On Symmetric Polyconvexity

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Abstract

Symmetric quasiconvexity plays a key role for energy minimization in geometrically linear elasticity theory. Due to the complexity of this notion, a common approach is to retreat to necessary and sufficient conditions that are easier to handle. This poster focuses on symmetric polyconvexity, which is a sufficient condition. We prove a new characterization of symmetric polyconvex functions in the two- and three-dimensional setting, and use it to investigate relevant subclasses like symmetric polyaffine functions and symmetric polyconvex quadratic forms. In particular, we provide an example of a symmetric rank-one convex quadratic form in 3d that is not symmetric polyconvex. The construction takes the famous work by Serre from 1983 on the classical situation without symmetry as inspiration. Beyond their theoretical interest, these findings may turn out useful for computational relaxation and homogenization.

Introduction

Elastic energy are of the form

$$E(v) = \int_\Omega W(\nabla v)dx,$$

where $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, is a bounded domain representing the reference configuration of the elastic body, $v : \Omega \to \mathbb{R}^d$ is the deformation of the body, and $W : \mathbb{R}^{d\times d} \to \mathbb{R}$ is an elastic energy density. The energy densities cannot be convex due to incompatibility with the concept of frame-indifference, see e.g. [5]. The weaker notion of polyconvexity, which was introduced by Ball in 1977 [1], has turned out to be particularly suitable here. We recall that a function $\mathbb{R}^{d\times d} \to \mathbb{R}$ is called polyconvex if it depends in a convex way on its minors.

We investigate functions that are polyconvex when composed with the linear projection of $\mathbb{R}^{d\times d}$ onto the subspace of symmetric matrices $\mathcal{S}^{d\times d}$, that is, $f : \mathcal{S}^{d\times d} \to \mathbb{R}$ such that $\mathbb{R}^{d\times d} \ni F \mapsto f(F^s)$ is polyconvex. We call such functions *symmetric polyconvex.* This notion has applications in the geometrically linear theory of elasticity, which results from nonlinear elasticity theory by replacing the requirement of frame-indifference with the assumption that the elastic energy density is invariant under infinitesimal rotations, see e.g., [2, 3, 5]. In this case, the energy densities depend only on the symmetric part of the deformation gradient, or on linear strains $e(u) = \frac{1}{2}(\nabla u + (\nabla u)^T)$, where $u : \Omega \to \mathbb{R}^d$, $u(x) = v(x) - x$, is the elastic displacement field. That is,

$$E(u) = \int_\Omega f(e(u))dx, \qquad (1)$$

with a density $f : \mathcal{S}^{d\times d} \to \mathbb{R}$. Here, we are particularly interested in the case when f is symmetric polyconvex. The overall aim is to identify necessary and sufficient conditions for symmetric polyconvexity in 2d and 3d, see [4].

The $2d$ case

In the $2d$ case we prove that any function $f : \mathcal{S}^{2\times 2} \to \mathbb{R}$ is symmetric polycovex if and only if it can be expressed as a convex function of the matrix and the determinant such that the dependance on the determinant is in a non increasing way.

$$f : \mathcal{S}^{2\times 2} \to \mathbb{R} \text{ is spc} \Leftrightarrow \exists g : \mathcal{S}^{2\times 2} \times \mathbb{R}, g(., \searrow) : f(\varepsilon) = g(\varepsilon, \det \varepsilon)$$

The function $\varepsilon \mapsto \det \varepsilon$ is not symmetric polyconvex and $\varepsilon \mapsto -\det \varepsilon$ is and in general a function $\varepsilon \mapsto h(\det \varepsilon)$ is symmetric polyconvex if and only if h is non increasing.

The 3d case

In the 3d case, we prove that any function is symmetric polyconvex iff it can be represented as a convex function of first and second order minors (hence, no dependence on the determinant) whose subdifferential with respect to its cofactor variable is negative semi-definite.

$$f(\varepsilon) = g(\varepsilon, \operatorname{cof} \varepsilon), \text{ g convex and } \partial_2 g \text{ is negative semi definite}$$

- The function $\varepsilon \mapsto \det \varepsilon$ is not symmetric polyconvex in the $3d$ case and in general any function $\varepsilon \in \mathcal{S}^{3\times 3} \mapsto h(\det \varepsilon)$ is symmetric polyconvex if and only if h is constant
- Any expression of the form $A : \operatorname{cof} \varepsilon$, $A \in \mathcal{S}^{3\times 3}$, is symmetric polyconvex if and only if A is negative semi definite.

Quadratic forms

In the case of quadratic forms, we we give explicit characterizations, both for $d = 2$ and $d = 3$. $f : \mathcal{S}^{d\times d} \to \mathbb{R}$ is symmetric polyconvex iff

$$f(\varepsilon) = h(\varepsilon) - \alpha \det \varepsilon \quad \text{and} \quad f(\varepsilon) = h(\varepsilon) - A : \operatorname{cof} \varepsilon$$

where $h : \mathcal{S}^{d\times d} \to \mathbb{R}$ is convex, $\alpha > 0$, and $A \in \mathcal{S}^{3\times 3}$ is positive semi-definite.

Polyconvexity$\not\Leftarrow$ rank one convexity for quadratic forms

For quadratic forms, we know by the example of Serre that in the classical framwork that rank one convexity is ot equivalente to polyconvexity. We prove that the statement is still true in the symmetric setting, our couterexample is motivated by the one given by Serre in the classical case [6]

There exists $\eta > 0$ such that the quadratic form $f : \mathcal{S}^{3\times 3} \to \mathbb{R}$ given by

$$f(\varepsilon) = (\varepsilon_{13} - \varepsilon_{23})^2 + (\varepsilon_{12} - \varepsilon_{13})^2 + (\varepsilon_{12} - \varepsilon_{23})^2 + \varepsilon_{11}^2 + \varepsilon_{22}^2 + \varepsilon_{33}^2 - \eta|\varepsilon|^2$$

is symmetric rank-one convex, but is not symmetric polyconvex. The proof that this function is not symmetric polyconvex is based on our 3d-characterization and on a careful study of the minimizers of $f + \eta|\varepsilon|^2$ in the set of compatible matrices with unit norm.

Symmetric polyaffinity $\Leftrightarrow$ Symmetric Affinity

Let $f : \mathcal{S}^{3\times 3} \to \mathbb{R}$. Then f is symmetric polyaffine (equivalently symmetric quasiaffine or symmetric rankone affine) if and only if it is affine, i.e. there are $B \in \mathcal{S}^{3\times 3}$ and $b \in \mathbb{R}^3$ such that

$$f(\varepsilon) = B : \varepsilon + b, \quad \varepsilon \in \mathcal{S}^{3\times 3}.$$

Symmetric polyconvex sets and polyconvex hulls

In progress, we are working on a characterization of symmetric polyconvex sets and polyconvex hulls in the cases $d = 2, 3$.

References

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