Asset Allocation with Multiple State Variables

Stochastic Optimal Control in Finance

Diploma Thesis
of
Tobias Neugebauer

Westfälische Wilhelms-Universität
Münster

Faculty of Mathematics

Author	Tobias Neugebauer born on 22.08.1981 in Bielefeld - Germany t_neug01@uni-muenster.de
Supervisor	Professor Martin Burger
Co-supervisor	Professor Nicole Branger
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Chapter 1

Introduction

1.1 Motivation

Many economic models include on the one hand stochastic dynamics and on the other hand they seek to maximise some utility. These problems, referred to as stochastic control problems, are the matter of interest of this thesis. One example of these problems is to optimise the expenditure of some income from stochastic sources. Other examples of stochastic control are optimal stopping times (e.g. when to exercise an American option), the allocation of a portfolio that includes risky investment opportunities or hedging options on non-traded underlayings. The variety of stochastic control problems is great, not only in Economics but also in Physics and other fields. This thesis outlines the theoretical mathematical background for problems that read *"find the optimal control that maximises some utility function (or minimises some cost function)"* . This kind of problems can often be reformulated as partial differential equations (PDEs). The main mathematical obstacles are to show the equivalence of the optimal control problem and the PDE formulation of the problem as well as the existence and uniqueness of the solution. Furthermore we discuss finite difference schemes to compute solutions for the problems in focus.

The thesis is structured as follows: In chapter two we will formulate a general stochastic control problem and define all its components. The principle of dynamic programming then supplies a method of deducing a PDE called Hamilton-Jacobi-Bellman equation (HJB) from this formulation. Since the setup includes stochastic dynamics, we will have to make use of Itō's formula (also called Itō's lemma), a main tool in stochastic control that causes the HJB to be a second order PDE. In addition it is non-linear. This is the matter of interest for us.

Solutions to HJBs are in general not unique. But there is only one solution that solves the optimal control problem and therefore it is the only economically relevant solution of the HJB. In chapter three we will define this solution, the so called viscosity solution, and then show that it solves the stochastic control problem. Moreover we show its existence and uniqueness.

Most of the time analytic solutions of the PDEs we will discuss are not available, so in general we have to fall back on numerical computations of the solution. As the complexity of economic models is increasing, various difficulties arise with regard to implementing numerical schemes. Most notably, highly non-linear PDEs and PDEs of higher dimensions pose problems. With using a numerical scheme to solve the PDE one wants to make sure to compute the correct solution. In chapter four we will show an unconditionally stable finite difference scheme (FDM) and prove its convergence to the economically relevant solution, while chapter five will focus on some technical issues that can occur from a practical point of view. We will discuss the problems along with concrete examples which will be introduced in this chapter. The examples are designed to check an implementation step by step and will have different layers of difficulty. Chapter six will be devoted to discussing the numerical results of the applied FDM scheme of the example problem as well as shortly assessing the closure of the solution to the economic reality.

1.2 Examples

We will now give some examples to introduce financial applications of optimal control theory in order to get an idea of the problems. We will focus on asset allocation problems though the range of applications is widely spread. The first examples are kept simple but cover the basic problems. Example 1 is the most basic version of an asset allocation introduced by Merton [15]. It leads to an ordinary differential equation (ODE). We are able to give an analytic solution that allows for good judgement of numerical results which we will provide in chapter 6. Example 2 is an extension of example 1 which leads to a two dimensional PDE. We build up to a relevant economic problem in section 1.3 that was recently discussed by Munk and Sørensen [16].

Example 1

As a first example we introduce an optimal consumption and portfolio selection problem. It was introduced by Merton [15] who provided an analytical solution (we present the form in which Kohn [12] picked it up). An individual invests the fraction $I \in [0;1]$ of her wealth in a risky asset, e.g. a stock s which evolves as

$$ds = s\mu_s dt + s\sigma_s d\omega_s \ ,$$

where ω_s is a geometric Brownian motion, μ_s a constant drift and σ_s the stock price volatility. The remaining wealth fraction $(1-I)$ will increase with interest rate r. The individual continuously consumes wealth with amount $c \geq 0$. For simplicity the individual will not have any utility from wealth after some time T and therefore she only has utility from consuming during a relevant time period $t \in [t_0;T)$. This leads to an initial value problem for the individual's wealth function

$$\begin{cases} dW = \Big[IW\mu_s + (1-I)Wr - c\Big]dt + IW\sigma_s d\omega & ,t \in [t_0;T) \\ W = w & ,t = t_0 \end{cases} . \tag{1.1}$$

