Existence and uniqueness result for a fluid-structure-interaction evolution problem in an unbounded 2D channel

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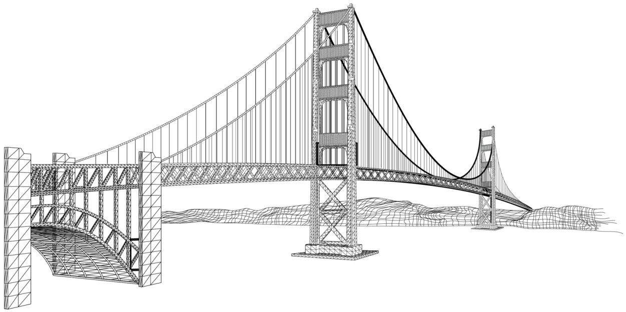


Dynamic response:

- one degree and **two degrees of freedom instability**
- buffeting
- vortex shedding

Main goal

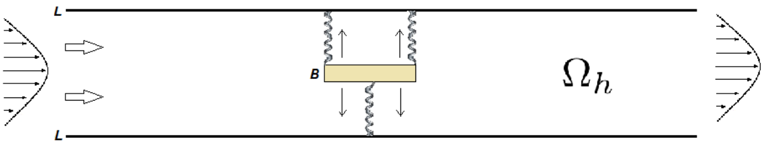
Stationary case:

Bonheure, D. and Galdi, G.P and Gazzola, F. *Equilibrium configuration of a rectangular obstacle immersed in a channel flow* CRAS Paris, 2020.

We study the following two-dimensional fluid-structure-interaction **evolution** problem

$$\begin{cases} u_t = \mu\,\Delta u - (u\cdot\nabla)u - \nabla p, \quad \operatorname{div} u = 0 \qquad \text{in } \Omega_h \times (0,T) \\ u = 0 \qquad \text{on } \Gamma = \mathbb{R}\times\{-L,L\}, \qquad u = h'\,\hat{e}_2 \qquad \text{on } \partial B \\ \lim_{|x_1|\to\infty} u(x_1,x_2) = q := \lambda(L^2 - x_2^2)\,\hat{e}_1 \end{cases} \tag{F}$$

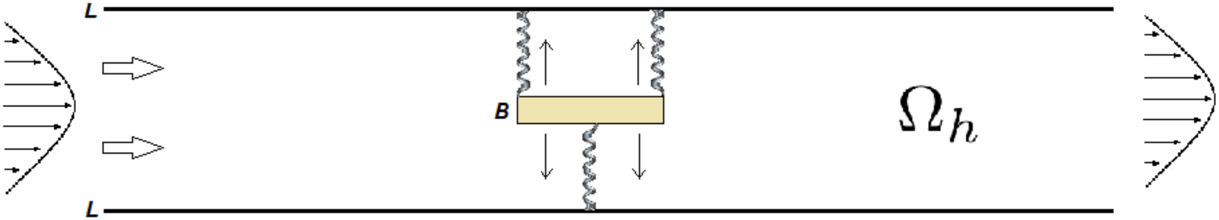
$$h'' + f(h) = -\hat{e}_2 \cdot \int_S \mathcal{T}(u,p)\cdot\hat{n} \qquad \text{in } (0,T) \tag{S}$$



$$B_h = B + h\hat{e}_2 \quad \forall\,|h| < L-\delta$$
$$\Omega_h = \mathbb{R}\times(-L,L)\setminus B_h = A\setminus B_h.$$

Question: Is (F)-(S) well-posed?

Patriarca, C. *Existence and uniqueness result for a fluid-structure-interaction evolution problem in an unbounded 2D channel* Preprint, 2021.



Theorem 1 (Main theorem)

Let q be a Poiseuille flow to which it is associated a prescribed flow rate magnitude Φ. Assume that $|h_0| < L - \delta$ and that u_0 satisfying

$$u_0(x) = \bar{u}_0(x) + \zeta(x_1)\,\lambda(L^2 - x_2^2)\hat{e}_1, \quad \textit{with} \quad \bar{u}_0(x) \in L^2(\Omega_h),$$

is such that $u_0|_{B_{h_0}} = h_1\hat{e}_2$. Then, problem (F)-(S) admits a unique weak solution (u, h), defined in a suitable sense, for any $T < \infty$. Moreover the energy of (u, h) is bounded.

