Inverting the attenuated vectorial Radon transform

F. Natterer
Institut für Numerische und instrumentelle Mathematik
Westf. Wilhelms-Universität Münster
Einsteinstrasse 62, D-48149 Münster, Germany
e-mail: nattere@math.uni-muenster.de

January 6, 2004

Abstract: We give a new derivation of the inversion formula of Bukgheim for the attenuated vectorial Radon transform.

1 Introduction

The attenuated vectorial Radon transform is defined to be

$$(R_a f)(\theta, s) = \theta \cdot \int_{x\cdot\theta=s} f(x) e^{-(Da)(x,\theta^\perp)} dx, \tag{1.1}$$

$$(Da)(x,\theta) = \int_0^\infty a(x+t\theta)dt.$$

Here, a is a compactly supported smooth function in $\mathbb{R}^2$, $\theta = \theta(\varphi) = (\cos\varphi, \sin\varphi)^\top$, $\phi^\perp = \theta(\varphi + \pi 12)$ are unit vectors, $s\epsilon\mathbb{R}^1$ and the dot indicates the natural inner product in R^2. f is a compactly supported smooth vector field in $\mathbb{R}^2$. The problem is to invert R_a for a given, i. e. to solve the equation $R_a f = g$ for f.
The first inversion formula for R_a was given by Bukgheim and Kazantsev [1] based on the theory of A-analytic functions. The very existence of such a formula was a great surprise since it is well known that for $a = 0$ the

vectorial Radon transform R_0 is not invertible: Only the irrotational part of the vector field f is determined by $g = R_0 f$. Subsequently a different proof of the Bukgheim-Kazantsev inversion formula was given by Bal [2] using an extension of the δ-approach of Novikov [3].
The present note is concerned with still another proof of the Bukgheim-Kazantsev formula that is based on the authors proof of Novikov's inversion formula in [4]; see also [6], Theorem 2.23

2 The inversion formula

Let R be the Radon transform, i.e.

$$(Rf)(\theta, s) = \int\limits_{x\cdot\theta=s} f(x)dx,$$

and let H be the Hilbert transform, i.e.

$$(Hf)(s) = \frac{1}{\pi}\int\limits_{\mathbb{R}^1} \frac{f(t)}{s-t}dt.$$

As in[4] we introduce the function $h = \frac{1}{2}(I + iH)Ra$.
Let f be a vector field in $\mathbb{R}^2$. The Helmholtz decomposition of f assumes the form

$$f = \text{grad } w + \text{curl } v, \text{curl } v = \begin{pmatrix} -\frac{\partial v}{\partial x_2} \\ \frac{\partial v}{\partial x_1} \end{pmatrix} \tag{2.1}$$

with certain scalar functions w, v; see e.g. [5], p. 53. grad w is the irrotational , curl v the solenoidal part of f. Note that (2.1) defines an orthogonal decomposition in the space $L_2(\mathbb{R}^2)^2$.

Our goal is to prove the following inversion formula for R_a.

Theorem 2.1 *Let $g = R_a f$ with f as in (2.1). Put $g_a = e^{-h}He^h g$. Then, for $a(x) \neq 0$,*

$$w(x) = \frac{1}{4\pi} Re \int\limits_{S^1} e^{(Da)(x,\theta^\perp)} g_a(\theta, x\cdot\theta)d\theta,$$
$$v(x) = -\frac{1}{4\pi a(x)} Im \text{ div} \int\limits_{S^1} \theta e^{(Da)(x,\theta^\perp)} g_a(\theta, x\cdot\theta)d\theta.$$

The proof will be given in the next section.