We call this stochastic differential equation (SDE) the state equation. Of course once the wealth hits zero, it will remain there until the rest of the time period. If that happens, the investment strategy becomes irrelevant and in addition we cannot consume anymore, thus both c and I will be zero from that time on, too. Along with the power utility function $u(c) = c^\gamma$ with risk aversion $\gamma \in (0,1)$ the optimal control problem is to find the consumption $c = c(t)$ and investment strategy $I = I(t)$ that maximise the individuals utility over the time period $[0;T]$. Furthermore we are interested in the value of that utility. For that purpose we define the indirect utility function J

$$J(w,t) = \max_{c,I} E_{(W(t)=w)}\Big[\int_t^T \exp(-\delta\tau)u\big(c(\tau)\big)\mathrm{d}\tau\Big] \ . \tag{1.2}$$

The indirect utility is discounted to time $t = 0$ to indicate a time preference of consumption denoted by δ.

From this problem we can deduce the following HJB equation

$$0 = J_t + \max_{c,I}\Big\{e^{-\delta t}u(c)^\gamma + \Big(Iw\mu_s + (1-I)wr - c\Big)J_w + \frac{1}{2}\sigma_s^2 J_{ww}\Big\} \ . \tag{1.3}$$

Please accept this equation for now. In the next chapter we will derive it step by step. At the final time $t = T$ the individual has no utility regardless of the control values. Hence, $J(w,T) = 0$ and we have to solve (1.3) backwards in time for $0 \leq t < T$. It is an ODE of second order. The optimal consumption can be derived by the standard first order condition for a maximum and we get

$$c^*(t) = \Big(\frac{e^{\delta t}}{\gamma} J_w\Big)^{\frac{1}{\gamma-1}} \ . \tag{1.4}$$

Accordingly the optimal investment is

$$I^*(t) = \frac{(r-\mu_s)J_w}{\sigma_s^2 w J_{ww}} \ . \tag{1.5}$$

In the reduced case of $\mu_s = 0$ we have a deterministic control problem with state equation (1.1) becoming the ODE $\dot{y} = l + ry - c$. The HJB is still given by (1.3) but with $\mu_s = 0$ it would be of first order only. The optimal consumption however will stay unchanged and still satisfy (1.4). This is a consequence of the fact that the expected value of a stochastic integral vanishes.

We can provide a closed form solution of the problem

$$\begin{aligned} J(w,t) &= w^\gamma g(t) \\ &= w^\gamma e^{-\delta t}\Bigg[-\frac{1-\gamma}{\delta-\nu\gamma}\Big(1-\exp\Big(-\frac{(\delta-\nu\gamma)(T-t)}{1-\gamma}\Big)\Big)\Bigg]^{1-\gamma} \end{aligned} \tag{1.6}$$

where

$$\nu = r + \frac{(\mu_s - r)^2}{2\sigma_s^2(1-\gamma)} \ .$$

The optimal controls c^* and I^* are

$$c^*(w,t) = w\Big(e^{\delta t} g(t)\Big)^{\frac{1}{\gamma-1}} \quad \text{and} \quad I^*(w,t) \equiv I^* = \frac{\mu_s - r}{\sigma_s^2(1-\gamma)} \ . \tag{1.7}$$

The fact that the function (1.6) solves the HJB can be seen by brute force differentiation. However, a derivation is given in Kohn [12]. The key is to see that the solution has the form $J(w,t) = w^\gamma g(t)$. To show this one has to proof that $J(Cw,t) = C^\gamma J(w,t)$. We solved equation (1.3) but since it appeared from nowhere we have to show that the solution formula (1.6) satisfies (1.2) and that the controls (1.7) are optimal. This can be shown by using a verification argument[1]: The idea is to show that no control can achieve a greater indirect utility than (1.6) and therefore that (1.6) is an upper bound for the value function. On the other hand we have to show that the controls (1.7) achieve the indirect utility of the solution formula. Again, we refer to Kohn [12] instead of doing that here.

