Asymptotic analysis of singularly perturbed elliptic functionals

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Winterschool on Analysis and Applied Mathematics,
Münster 22-26 February 2021

The Mumford-Shah functional and the problem of image-segmentation

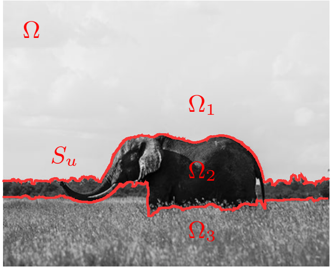
Let $\Omega \subset \mathbb{R}^2$ be the image domain and $g\colon \Omega \to \mathbb{R}$ be a grey-scale image.

P. Compute a an optimal partition $\Omega_1, ..., \Omega_n$ of Ω and $u\colon \Omega \to \mathbb{R}$ such that u is smooth in Ω_i $i = 1, ..., n$, it jumps on $S_u = (\partial\Omega_1 \cup ... \cup \partial\Omega_n) \cap \Omega$ and approximates g.

A. Minimize the functional

$$\underbrace{\int_\Omega \alpha |\nabla u|^2 dx + \beta \mathcal{H}^1(S_u)}_{\textsf{MS}(u)} + \underbrace{\int_\Omega |u - g|^2 dx}_{\text{fidelity term}}$$

over all $u \in SBV(\Omega)$.



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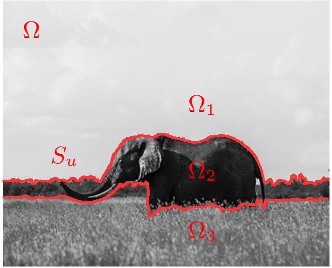

A regularisation of **MS** in terms of Γ-convergence is given by the Ambrosio-Tortorelli functional, i.e.,

$$\mathbf{AT}_\varepsilon(u, v) := \underbrace{\int_\Omega \alpha v^2 |\nabla u|^2 dx}_{\text{bulk term}} + \underbrace{\int_\Omega \frac{\beta}{2} \left(\frac{(1-v)^2}{\varepsilon} + \varepsilon |\nabla v|^2 \right) dx}_{\text{surface term}}$$

with $u, v \in W^{1,2}(\Omega)$.

Brittle energies in Fracture Mechanics

Besides image segmentation, the Mumford-Shah functional describes the so called *brittle energy* in Theory of Fracture Mechanics.

The brittle energy is the energy necessary to the production of a crack:

$$\mathcal{E}(u) = \underbrace{\int_\Omega W(\nabla u)dx}_{\text{elastic energy}} + \underbrace{\lambda \mathcal{H}^2(S_u)}_{\text{surface term}}$$

where $\Omega \subset \mathbb{R}^3$ and $u \in SBV(\Omega; \mathbb{R}^3)$ denotes the deformation. A feature of $\mathcal{E}$ is that it does not depend on the jump opening

$$[u] = u^+ - u^-.$$

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More in general

$$\mathcal{E}(u) = \underbrace{\int_\Omega W(x, \nabla u)dx}_{\text{elastic energy}} + \underbrace{\int_{S_u} \varphi(x, \nu_u)d\mathcal{H}^2}_{\text{surface term}}.$$

Approximation of brittle energies with elliptic functionals

Aim: Study the Γ-convergence of $\mathcal{F}_\varepsilon$ satisfying

$$C_1 \mathbf{AT}_\varepsilon(u,v) \le \mathcal{F}_\varepsilon(u,v) \le C_2 \mathbf{AT}_\varepsilon(u,v)$$

where

$$\mathbf{AT}_\varepsilon(u,v) := \underbrace{\int_\Omega \psi(v)|\nabla u|^p dx}_{\text{bulk term}} + \underbrace{\int_\Omega \left(\frac{(1-v)^p}{\varepsilon} + \varepsilon^{p-1}|\nabla v|^p \right) dx}_{\text{surface term}}$$

$\Omega \subset \mathbb{R}^n$, $(u,v) \in W^{1,p}(\Omega;\mathbb{R}^m) \times W^{1,p}(\Omega)$, $p > 1$ and $\psi \colon \mathbb{R} \to [0,+\infty)$, increasing on $[0,+\infty)$, decreasing on $(-\infty,0)$, $\psi(1)=1$, $\psi(0)=0$.

