Constant rank differential operators and homogenization

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Constant rank differential operators

Given $k, N, M \in \mathbb{N} \setminus \{0\}$ and linear maps $A^{(i)}\colon \mathbb{R}^N \to \mathbb{R}^M$ for all d-dimensional multi-indices i with $|i| = k$, consider

$$\mathscr{A} u := \sum_{|i|=k} A^{(i)} \partial_i u \quad \text{with } u\colon \mathbb{R}^N \to \mathbb{R}^M.$$

It is a linear, k-th order, homogeneous differential operators with constant coefficients.
The crucial requirement is: $\mathscr{A}$ is of constant rank in the sense of F. Murat, i.e. there exists $r \in \mathbb{N}$ such that the rank of $\mathbb{A}[\omega]$ equals r for all $\omega \in \mathbb{R} \setminus \{0\}$, where $\mathbb{A}$ is the symbol of $\mathscr{A}$.

Examples

Divergence, curl, curl of the curl, magnetostatic equations, 'higher order gradients', ...

This class of operators was used in a variational framework by I. Fonseca & S. Müller to introduce a generalization of the notion of quasiconvexity.

$\mathscr{A}$-free maps

$\mathscr{A}$ may be regarded as an operator from $L^p(\Omega;\mathbb{R}^N)$ to $W^{-k,p}(\Omega;\mathbb{R}^M)$: for $u \in L^p(O;\mathbb{R}^N)$ we define the pairing

$$\langle \mathscr{A}u, v\rangle := \int_\Omega u \cdot \mathscr{A}^* v \,\mathrm{d}x \quad \text{for all } v \in W_0^{k,p'}(\Omega;\mathbb{R}^M),$$

with $\mathscr{A}^*$ the formal adjoint of $\mathscr{A}$.
In particular, we say that $u \in L^p(\Omega;\mathbb{R}^N)$ is $\mathscr{A}$-free if $\mathscr{A}u = 0$ in $W^{-k,p}(\Omega;\mathbb{R}^M)$.

Our questions

Given u that is $\mathscr{A}$-free on Ω,

1. can we find a second differential operator $\mathscr{B}$ and $w \in W^{k,p}(\Omega;\mathbb{R}^M)$ such that $u = \mathscr{B}w$?
2. can we extend u to the whole space in such a way that the extension is still $\mathscr{A}$-free?

Existence of potentials for $\mathscr{A}$-free maps, I

Q: given an $\mathscr{A}$-free u on Ω, are there a differential operator $\mathscr{B}$ and $w \in W^{k,p}(\Omega;\mathbb{R}^M)$ such that $u = \mathscr{B}w$?
A: yes, if we can extend u to the whole space, i.e. if our second question has a positive answer.

Theorem (Davoli, Kružík, & P., in preparation)

Let $\mathscr{A}$ be a linear, k-th order, homogeneous differential operator with constant coefficients and constant rank. If $\Omega \subset \mathbb{R}^d$ is a bounded, connected, open set with Lipschitz boundary which is also an $\mathscr{A}$-extension domain, then there exists a differential operator $\mathscr{B}$ of order ℓ satisfying the following: for all $\mathscr{A}$-free maps $u \in L^p(\Omega;\mathbb{R}^N)$ there is a function $w \in W^{\ell,p}(\Omega;\mathbb{R}^M)$ such that $u = \mathscr{B}w$ almost everywhere in Ω.

Existence of potentials for $\mathscr{A}$-free maps, II

The result relies heavily on some properties of constant rank operators that have been recently investigated by B. Raiţă and co-authors:

- B. Raiţă: if $\mathscr{A}$ is as above, there exists $\mathscr{B}$ of constant rank such that for all $\mathscr{A}$-free $u \in \mathscr{S}(\mathbb{R}^d;\mathbb{R}^N)$ there exists $w \in \mathscr{S}(\mathbb{R}^d;\mathbb{R}^M)$ such that $u = \mathscr{B}w$.
- A. Guerra & B. Raiţă: $\mathscr{B}$ is of constant rank if and only if

$$\|\nabla^\ell(\phi - \Pi_{\mathscr{B}}\phi)\|_{L^p(\mathbb{R}^d;\mathbb{R}^{N\times d^\ell})} \leq c\|\mathscr{B}\phi\|_{L^p(\mathbb{R}^d;\mathbb{R}^N)} \quad \text{for all } \phi \in C_c^\infty(\mathbb{R}^d;\mathbb{R}^M).$$

Outline of the proof of our result:

1. The assumptions yield $\tilde{u}$ $\mathscr{A}$-free extension of u to the whole space.
2. Approximate $\tilde{u}$ by smooth functions and use the existence of potentials for smooth functions.
3. Use the Korn inequality on $\mathscr{B}$ to get some compactness on the sequence of smooth potentials.