[1] For more about verification theorems see [Verification Theorems within the Framework of Viscosity Solutions, XUN YU ZHOU, Journal of Mathematical Analysis and Applications, 176 (1993)] and [On some recent aspects of stochastic control and their applications, HUYEN PHAM, Probability Surveys Vol. 2 (2005), pp. 506–549]

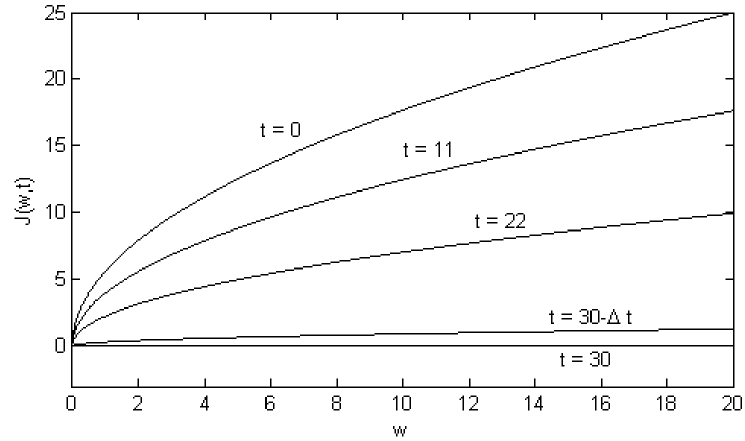


Figure 1.1: Analytic solution of example 1

Figure 1.1 shows the analytic solution for different time steps. This behaviour of the solution is typical for power utility functions like u defined above.

Example 2

Now we introduce an example of pure asset allocation in two variables: Wealth W and interest r. The underlaying stochastic processes are the stock market, which follows the exact same process as in example one, and the interest rate, which follows a mean reversion process

$$\mathrm{d}s = \mu_s s \mathrm{d}t + \sigma_s s \mathrm{d}\omega_s$$

$$\mathrm{d}r = \mu_r(\bar{r} - r)\mathrm{d}t + \sigma_r \mathrm{d}\omega_r \ .$$

Here $\bar{r}$ is the long term interest rate and μ_r is the speed of adjustment in time at which the interest returns to its long term value from any disturbance. The goal is again to maximise the financial wealth at the

$$\begin{cases} \mathrm{d}W = \big[(1-I)\mu_r(\bar{r}-r) + I\mu_s\big]\mathrm{d}t + I\sigma_s \mathrm{d}\omega_s + (1-I)\sigma_r \mathrm{d}\omega_r \ , & t \in [0;T) \\ W = 0 \ , & t = T \end{cases} .$$

Like in example 1 we give the HJB but refer to the next chapter for the justification.

$$0 = J_t + \max_c \left\{ (l + a\mu + br - c)J_y + \frac{1}{2}(\mu + a\sigma)^2 J_{yy} + c^\gamma \right\} \tag{1.8}$$

where

$$B = \begin{pmatrix} \sigma_r & 0 \\ (1-I)\sigma_r & I\sigma_s \end{pmatrix} \in \mathbb{R}^{2\times 2} \ , \qquad a = \begin{pmatrix} \mu_r \\ (1-I)\mu_r(\bar{r}-r) + I\mu_s \end{pmatrix} \in \mathbb{R}^2 \ .$$

1.3 Example 3: Asset allocation with stochastic dynamics

Now we introduce an allocation problem which is designed to show the impact of uncertain labour income on investment and consumption decisions for the life-cycle of a human being. This specific setup was introduced by Munk and Sørensen [16].

The setup of example 1 and 2 can be seen as a special case of this example. An individual has a certain amount of financial wealth plus her future labour income. By consuming financial wealth she has utility specified by a utility function u. The individual wants to maximise her utility over her lifetime. The financial wealth not yet consumed can be divided between risky and riskless assets to earn income. We allow for investments in one bond and one stock.

The underlaying stochastic variables are the interest rate r, the individual's labour income y and wealth W.

The interest rate follows a mean reversion process. After any perturbation through $-\sigma_r d\omega_r$ it tends to go back to the long run equilibrium $\bar{r}$

$$dr_t = \kappa(\bar{r} - r_t)dt - \sigma_r d\omega_r \ .$$

κ is the speed of adjustment while turning back to the long term mean $\bar{r}$ and σ_r the interest rate volatility. The market price of interest rate risk is denoted by λ_r and is assumed to be constant. All constants, namely κ, $\bar{r}$, σ_r and λ_r, have to be nonnegative. ω_r is a Brownian motion.