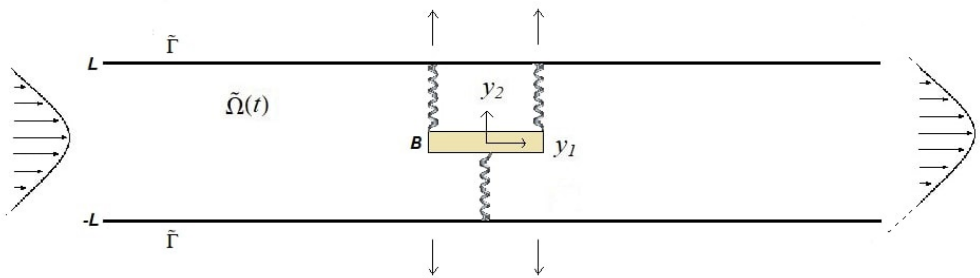
Appendix: Steps of the proof

- Preliminary results
  1. Reformulation into an equivalent problem
  2. Definition of weak solution
- Proof of the main result. Existence through a penalized problem

Conca, C. and San Martin, J. and Tucsnak, M. *Existence of solutions for the equations modelling the motion of a rigid body in a viscous fluid.* Communications in Partial Differential Equations, 2000.

- Proof of the main result. Uniqueness

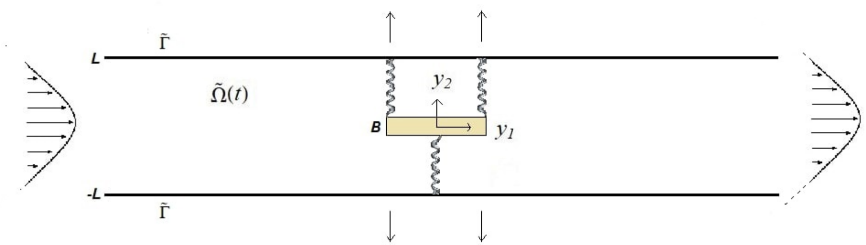
Glass, O. and Sueur, F. *Uniqueness Results for Weak Solutions of Two-Dimensional Fluid-Solid Systems.* Arch. Rational Mech. Anal., 2015.



(F)-(S) becomes (F')-(S')

$$\begin{cases} v_t = \mu\,\Delta v - (v\cdot\nabla)\,v - \nabla\mathfrak{p} + (h'\hat{e}_2\cdot\nabla)\,v \quad \operatorname{div} v = 0 \qquad \text{in } \tilde{\Omega}(t)\times(0,T) \\ v = 0 \qquad \text{on } \tilde{\Gamma}(t), \qquad v = h'\hat{e}_2 \qquad \text{on } \partial B = S \\ \lim_{|y_1|\to\infty} v(y) = \tilde{q}(y) = \tilde{q}_h(y) := \lambda(L^2 - (y_2 + h(t))^2)\,\hat{e}_1 \end{cases} \tag{F'}$$

$$h'' + f(h) = -\hat{e}_2 \cdot \int_S \mathcal{T}(v,\mathfrak{p})\cdot\hat{n} \qquad \text{in } (0,T). \tag{S'}$$



We look for solutions to the problem (F')-(S') of the form

$$v = \hat{v} + a,$$

where a is a function such that:

$$\nabla \cdot a = 0 \quad \text{in } \Omega_h, \qquad a = \tilde{q}_h \quad \text{in } \Omega_{h,i}. \tag{1}$$

