Optimal stability estimates for advection-diffusion equations

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Objectives

This work presents two main objectives:

1. To develop optimal stability estimates for advection-diffusion equations with Sobolev regular velocity fields.
2. To extend the estimates to advection-diffusion equations with velocity fields whose gradients are singular integrals of L^1 functions, and entail a new well-posedness result.

Introduction

Consider a scalar parameter $\theta : (0,T)\times\mathbb{R}^d \to \mathbb{R}$ driven by a vector field $u:(0,T)\times\mathbb{R}^d\to\mathbb{R}^d$. The Cauchy problem for the advection-diffusion equation is the following

$$\begin{cases} \partial_t\theta + \nabla\cdot(u\theta) = \kappa\Delta\theta, \\ \theta(0,\cdot) = \theta^0, \end{cases} \tag{1}$$

in $(0,T)\times\mathbb{R}^d$ for a given initial datum θ^0 and with diffusion coefficient $\kappa > 0$. There are plenty of physical phenomena modelled with this equation, including drift-diffusion processes in semiconductor physics, heat transmission through a fluid layer, or mixing by stirring in industrial applications.

The mathematical theory in the setting of smooth vector fields is contained in the general classical theory for parabolic equations. For rough vector fields there are two main approaches: the Ladyzenskaya-Prodi-Serrin setting (ceases to hold when $\kappa = 0$, so it is not so convenient for this case) and the DiPerna-Lions setting (see block 1). In addition, it is also interesting to study the case when the gradient of u is a singular integral of an L^1 function, for instance, when studying the 2D or 3D Navier-Stokes equation this implies to work with merely integrable vorticity $\omega = \nabla\times u$.

This work extends the theory for the transport equation in [1] and [3] to the diffusive case. All the results here are developed in [2].

Kantorovich–Rubinstein distances

Consider the transport of nonnegative densities $\mu_1, \mu_2 \in L^1_+(\mathbb{R}^d)$. $\Pi(\mu_1,\mu_2)$ denotes the set of all transport plans between μ_1 and μ_2 with $\mu_1[\mathbb{R}^d] = \mu_2[\mathbb{R}^d] \in \mathbb{R}$. Consider a continuous nondecreasing cost function $c:[0,\infty)\to[0,\infty)$. The optimal transportation problem can be stated as a minimization problem (left) or on its dual representation maximizing over a set of functions (right):

$$\mathcal{D}_c(\mu_1,\mu_1) = \inf_{\pi\in\Pi(\mu_1,\mu_2)} \iint_{\mathbb{R}^d\times\mathbb{R}^d} c(|x-y|)d\pi(x,y) \equiv \sup_{|\zeta(x)-\zeta(y)|\le c(|x-y|)} \int_{\mathbb{R}^d} \zeta(x)d(\mu_1-\mu_2)(x)$$

Since $\mathcal{D}_c(\cdot,\cdot)$ only depends on the difference $\mu_1-\mu_2$ we can consider densities of same mass instead of nonnegative. Moreover $\mathcal{D}_c(\cdot)$ is a norm on the space of mean zero densities:

$$\mathcal{D}_c(\mu) := \mathcal{D}_c(\mu,0) = \mathcal{D}_c(\mu^+,\mu^-), \quad \text{if } \int_{\mathbb{R}^d}\mu(x)dx = 0.$$

Those distances are particularly interesting because they metrize weak convergence. Stability estimates will be given by means of the distance associated to the concave cost $c(z) = \log(z/\delta+1)$ with $\delta>0$.

1. DiPerna-Lions setting

Let $1<p\le\infty$ and then consider an advection field

$$u\in L^1((0,T);W^{1,p}(\mathbb{R}^d)),$$
$$(\nabla\cdot u)^- \in L^\infty((0,T)\times\mathbb{R}^d).$$

Let the initial datum $\theta^0\in(L^1\cap L^q)(\mathbb{R}^d)$ with $1/p + 1/q \le 1$ and with finite first moments,

$$\int_{\mathbb{R}^d}|x||\theta^0(x)|dx < \infty.$$

Then there is a unique solution of the advection-diffusion equation (1),

$$\theta\in L^\infty((0,T);(L^1\cap L^q)(\mathbb{R}^d)),$$
$$\nabla\theta\in L^1((0,T)\times\mathbb{R}^d).$$

2. Integral kernels

Let $u = k*\omega$ where $\omega\in L^1((0,T)\times\mathbb{R}^d)$ and k a nonsingular kernel:

- $k\in\mathcal{S}'(\mathbb{R}^d)$;
- $k|_{\mathbb{R}^d\setminus\{0\}} \in C^2(\mathbb{R}^d\setminus\{0\})$;
- $|D^\alpha k(x)| \lesssim |x|^{-d+1-|\alpha|}$, for $\alpha\in\mathbb{N}_0^d$, $|\alpha|\le 2$ and for all $x\neq 0$;
- $\left|\int_{R_1<|x|<R_2}\nabla k(x)dx\right| \lesssim 1$ for all $0<R_1<R_2<\infty$.

Then ∇u is a singular integral of ω and

$$u\in L^{p,\infty}((0,T)\times\mathbb{R}^d)$$

for $p = d/(d-1) > 0$.

Stability estimates

1. *u is in the DiPerna-Lions setting*:
If θ_1, θ_2 are two solutions of (1) with u_1, κ_1 and u_2, κ_2 respectively, then

$$\sup_{t\in(0,T)} \mathcal{D}_\delta(\theta_1,\theta_2) \lesssim 1 + \mathcal{D}_\delta(\theta_1^0,\theta_2^0) + \frac{\|u_1-u_2\|_{L^1(L^p)} + |\kappa_1-\kappa_2|}{\delta} \tag{2}$$

2. *u is such that its gradient is a singular integral of an L^1 function*:
Let $\theta\in L^\infty((0,T);(L^1\cap L^\infty)(\mathbb{R}^d))$, $\theta\not\equiv 0$ be a solution of (1), then $\forall\varepsilon>0$

$$\sup_{0\le t\le T}\mathcal{D}_\delta(\theta) \lesssim \mathcal{D}_\delta(\theta^0) + \varepsilon\|\theta\|_{L^1}\left[1+\log\left(\frac{1}{\varepsilon\delta}\left(\frac{\|\theta\|_{L^1}}{\|\theta\|_{L^\infty}}\right)^{1-\frac{1}{p}}\|u\|_{L^{p,\infty}}\right)\right] + C_\varepsilon\|\theta\|_{L^\infty(L^2)}. \tag{3}$$

A consequence

Theorem
Under the following assumptions

- *$u = k*\omega$ with $\omega\in L^1((0,T)\times\mathbb{R}^d)$ and k nonsingular integral,*
- *$(\nabla\cdot u)^-\in L^\infty((0,T)\times\mathbb{R}^d)$,*
- *$\theta^0\in(L^1\cap L^\infty)(\mathbb{R}^d)$,*

there is a unique distributional solution

$$\theta\in L^\infty((0,T);(L^1\cap L^\infty)(\mathbb{R}^d))$$

to the advection-diffusion equation (1).

Uniqueness: The novel result here concerns uniqueness for equation (1). For it we only need to assume $\nabla u = K*\omega$ we K is a singular integral kernel. Then the result comes as a direct consequence of the stability estimate (3).

Optimality: The stability estimate (2) is shown to be optimal for both the distance of the velocity fields $\|u_1-u_2\|_{L^1(L^p)}$ and the difference of diffusion coefficients $|\kappa_1-\kappa_2|$.

Main references

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