Further we consider a world with two traded assets: A bond B and a stock S with dynamics

$$dB = B\left[(r_t + \sigma_B(t)\lambda_r)\, dt + \sigma_B(t) d\omega_r\right] \quad \text{and}$$

$$dS = S\left[(r_t + \psi)dt + \sigma_S\left(\rho_{SB} d\omega_r + \sqrt{1-\rho_{SB}^2}\, d\omega_S\right)\right] \ .$$

The economic reasoning behind this can be found for example in Hull [10]. We do not go into further detail here. It is most likely that the bond price volatility $\sigma_B(t) = \sigma_B(t, r)$ also depends on the interest rate level but for a zero-coupon bond we have the volatility $\sigma_B(t) = \frac{\sigma_r}{\kappa}(1 - exp(-\kappa(\bar{T} - t)))$. Further a zero-coupon bond implies a perfect correlation with the interest rate, i.e. $\rho_{Br} \equiv 1$. ω_S is a Brownian motion independent of ω_r.

The financial wealth is now determined by the earnings and expenses: The interest income $\theta_r r$, which is due to putting the amount π_r in a bank account, the returns from investment in assets given by the invested amount θ_B and θ_S and the above asset dynamics, the labour income stream y and finally by substracting the consuption c. In total we get

$$\begin{aligned} dW_t &= \theta_r r dt + \theta_B dB + \theta_S dS + y dt - c dt \\ &= \Big(y - c + \theta_r r + \theta_B(r + \sigma_B \lambda_r) + \theta_S(r + \psi)\Big)dt \\ &\quad + \Big(\theta_B \sigma_B + \theta_S \sigma_S \rho_{SB}\Big)d\omega_r + \Big(\sigma_S \sqrt{1 - \rho_{SB}^2}\Big)d\omega_S \ . \end{aligned}$$

Since the wealth is split between investments in the assets and the bank account, we obtain the relation $W = \theta_r + \theta_B + \theta_S$. Thus the dynamics for the wealth become

$$\begin{aligned} dW_t = &\Big(y - c + Wr + \theta_B \sigma_B \lambda_r + \theta_S \psi\Big)dt \\ &+ \Big(\theta_B \sigma_B + \theta_S \sigma_S \rho_{SB}\Big)d\omega_r + \Big(\sigma_S \sqrt{1 - \rho_{SB}^2}\Big)d\omega_S \ . \end{aligned}$$

Though one can influence one's own labour income to a certain extent, for the dynamics we take the point of view of an investment adviser, i.e. labour income is exogene.

$$\frac{dy}{y} = \Big[(\xi_0(t) + \xi_1 r_t)dt + \sigma_y(t)\Big\{\rho_{By}d\omega_B + \rho_{Sy}d\omega_S + \sqrt{1 - \rho_{yB}^2 - \rho_{yS}^2}d\omega_y\Big\}\Big]$$

The individual receives a steady stream of income from non-financial sources. This stream may be influenced by the interest rate and stock market perturbations to reflect the dependence on booming periods and recessions. The time-dependence of the coefficients is due to approaching the individual's salary variations during her life-cycle. ω_y is a standard Brownian motion independent of ω_r and ω_S.

As direct utility function we use a common power utility function with constant relative risk aversion $\gamma > 0$.

$$U(c) = \frac{c^{1-\gamma}}{1-\gamma} \ , \tag{1.9}$$

with $\gamma \neq 1$. The proxy will be $\gamma = 4$, so $U(c)$ is a strictly negative monotone increasing function. Observe that it is defined for positive values of consumption only, i.e. $c > 0$. Utility is assumed to be time additive but we presume an individual time preference of consumption δ. Hence we formulate an indirect utility function

$$\int_0^T e^{-\delta t} u(c(t))dt + \epsilon e^{-\delta T} u(W_T) \ ,$$

where W_T is the final time financial wealth. The second part is associated with utility from passing wealth on to heirs. We set $\epsilon = 0$ if there is no associated utility and otherwise choose $\epsilon = 1$ or more general $\epsilon \geq 0$.

Now we can formulate the objective: Find the controls θ_B, θ_S and c that maximise the utility over the individuals life-cycle. In order to do so we define a value function

$$J(W, r, y, t) = \sup_{\theta_B, \theta_S, c} E\left[\int_t^T e^{-\delta\tau} u(c(\tau))d\tau + \epsilon e^{-\delta T} u(W_T)\right] \ . \tag{1.10}$$

Since we have uncertain dynamics, we need to take the expected value of the utility in order to have the functional J well-defined.

