Department MATHEMATIK

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Continuum Limit of Lipschitz Learning on Graphs

Γ-Convergence and Compactness of Supremal Graph Functionals

Semi-Supervised Learning on Graphs

Semi-supervised Learning

Given a finite set of points $\Omega_n \subset \Omega \subset \mathbb{R}^d$ with labels $g : \mathcal{O}_n \subset \Omega_n \to \mathbb{R}$, find a function

$$u_n : \Omega_n \to \mathbb{R}, \quad \text{s.t. } u = g \text{ on } \mathcal{O}_n.$$

Smoothness assumption: Points that are close to each other are more likely to share a label.

p-Laplacian Learning: The objective of p-Laplacian learning [ST19] for $p < \infty$ is to find $u : \Omega_n \to \mathbb{R}$ that solves

$$\min_{u:\Omega_n\to\mathbb{R}} \sum_{x,y\in\Omega_n} w(x,y)^p |u(y) - u(x)|^p, \text{ s.t. } u = g \text{ on } \mathcal{O}_n.$$

Lipschitz Learning

We want to find a function $u : \Omega_n \to \mathbb{R}$ that solves the following minimization problem,

$$\min_{u:\Omega_n\to\mathbb{R}} \max_{x,y\in\Omega_n} w_n(x,y) |u(y) - u(x)|, \quad \text{s.t. } u = g \text{ on } \mathcal{O}_n,$$

$$\Big\downarrow \; n \to \infty$$

$$\min_{u\in W^{1,\infty}(\Omega)} \|\nabla u\|_{L^\infty(\Omega)}, \quad \text{s.t. } u = g \text{ on } \mathcal{O}.$$

Our convergence result [RB21] states that discrete minimizers convergence towards minimizers of the continuum problem above in $L^\infty(\Omega)$.

Weighted Graphs

We model the data as a weighted graph (Ω_n, w_n) with edge weights given by

$$w_n(x,y) := \eta(|x-y| / s_n), \quad x, y \in \Omega_n.$$

Here $s_n > 0$ is a scaling parameter and

$$\eta : [0,\infty) \to [0,\infty)$$

is a kernel function.

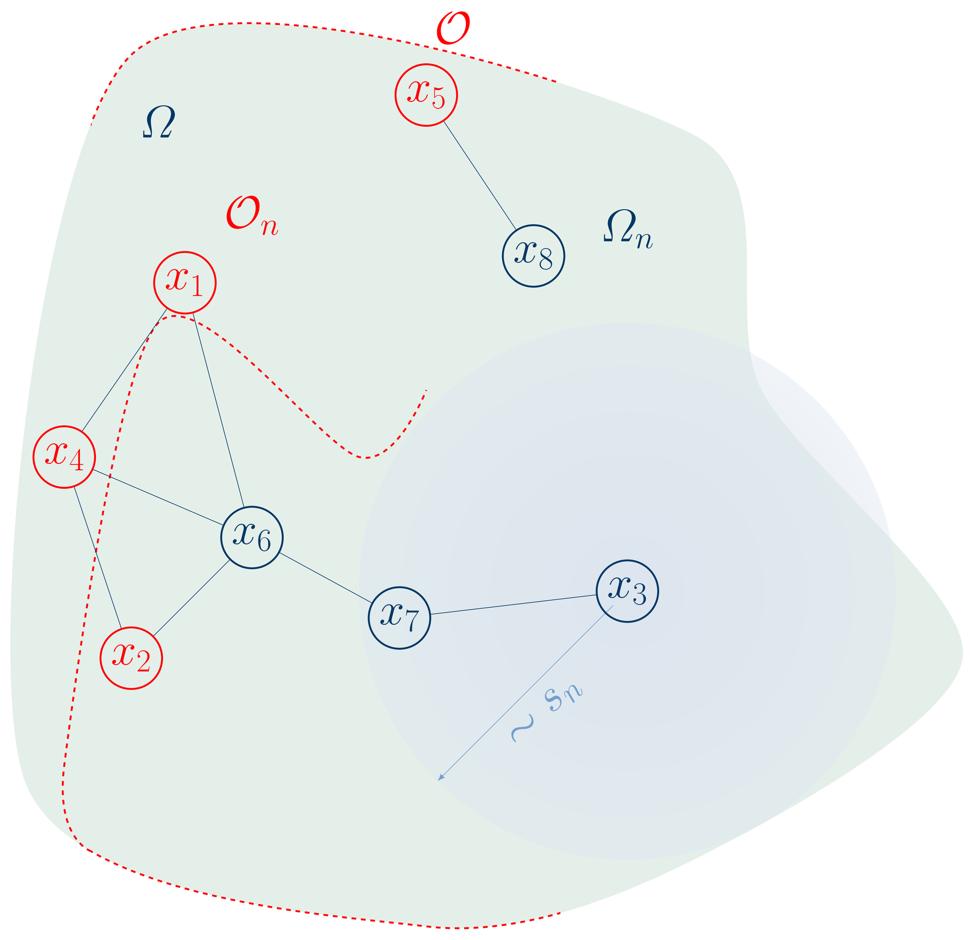
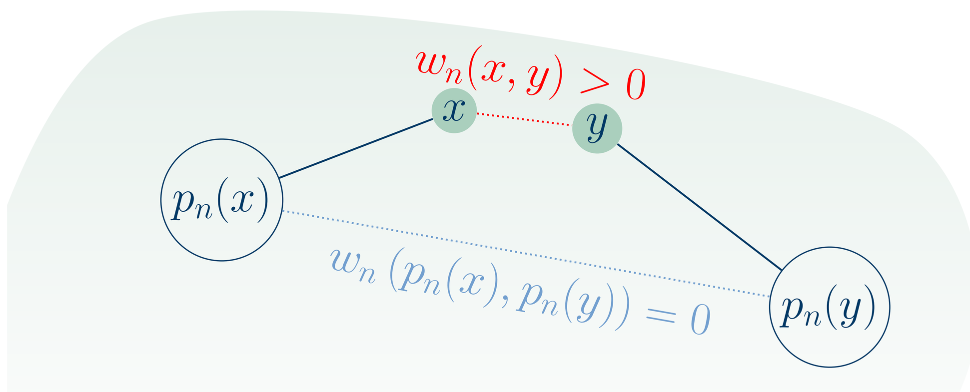


