Part IV: Equilibrium patterns of dislocations

(w M. G. Mora & L. Rondi; & J.A. Carrillo, J. Mateu, J. Verdera)

$$I_n(x_1, \dots, x_n) = \frac{1}{n^2} \sum_{i,j=1}^{n} V(x_i - x_j) + \frac{1}{n} \sum_{i=1}^{n} F(x_i), \quad x_i \in \mathbb{R}^2$$

- $V(x) = -\log|x| + \dfrac{x_1^2}{|x|^2}$ anisotropic interaction
- F: confinement

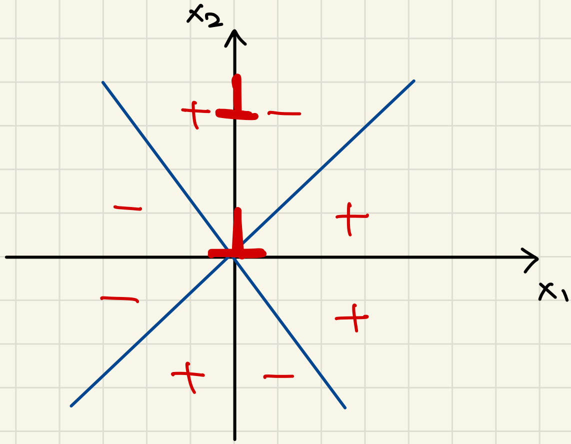
$V(0) = +\infty$
$V(+\infty) = -\infty$ $\rightarrow$ F needed!

* Positive edge dislocations with B.v. e_1.

* Related to Green's function of linear, isotropic elasticity

$$\begin{cases} \operatorname{div} C K = 0 & \mathbb{R}^2 \\ \operatorname{curl} K = e_1 \delta_0 & \mathbb{R}^2 \end{cases} \rightarrow K(x) = \frac{1}{2\pi} e_1 \otimes \frac{x^\perp}{|x|^2} + \nabla v$$

- K: dislocation strain; $\sigma = CK$: stress
- $F_{PK} = (\sigma e_1) \times e_3$
- $F_{PK} \cdot e_1 \cong x_1 \dfrac{(x_1^2 - x_2^2)}{|x|^4}$ glide force



$F_{P_k} \overset{\sim}{=} -\nabla V$

<u>Conjecture</u> : Vertical walls !

Prove it for upscaled energy (Γ-limit)

$$I(\mu) = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} V(x-y)\, d\mu(x)\, d\mu(y) + \int_{\mathbb{R}^2} F(x)\, d\mu(x)$$

$\hookrightarrow$ min of $I(\mu)$ on $\mathcal{P}(\mathbb{R}^2)$: equilibrium pattern at mesoscale

$F(x) = |x|^2$

<u>Dislocation patterns,</u> LEDS, LAGB

Lauteri-Luckhaus ; Fanzon-Palombaro-Ponsiglione

Scala; Fonseca-Morini-Fusco-Leoni; van Meurs ;

De Luca etc.

$$I(\mu) = \iint_{\mathbb{R}^2 \times \mathbb{R}^2} V(x-y)\,d\mu(x)\,d\mu(y) + \int_{\mathbb{R}^2} |x|^2\,d\mu(x)$$

$$V(x) = \underbrace{-\log|x|}_{\text{coulomb}} + \underbrace{\cancel{\frac{x_1^2}{|x|^2}}}_{\text{anisotropy}}$$

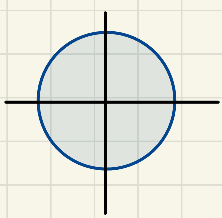
$V(x) = -\log|x|$: classical case (Frostman, Wigner)

I_c

Random matrices
coulomb gases
interpolation theory

KNOWN FACTS:

- $\exists$ min, cp support ✓
- Uniqueness ✓
- Symmetries ✓ RADIAL
- Dimension of support 2 (full)
- Explicit min ✓ $\mu_0 = \frac{1}{\pi} \chi_{B_1(0)}$



circle law

$$I(\mu) = \iint_{\mathbb{R}^2\times\mathbb{R}^2} V(x-y)\,d\mu(x)\,d\mu(y) + \int_{\mathbb{R}^2} |x|^2 d\mu(x)$$

$$V(x) = \underbrace{-\log|x|}_{\text{coulomb}} + \underbrace{\frac{x_1^2}{|x|^2}}_{\text{anisotropy}}$$

$\exists$: $I(\mu) \geq c \int_{\mathbb{R}^2} |x|^2 d\mu(x)$

↪ tightness

I lsc

$\exists$ ✓

cp supp : due to confinement ! ✓

same as for I_c !