Chapter 2

Stochastic Optimal Control

In this chapter we define stochastic control problems and deduce a PDE from the problem formulation. We also pose some restrictions that we will need for the results in the following chapters.

2.1 The definition of Stochastic Optimal Control problems

A stochastic control problem consists of dynamics y, called the state equation determined by a and b, the initial state y_0, the final time T, the running utility g and the final time utility $\bar{g}$. We define these terms:

2.1.1 Definition (state equation)**.** *The state $y(t) \in \mathbb{R}^d$ of a dynamic system in a continuous time setup is described by a stochastic differential equation (SDE):*

$$\begin{cases} dy(t) &= a(y(t), \alpha(t))\ dt\ + b(y(t), \alpha(t))\ d\omega_t \\ y(0) = y_0 \end{cases} \tag{2.1}$$

where the given specific dynamics y is determined by $a : \mathbb{R}^d \times A \to \mathbb{R}^d$ and $b : \mathbb{R}^d \times A \to \mathbb{R}^{d \times m}$. y_0 is the initial condition. One can influence these dynamics via a control vector $\alpha(t)$. ω is a m-dimensional standard Brownian motion where each component ω_i is independent of ω_j for all $j \neq i$. We write $\phi(t, y_0, \alpha, \omega)$ for the solution $y(t)$ of the state equation with initial condition $y(0) = y_0$ and abbreviate it with $\phi(t)$ if α and y_0 are understood.

The solution ϕ of (2.1) depends on ω and is therefore a set for every initial condition and control. For the controls α we define:

$$\mathfrak{A} := \left\{\alpha : \mathbb{R}_0^+ \to A \mid \alpha \text{ is measurable and essentially bounded}\right\}$$

We now want to find the control α that maximises some utility function g (or minimise some kind of cost which is mathematically equivalent) over a time period. The resulting function v is called value function or indirect utility function and has the form

$$\begin{aligned} v(t,y) = \sup_{\alpha \in \mathfrak{A}} \Big[\mathbb{E}_{\phi(t)=y} \int_t^T & e^{-\delta(\tau - t)} g(\phi(\tau, y, \alpha, \omega), \alpha(\tau)) \, \mathrm{d}\tau \\ & + e^{-\delta(T-t)} \bar{g}(\phi(T, y, \alpha, \omega)) \Big] \end{aligned} \tag{2.2}$$

at each time $t \in [0, T]$. Furthermore we are interested in finding the indirect utility function v. The utility g is also called the direct or running utility function. It is discounted with discount rate δ to get the present value of future utility and depends on the stochastic set ϕ. Hence to have a well defined value function we take the expected value of the direct utility over the time period. The function $\bar{g}$ corresponds to some final time utility of wealth not consumed (e.g. benefit from leaving financial wealth to heirs). Observe that y now denotes the initial condition of (2.1).

The direct utility g in (3.3.1) is discounted to time t, which is also used in 1.3. It is also possible to discount to time 0, as in examples 1-3. The value function (3.3.1) in this case reads

$$\begin{aligned} v(t,y) = \sup_{\alpha \in \mathfrak{A}} \Big[\mathbb{E}_{\phi(t)=y} \int_t^T & e^{-\delta \tau} g(\phi(\tau, y, \alpha, \omega), \alpha(\tau)) \, \mathrm{d}\tau \\ & + e^{-\delta T} \bar{g}(\phi(T, y, \alpha, \omega)) \Big] \ . \end{aligned}$$

This will make a difference later when we derive the HJB. Through the paper we mostly work with discounting to time t though the changes to be made for discounting to time 0 are little. All results can be applied to both cases. If we allow the final time T to be infinity we face a so called infinit horizon problem. Discounting is mandatory then to make the integral meaningful. We do not go into detail about this but refer to Pham [18], Kohn [12] and only allow finite final times T.

The control $\alpha(t)$ is arbitrary. In a discrete setting one can influence the dynamics at certain points of time and α is a control vector that is piecewise constant in time. We now assume that even in a continuous time setup α is piecewise constant in time, i.e. we have to optimise over a vector of control states.

In order to ensure that a solution exists we have to make some assumptions.

