On vortex dynamics for the 2D Euler equations

Stefano Ceci and Christian Seis

Institut für Analysis und Numerik, Westfälische Wilhelms-Universität Münster

Introduction

We study the interaction of vortex patches in an incompressible, inviscid fluid moving in a bounded two-dimensional domain Ω. The name "vortex patches" refers to areas of the fluid that present a high vorticity, which is responsible for generating the velocity field that transports the fluid particles. The resulting motion is due to the interactions of the vortices with each other and with the boundary of the domain Ω.

The interest in the study of vortex interaction arises from the need to validate the mathematical model by observing whether it presents phenomena observed in reality.

The fluid is described by the Euler vorticity equation

$$\partial_t \omega + u \cdot \nabla \omega = 0, \tag{1}$$

where

$$u = K * \omega + \text{boundary interaction},$$

and K is the Biot–Savart kernel

$$K(z) = \frac{1}{2\pi} \frac{z^\perp}{|z|^2}.$$

The scalar $\omega : (0, +\infty) \times \Omega \to \mathbb{R}$ is the vorticity (the tendency of the fluid to rotate), while the vector field $u : (0, +\infty) \times \Omega \to \mathbb{R}^2$ is the velocity of the fluid.

It is well-known that the dynamics of vortex patches obeying the Euler equation can be described by the so-called Kirchhoff–Routh point-vortex system: given N points $Y_1, \ldots, Y_N$, their evolution is expressed as

$$\begin{cases} \frac{d}{dt} Y_i(t) = \sum_{\substack{j=1 \\ j \neq i}}^N a_j K(Y_i(t) - Y_j(t)) \\ \qquad\qquad + \text{boundary interaction} \\ Y_i(0) = \bar{Y}_i; \end{cases} \tag{2}$$

here $a_1, \ldots, a_N \in \mathbb{R}$ are the intensities.

The general idea, dating back to the 19-th century and the work of Helmholtz [2], is to substitute the vortex patches with points and consider their evolution via (2).

Objective

We assume that the initial configuration of the vorticity is made up of several vortex patches concentrated around some points. We endeavour to prove that the concentration of the vortex patches around the point vortices holds also for later times.

Mathematical setting

We begin with an initial datum of the form

$$\bar{\omega} = \sum_{i=1}^N \bar{\omega}_i, \tag{3}$$

where the $\bar{\omega}_i$'s represent the different vortex patches and are assumed to satisfy the following assumptions:

- Initial supports far from each other.
- The total vorticity of the initial i-th vortex patch equals the intensity a_i of the i-th point vortex in (2), i.e.
$$\int \bar{\omega}_i \, dx = a_i.$$
- The patches are initially concentrated around the point-vortices, in the following (mild) sense:
$$W_2\left(\frac{\bar{\omega}_i}{a_i}, \delta_{\bar{Y}_i}\right) \leq \varepsilon.$$
Here W_2 is the 2-Wasserstein distance.
- $\bar{\omega} \in L^p$, for some $p > 2$, and the following scaling assumption holds:
$$\|\bar{\omega}\|_{L^p} \lesssim \frac{1}{\varepsilon^{2(1-\frac{2}{p})}}.$$

The solution to Euler (1) with initial datum (3) will have the form

$$\omega(t,x) = \sum_{i=1}^N \omega_i(t,x),$$

where ω_i solves the linear transport equation

$$\partial_t \omega_i + u \cdot \nabla \omega_i = 0, \quad \omega_i(0,x) = \bar{\omega}_i(x).$$

Result

There exists a time $T > 0$ independent of the concentration parameter $\varepsilon \ll 1$ and a constant $C > 0$, also independent of ε, such that, for any $i = 1, \ldots, N$,

$$W_2\left(\frac{\omega_i(t)}{a_i}, \delta_{Y_i(t)}\right) \lesssim e^{Ct}\varepsilon \quad \text{for all } t < T.$$

Moreover, there hold also

$$|X_i(t) - Y_i(t)| \lesssim e^{Ct}\varepsilon, \quad \left|\frac{d}{dt}X_i(t) - \frac{d}{dt}Y_i(t)\right| \lesssim e^{Ct}\varepsilon$$

for all $t < T$, where

$$X_i(t) = \frac{1}{a_i}\int x \, \omega_i(t,x) \, dx$$

is the center of vorticity of the i-th vortex patch ω_i, and Y_i is the solution to the point-vortex model (2).