Theorem 2.1 is surprising since it is well known and obvious that for $a = 0$ only w in (2.1) is determined by the data g; see e.g. [6], Theorem 2.25. Therefore the question arises what happens as $a \to 0$. We assume $a(x) = \varepsilon a_0(x)$ where $a_0(x) \neq 0$ whenever $f(x) \neq 0$ and study the limit $\varepsilon \to 0$. For w we simply obtain

$$\lim_{\varepsilon \to 0} w(x) = \frac{1}{4\pi} \int_{S^1} \theta (HRf)(\theta, x \cdot \theta) d\theta, \tag{2.2}$$

hence

$$\lim_{\varepsilon \to 0} grad\ w(x) = \frac{1}{4\pi} \int_{S^1} (HR'f)(\theta, x \cdot \theta) d\theta$$

where the prime denotes derivation with respect to the second argument of Rf. Thus (2.2) is just a well known inversion formula for the Radon normal transform; see e.g. Theorem 2.28 of [6].

For v the situation is more difficult. By Taylor expansion around $\varepsilon = 0$ we obtain up to $0(\varepsilon^2)$

$$Im \int_{S^1} \theta e^{(Da)(x,\theta^\perp)} g_a(\theta, x \cdot \theta) d\theta = -\frac{\varepsilon}{2} \int_{S^1} \theta ((HRa)R_0 f)(\theta, x \cdot \theta) d\theta$$

$$+\frac{\varepsilon}{2} \int_{S^1} \theta (H(RaR_0 f))(\theta, x \cdot \theta) d\theta + \varepsilon \int_{S^1} \theta (H \frac{\partial}{\partial \varepsilon} (R_a f)|_{\varepsilon=0})(\theta, x \cdot \theta) d\theta.$$

3 Proof of Theorem 2.1

We start with some lemmas, the proofs of which can be found in [4].

Lemma 3.1 *Let* $\theta = \begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix}$ *and* $u(x,\theta) = h(\theta, x \cdot \theta) - (Da)(x, \theta^\perp)$. *Then, with certain functions* $u_\ell(x)$, $u(x,\theta) = \sum_{l>0\ odd} u_\ell(x) e^{i\ell\varphi}$.

Lemma 3.2 *Let* $\theta = \begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix}$, $\frac{x^\perp}{|x|} = \begin{pmatrix} \cos\psi \\ \sin\psi \end{pmatrix}$. *Then,*

$$\int_{-\pi}^{\pi} \frac{\theta}{x\cdot\theta} e^{i\ell\varphi} d\varphi = \begin{cases} 0, & \ell odd, \\ 2\pi x/|x|^2, & \ell = 0, \\ -2\pi i e^{i\ell\psi}, x^\perp/|x|^2, & \ell > 0 even. \end{cases}$$

Lemma 3.3 *With the Dirac function* δ, *we have*

$$\mathrm{div}\frac{x}{|x|^2} = 2\pi\delta(x),\ \mathrm{div}\frac{x^\perp}{|x|^2} = 0.$$

For the proof of Theorem 2.1 we proceed very much as in [4]. First we compute

$$\begin{aligned}(He^h g)(\theta, 1) &= \frac{1}{\pi}\int_{\mathbf{R}^1} \frac{e^{h(\theta,t)}}{s-t} \int_{y\cdot\theta=t} \theta\cdot f(y) e^{-(Da)(y,\theta^\perp)} dy dt \\ &= \frac{1}{\pi}\int_{\mathbf{R}^2} \frac{\theta\cdot f(y)}{s-y\cdot\theta} e^{-(Da)(y,\theta^\perp)+h(\theta,y\cdot\theta)} dy,\end{aligned}$$

hence

$$e^{(Da)(x,\theta^\perp)} g_a(\theta, x\cdot\theta) = \frac{1}{\pi}\int_{\mathbf{R}^2} \frac{\theta\cdot f(y)}{(x-y)\cdot\theta} e^{u(y,\theta)-u(x,\theta)} dy. \tag{3.1}$$