For now let

$$
\begin{aligned}
&\text{(i)} \quad \text{the set of control values } A \text{ be compact and} \\
&\text{(ii)} \quad a(y,\alpha),\ b(y,\alpha) \text{ and } g(y,\alpha) \text{ be bounded and satisfy a} \\
&\qquad \text{Lipschitz condition with respect to } y. \\
&\text{(iii)} \quad b(y,\alpha) \geq 0 \\
&\text{(iv)} \quad \delta \geq 0
\end{aligned}
\tag{2.3}
$$

In a financial context point (iii) referes to non-negative volatilities and (iv) to non-negative discount rates.

2.2 Itō's formula and the dynamic programming principle

Now that we have formally defined all ingredients of the problem, we need some tools in order to deduce a PDE called HJB. The two main tools for deriving the HJB are Itō's formula and the dynamic programming principle.

2.2.1 Lemma (Itō's formula)**.** *Let y satisfy the stochastic process* (2.1)*. Any function $f(t, y)$ with $f : (\mathbb{R}_0^+ \times \Omega) \to \mathbb{R}$ which is once differentiable in the first argument and twice in the second is a stochastic process as well and satisfies*

$$df(t,y) = \frac{\partial f}{\partial t}dt + \sum_i \frac{\partial f}{\partial y_i}dy_i + \frac{1}{2}\sum_{(i,j)} \frac{\partial^2 f}{\partial y_i \partial y_j}dy_i dy_j \ . \tag{2.4}$$

Starting of from (2.4) we insert the process (2.1) and deduce

$$
\begin{aligned}
df(t,y) &= \frac{\partial f}{\partial t}dt + \sum_{i=1}^{d} a_i \frac{\partial f}{\partial y_i}dt + \sum_{i=1}^{d}\sum_{k=1}^{m} b_{i,k}\frac{\partial f}{\partial y_i}d\omega_k \\
&\quad + \frac{1}{2}\sum_{i=1}^{d}\sum_{j=1}^{d}\sum_{k=1}^{m}\sum_{l=1}^{m} b_{i,k}b_{j,l}\frac{\partial^2 f}{\partial y_i \partial y_j}d\omega_k d\omega_l \\
&= \frac{\partial f}{\partial t}dt + \sum_{i=1}^{d} a_i \frac{\partial f}{\partial y_i}dt + \sum_{i=1}^{d}\sum_{k=1}^{m} b_{i,k}\frac{\partial f}{\partial y_i}d\omega_k \\
&\quad + \frac{1}{2}\sum_{i,j=1}^{d}\sum_{k=1}^{m} b_{i,k}b_{j,k}\frac{\partial^2 f}{\partial y_i \partial y_j}dt \\
&= \Big(f_t + a\nabla f + \frac{1}{2}Tr(bb^T\Delta f)\Big)dt + (\nabla f)^T b d\omega \ ,
\end{aligned}
$$

where we dropped the arguments of f for notational simplicity. We will do so sometimes without further mentioning. Also, we did not look into more detail with regard to stochastic calculus. For further reference concerning this as well as a proof of 2.2.1 we refer to Karatzas and Shreve [11]. We only point out some facts about standard Brownian motion we used in the above derivation: on the one hand $d\omega_i d\omega_j = 0$ for $i \neq j$, $dtdt = 0$ and $d\omega_i dt = 0$ and on the other hand $d\omega_i d\omega_i = dt$ [1]. After the first equality sign we already dropped terms including $dtdt$ and $dtd\omega_i$; after the second we dropped terms including $d\omega_i d\omega_j = 0$ with $i \neq j$. The above equation can be rephrased as:

$$\begin{aligned} &f(t_2, y(t_2)) - f(t_1, y(t_1)) \\ &= \int_{t_1}^{t_2} \Big(f_t(t, y(t)) + a\nabla f(t, y(t)) + \frac{1}{2} b^2 \Delta f(t, y(t)) \Big) dt \\ &\quad + \int_{t_1}^{t_2} b\nabla f(t, y(t)) d\omega \ . \end{aligned} \tag{2.5}$$

We will use Itō's formula henceforth in that manner. We abused notations here to keep the formulas well-arranged. The term b^2 should actually be bb^T and we shifted the matrices in the second integral so this notation is intuitive.

