Part V : Equilibrium patterns of dislocations

(w M. G. Mora & L. Rondi ; & J.A. Carrillo,
J. Mateu , J. Verdera)

$$\underline{I(\mu)} = \iint_{\mathbb{R}^2\times\mathbb{R}^2} V(x-y)\,d\mu(x)\,d\mu(y) + \int_{\mathbb{R}^2} |x|^2 d\mu(x)$$

$$V(x) = \underbrace{-\log|x|}_{\text{coulomb}} + \underbrace{\frac{x_1^2}{|x|^2}}_{\text{anisotropy}}$$

min of $I(\mu)$ on $P(\mathbb{R}^2)$: equilibrium pattern at mesoscale

So FAR: $\exists !$ min with cp support !

↳ is it a wall ?
LAGB

$$I(\mu) = \iint_{\mathbb{R}^2\times\mathbb{R}^2} V(x-y)\,d\mu(x)\,d\mu(y) + \int_{\mathbb{R}^2} |x|^2\,d\mu(x)$$

$$V(x) = -\log|x| + \frac{x_1^2}{|x|^2}$$

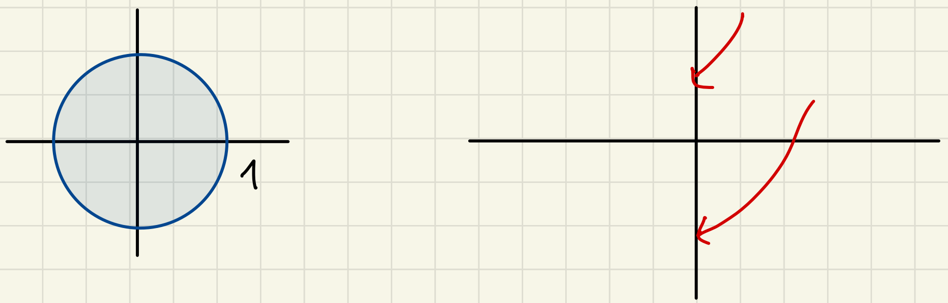
Coulomb anisotropy

Note

$$I(\mu) = I_c(\mu) + \iint_{\mathbb{R}^2\times\mathbb{R}^2} \frac{(x_1-y_1)^2}{|x-y|^2}\,d\mu(x)\,d\mu(y)$$