From Lemma 3.1 we obtain

$$\sinh\left(u(y,\theta) - u(x,\theta)\right) = \sum_{\ell>0\ odd} u_\ell(x,y) e^{i\ell\varphi}$$

with certain functions $u_\ell(x,y)$. Hence,

$$\begin{aligned}\int_{-\pi}^{\pi} \frac{\theta}{(x-y)\cdot\theta} e^{\pm i\varphi} e^{u(y,\theta)-u(x,\theta)} d\varphi &= i\int_{-\pi}^{\pi} \frac{\theta}{(x-y)\cdot\theta} e^{\pm i\varphi} \sinh(u(y,\theta) - u(x,\theta) d\varphi \\ &= i\sum_{\ell>\ odd} \int_{-\pi}^{\pi} \frac{\theta}{(x-y)\cdot\theta} e^{i(\ell\pm 1)\varphi} d\varphi u_\ell(x,y).\end{aligned}$$

Using Lemma 3.2 we obtain

$$\int_{-\pi}^{\pi} \frac{\theta}{(x-y\cdot\theta} e^{i\varphi} e^{u(y,\theta)-u(x,\theta)} d\varphi = 2\pi \sum_{\ell>0\ odd} e^{i(\ell+1)\psi} \frac{(x-y)^{\perp}}{|x-y|^2} u_\ell(x,y)$$

$$= 2\pi e^{i\psi} \left(\sinh u(y,w) - u(x,w)\right) \frac{(x-y)^{\perp}}{|x-y|^2}$$

where $w = \frac{(x-y)^{\perp}}{|x-y|} = \begin{pmatrix} \cos\psi \\ \sin\psi \end{pmatrix}$. Likewise,

$$\int_{-\pi}^{\pi} \frac{\theta}{(x-y)\cdot\theta} e^{-i\varphi} e^{(u(y,\theta)-u(x,\theta))} d\varphi = 2\pi i \frac{x-y}{|x-y|^2} u_1(x,y)$$

$$+ \ 2\pi \sum_{\ell>2\ odd} e^{i(\ell-1)\psi} \frac{(x-y)^{\perp}}{|x-y|^2} u_\ell(x,y)$$

$$= 2\pi \left(i\frac{x-y}{|x-y|^2} - \frac{(x-y)^{\perp}}{|x-y|^2} \right) u_1(x,y) + 2\pi e^{-i\psi} \sinh\left(u(y,w) - u(x,w)\right) \frac{(x-y)^{\perp}}{|x-y|^2}.$$

It follows that

$$\int_{-\pi}^{\pi} \frac{\theta}{(x-y)\cdot\theta} \cos\varphi e^{u(y,\theta)-u(x,\theta)} d\varphi \tag{3.2}$$

$$= \pi \left(i(x-y) - (x-y)^{\perp} \right) \frac{u_1(x,y)}{|x-y|^2} + 2\pi\cos\psi \sinh\left(u(y,w) - u(x,w)\right) \frac{(x-y)^{\perp}}{|x-y|^2}$$

and

$$\int_{-\pi}^{\pi} \frac{\theta}{(x-y)\cdot\theta} \sin\varphi e^{u(y,\theta)-u(x,\theta)} d\varphi \tag{3.3}$$

$$= -\pi \left((x-y) + i(x-y)^{\perp} \right) \frac{u_1(x,y)}{|x-y|^2} + 2\pi\sin\psi \sinh\left(u(y,w) - u(x,w)\right) \frac{(x-y)^{\perp}}{|x-y|^2}.$$

Note that $u(y,w) - u(x,w)$ is real (see[4]). So the sinh-terms drop out when (see below) the imaginary parts are taken. We put

