Regularity and existence of solutions for a Neumann p-Laplacian problem in the metric setting

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Overview of the results obtained

- Existence of a solution and a weaker uniqueness property.
- Minimizers of the Neumann p-Laplacian problem satisfy a De Giorgi type inequality and consequently we give boundedness properties for them.

Hypotheses on the metric measure space X

Let (X, d, μ) be a metric measure space, where μ is a Borel regular measure. Let $B(x, \rho) \subset X$ be a ball with the center $x \in X$ and the radius $\rho > 0$.

- Doubling measure. A measure μ on X is said to be doubling if there exists a constant K, called the doubling constant, such that
$$0 < \mu(B(x, 2\rho)) \le K\mu(B(x, \rho)) < +\infty,$$
for all $x \in X$ and $\rho > 0$.
- $(1,p)$-Poincaré inequality. Let $p \in [1, +\infty[$. A metric measure space X supports a $(1,p)$-Poincaré inequality if there exist $K > 0$ and $\lambda \ge 1$ such that
$$\frac{1}{\mu(B(x,r))}\int_{B(x,r)} |u - u_{B(x,r)}| d\mu \le Kr\left(\frac{1}{\mu(B(x,\lambda r))}\int_{B(x,\lambda r)} g_u^p\, d\mu\right)^{\frac{1}{p}}$$
for all $B(x,r) \subset X$, $u \in L^1_{loc}(X)$, where $u_{B(x,r)} = \frac{1}{\mu(B(x,r))}\int_{B(x,r)} u d\mu$.

Hypotheses on $\Omega \subset X$

(H_1) There exists a constant $K \ge 1$ such that for all $y \in \Omega$ and $0 < \rho \le \operatorname{diam}(\Omega)$, we have
$$\mu(B(y,\rho) \cap \Omega) \ge \frac{1}{K}\mu(B(y,\rho)).$$

(H_2) (Ahlfors codimension 1 regularity of P_Ω) For all $y \in \partial\Omega$ we have that
$$\frac{1}{K\rho}\mu(B(y,\rho)) \le P_\Omega(B(y,\rho)) \le \frac{K}{\rho}\mu(B(y,\rho)),$$
where K and ρ are as in (H_1).

(H_3) $(\Omega, d_{|\Omega}, \mu_{|\Omega})$ admits a $(1,p)$-Poincaré inequality with $\lambda = 1$, where $p \in]1, +\infty[$.

Upper gradients

- A non negative Borel measurable function g is said to be an upper gradient of function $u : X \to [-\infty, +\infty]$ if, for all compact rectifiable arc length parametrized paths γ connecting x and y, we have
$$|u(x) - u(y)| \le \int_\gamma g\, ds \qquad (1)$$
whenever $u(x)$ and $u(y)$ are both finite and $\int_\gamma g\, ds = +\infty$ otherwise.

The minimal p-weak upper gradient

- Let $p \in [1, +\infty[$. Let Γ be a family of paths in X. We say that
$$\inf_\phi \int_X \phi^p d\mu$$
is the p-modulus of Γ, where the infimum is taken among all non negative Borel measurable functions ϕ satisfying $\int_\gamma \phi\, ds \ge 1$, for all rectifiable paths $\gamma \in \Gamma$.
- If (1) is satisfied for p-almost all paths γ in X, that is the set of non constant paths that do not satisfy (1) is of zero p-modulus, then g is said a p-weak upper gradient of u.
- The family of weak upper gradients has a minimal element g_u, that is called the minimal p-weak upper gradient of u.

The Newtonian space

The Newtonian space $N^{1,p}(X)$ is defined by
$$N^{1,p}(X) = V^{1,p}(X) \cap L^p(X), \quad p \in [1, +\infty[,$$
where $V^{1,p}(X) = \{u : u \text{ is measurable and } g_u \in L^p(X)\}$. We consider $N^{1,p}(X)$ equipped with the norm
$$\|u\|_{N^{1,p}(X)} = \|g_u\|_{L^p(X)} + \|u\|_{L^p(X)}.$$
We denote with $N^{1,p}_*(X) = \{u \in N^{1,p}(X) : \int_X u\, dx = 0\}$.

