Rigidity estimates for isometric and conformal maps from $\mathbb{S}^{n-1}$ to $\mathbb{R}^n$

Konstantinos Zemas
WWU Münster

Winterschool on Analysis and Applied Mathematics

22-26 February, 2021, Münster, Germany

Reference:

Luckhaus - Z.: Rigidity estimates for isometric and conformal maps from $\mathbb{S}^{n-1}$ to $\mathbb{R}^n$,
submitted, Arxiv preprint, January 2021

The spherical version of Liouville's theorem

(*i*) Let $n \geq 2$ and $1 \leq p \leq \infty$. A G.O.P. (\-R.) $u \in W^{1,p}(\mathbb{S}^{n-1};\mathbb{S}^{n-1})$ is isometric iff $\exists\ O \in O(n)$ so that $\forall\ x \in \mathbb{S}^{n-1}$,

$$u(x) = Ox.$$

(*ii*) Let $n \geq 3$. A G.O.P. (\-R.) $u \in W^{1,n-1}(\mathbb{S}^{n-1};\mathbb{S}^{n-1})$ of degree 1 (\ -1) is conformal iff $\exists\ O \in O(n)$, $\xi \in \mathbb{S}^{n-1}$ and $\lambda > 0$ so that $\forall\ x \in \mathbb{S}^{n-1}$,

$$u(x) = O\phi_{\xi,\lambda}(x).$$

Here, $\phi_{\xi,\lambda} := \sigma_\xi^{-1} \circ i_\lambda \circ \sigma_\xi$, where σ_ξ is the stereographic projection of $\mathbb{S}^{n-1}$ onto $\overline{T_\xi\mathbb{S}^{n-1}}$ and $i_\lambda : T_\xi\mathbb{S}^{n-1} \mapsto T_\xi\mathbb{S}^{n-1}$ is the dilation in $T_\xi\mathbb{S}^{n-1}$ by factor $\lambda > 0$.

- ◇ New short 'intrinsic" proof available that can also be perturbed to give

A qualitative analogue for degree $(\pm)1$ maps on $\mathbb{S}^{n-1}$

Proposition

Let $n \geq 3$ and $(u_j)_{j\in\mathbb{N}} \in W^{1,n-1}(\mathbb{S}^{n-1};\mathbb{S}^{n-1})$ be a sequence of G.O.P. maps of degree 1, such that

$$\lim_{j\to\infty} \fint_{\mathbb{S}^{n-1}} \left(\left(\frac{|\nabla_T u_j|^2}{n-1} \right)^{\frac{n-1}{2}} - \sqrt{\det\left(\nabla_T u_j^t \nabla_T u_j\right)} \right) = 0$$

$$\Updownarrow$$

$$\lim_{j\to\infty} \fint_{\mathbb{S}^{n-1}} \left(\frac{|\nabla_T u_j|^2}{n-1} \right)^{\frac{n-1}{2}} = 1.$$

Then there exist $(\phi_j)_{j\in\mathbb{N}} \in Conf_+(\mathbb{S}^{n-1})$ and $R \in SO(n)$ so that up to a non-relabeled subsequence

$$u_j \circ \phi_j \to R\,\mathrm{id}_{\mathbb{S}^{n-1}} \quad \text{strongly in} \quad W^{1,n-1}(\mathbb{S}^{n-1};\mathbb{S}^{n-1}).$$

A sharp quantitative analogue for degree $(\pm)1$ maps on $\mathbb{S}^2$

Theorem [A.B. Mantel-C.B. Muratov-T.M. Simon, Hirsch-Z., Topping]

There exists $c > 0$ so that for every

$$u \in \mathcal{A}_{\mathbb{S}^2} := \left\{ v \in W^{1,2}(\mathbb{S}^2, \mathbb{S}^2) : \deg v := \fint_{\mathbb{S}^2} \langle v, \partial_{\tau_1} v \wedge \partial_{\tau_2} v \rangle = 1 \right\}$$

there exists $\phi \in Conf_+(\mathbb{S}^2)$ such that

$$\fint_{\mathbb{S}^2} |\nabla_T u - \nabla_T \phi|^2 \leq c \left(\frac{1}{2} \fint_{\mathbb{S}^2} |\nabla_T u|^2 - 1 \right).$$

- **Question\ Work in progress:** What happens in the class of maps of degree $k \geq 2$ (bubbling phenomena for almost energy minimizers of higher degree)?