The Bellman principle of optimality states that if we have an optimal control $\hat{\alpha}(t)$ for an optimal control function $\sup \int f dt$ and restrict it to a subinterval $t_0 \leq t \leq t_1$, the control is still optimal for this interval with regard to the state resulting from the previous decisions[2]. Indeed if it was not like this, there would be a control $\alpha_0(t)$ which implies a higher utility for $t_0 \leq t \leq t_1$. But then

$$\begin{aligned} \int_0^T g(\hat{\alpha}) &= \int_0^{t_1} g(\hat{\alpha}) + \int_{t_1}^{t_2} g(\hat{\alpha}) + \int_{t_2}^T g(\hat{\alpha}) \\ &< \int_0^{t_1} g(\hat{\alpha}) + \int_{t_1}^{t_2} g(\alpha_0) + \int_{t_2}^T g(\hat{\alpha}) \end{aligned}$$

defines a new control that indicates a higher utility than $\hat{\alpha}$ which is a contradiction. In stochastic control theory this is called the dynamic programming principle:

2.2.2 Theorem (Dynamic programming principle). *Let* $(t, y) \in [0, T] \times \mathbb{R}^n$

[1] the conventional "multiplicational table", cf. [11, p. 154]

[2] "An optimal policy has the property that whatever the initial state and initial decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision.", Bellman, R.: Dynamic Programming. Princeton, 1957

and v as in (3.3.1). *Then*

$$v(t,y) = \sup_{\alpha\in\mathfrak{A}} \sup_{\theta\in(t,T)} \mathbb{E}\left[\int_t^\theta e^{-\delta\tau} g(\phi(\tau),\alpha(\tau))\ d\tau\ + e^{-\delta\theta} v(\theta,\phi(\theta))\right]$$
$$= \sup_{\alpha\in\mathfrak{A}} \inf_{\theta\in(t,T)} \mathbb{E}\left[\int_t^\theta e^{-\delta\tau} g(\phi(\tau),\alpha(\tau))\ d\tau\ + e^{-\delta\theta} v(\theta,\phi(\theta))\right] .$$

A proof can be found in Pham [18]. A more intuitive version of 2.2.2 is:

2.2.3 Remark (Dynamic programming principle (intuitive version)). *With the above asumptions and notations the following statements hold true:*
(i) For all $\alpha\in\mathfrak{A}$ and $t<\theta<T$ we have

$$v(t,y) \geq \mathbb{E}\left[\int_t^\theta e^{-\delta\tau} g(\phi(\tau),\alpha(\tau))\ d\tau\ + e^{-\delta\theta} v(\theta,\phi(\theta))\right] .$$

(ii) For all $\epsilon>0$, there exists an $\alpha\in\mathfrak{A}$, such that for all $t<\theta<T$ we have

$$v(t,y)-\epsilon \leq \mathbb{E}\left[\int_t^\theta e^{-\delta\tau} g(\phi(\tau),\alpha(\tau))\ d\tau\ + e^{-\delta\theta} v(\theta,\phi(\theta))\right] .$$

We used discounting to time 0 here but of course both 2.2.2 and 2.2.3 also hold for discounting to time t with the obvious changes. The latter formulation 2.2.3 however is only valid in the finite horizon case while 2.2.2 also holds in the infinite horizon case.

2.3 The Hamilton-Jacobi-Bellman Equation

We now take the optimal value function defined by (3.3.1) and apply the dynamic programming principle 2.2.2 with choosing the interval to be $[t;t']$ with $t<t'$ and $t'-t=\Delta t$. That yields with $\phi(\tau)=\phi(\tau,y,\alpha,\omega)$

$$\begin{aligned}
v(t,y) &= \sup_{\alpha\in\mathfrak{A}} \mathbb{E}_{\phi(t)=y}\left[\int_t^T e^{-\delta(\tau-t)} g(\phi(\tau),\alpha)\ \mathrm{d}\tau\ + e^{-\delta(T-t)}\bar{g}(\phi(T))\right] \\
&= \sup_{\alpha\in\mathfrak{A}} \mathbb{E}_{\phi(t)=y}\Bigg[\int_t^{t'} e^{-\delta(\tau-t)} g(\phi(\tau),\alpha)\ \mathrm{d}\tau \\
&\qquad + e^{-\delta\Delta t}\left(\int_{t'}^T e^{-\delta(\tau-t)} g(\phi(\tau),\alpha)\ \mathrm{d}\tau\ + e^{-\delta(T-t)}\bar{g}(\phi(T))\right)\Bigg] \\
&= \sup_{\alpha\in\mathfrak{A}} \mathbb{E}_{\phi(t)=y}\left[\int_t^{t'} e^{-\delta(\tau-t)} g(\phi(\tau),\tau),\alpha) d\tau + e^{-\delta\Delta t} v(\phi(t'),t')\right] \\
&\approx \sup_{\alpha\in\mathfrak{A}} \left(e^{-\delta\Delta t} g(y,\alpha)\Delta t + e^{-\delta\Delta t}\ \mathbb{E}_{\phi(t)=y}\left[v(\phi(t'),t')\right]\right)
\end{aligned}$$