Figure: Visualization of the graph design.

Scaling and Domain

For the continuum limit one has to avoid situations, where points $x, y \in \Omega$ have significant weight while their closest vertices $p_n(x), p_n(y) \in \Omega_n$ do not.

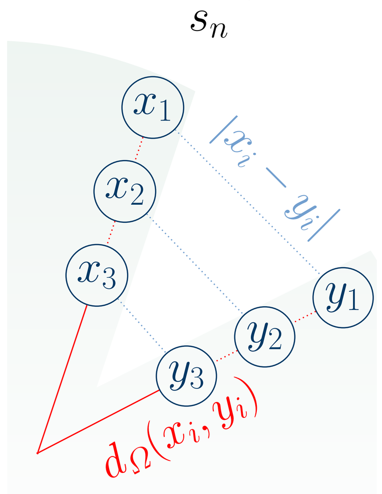


Using the Hausdorff distance

$$d_H(A,B) = \sup_{x\in A} \operatorname{dist}(x,B) \vee \sup_{x\in B} \operatorname{dist}(x,A)$$

we require the weakest scaling for asymptotic connectedness:

$$\frac{r_n}{s_n} := \frac{d_H(\Omega_n, \Omega) \vee d_H(\mathcal{O}_n, \mathcal{O})}{s_n} \longrightarrow 0.$$



We also need local convexity of the domain, expressed via the geodesic distance

$$\lim_{\delta\downarrow 0} \sup_{|x-y|\leq\delta} \frac{d_\Omega(x,y)}{|x-y|} = 1.$$

This prevents sharp internal corners as displayed on the left.

Continuum limit via Γ-Convergence

Graph and Continuum Functional

- The mapping $p_n : \Omega \to \Omega_n$ denotes a closest point projection.
- The graph functional is extended to $L^\infty(\Omega)$ via

$$\operatorname{dom}(E_n) := \{u \in L^\infty(\Omega) : \exists \bar{u} : \Omega_n \to \mathbb{R},\ u = \bar{u} \circ p_n,\ \bar{u}|_{\mathcal{O}_n} = g\},$$
$$E_n(u) := \frac{1}{s_n} \max_{x,y\in\Omega_n} w_n(x,y) |\bar{u}(x) - \bar{u}(y)|.$$