The function $\hat{v}$ solves the following problem:

$$\begin{cases} \hat{v}_t - \mu\,\Delta\hat{v} + (\hat{v}\cdot\nabla)\,\hat{v} + \nabla\mathfrak{p} - (h'\hat{e}_2\cdot\nabla)\,\hat{v} - (h'\hat{e}_2\cdot\nabla)\,a + (\hat{v}\cdot\nabla)\,a + (a\cdot\nabla)\,\hat{v} = \hat{g} \\ \operatorname{div}\hat{v} = 0 \qquad \text{in } \tilde{\Omega}(t)\times(0,T) \\ \hat{v} = 0 \qquad \text{on } \tilde{\Gamma}(t), \qquad \hat{v} = h'\hat{e}_2 \qquad \text{on } S, \qquad \lim_{|y_1|\to\infty}\hat{v} = 0 \end{cases} \tag{F''}$$

where

$$\hat{g} := \mu\,\Delta a - (a\cdot\nabla)\,a.$$

$$h'' + f(h) = -\hat{e}_2 \cdot \int_S \mathcal{T}(\hat{v}+a,\mathfrak{p})\cdot\hat{n} \qquad \text{in } (0,T). \tag{S''}$$

Steps of the proof

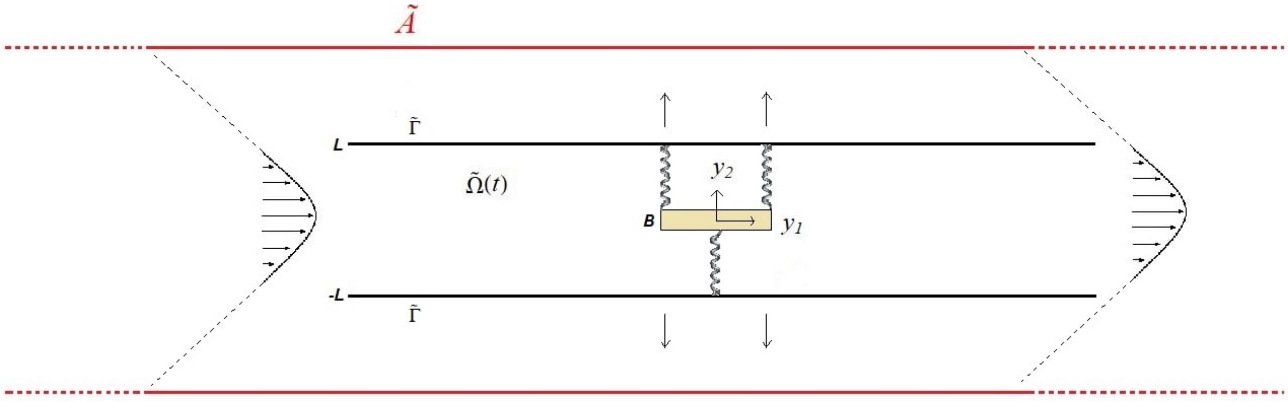
- Preliminary results
  1. Reformulation into an equivalent problem ✓
  2. Definition of weak solution
- Proof of the main result. Existence through a penalized problem
- Proof of the main result. Uniqueness

Fujita, H. and Sauer, N. *On existence of weak solutions of the Navier-Stokes equations in regions with moving boundaries.* Journal of the Faculty of Science. Section I A., 1970.

The crucial idea of the method implies introducing an auxiliary **fixed**, infinite domain $\tilde{A}$ given by:

$$\tilde{A} = A - A = \{x - y \,|\, x \in A, y \in A\},$$

such that $\tilde{\Omega}(t) \subset A_{h(t)} \subset \tilde{A}$. Thus, we choose $\tilde{A} = \mathbb{R} \times (-2L + \delta, 2L - \delta)$.



Question: Definition of weak solutions to (F')-(S')?

Classical functional spaces:

$$\mathcal{V}(\tilde{A}) = \{v \in \mathcal{D}(\tilde{A}) \,|\, \mathrm{div}\, v = 0\},$$
$$H(\tilde{A}) = \text{closure of } \mathcal{V} \text{ w.r.t. the norm } \|\cdot\|_{L^2(\tilde{A})},$$
$$V(\tilde{A}) = \text{closure of } \mathcal{V} \text{ w.r.t. the norm } \|\nabla\cdot\|_{L^2(\tilde{A})}.$$