Existence of $\mathscr{A}$-free extensions

Q: given an $\mathscr{A}$-free u on Ω, can we extend u to the whole space in such a way that the extension is still $\mathscr{A}$-free?

A: yes, if there is a potential $\mathscr{B}$ for $\mathscr{A}$ on Ω for which a Korn-type inequality holds, i.e. if our first question has a positive answer and $\mathscr{B}$ fulfils (1) below.

Theorem (Davoli, Kružík, & P., in preparation)

Let Ω be an open set with bounded Lipschitz boundary and $\mathscr{A}$ be as before. Let also $\mathscr{B}$ be a linear, ℓ-th order, homogeneous differential operator with constant coefficients that is a potential for $\mathscr{A}$ on Ω and that satisfies

$$\|\nabla^\ell (w - \Pi_{\mathscr{B}} w)\|_{L^p(\Omega;\mathbb{R}^{N\times d^\ell})} \leq c\|\mathscr{B}w\|_{L^p(\Omega;\mathbb{R}^M)} \tag{1}$$

for all $w \in W^{\ell,p}(\Omega;\mathbb{R}^N)$. Then, there exist a linear extension operator $\mathrm{E}_{\mathscr{A}} : L^p(\Omega;\mathbb{R}^N) \to L^p(\mathbb{R}^d;\mathbb{R}^N)$ and a constant $c := c(d,p,\mathscr{A},\Omega)$ such that, for all $\mathscr{A}$-free $u \in L^p(\Omega;\mathbb{R}^N)$

1. $\mathrm{E}_{\mathscr{A}} u = u$ *a.e. in* Ω,
2. $\|\mathrm{E}_{\mathscr{A}} u\|_{L^p(\mathbb{R}^d;\mathbb{R}^N)} \leq c\|u\|_{L^p(\Omega;\mathbb{R}^N)}$, *and*
3. $\mathscr{A}(\mathrm{E}_{\mathscr{A}} u) = 0$ *on* $\mathbb{R}^d$.

Existence of $\mathscr{A}$-free extensions, II

The requirement that $\mathscr{A}$-free maps on Ω admit potentials allows us to recast the problem in terms of extensions of Sobolev maps.

Outline of the proof of our result: By assumption, for $u \in L^p(\Omega;\mathbb{R}^N)$ such that $\mathscr{A}u = 0$, there exists $w \in W^{\ell,p}(D;\mathbb{R}^M)$ satisfying $u = \mathscr{B}w$. Hence, we may set

$$\mathrm{E}_{\mathscr{A}}u := \mathscr{B}\big(\mathrm{E}(w - \Pi_{\mathscr{B}}w)\big).$$

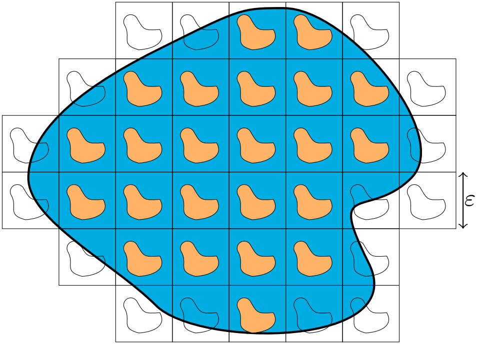
Here, E is the standard extension operator for Sobolev maps.

The estimate on the norm of $\mathrm{E}_{\mathscr{A}}u$ follows from the Korn inequality for $\mathscr{B}$ and from the properties of E.

Homogenization under differential constraints

Asymptotic analysis of a high-contrast composite: Ω = reference configuration, $\Omega_{0,\varepsilon}$ = 'soft' inclusions, $\Omega_{1,\varepsilon}$ = 'stiff' matrix. For $u\colon \mathbb{R}^d \supset \Omega \to \mathbb{R}^N$ $\mathscr{A}$-free

$$\mathscr{F}_\varepsilon(u) := \int_{\Omega_{0,\varepsilon}} f_{0,\varepsilon}(\varepsilon u)\,\mathrm{d}x + \int_{\Omega_{1,\varepsilon}} f_1\left(\frac{x}{\varepsilon}, u\right)\mathrm{d}x.$$



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