U : related to strict convexity ! (in μ)

- confinement linear $\Rightarrow$ convex
- nonlocal term "quadratic"

↪ strictly cx ?

TRICKY ! E.g. $V(x) = |x|^2$

$$J(\mu) = \iint_{\mathbb{R}^2 \mathbb{R}^2} |x-y|^2 \, d\mu(x) \, d\mu(y)$$

NOT CONVEX!

$$= \iint \left(|x|^2 + |y|^2\right) d\mu(x) \, d\mu(y)$$

$$- 2 \iint x \cdot y \, d\mu(x) \, d\mu(y)$$

$$= \underbrace{2 \int_{\mathbb{R}} |x|^2 \, d\mu(x)}_{\text{linear}} - \underbrace{2 \left| \int_{\mathbb{R}} x \, d\mu(x) \right|^2}_{\text{convex}}$$

concave !!!

Result (w Mora & Ronchi)

If $\mu_1, \mu_2 \in \mathcal{P}(\mathbb{R}^2)$, $\mu := \mu_1 - \mu_2$ $\left(\begin{array}{l}\text{cp support} \\ \text{finite energy}\end{array}\right)$

$\mu_1 \neq \mu_2$

$$\Rightarrow \quad Q(\mu,\mu) = \int_{\mathbb{R}^2} (V * \mu)(x)\, d\mu(x) > 0$$

- nontrivial : μ signed , V non-positive
- Result $\Rightarrow$ strict convexity of I $\Rightarrow$ U !

Why ? $\quad \nu = \lambda \nu_1 + (1-\lambda)\nu_2, \quad \nu_1 \neq \nu_2$

$$Q(\nu,\nu) = \lambda^2 Q(\nu_1,\nu_1) + (1-\lambda)^2 Q(\nu_2,\nu_2) + 2\lambda(1-\lambda) Q(\nu_1,\nu_2)$$

Result $\Rightarrow \quad Q(\nu_1-\nu_2, \nu_1-\nu_2) > 0$

$\Leftrightarrow \quad Q(\nu_1,\nu_2) + Q(\nu_2,\nu_2) - 2Q(\nu_1,\nu_2) > 0$

$$< \lambda^2 Q(\nu_1,\nu_1) + (1-\lambda)^2 Q(\nu_2,\nu_2)$$
$$+ \lambda(1-\lambda)\left[Q(\nu_1,\nu_1) + Q(\nu_2,\nu_2)\right]$$
$$= \lambda Q(\nu_1,\nu_1) + (1-\lambda) Q(\nu_2,\nu_2).$$

Result (w Mora & Ronchi)

If $\mu_1, \mu_2 \in \mathcal{P}(\mathbb{R}^2)$, $\mu = \mu_1 - \mu_2$ (cp support, finite interaction)
$\mu_1 \neq \mu_2$

$$\Rightarrow \quad Q(\mu, \mu) = \int_{\mathbb{R}^2} (V * \mu)(x)\, d\mu(x) > 0$$

Idea of the proof : undo the convolution

How ? Work in Fourier space

$$\int_{\mathbb{R}^2} (V * \mu)\, d\mu = \int_{\mathbb{R}^2} \hat{V}\, \underbrace{|\hat{\mu}|^2}_{>0}\, d\xi \overset{?}{>} 0$$

If $\hat{V} > 0 \Rightarrow$ ok!

Compute $\hat{V}$:

$$\hat{V}(\xi) = \frac{1}{\pi} \frac{\xi_2^2}{|\xi|^4} \qquad \xi \neq 0$$

$$\Rightarrow \langle \hat{V}, \varphi \rangle = \frac{1}{\pi} \int \frac{\xi_2^2}{|\xi|^4} \varphi(\xi)\, d\xi . \qquad \varphi(0) = 0$$

$$> 0 \qquad \varphi > 0$$

$\varphi = |\hat{\mu}|^2 > 0$ ✓

$\mu = \mu_1 - \mu_2 \Rightarrow \int \mu = 0 \Rightarrow \hat{\mu}(0) = 0$

$\hookrightarrow \quad V(x) = -\log|x| + \frac{x_1^2}{|x|^2}$

by approximation
with Riesz
potentials

$\hookrightarrow$ known F.T.

$$\frac{1}{2}\frac{x_1^2 - x_2^2}{|x|^2} + \frac{1}{2}$$

$$= \frac{1}{2}\frac{P_2(x)}{|x|^2} + \frac{1}{2}$$

harmonic polynomial
homogeneous

known F.T.

SO FAR: $\exists !$ min with compact supp.

it is a wall?

LAGB ?