Non-standard functional spaces:

$$\mathcal{W}(\tilde{A}) = \{(v,l) \in \mathcal{V}(\tilde{A}) \times \mathbb{R} \,|\, v|_B = l\,\hat{e}_2\},$$
$$\mathbb{H}(\tilde{A}) = \text{closure of } \mathcal{W} \text{ in } L^2(\tilde{A}) \times \mathbb{R}, \qquad \mathbb{V}(\tilde{A}) = \text{closure of } \mathcal{W} \text{ in } H^1_0(\tilde{A}) \times \mathbb{R}$$

to which we associate the scalar products

$$\langle (v_1,l_1),(v_2,l_2)\rangle_{\mathbb{H}(\tilde{A})} = \int_{\tilde{A}\setminus B} v_1\cdot v_2\, dy + l_1 l_2, \quad \langle (v_1,l_1),(v_2,l_2)\rangle_{\mathbb{V}(\tilde{A})} = \int_{\tilde{A}\setminus B} \nabla v_1\cdot\nabla v_2\, dy + l_1 l_2.$$

Functional spaces for the weak formulation of problem (F')-(S')

$$\mathcal{W}_h = \{(v,l) \in \mathcal{W}(\tilde{A}) \,|\, \mathrm{supp}\, v \in A_h\},$$
$$\mathbb{H}_h = \text{closure of } \mathcal{W}_h \text{ in } L^2(\tilde{A}) \times \mathbb{R}, \qquad \mathbb{V}_h = \text{closure of } \mathcal{W}_h \text{ in } H^1_0(\tilde{A}) \times \mathbb{R}.$$

Proposition

If a couple (v, h) is a classical solution to (F')-(S'), then, given the extension a, the function $\hat{v} = v - a$ satisfies

$$- \int_0^T \{ (\hat{v}, \phi_t)_{L^2(\tilde{\Omega}(t))} + h'\, l' - f(h)\, l \} + \mu \int_0^T (\nabla \hat{v}, \nabla \phi)_{L^2(\tilde{\Omega}(t))} + \int_0^T \{ \psi(\hat{v}, \hat{v}, \phi) + \psi(\hat{v}, a, \phi) + \psi(a, \hat{v}, \phi) - \psi(h' \hat{e}_2, a, \phi) - \psi(h' \hat{e}_2, \hat{v}, \phi) \} = \int_0^T \langle \hat{g}, \phi \rangle + h_1\, l(0) + (\hat{v}_0, \phi(0))_{L^2(\tilde{\Omega}(0))} \tag{W}$$

for every $(\phi, l) \in C^1([0, T]; \mathbb{V}_h)$ such that $\phi(\cdot, T) = l(T) = 0$.

Definition

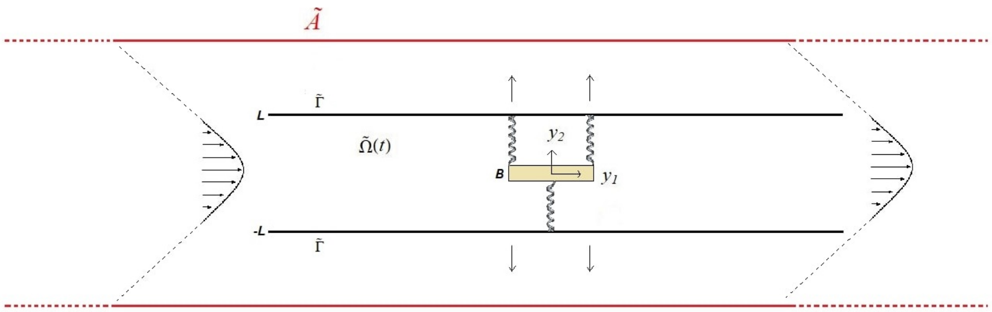
A couple (v, h) is called a weak solution of (F')-(S') if, given $\hat{v} = v - a$, where a is the solenoidal extension of the Poiseuille flow:

$$h \in W^{1,\infty}(0, T; \mathbb{R}),$$

$$(\hat{v}, h') \in L^2(0, T; \mathbb{V}_h) \cap L^\infty(0, T; \mathbb{H}_h),$$

$$(\hat{v}, h) \text{ satisfies (W) for every } (\phi, l) \in C^1([0, T]; \mathbb{V}_h) \text{ such that } \phi(\cdot, T) = l(T) = 0.$$