$$A(x) = \int_{\mathbb{R}^2} a(x+y) \frac{y}{|y|^2} dy.$$

From Lemma 3.3 we obtain

$$div A = -2\pi a,\ div A^{\perp} = 0 \tag{3.4}$$

and by straight computation and Lemma 3.2

$$\begin{aligned}
\int_{-\pi}^{\pi} (Da)(x,\theta^{\perp}) e^{i\varphi} d\varphi &= \int_{-\pi}^{\pi}\int_{0}^{\infty} a\left(x + t\theta(\varphi + \pi 12)\right) e^{-i\varphi} d\varphi dt \\
&= i \int_{-\pi}^{\pi}\int_{0}^{\infty} a\left(x + t\theta(\varphi)\right) e^{-i\varphi} d\varphi dt \\
&= i \int_{\mathbf{R}^2} a(x+y) \frac{y_1 - i y_2}{|y|^2} dy \\
&= \begin{pmatrix} i \\ 1 \end{pmatrix} \cdot A(x), \\
\int_{-\pi}^{\pi} (HRa)(\theta, x\cdot\theta) e^{i\varphi} d\varphi &= \frac{1}{\pi} \int_{-\pi}^{\pi}\int_{\mathbf{R}^1} \frac{(Ra)(\theta,s)}{x\cdot\theta - s} ds e^{-i\varphi} d\varphi
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{\pi} \int_{-\pi}^{\pi}\int_{\mathbf{R}^1}\int_{\mathbf{R}^1} a(s\theta + t\theta^{\perp}) dt \frac{ds}{x\cdot\theta - s} e^{-i\varphi} d\varphi \\
&= -\frac{1}{\pi} \int_{-\pi}^{\pi}\int_{\mathbf{R}^2} a(y+x) \frac{dy}{y\cdot\theta} e^{i\varphi} d\varphi \\
&= -\frac{1}{\pi} \begin{pmatrix} 1 \\ -i \end{pmatrix} \cdot \int_{\mathbf{R}^2} a(y+x) \int_{-\pi}^{\pi} \frac{\theta}{y\cdot\theta} d\varphi dy \\
&= -2 \begin{pmatrix} 1 \\ -i \end{pmatrix} \cdot \int_{\mathbf{R}^2} a(y+x) \frac{y}{|y|^2} dy \\
&= 2 \begin{pmatrix} -1 \\ i \end{pmatrix} \cdot A(x).
\end{aligned}$$

It follows that

$$u_1(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} u(x,\theta) e^{-i\varphi} d\varphi$$

$$
\begin{aligned}
&= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(h(\theta, x\cdot\theta) - (Da)(x,\theta^{\perp})\right)e^{i\varphi}d\varphi \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\frac{1}{2}(Ra)(\theta, x\cdot\theta) + \frac{i}{2}(HRa)(\theta, x\cdot\theta) - (Da)(x,\theta^{\perp})\right)e^{-i\varphi}d\varphi \\
&= \frac{1}{2\pi}\left(\frac{i}{2}2\begin{pmatrix}-1\\ i\end{pmatrix} - \begin{pmatrix}i\\ 1\end{pmatrix}\right)\cdot A(x) = -\frac{1}{\pi}\begin{pmatrix}i\\ 1\end{pmatrix}\cdot A(x)
\end{aligned}
$$

where we have used that the function $\theta \longrightarrow (Ra)(\theta, x\cdot\theta)$ is even. Combining this formula with (3.2), (3.3) we obtain

$$
\begin{aligned}
& Im\int_{-\pi}^{\pi}\frac{\theta}{(x-y)\cdot\theta}\cos\varphi e^{u(y,\theta)-u(x,\theta)}d\theta \\
&= \frac{(A_1(x)-A_1(y))\,(x-y)^{\perp} - (A_2(x)-A_2(y))\,(x-y)}{|x-y|^2}, \qquad (3.5)
\end{aligned}
$$

$$
\begin{aligned}
& Im\int_{-\pi}^{\pi}\frac{\theta}{(x-y)\cdot\theta}\sin\varphi e^{u(y,\theta)-u(x,\theta)}d\varphi \\
&= \frac{(A_2(x)-A_2(y))\,(x-y)^{\perp} + (A_1(x)-A_1(y))\,(x-y)}{|x-y|^2}. \qquad (3.6)
\end{aligned}
$$

Very much in the same way one can derive (see[4])

$$
Re\int_{-\pi}^{\pi}\frac{\theta}{(x-y)\cdot\theta}e^{u(y,\theta)-u(x,\theta)}d\varphi = 2\pi\frac{x-y}{|x-y|^2} \qquad (3.7)
$$

