A simple method for finding the support of a scatterer

F. Natterer
Institut für Numerische und instrumentelle Mathematik
Westf. Wilhelms-Universität Münster
Einsteinstrasse 62, D-48149 Münster, Germany
e-mail: nattere@math.uni-muenster.de

November 2003

Abstract

We consider the acoustic inverse scattering problem at fixed frequency for one incoming wave. We try to determine the shape of a medium scatterer. Our method works entirely in Fourier space.

1 Introduction

Finding the shape of a scatterer from the far field for just one irradiating wave has found much interest resently; see [1]. Let the scatterer be described by a function f in $\mathbb{R}^2$ whose support is sought. Let the model be

$$\begin{aligned} \triangle u + k^2(1+f)u &= 0 \\ u &= u_i + u_s,\ u_i = e^{ikx\cdot\theta} \end{aligned} \tag{1.1}$$

where u_s satisfies the Sommerfeld radiation condition at infinity, $\theta \in S^1$ is the direction and k the wave number of the incoming wave. Assume that f is supported in $|x| < \rho$. We want to determine the support of f from the far field.

2 The method

Putting $f_\theta = f(1 + u_s/u_i)$ we can rewrite (1.1) as

$$\triangle u_s + k^2 u_s = -k^2 f_\theta u_i. \tag{2.1}$$

Since f, f_θ have the same support we can as well determine the support of f_θ. This is facilitated by the fact that (2.1) is linear in f_θ. In fact (2.1) is identical to the equation for the Born approximation. It is well known that the far field of (2.1) gives rise to the function

$$g_\theta(\omega) = \hat{f}_\theta(k(\omega - \theta)), \omega \in S^1. \tag{2.2}$$

Thus the problem is to find the support of the function f_θ from (2.2). A simple reconstruction f_θ^* for f_θ is

$$f_\theta^*(x) = \int_{S^1} g_\theta(\omega) e^{ik(\omega-\theta)\cdot x} K(\omega) d\omega \tag{2.3}$$

where K is chosen suitably. We have

$$\begin{aligned} f_\theta(x) &= \int_{S^1} \hat{f}(k(\omega - \theta)) e^{ik(\omega-\theta)\cdot x} K(\omega) d\omega \\ &= \frac{1}{2\pi} \int_{S^1} \int_{\mathrm{R}^2} f(y) e^{-iy\cdot k(\omega-\theta)} dy e^{ik(\omega-\theta)\cdot x} K(\omega) d\omega \\ &= \int_{\mathrm{R}^2} f(y) \phi(x-y) dy, \end{aligned}$$

$$\phi(x) = \frac{1}{2\pi} \int_{S^1} K(\omega) e^{ik(\omega-\theta)\cdot x} d\omega. \tag{2.4}$$

Thus our reconstruction algorithm has the point spread function ϕ.

3 The point spread function

Now we put $\theta = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, i.e. the irradiating plane wave is falling in in the direction of the x_1 axis. We adapt the jargon of radar, speaking of the

x_1, x_2 direction as down-range and cross-range direction, respectively. With $\omega = \begin{pmatrix} \cos\psi \\ \sin\psi \end{pmatrix}$ we have

$$\phi(x_1, x_2) = \frac{1}{2\pi} \int_0^{2\pi} K(k(\cos\psi - 1), k\sin\psi) e^{ikx_1(\cos\psi - 1) + ikx_2 \sin\psi d\psi}. \tag{3.1}$$

For $K = 1$ we have

$$\phi_1(x_1, x_2) = e^{-ikx_1} J_0(k|x|)$$

with the zero order Bessel function of the first kind. This is an oscillating point spread function as studied in [2].
In order to improve resolution in the x_1 direction we choose K so as to minimize

$$\int_{-\infty}^{+\infty} x_1^2 |\phi(x_1, 0)|^2 dx_1$$

among all K with $\phi(0,0)$ a prescribed positive constant. Substituting $t = 1 - \cos\psi$ we have

$$\begin{aligned} \phi(x_1, 0) &= \frac{1}{2\pi} \int_0^2 e^{-ix_1 kt} K(t) dt, \\ K(t) &= \left(\left(K(-kt, k\sqrt{t(2-t)} \right) + K\left(-kt, -k\sqrt{t(2-t)} \right) \right) / \sqrt{t(2-t)}. \end{aligned}$$