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More precisely

$$\mathcal{F}_\varepsilon(u,v) := \underbrace{\int_\Omega \psi(v) f_\varepsilon(x,\nabla u) dx}_{\text{bulk term}} + \underbrace{\frac{1}{\varepsilon}\int_\Omega g_\varepsilon(x,v,\varepsilon\nabla v) dx}_{\text{surface term}},$$

with

$$c_1|\xi|^p \le f_\varepsilon(x,\xi) \le c_2|\xi|^p \quad \text{for all} \quad (x,\xi) \in \mathbb{R}^n \times \mathbb{R}^{m\times n},$$

$$c_3(|1-v|^p+|w|^p) \le g_\varepsilon(x,v,w) \le c_4(|1-v|^p+|w|^p) \quad \text{for all} \quad (x,v,w) \in \mathbb{R}^n\times\mathbb{R}\times\mathbb{R}^n.$$

Γ-convergence result

Theorem (Bach, M., Zeppieri (2021))

For suitable integrands f_ε *and* g_ε *let*

$$\mathcal{F}_\varepsilon(u,v,\Omega) := \int_\Omega \psi(v) f_\varepsilon(x,\nabla u)dx + \frac{1}{\varepsilon}\int_\Omega g_\varepsilon(x,v,\varepsilon\nabla v)dx.$$

Then (u.t.s.) the functionals $\mathcal{F}_\varepsilon(\cdot\,,\cdot\,,\Omega)$ Γ*-converge to*

$$\mathcal{F}(u,v,\Omega) := \int_\Omega f_\infty(x,\nabla u)dx + \int_{S_u\cap\Omega} g_\infty(x,[u],\nu_u)d\mathcal{H}^n$$

with $u \in GSBV^p(\Omega;\mathbb{R}^m), v = 1$ *a.e. in* Ω *and* $f_\infty : \mathbb{R}^n \times \mathbb{R}^{m\times n} \to [0,+\infty)$ *and* $g_\infty : \mathbb{R}^n \times \mathbb{S}^{n-1} \to [0,+\infty)$ *are Borel functions.*

Tools:

- Localisation method;
- Integral representation (Bouchittè, Fonseca, Leoni, and Mascarenhas, 2002).

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Characterization of f_∞ and g_∞

$$\mathcal{F}(u,v,\Omega) := \int_\Omega f_\infty(x,\nabla u)dx + \int_{S_u\cap\Omega} g_\infty(x,\nu_u)d\mathcal{H}^n$$

g_∞ does not depend on the jump opening $[u]$, that means $\mathcal{F}$ is a *brittle energy*. This is one of the effects of the volume-surface decoupling. We show in fact that f_∞ and g_∞ are obtained by

- $$f_\infty(x,\xi) = \limsup_{\rho\to 0}\lim_{\varepsilon\to 0}\frac{1}{\rho^n}\left(\inf\int_{Q_\rho(x)} f_\varepsilon(x,\nabla u)dx\right)$$

  where the infimum is taken over all functions $u \in W^{1,p}(Q_\rho(x);\mathbb{R}^m)$ with $u(x) = \xi x$ near $\partial Q_\rho(x)$;
- $$g_\infty(x,\nu) = \limsup_{\rho\to 0}\lim_{\varepsilon\to 0}\frac{1}{\rho^{n-1}}\left(\inf\frac{1}{\varepsilon}\int_{Q^\nu_\rho(x)} g_\varepsilon(x,v,\varepsilon\nabla v)dx\right)$$

  where the infimum is taken among all $v \in W^{1,p}(Q^\nu_\rho(x))$, with $0 \le v \le 1$, for which there exists $u \in W^{1,p}(Q^\nu_\rho(x);\mathbb{R}^m)$ such that $v\nabla u = 0$ a.e. in $Q^\nu_\rho(x)$ and $(u,v) = (u^\nu_x,1)$ near $\partial Q^\nu_\rho(x)\cap\{|(y-x)\cdot\nu| > \varepsilon\}$ where u^ν_x is the jump function given by

  $$u^\nu_x(y) = \begin{cases} e_1 & \text{if } (y-x)\cdot\nu \ge 0\,, \\ 0 & \text{if } (y-x)\cdot\nu < 0\,. \end{cases}$$

An application to homogenisation of damage models

Our analysis applies to the case of homogenisation of damage models, where

$$f_\varepsilon(x,\xi) = f\left(\frac{x}{\varepsilon},\xi\right) \quad \text{and} \quad g_\varepsilon(x,v,w) = g\left(\frac{x}{\varepsilon},v,w\right). \tag{1}$$

In particular if the two limits

$$\lim_{r\to+\infty} \frac{1}{r^n}\left(\inf \int_{Q_r(rx)} f(x,\nabla u)dx\right) =: f_{hom}(\xi),$$

$$\lim_{r\to+\infty} \frac{1}{r^{n-1}}\left(\inf \int_{Q_r^\nu(rx)} g(x,v,\nabla v)dx\right) =: g_{hom}(\nu)$$

where the infimum are taken in suitable classes of functions similar to that chosen for f_∞ and g_∞, exist and are independent of x, then the Γ-limit of $\mathcal{F}_\varepsilon(\cdot\,,\cdot\,,\Omega)$ with f_ε and g_ε as in (1) is given by

$$\mathcal{F}_{hom}(u,v,\Omega) := \int_\Omega f_{hom}(\nabla u)dx + \int_{S_u} g_{hom}(\nu_u)d\mathcal{H}^{n-1}$$

with $u \in GSBV^p(\Omega;\mathbb{R}^m), v = 1$ a.e. in Ω.

Bach, A., Marziani, R., Zeppieri, C., I., Γ-*convergence and stochastic homogenisation of singularly perturbed elliptic functionals*. Submitted, (2021).