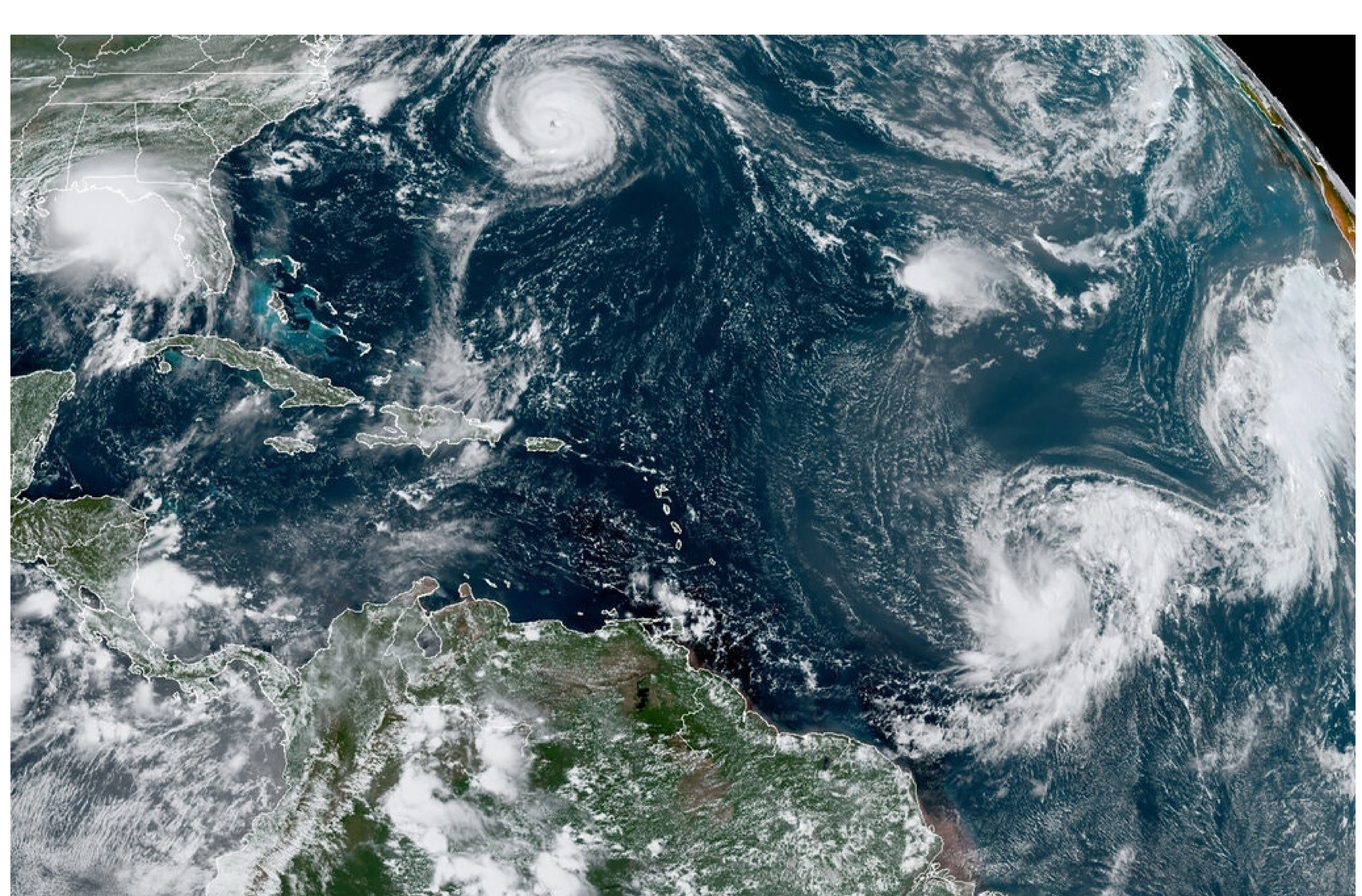

Figure 1:Storms in the Atlantic basin. Image taken from The New York Times website. Credits to NOAA, via Associated Press.

Difference from previous works

Previous works:

- Bounded vorticity, i.e. $\omega \in L^\infty$.
- Strong notion of concentration.

Our contribution:

- Possibly unbounded vorticity: $\omega \in L^p$ for some $p > 2$.
- Milder notion of concentration in terms of the 2-Wasserstein distance.
- Stability-type estimate between solutions to (1) and to (2).

On the condition $p > 2$

The constraint $p > 2$ has two reasons:

- It guarantees that the velocity is bounded.
- It ensures that the Lebesgue norms of ω are preserved in time, and that there exists a lagrangian representation in terms of the flow.

Comments

Our result provides a stability-type estimate between solutions to Euler (1) with initial vortex patch configuration and solutions to the point vortex system (2).

Note that uniqueness for solutions to (1) is not known in the setting $\omega \in L^p$ for $p < \infty$, but we recover it in the limit.

References

[1] Stefano Ceci and Christian Seis. Vortex dynamics for 2D Euler flows with unbounded vorticity. *Rev. Math. Iberoam.*, 2020. In press.

[2] H. Helmholtz. Über Integrale der hydrodynamischen Gleichungen, welche den Wirbelbewegungen entsprechen. *J. Mathematik*, 55:25–55, 1858.

Contact

email: ceci@wwu.de