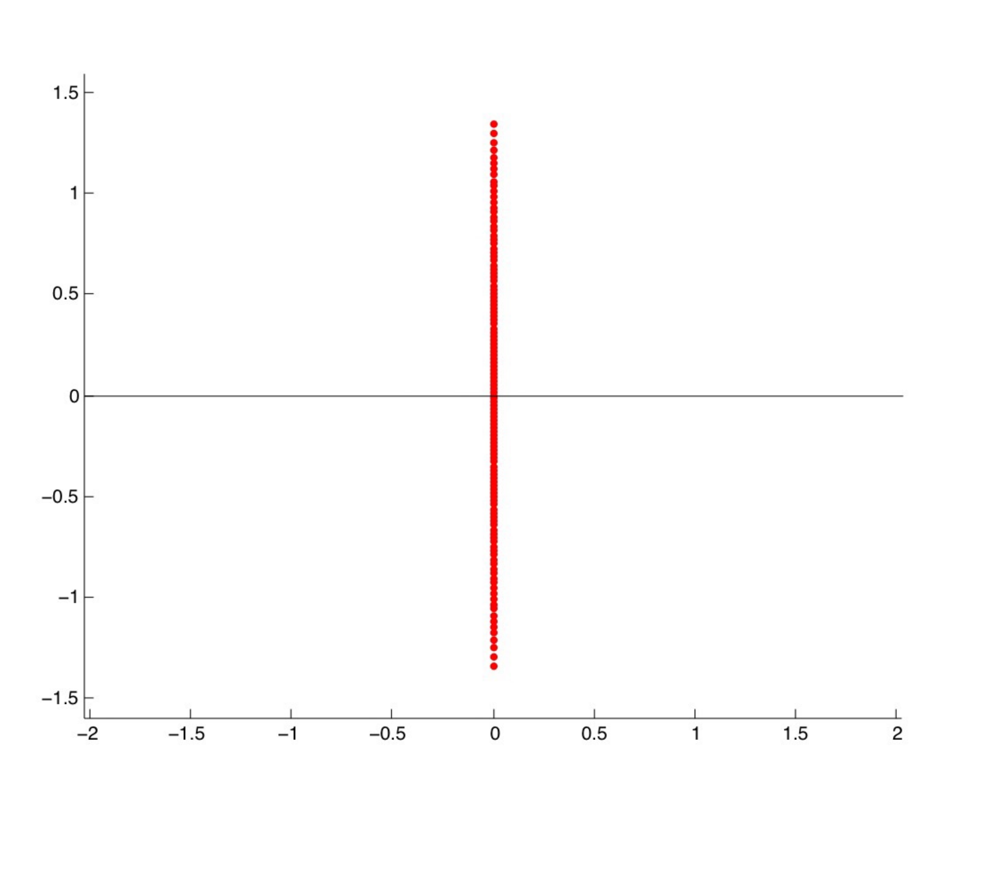
$\mu \in \mathcal{P}(\mathbb{R}^2)$

unique min
circle law



Combined effect?

What do numerics say?



1 d, vertical support. True?

↪ If true, then what?

$$\underline{I(\mu)} = \iint_{\mathbb{R}^2\times\mathbb{R}^2} V(x-y)\,d\mu(x)\,d\mu(y) + \int_{\mathbb{R}^2} |x|^2 d\mu(x)$$

$$V(x) = -\log|x| + \frac{x_1^2}{|x|^2}$$

- If $\underline{\mu = \delta_0(x_1) \otimes \mu_V(x_2)}$

$*$ anisotropic term $= 0 \Rightarrow I(\mu) = I_c(\mu)$

$*$ $$\underbrace{I_c(\mu)}_{= I_{\log}(\mu_V)} = \iint_{\mathbb{R}\times\mathbb{R}} -\log|x-y|\,d\mu_V(x)\,d\mu_V(y) + \int_{\mathbb{R}} |x|^2 d\mu_V(x)$$

$$\Rightarrow I(\mu) = \underbrace{I_{\log}(\mu_V)}_{\text{unique min : semi-circle law}}$$ Log-gas energy

unique min : semi-circle law (Wigner, Mehta)

$$\mu_V(x) = \frac{1}{\pi}\sqrt{2-x^2}\,dx \llcorner_{(-\sqrt{2},\sqrt{2})}$$

- If $\mu = \delta_0(x_1) \otimes \mu_V(x_2)$

$\Rightarrow \mu =$ S.C. law !

Theorem (w Mora & Rondi)

The S.C. law $\mu = \frac{1}{\pi} \delta_0(x_1) \otimes \sqrt{2 - x_2^2}\, dx_2 \llcorner (-\sqrt{2}, \sqrt{2})$

satisfies the E-L conditions

$$\begin{cases} (V * \mu)(x) + \dfrac{|x|^2}{2} = c \quad \text{on supp}\,\mu & \text{(EL1)} \\ (V * \mu)(x) + \dfrac{|x|^2}{2} \geq c \quad \text{in } \mathbb{R}^2 & \text{(EL2)} \end{cases}$$

$$c = I(\mu) - \frac{1}{2} \int_{\mathbb{R}^2} |x|^2 \, d\mu(x)$$

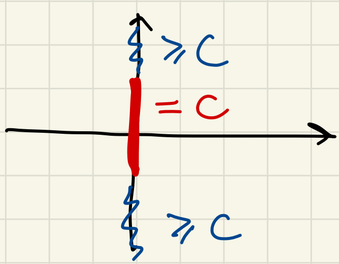
and hence is the unique min of I.