Assuming $K(0) = K(2) = 0$ we easily obtain

$$\int_{-\infty}^{+\infty} x_1^2 |\phi(x_1, 0)|^2 dx_1 = \frac{1}{2\pi k^3} \int_0^2 |K'(t)|^2 dt.$$

This has to be minimized subject to the value of

$$\int_0^2 K(t) dt$$

being prescribed. The solution - up to an irrelevant constant - is

$$K(t) = t(2-t),$$

yielding

$$K\left(-kt, k\sqrt{t(2-t)}\right) + K\left(-kt, -k\sqrt{t(2-t)}\right) = (t(2-t))^{3/2}.$$

Going back to the original coordinates this means

$$K\left(k(\cos\psi - 1), k\sin\psi\right) + K\left(k(\cos\psi - 1), -k\sin\psi\right) = |\sin\psi|^3,$$

and the point spread function is going to be

$$\phi_2(x_1, x_2) = e^{-ikx_1} \int_0^{2\pi} |\sin\psi|^3 e^{ikx_1\cos\psi + ikx_2\sin\psi} d\psi.$$

The reconstruction can be done simply by formula (2.3), i.e. inverse Fourier transform of the data along the Ewald circle $\{k(\omega - \theta), \omega \varepsilon S^1\}$ and multiplication by K.

4 Discussion

The point spread functions ϕ_1 and ϕ_2 are markedly different. Their real parts are displayed in Fig. 1. The wave is falling in from top. So vertical is down range, horizontal is cross range. Both functions are oscillating and peak at the origin. ϕ_1 oscillates cross range around zero, leading to good cross range resolution. However, in down range ϕ_1 oscillates around a slowly decreasing function, yielding poor down range resolution. ϕ_2 decays quickly down range but slowly cross range, indicating good down range and poor cross range resolution. In Fig. 2 we display a reconstruction of a circle. ϕ_1 provides a reconstruction with poor down range resolution, ϕ_2 one with poor cross range resolution. The combined reconstruction has good resolution in both directions.

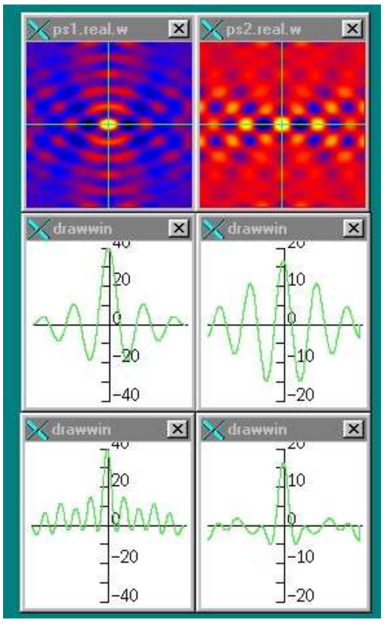

Fig. 1

Fig. 1: Point spread functions $\phi_1(left)$ and $\phi_2(right)$ Top: 2 D display of real part. Middle: Horizontal (cross range) cross section. Bottom: Vertical (down range) cross section.

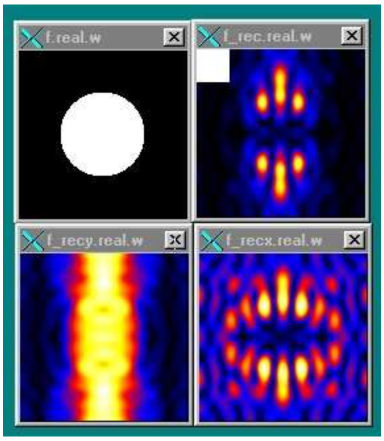

Fig. 2

Fig. 2: Reconstruction of disk. Top left: Original. Bottom: Reconstructions with $\phi_1(left)$ and $\phi_2(right)$. Top right: Combined reconstruction. Wave length indicated by square in the upper left corner.

References

[1] Potthast, R., Sylvester, J., and Kusiak, S.: A 'range test' for determining matters with unknown physical properties, Preprint 2003

[2] Natterer, F., Cheney, M. and Borden , B.: Resolution for radar and X-ray tomography, to appear in Inverse Problems.