved along the axis of the scanner with a constant speed as the x-ray source spins around the patient, allowing for 3D imaging [2]. We give the conditions on the scanning parameters (i.e. detector width, angular speed ω, translation speed v) for reliable reconstruction of a function of essential bandwidth Ω (i.e. spatial resolution $2\pi/\Omega$ in all directions), and we describe an algorithm which actually achieves this resolution.

Assume that the patient is lying in a cylinder C_ρ of radius ρ around the axes (the x_1 axes) of the scanner, and that the source sits on a concentric circle of radius $r \geq \rho$. Let f be a function which is supported in C_ρ. We define

$$g(x_1, \alpha, \beta) = \int f(x_1 e_1 + r\theta(\alpha) - s\theta(\beta))ds \qquad (1)$$

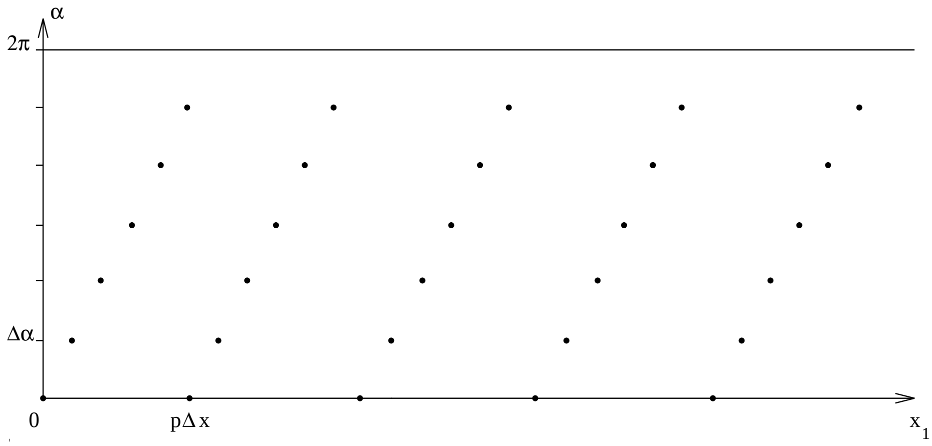
were e_1 is the first unit vector, and

$$\theta(\alpha) = \begin{pmatrix} 0 \\ \cos\alpha \\ \sin\alpha \end{pmatrix} = \begin{pmatrix} 0 \\ \theta'(\alpha) \end{pmatrix} .$$

In helical scanning the function g is sampled at

$$\begin{array}{lllllllll} x_1 & = & k\Delta x & , & \Delta x & = & v\Delta t & , & \\ \alpha & = & k\Delta\alpha & , & \Delta\alpha & = & \frac{2\pi}{p} = \omega\Delta t & , & k = 0, 1, \ldots \\ \beta & = & \ell\Delta\beta & , & \ell & = & -q, \ldots, q & , & \Delta\beta q = \arcsin(\rho/r) . \end{array} \qquad (2)$$

Here, $\Delta\beta$ is the detector spacing and $\Delta\alpha$ the angle between two source positions. The condition that $\Delta\alpha = 2\pi/p$ with an integer p makes the sampling lattice periodic in α. We sketch the sampling points in the x_1-α-plane:



Thus the sampling points (2) are precisely the lattice

$$WZ^3\,, \qquad W = \begin{pmatrix} p\Delta x & \Delta x & 0 \\ 0 & \Delta\alpha & 0 \\ 0 & 0 & \Delta\beta \end{pmatrix}\,. \tag{3}$$

The reciprocal lattice is generated by the matrix

$$2\pi W^{-T} = \begin{pmatrix} 2\pi/p\Delta x & 0 & 0 \\ -1 & 2\pi/\Delta\alpha & 0 \\ 0 & 0 & 2\pi/\Delta\beta \end{pmatrix}\,.$$

Determining the essential support of the Fourier transform of g and using the Petersen-Middleton sampling theorem would possibly lead to an efficient sampling scheme. In this note we pursue a less ambitious goal. We use the inversion formula for the 2D fan-beam transform ([1], p. 368)

$$V_\Omega \star f(x) = r \int_0^{2\pi} \int_{-\pi/2}^{\pi/2} v_\Omega(x' \cdot \theta'(\alpha+\beta) - r\sin\beta) g(x_1, \alpha, \beta) \cos\beta d\beta d\alpha\,. \tag{4}$$