- The continuum functional is defined as

$$\operatorname{dom}(\mathcal{E}) := \{u \in W^{1,\infty}(\Omega) : u = g \text{ on } \mathcal{O}\},$$
$$\mathcal{E}(u) := \|\nabla u\|_{L^\infty(\Omega)}.$$

Γ-Convergence and Compactness

Let Ω be weakly convex and s_n a null sequence with

$$\frac{r_n}{s_n} \longrightarrow 0.$$

Then it holds in the topology of $L^\infty(\Omega)$:

- $E_n \xrightarrow{\Gamma} \sigma_\eta \mathcal{E}$,
- $\sup_{n\in\mathbb{N}} E_n(u_n) < \infty$ implies that $(u_n)_{n\in\mathbb{N}}$ is relatively compact.

Minimizers and Ground States

Using Γ-convergence and compactness:

Minimizers / ground states of $E_{n,\text{cons}}$ converge to minimizers / ground states of $\mathcal{E}_{\text{cons}}$.

Ground states of E_n and $\mathcal{E}$ are solutions of

$$\min_{u\in L^\infty(\Omega)} \frac{F(u)}{\|u\|_{L^p(\Omega)}}, \quad F \in \{E_n, \mathcal{E}\},$$

and are geodesic distance functions [BKB20].

Numerics

Lipschitz Learning and Ground States

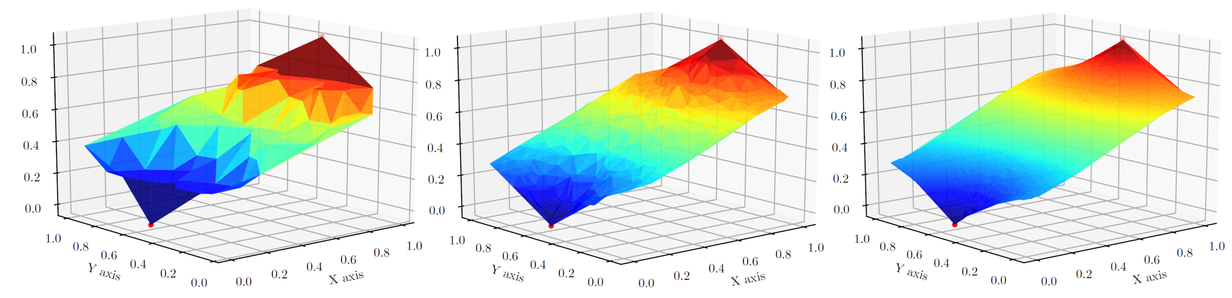


Figure: Continuum limit of Lipschitz Learning on the square with two fixed labels.

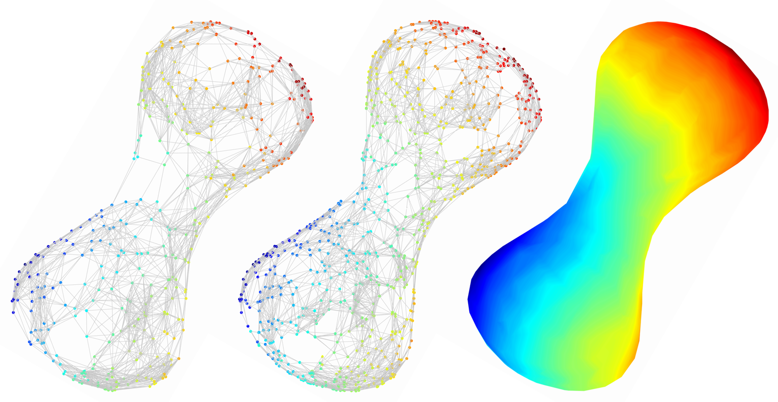


Figure: Continuum limit of ground states on a manifold with one fixed label.

References

[RB21] T. Roith and L. Bungert. *Continuum Limit of Lipschitz Learning on Graphs.* 2021. arXiv: 2012.03772 [cs.LG].

[BKB20] L. Bungert, Y. Korolev, and M. Burger. "Structural analysis of an L-infinity variational problem and relations to distance functions". In: *Pure and Applied Analysis* 2.3 (2020).

[ST19] D. Slepcev and M. Thorpe. "Analysis of p-Laplacian regularization in semisupervised learning". In: *SIAM J. Math. Anal.* 51.3 (2019).

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