Now we finish the proof by putting

$$
\begin{aligned}
W(x) &= Re\int_{-\pi}^{\pi} e^{(Da)(x,\theta^{\perp})}g_a(\theta, x\cdot\theta)d\varphi, \\
V(x) &= Im\int_{-\pi}^{\pi} \theta e^{(Da)(x,\theta^{\perp})}g_a(\theta, x\cdot\theta)d\varphi.
\end{aligned}
$$

By (3.1),(3.5)-(3.7) we obtain

$$
W(x) = \frac{1}{\pi}Re\int_{-\pi}^{\pi}\int_{R^2}\frac{\theta\cdot f(y)}{(x-y)\cdot\theta}e^{u(y,\theta)-u(x,\theta)}dyd\varphi
$$

$$
\begin{aligned}
&= 2\int\limits_{\mathbf{R}^2} f(y)\cdot\frac{x-y}{|x-y|^2}dy, \\
V(x) &= \frac{1}{\pi}Im\int\limits_{-\pi}^{\pi}\int\limits_{\mathbf{R}^2}\frac{\theta\cdot f(y)}{(x-y)\cdot\theta}\theta e^{u(y,\theta)-u(x,\theta)}dyd\varphi \\
&= \frac{1}{\pi}\int\limits_{\mathbf{R}^2} f_1(y)\frac{(A_1(x)-A_1(y))\,(x-y)^{\perp}-(A_2(x)-A_2(y))\,(x-y)}{|x-y|^2}dy \\
&+ \frac{1}{\pi}\int\limits_{\mathbf{R}_2} f_2(y)\frac{(A_2(x)-A_2(y))\,(x-y)^{\perp}+(A_1(x)-A_2(y))\,(x-y)}{|x-y|^2}dy
\end{aligned}
$$

and, by Lemma 3.3,

$$
\begin{aligned}
div\ V(x) &= \frac{1}{\pi}\int\limits_{\mathbf{R}^2} f_1(y)\frac{gradA_1(x)\cdot(x-y)^{\perp}-gradA_2(x)\cdot(x-y)}{|x-y|^2}dy \\
&+ \frac{1}{\pi}\int\limits_{\mathbf{R}^2} f_2(y)\frac{gradA_2(x)\cdot(x-y)^{\perp}+gradA_1(x)\cdot(x-y)}{|x-y|^2}dy \\
&= \frac{1}{\pi}\int\limits_{\mathbf{R}^2} f_1(y)\frac{-(x_2-y_2)divA(x)+(x_1-y_1)divA^{\perp}(x)}{|x-y|^2}dy \\
&+ \frac{1}{\pi}\int\limits_{\mathbf{R}^2} f_2(y)\frac{(x_1-y_1)divA(x)-(x_2-y_2)divA^{\perp}(x)}{|x-y|^2}dy \\
&= -2a(x)\int\limits_{\mathbf{R}^2} f(y)\cdot\frac{(x-y)^{\perp}}{|x-y|^2}dy.
\end{aligned}
$$

Now we use the Helmholtz decomposition f=grad w+curl v to obtain

$$
\begin{aligned}
W &= 4\pi w, \\
divV &= -4\pi a v
\end{aligned}
$$

and hence the theorem. □

References

[1] Buckgheim, A. A. - Kazantsev, S. G.: Inversion formula for the fan-beam attenuated Radon transform in a unit disk, Preprint No. 99, August 2002, The Sobolev Institute of Mathematics of SB RAS.

[2] Bal, G.: On the attenuated Radon transform with full and partial measurements, Preprint 2003.

[3] Novikov, R. G.: An inversion formula for the attenuated X-ray transform, Preprint 2000.

[4] Natterer, F.: Inversion of the attenuated Radon transform, Inverse Problems 17, 113-119 (2001).

[5] Morse, P. M.:-Feshbach, H.:Methods of Theorectical Physics. McGraw-Hill, New York 1953.

[6] Natterer, F.-Wübeling, F.: Mathematical Methods in Image Reconstruction. SIAM, Philadelphia 2001.