Flexibility (vs Rigidity) of Isometric and Conformal maps from $\mathbb{S}^{n-1}$ to $\mathbb{R}^n$

- Wide variety of such maps from $\mathbb{S}^{n-1}$ into $\mathbb{R}^n$.
- Classical rigidity in the Weyl problem for isometric embeddings (C^2 or even $C^{1,\alpha}$ for $\alpha > \frac{2}{3}$).
- Flexibility via the celebrated Nash-Kuiper theorem.
- For conformal maps from $\mathbb{S}^{n-1}$ to $\mathbb{R}^n$, other examples that are not Möbius transformations are also (at least when $n = 3$) used in cartography, others (for $n = 3$) are provided by the Uniformization theorem, ...
- Liouville's rigidity theorem on $\mathbb{S}^{n-1}$ on the one hand, and the above flexibility phenomena on the other, indicate that an extra (isoperimetric-like) deficit measuring the deviation of $u(\mathbb{S}^{n-1})$ from being a round sphere is necessary when one seeks stability of the isometry (resp. conformal) group of $\mathbb{S}^{n-1}$ among low regularity (say Sobolev) maps from $\mathbb{S}^{n-1}$ into $\mathbb{R}^n$.

Stability in the isometric case, $n \geq 2$

Theorem [Luckhaus-Z.]

Let $n = 2, 3$. There exists $c_n > 0$ so that for every $u \in W^{1,2}(\mathbb{S}^{n-1}; \mathbb{R}^n)$ there exists $O \in O(n)$ such that

$$\fint_{\mathbb{S}^{n-1}} |\nabla_T u - OP_T|^2 \leq c_n \Big(\delta(u) + \varepsilon(u) \Big).$$

Here,

$$\delta(u) := \|(\sigma_{n-1} - 1)_+\|_{L^2}$$

(where $0 \leq \sigma_1 \leq \cdots \leq \sigma_{n-1}$ are the eigenvalues of $\sqrt{\nabla_T u^t \nabla_T u}$) is the L^2-*isometric deficit* of u (penalizing stretches) and

$$\varepsilon(u) := (1 - |V_n(u)|)_+ := \left(1 - \left| \fint_{\mathbb{S}^{n-1}} \left\langle u, \bigwedge_{i=1}^{n-1} \partial_{\tau_i} u \right\rangle \right| \right)_+$$

is (the positive part of) the *generalized isoperimetric deficit (or excess in volume)* of u.

- For $n \geq 4$ and $M > 0$ given, the previous estimate holds true as well for $u : \mathbb{S}^{n-1} \mapsto \mathbb{R}^n$ for which $\|\nabla_T u\|_{L^{2(n-2)}} \leq M$. The constant in this case will depend both on n and (possibly on) M.

- Notice that

$$\delta(u) \leq \left\|\sigma_{n-1} - 1\right\|_{L^2} \leq \left\|\sqrt{\nabla_T u^t \nabla_T u} - I_x\right\|_{L^2} \leq \sqrt{n-1}\left\|\sigma_{n-1} - 1\right\|_{L^2},$$

so $\delta(u)$ is sharper than the *full* L^2*-isometric deficit* $\delta_{\rm isom}(u)$, since it only *penalizes stretches* under u.

- The previous estimate is optimal in its setting, in the sense that the exponents with which the two deficits appear cannot generically be improved, as can be checked even by one-dimensional examples.