Here, v is defined by

$$\hat{v}_\Omega(\sigma) = \frac{1}{2}(2\pi)^{-3/2}|\sigma|\hat{\phi}(\frac{|\sigma|}{\Omega})$$

where $\hat{\phi}$ is a filter factor vanishing outside $[0, 1]$, and the convolution in (4) with the function V_Ω defined by

$$\hat{V}_\Omega(\xi) = \frac{1}{2\pi}\hat{\phi}(\frac{|\xi|}{\Omega}) \ , \quad \xi \in \mathbb{R}^2$$

is with respect to the 2D variable $x' = \begin{pmatrix} x_2 \\ x_3 \end{pmatrix}$. Convolving (4) in the variable x_1 with the filter

$$\delta_\Omega(x_1) = (2\pi)^{-1/2} \int_{-\Omega}^{\Omega} e^{ix_1\psi_1} d\psi_1$$

(or any other filter with bandwidth Ω) gives a reconstruction formula for helical CT, namely

$$\begin{aligned} f_\Omega(x) &= r \int_{-\infty}^{+\infty} \int_0^{2\pi} \int_{-\pi/2}^{\pi/2} \delta_\Omega(x_1 - y_1) v_\Omega(x' \cdot \theta'(\alpha + \beta) - r \sin \beta) \\ & \qquad g(y_1, \alpha, \beta) \cos \beta d\beta d\alpha dy_1 \ . \end{aligned} \tag{5}$$

f_Ω deviates from the true f only in details of size $\leq \frac{2\pi}{\Omega}$. Applying the Petersen-Middleton sampling theorem to (5) as in [1] one obtains

$$\begin{aligned} f_\Omega(x) &= r \det(W) \sum_{k,\ell} \delta_\Omega(x_1 - k\Delta x) v_\Omega(x' \cdot \theta'(k\Delta\alpha + \ell\Delta\beta) - r \sin(\ell\Delta\beta)) \\ & \qquad g(k\Delta x, k\Delta\alpha, \ell\Delta\beta) \cos(\ell\Delta\beta) \ . \end{aligned} \tag{6}$$

Arranging the k-sum into complete cycles of length p and putting $x' = d\theta'(j\Delta\alpha)$ we obtain

$$\begin{aligned} f_\Omega(x_1, d\theta'(j\Delta\alpha)) &= r \det(W) \sum_{\ell=-q}^{q} \sum_{L=0}^{M} \sum_{k=Lp}^{(L+1)p-1} \delta_\Omega(x_1 - k\Delta x) \\ & v_\Omega(d \cos((k-j)\Delta\alpha + \ell\Delta\beta) - r \sin(\ell\Delta\beta)) g(k\Delta x, k\Delta\alpha, \ell\Delta\beta) \ . \end{aligned} \tag{7}$$

For each x_1, ℓ, d, L the k-sum is now a cyclic convolution of length p, making the evaluation amenable to FFT techniques. For each x_1, only those values of k for which $k\Delta x$ is close to x_1 contribute significantly. This means that only a few values of L have to be taken into account. Thus the evaluation of (7) for a fixed x_1 takes about the same number of operations as in 2D fan-beam tomography.

From [1] we know that the error in (6) is negligible provided that

$$\Delta\alpha \leq \frac{r+\rho}{\rho}\frac{\pi}{\Omega r}\,, \quad \Delta\beta \leq \frac{\pi}{\Omega r}\,, \quad \Delta x \leq \frac{\pi}{\Omega p}\,, \tag{8}$$

the last condition being obvious. In view of (2) this translates into

$$\omega\Delta t \leq \frac{r+\rho}{\rho}\,\frac{\pi}{\Omega r}\,, \quad \Delta\beta \leq \frac{\pi}{\Omega r}\,, \quad v \leq \frac{\omega}{2\Omega}\,. \tag{9}$$

If these conditions are met, no aliasing occurs in (6).

If one wants a different resolution in the x-direction, characterized by a bandwidth Ω_1, one simply has to replace Ω by Ω_1 in the last inequalities of (8) and (9).

References

[1] Natterer, F.: *Sampling in fan-beam tomography*, SIAM J. Appl. Math. **53**, 358-380 (1993).

[2] Kalender, W.A. - Seissler, W. - Klotz, E. - Vock, P.: Spiral Volumetric CT with Single-Breath-Hold Technique, Continuous Transport and Continuous Scanner Rotation, *Radiology* **176**, 181-183 (1990).

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