Sketch: On $x_1 = 0$

$$(V * \mu)(0, x_2) = (-\log|\cdot| * \mu)(0, x_2)$$ Log in 1d

$$\begin{cases} \text{(EL1) by Wigner } \checkmark \\ \text{(EL2) on } x_1 = 0 \text{ by Wigner } \checkmark \end{cases}$$

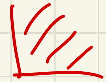
(EL2) on $\mathbb{R}^2$?



strategy:

$$x_1 \longmapsto (V * \mu)(x) + \frac{|x|^2}{2}$$

minimal at $x_1 = 0$.

Show: $\partial_{x_1}\left(V * \mu + \frac{|x|^2}{2} \right) \geq 0$ 

Comments :

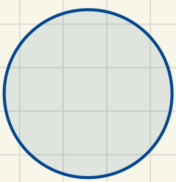
$$\underline{I(\mu)} = \iint_{\mathbb{R}^2\times\mathbb{R}^2} V(x-y)\,d\mu(x)\,d\mu(y) + \int_{\mathbb{R}^2} |x|^2\,d\mu(x)$$

Coulomb

$$V(x) = -\cancel{\log}|x| + \frac{x_1^2}{|x|^2} \rightarrow \frac{x_1^2}{|x|^{\square}} \text{ - dim}$$

- min of I is wall-like → conjecture !
- dramatic effect of anisotropy on min !

I_c



I

$\alpha < 0$

$$|\alpha| \frac{x_2^2}{|x|^2} + C$$

- tune the anisotropy. → weight

$$V_\alpha(x) = -\log|x| + \alpha \frac{x_1^2}{|x|^2}$$

$\underline{\alpha = 0}$: circle law

$\underline{\alpha = 1}$: s.c. law → μ_α

$\alpha \in (0,1)$

<u>Note</u> : for $\alpha > 1$ the s.c. law is still the unique min, since

s.c. unique min of I_1

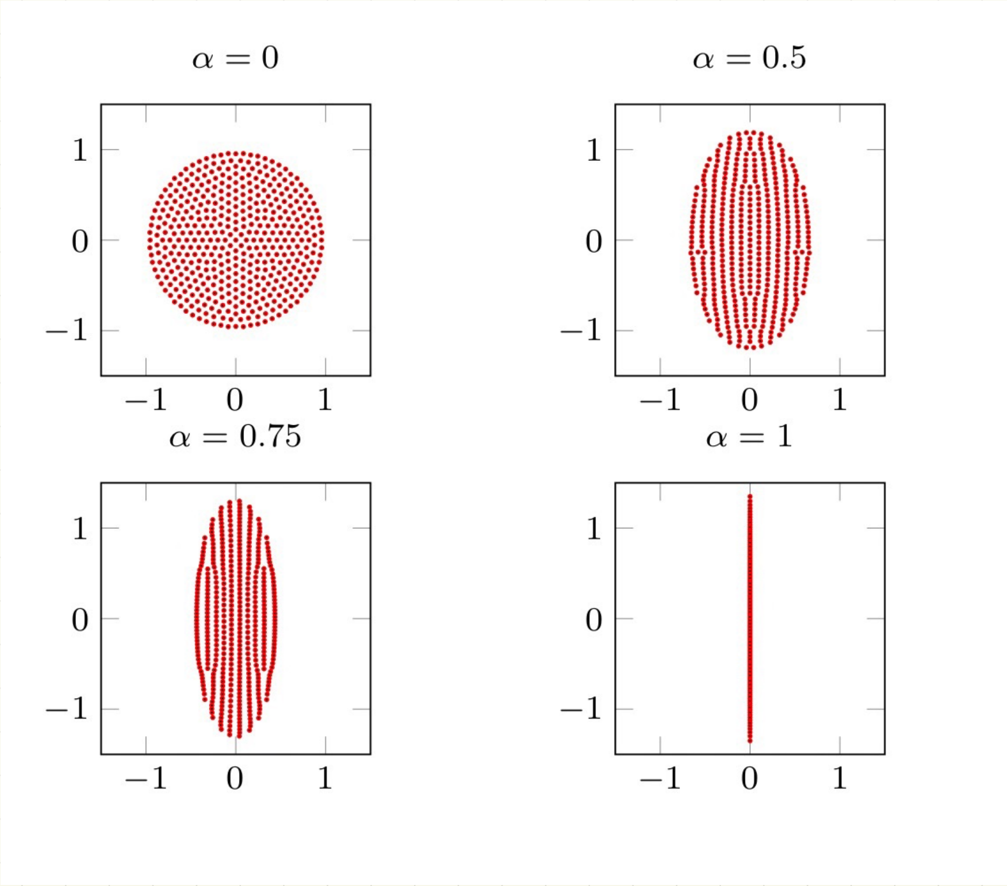
$$I_\alpha(\text{s.c.}) = I_1(\text{s.c.}) < I_1(\mu) \quad \mu \neq \text{s.c.}$$