The problem

Let X be a complete metric space equipped with a doubling measure supporting a $(1,p)$-Poincaré inequality $(1 < p < +\infty)$.
Given a Neumann boundary value problem with boundary data $f \neq 0$ and reaction term G, we consider the following functional
$$J(u) = \int_\Omega g_u^p d\mu - \int_\Omega G(u) d\mu + \int_{\partial\Omega} Tu\, f\, dP_\Omega \quad \text{for all } u \in N^{1,p}(\Omega). \qquad (2)$$
where

- Ω is a bounded domain (non empty, connected open set) in X with $X \setminus \Omega$ of positive measure such that Ω is of finite perimeter with perimeter measure P_Ω;
- $G : \Omega \to \mathbb{R}$ is defined as
$$G(u) = c - |u|^\gamma \text{ for all } u \in N^{1,p}(\Omega), \qquad (3)$$
for some $c > 0$ and $1 < \gamma < p^* = \frac{ps}{s-p}$ if $p < s$, $1 < \gamma < +\infty$ else;
- $f : \partial\Omega \to \mathbb{R}$ is a bounded P_Ω-measurable function with $\int_{\partial\Omega} f dP_\Omega = 0$.

Solution to the Neumann boundary value problem

A function $u_0 \in N^{1,p}_*(\Omega)$ is a p-harmonic solution to the Neumann boundary value problem with boundary data $f \neq 0$ and reaction term G if
$$J(u_0) = \int_\Omega g_{u_0}^p d\mu - \int_\Omega G(u_0) d\mu + \int_{\partial\Omega} Tu_0 f dP_\Omega$$
$$\le \int_\Omega g_v^p d\mu - \int_\Omega G(v) d\mu + \int_{\partial\Omega} Tv f dP_\Omega = J(v)$$
for every $v \in N^{1,p}_*(\Omega)$, where g_{u_0}, g_v are the minimal p-weak upper gradients of u_0 and v in Ω, respectively, and Tu_0 and Tv are the traces of u_0 and v on $\partial\Omega$, respectively.

Main results

Existence and a weaker uniqueness result

- J has a minimizer in $N^{1,p}_*(\Omega)$.
- If $u_1, u_2 \in N^{1,p}_*(\Omega)$ are two minimizers of J, then $g_{u_1} = g_{u_2}$ a.e. in Ω.

Boundedness result

Let $0 < R < \frac{\operatorname{diam}(\Omega)}{4}$ and $\Omega_R = \{y \in \Omega : d(y, \partial\Omega) < \frac{R}{2}\}$. If $u \in N^{1,p}_*(\Omega)$ is a minimizer of J and $f \in L^\infty(\partial\Omega)$, then $u \in L^\infty(\Omega_R)$ and $Tu \in L^\infty(\partial\Omega_R)$.

Proof methods based on a De Giorgi type inequality

Let $u \in N^{1,p}_*(\Omega)$ be a minimizer of J and $f \in L^\infty(\partial\Omega)$. If $y \in \partial\Omega$, $0 < \rho < R < \frac{\operatorname{diam}(\Omega)}{10}$ and $\alpha \in \mathbb{R}$, then there is $K \ge 1$ such that the following De Giorgi type inequality
$$\int_{\Omega\cap B(y,\rho)} g^p_{(u-\alpha)_+} d\mu \le \frac{K}{(R-\rho)^p}\int_{\Omega\cap B(y,R)} (u-\alpha)^p_+ d\mu + K\int_{\partial\Omega\cap B(y,R)} |f|(u-\alpha)^p_+ dP_\Omega \qquad (4)$$
is satisfied.

Some references

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Other results and forthcoming research

- Neumann p-Laplacian problem with zero boundary data;
- Extending the results to the (p,q)- Laplacian problem in the metric setting.