A penalized problem



PP:
Let $n \geq 1$ be fixed. Find $(\hat{v}, h') \in L^2(0,T;\mathbb{V}(\tilde{A})) \cap L^\infty(0,T;\mathbb{H}(\tilde{A})), h \in W^{1,\infty}(0,T;\mathbb{R})$ satisfying

$$-\int_0^T \{(\hat{v}, \phi_t)_{L^2(\tilde{A}\setminus B)} + h' \, l' - f(h) \, l\} + \mu \int_0^T (\nabla \hat{v}, \nabla \phi)_{L^2(\tilde{A}\setminus B)} + \int_0^T \{\psi(\hat{v}, \hat{v}, \phi) + \psi(\hat{v}, a, \phi) +$$

$$\psi(a, \hat{v}, \phi) - \psi(h'\hat{e}_2, a, \phi) - \psi(h'\hat{e}_2, \hat{v}, \phi)\} + n \int_0^T (\chi_{E_h} \hat{v}, \phi)_{L^2(\tilde{A})} =$$

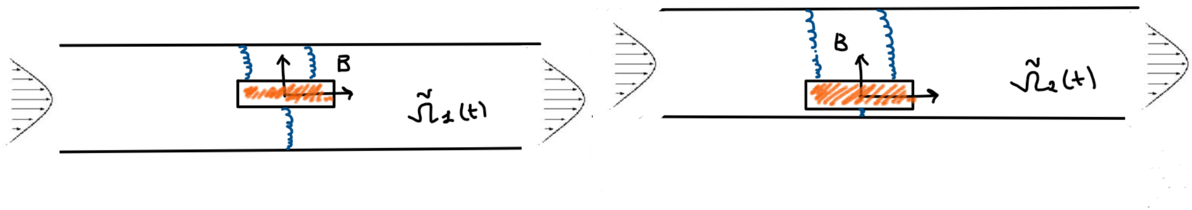
$$\int_0^T \langle \hat{g}, \phi \rangle + h_1 \, l(0) + (\hat{v}_0, \phi(0))_{L^2(\tilde{A}\setminus B)} \quad \forall \, (\phi, l) \in C^1([0,T], \mathbb{V}(\tilde{A})) \text{ s. t. } \phi(\cdot, T) = l(T) = 0.$$

How do we exploit the penalized problem PP?

1. Prove existence of solutions to PP
2. Prove an energy estimate
3. Let $n \to \infty$

Proof of the main result. Uniqueness

Difficulty? Let us consider two weak solutions of problem (F')-(S'), (v_1, h_1) and (v_2, h_2).



We introduce

$$\hat{v}_1 = v_1 - a_{h_1}, \qquad \hat{v}_2 = v_2 - a_{h_2}$$

Glass, O. and Sueur, F. *Uniqueness Results for Weak Solutions of Two-Dimensional Fluid-Solid Systems.* Arch. Rational Mech. Anal., 2015.

Idea? To build

$$\psi_t : \tilde{\Omega}_2(t) \to \tilde{\Omega}_1(t), \qquad \varphi_t = \psi_t^{-1} : \tilde{\Omega}_1(t) \to \tilde{\Omega}_2(t)$$

and to define the pullback of $\hat{v}_2, a_{h_2}$ by such map, $\hat{\mathfrak{v}}_2, \mathfrak{a}_2$. For any given $y = (y_1, y_2) \in \tilde{\Omega}_1(t)$:

$$\hat{\mathfrak{v}}_2 = \nabla\psi_t(y) \cdot \hat{v}_2(t, \varphi_t(y))$$
$$\mathfrak{a}_2 = \nabla\psi_t(y) \cdot a_{h_2}(\varphi_t(y)).$$

Then, one can define

$$w := \hat{v}_1 - \hat{\mathfrak{v}}_2, \qquad \hat{h} := h_1 - h_2.$$


Grazie per la vostra attenzione