Stability in the conformal case, $n \geq 3$: If $u \in W^{1,n-1}(\mathbb{S}^{n-1};\mathbb{R}^n)$ then

$$\fint_{\mathbb{S}^{n-1}} \left(\frac{|\nabla_T u|^2}{n-1}\right)^{\frac{n-1}{2}} \geq \fint_{\mathbb{S}^{n-1}} \sqrt{\det\left(\nabla_T u^t \nabla_T u\right)} \geq \left|\fint_{\mathbb{S}^{n-1}} \left\langle u, \bigwedge_{i=1}^{n-1} \partial_{\tau_i} u\right\rangle\right|^{\frac{n-1}{n}},$$

or equivalently,

$$[D_{n-1}(u)]^{\frac{n}{n-1}} \geq [P_{n-1}(u)]^{\frac{n}{n-1}} \geq |V_n(u)|.$$

- Thus, the combined deficit

$$\mathcal{E}_{n-1}(u) := \frac{[D_{n-1}(u)]^{\frac{n}{n-1}}}{|V_n(u)|} - 1$$

  provides the correct notion of deficit when one seeks stability of the conformal group of $\mathbb{S}^{n-1}$ among maps from $\mathbb{S}^{n-1}$ to $\mathbb{R}^n$.

- $\mathcal{E}_{n-1}$ is translation, rotation and scaling invariant, as well as invariant under precompositions with conformal diffeomorphisms of $\mathbb{S}^{n-1}$.
- $\mathcal{E}_{n-1}(u)$ is nonnegative and vanishes iff u is a degree ± 1 Möbius transformations of $\mathbb{S}^{n-1}$, up to a translation vector and a scaling factor.

Theorem [L.-Z.]

(i) There exists $c > 0$ such that $\forall u \in W^{1,2}(\mathbb{S}^2;\mathbb{R}^3)$ with $V_3(u) \neq 0$ $\exists$ a Möbius transformation ϕ of $\mathbb{S}^2$ and $\lambda > 0$ such that

$$\fint_{\mathbb{S}^2} \left|\frac{1}{\lambda}\nabla_T u - \nabla_T \phi\right|^2 \leq c\mathcal{E}_2(u).$$

(ii) Let $n \geq 4$. There exist constants $\theta \in (0,1)$ (sufficiently small) and $c_{n-1} > 0$ such that $\forall$ $u \in W^{1,\infty}(\mathbb{S}^{n-1};\mathbb{R}^n)$ with $\|\nabla_T u - P_T\|_{L^\infty} \leq \theta$, $\exists$ a Möbius transformation ϕ of $\mathbb{S}^{n-1}$ and $\lambda > 0$ such that

$$\fint_{\mathbb{S}^{n-1}} \left|\frac{1}{\lambda}\nabla_T u - \nabla_T \phi\right|^2 \leq c_{n-1}\mathcal{E}_{n-1}(u).$$

- The estimate is again optimal in its framework (consider $u_\sigma(x) := A_\sigma x : \mathbb{S}^{n-1} \mapsto \mathbb{R}^n$, where $A_\sigma := \mathrm{diag}(1,\ldots,1+\sigma) \in \mathbb{R}^{n\times n}$ as $\sigma \to 0^+$).
- Can the local result in (ii) be generalized to a more global one, possibly via a PDE approach?

Linear stability in the conformal case; $n \geq 3$

If $u = \mathrm{id}_{\mathbb{S}^{n-1}} + w$, then a formal Taylor expansion gives

$$\mathcal{E}_{n-1}(u) = Q_n(w) + (\text{higher order terms in } \nabla_T w),$$

where

$$Q_n(w) := \frac{n}{2(n-1)} \fint_{\mathbb{S}^{n-1}} \left(|\nabla_T w|^2 + \frac{n-3}{n-1} (\mathrm{div}_{\mathbb{S}^{n-1}} w)^2 \right) - \frac{n}{2} \fint_{\mathbb{S}^{n-1}} \langle w, A(w) \rangle;$$

$$A(w) := (\mathrm{div}_{\mathbb{S}^{n-1}} w) x - \sum_{j=1}^{n} x_j \nabla_T w^j,$$

defined on the space

$$H_n := \left\{ w \in W^{1,2}(\mathbb{S}^{n-1}; \mathbb{R}^n) : \fint_{\mathbb{S}^{n-1}} w = 0 \, , \, \fint_{\mathbb{S}^{n-1}} \langle w, x \rangle = 0 \right\}.$$

