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EINDHOVEN UNIVERSITY OF TECHNOLOGY

Asymptotic limits of random walks via generalized gradient flows

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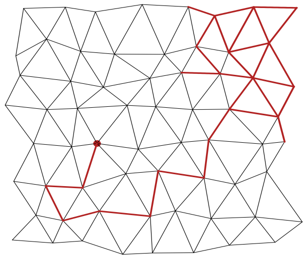
Random walks as generalized gradient flows



Let $G^h = (V^h, E^h)$ be a graph, where $h = \max_{e\in E^h} |e|$. The distribution $\rho^h \in \mathscr{P}(V^h)$ of the random walk satisfies:

$$\partial_t \rho_t^h = Q^* \rho_t^h,$$

$$(Q\varphi^h)(x) = \sum_{y\sim x} \big(\varphi^h(y) - \varphi^h(x)\big)\,\kappa^h(y,x),$$

with jump rates κ^h such that $\kappa^h(y,x)\pi^h(y) = \kappa^h(x,y)\pi^h(x) =: \theta^h(x,y)$.

The random walk possesses a *generalized gradient structure* with entropy $\mathscr{E}_h$ and dissipation potential $\mathscr{R}_h$, i.e. there exists j^h such that the pair (ρ^h, j^h) satisfies:

$$(\mathrm{CE}_h) \quad \partial_t \rho_t^h + \overline{\nabla}\cdot j_t^h = 0, \qquad (\mathrm{EDP}_h) \quad \mathscr{I}_h(\rho^h, j^h) = 0;$$

$$\mathscr{I}_h(\rho^h, j^h) := \int_0^T \mathscr{R}_h(\rho_t^h, j_t^h) + \mathscr{D}_h(\rho_t^h)\,dt + \mathscr{E}_h(\rho_T^h) - \mathscr{E}_h(\rho_0^h).$$

Diffusion processes as Otto/Wasserstein gradient flows

The diffusion equation in $\Omega \subset \mathbb{R}^d$

$$\partial_t \rho_t = \nabla\cdot(\nabla\rho_t + \rho_t \nabla V)$$

can be interpreted as a gradient flow of the relative entropy $\mathscr{E}(\rho) = Ent(\rho|\pi)$, $\pi \simeq e^{-V}dx$ with respect to the Wasserstein metric in $\mathscr{P}(\Omega)$. Then the solution can characterized by EDP:

$$(\mathrm{CE}) \quad \partial_t \rho_t + \nabla\cdot j_t = 0, \qquad (\mathrm{EDP}) \quad \mathscr{I}_0(\rho, j) = 0.$$

Question: Under what assumptions on the discretization does the random walk converge to a diffusion process while preserving the gradient structure?

Assumptions

- ζ-regular finite-volume discretization;
- Lower bound: $\kappa^h(K,L) \gtrsim \dfrac{|(K|L)|}{h|K|}$;
- Upper bound: $\sup_{h>0} \sup_{K\in\mathscr{T}^h} h^2 \sum_{L\in\mathscr{T}_K^h} \kappa^h(K,L) < \infty$;
- Local martingale: $\sum_{L\in\mathscr{T}_K^h} \kappa^h(K,L)(x_K - x_L) = 0$.

Convergence results

- $(\rho^h, j^h) \in (\mathrm{CE}_h) \to (\rho, j) \in (\mathrm{CE})$, $\quad u^h := \dfrac{d\rho^h}{d\pi^h} \xrightarrow{L^1} u := \dfrac{d\rho}{d\pi}$;
- Entropy: $\mathscr{E}_h(\rho^h) = Ent(\rho^h|\pi^h) \longrightarrow \mathscr{E}(\rho) = Ent(\rho|\pi)$;
- Dissipation potential:

$$\liminf_{h\to 0} \mathscr{R}_h(\rho^h, j^h) \ge \mathscr{R}(\rho, j) = \int_\Omega \left|\frac{dj}{d\rho}\right|^2_{\mathbb{T}^{-1}} d\rho,$$

where $\mathscr{R}_h^*(\rho^h, \xi^h) = \dfrac{1}{2} \sum_{(K,L)\in E^h} \Psi^*\big(\xi^h(K,L)\big)\sqrt{u^h(K)u^h(L)}\,\theta^h(K,L)$;

- Fisher information:

$$\mathscr{D}_h(\rho^h) = \sum_{(K,L)\in E^h} \left|\overline{\nabla}\sqrt{u^h}\right|^2 \theta^h(K,L) \xrightarrow{\Gamma-\lim} \mathscr{D}(\rho) = \int_\Omega \langle \sqrt{u}, \mathbb{T}\sqrt{u}\rangle\,d\pi.$$