since anisotropy = 0

$$\leq I_\alpha(\mu) \quad \mu \neq \text{s.c.}$$

since $\alpha > 1$ and anisotropy ≥ 0.

what do numerics say ?



Is μ_α an ellipse ??

Theorem (w Mora & Rondi, Carrillo, Mateu, Verdera)

The "ellipse law"

$$\mu_\alpha = \frac{1}{\pi\sqrt{1-\alpha^2}} \chi_{\Omega_\alpha}, \quad \Omega_\alpha \text{ ellipse semi-axes } \sqrt{1-\alpha}, \ \sqrt{1+\alpha}, \ \alpha \in (0,1)$$

satisfies the E-L conditions

$$\begin{cases} (V_\alpha * \mu_\alpha)(x) + \frac{|x|^2}{2} = C_\alpha \text{ on } \Omega_\alpha & \text{(EL1)} \\ (V_\alpha * \mu_\alpha)(x) + \frac{|x|^2}{2} \geq C_\alpha \text{ in } \mathbb{R}^2 & \text{(EL2)} \end{cases}$$

Sketch.

- EL1 by explicitly computing $V_\alpha * \mu_{\text{ellipse}}$

$$V_\alpha = -\log|\cdot| + \alpha \frac{x_1^2}{|x|^2}$$

- for EL2

$$P_\alpha(x) := (V_\alpha * \mu_\alpha)(x) + \frac{|x|^2}{2}$$

By EL1: $P_\alpha = C_\alpha$ on Ω_α

Want : $P_\alpha \geq c_\alpha$ in $\mathbb{R}^2 \setminus \Omega_\alpha$

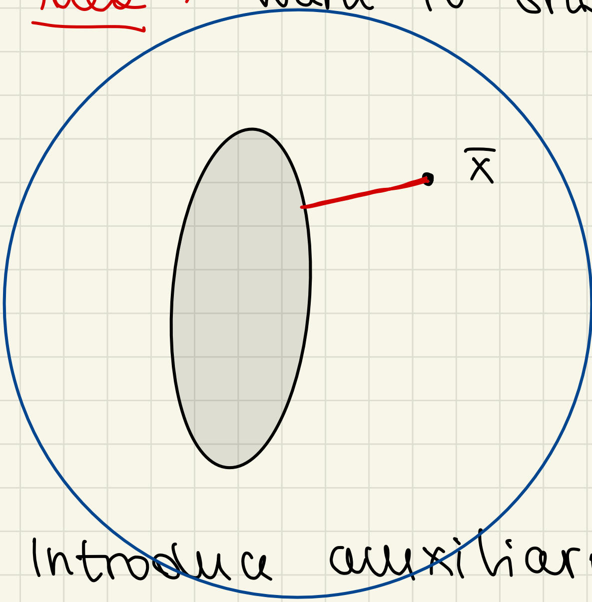
idea: P_α minimised on Ω_α

can we use a max-principle argument?

Issue : $\Delta^2 P_\alpha = 0$ outside Ω_α

biharmonic

Idea : want to show : $\partial_\nu P_\alpha(\bar{x}) \geq 0$ outside Ω_α



Introduce auxiliary function g_α:

- g_α harmonic outside Ω_α
- $g_\alpha(\bar{x}) = \partial_\nu P_\alpha(\bar{x})$
- $g_\alpha \geq 0$ on the boundary of domain