The associated Euler-Lagrange operator is

$$\mathcal{L}(w) := -\frac{1}{n-1} \Delta_{\mathbb{S}^{n-1}} w - \frac{n-3}{(n-1)^2} \left(\nabla_T \mathrm{div}_{\mathbb{S}^{n-1}} w - (n-1)(\mathrm{div}_{\mathbb{S}^{n-1}} w) x \right) - A(w).$$

Via a decomposition of the subspaces of (k-th order) spherical harmonics into eigenspaces of A, we obtain

Theorem [L.-Z.]

Let $n \geq 3$. There exists a constant $C_n > 0$ such that for every $w \in H_n$,

$$Q_n(w) \geq C_n \fint_{\mathbb{S}^{n-1}} \left|\nabla_T w - \nabla_T(\Pi_{n,0} w)\right|^2,$$

where $\Pi_{n,0} : H_n \mapsto H_{n,0}$ is the $W^{1,2}$-orthogonal projection on the kernel $H_{n,0}$ of Q_n in H_n. Actually $H_{n,0} \cong \mathfrak{moeb}(n-1)$ (of dimension $\frac{n(n+1)}{2}$).

- When $n = 3$, the optimal constant can be calculated explicitely, when $n \geq 4$ one can give an explicit lower bound on it.
- Note that $Q_n = Q_{n,\mathrm{conf}} + Q_{n,\mathrm{isop}}$. Both $Q_{n,\mathrm{conf}}$ and $Q_{n,\mathrm{isop}}$ have infinite-dimensional kernels, but what the previous Theorem says (in a quantitative fashion) is that the intersection of both is finite-dimensional and actually isomorphic to the Lie algebra of infinitesimal Möbius transformations of $\mathbb{S}^{n-1}$.

Linear stability in the isometric case, $n \geq 2$

Theorem [L.-Z.]

Let $n \geq 2$. For every $\alpha > 0$ $\exists$ a constant $C_{n,\alpha} > 0$ such that $\forall\ w \in W^{1,2}(\mathbb{S}^{n-1}; \mathbb{R}^n)$,

$$\alpha Q_{n,\mathrm{isom}}(w) + Q_{n,\mathrm{isop}}(w) \geq C_{n,\alpha} \fint_{\mathbb{S}^{n-1}} \left| \nabla_T w - [\nabla w_h(0)]_{\mathrm{skew}} P_T \right|^2,$$

$$Q_{n,\mathrm{isom}}(w) := \fint_{\mathbb{S}^{n-1}} \left| \frac{P_T^t \nabla_T w + (P_T^t \nabla_T w)^t}{2} \right|^2$$

is the quadratic form associated to the (full) isometric deficit $\delta^2_{\mathrm{isom}}(u)$ and

$$Q_{n,\mathrm{isop}}(w) := \frac{n}{n-1} \left[\fint_{\mathbb{S}^{n-1}} \frac{|\nabla_T w|^2 + (\mathrm{div}_{\mathbb{S}^{n-1}} w)^2}{2} - \left| (P_T^t \nabla_T w)_{\mathrm{s}} \right|^2 \right] - Q_{V_n}(w).$$

is the one with respect to $[P_{n-1}(u)]^{\frac{n}{n-1}} - V_n(u) \geq 0$, i.e. to the isoperimetric deficit.

Although (for $n \geq 3$) $\ker Q_{n,\mathrm{isom}}(w) \cong \mathfrak{so}(\mathfrak{n})$, an estimate of the previous type cannot hold for $Q_{n,\mathrm{isom}}$ alone, as can be easily seen by considering purely normal test vector fields.