where g is assumed to be smooth enough so that $e^{-\delta\Delta t}g(y,\alpha)\Delta t$ approximates $\int_t^{t'} e^{-\delta(\tau-t)}g(\phi(\tau),\alpha(\tau))d\tau$. Now we apply Itō's formula to the term $v(y(t),t)$. We know that ϕ follows an Itō process. Thus Itō's lemma states that $v(\phi)$ follows an Itō process as well and we deduce according to (2.5)

$$\begin{aligned} & v(\phi(t+\Delta t),t+\Delta t) - v(\phi(t),t) \\ &= \int_t^{t+\Delta t} \Big[v_t(\phi(\tau),\tau) - \delta v(\phi(\tau),\tau) \\ &\quad + a\cdot\nabla v(\phi(\tau),\tau) + \frac{1}{2}b^2\Delta v(\phi(\tau),\tau)\Big] d\tau \\ &\quad + \int_t^{t+\Delta t} b\nabla v(\phi(\tau),\tau)d\omega \ . \end{aligned}$$

Next we take the expected value of both sides. The expected value of the second integral vanishes. In addition we assume that we know the value of ϕ at time t. Hence we substitute the expected value of $\phi(t)$ by its known value y. Thus we derive

$$\begin{aligned} & \mathbb{E}_{\phi(t)=y}\Big[v(\phi(t+\Delta t),t+\Delta t)\Big] - v(\phi(t),t)) \\ &= \mathbb{E}_{\phi(t)=y}\int_t^{t+\Delta t} \Big[v_t(\phi(\tau),\tau) - \delta v(\phi(\tau),\tau) \\ &\quad + a\cdot\nabla v(\phi(\tau),\tau) + \frac{1}{2}b^2\Delta v(\phi(\tau),\tau)\Big] \ \mathrm{d}\tau \\ &\approx \Delta t\ \Big[v_t(y,t) - \delta v(y,t) + a\cdot\nabla v(y,t) + \frac{1}{2}b^2\Delta v(y,t)\Big] \ . \end{aligned}$$

By combining the above equations we conclude all in all

$$\begin{aligned} v(y,t) &= \sup_{\alpha\in\mathfrak{A}}\Big(\Delta t\ e^{-\delta\Delta t}g(y,\alpha) + \mathbb{E}\Big[e^{-\delta\Delta t}\ v(\phi(\Delta t),\Delta t)\Big]\Big) \\ &= \sup_{\alpha\in\mathfrak{A}}\Big(\Delta t\ e^{-\delta\Delta t}g(y,\alpha) + \Delta t\ e^{-\delta\Delta t}\Big[v_t(y,t) - \delta v(y,t) \\ &\quad + a\cdot\nabla v(y,t) + \frac{1}{2}b^2\Delta v(y,t)\Big] + v(y,t)\Big) \ . \end{aligned}$$

Since $v(y)$, $\delta v(y)$ and $v_t(y)$ do not depend on α we can extract them from the supremum. By dividing by Δt and thereafter taking the limit $\Delta t \to 0$ we obtain:

$$0 = v_t(y) - \delta v(y) + \sup_{\alpha\in\mathfrak{A}}\Big(g(y,\alpha) + a\cdot\nabla v(y) + \frac{1}{2}b^2\Delta v(y)\Big) \ .$$

This is the Hamilton-Jacobi-Bellman equation. We make a formal definiton:

2.3.1 Definition (Hamilton-Jacobi-Bellman Equation)**.**

$$v_t(y) - \delta v(y) + H(y, Dv(y), D^2v(y)) = 0 \tag{2.6}$$

is called Hamilton-Jacobi-Bellman equation (HJB) with

$$H(y,p,X) := \sup_{\alpha\in\mathfrak{A}} \left\{ \frac{1}{2} Tr(B(y,\alpha)X) + a(y,\alpha)\cdot p + g(y,\alpha) \right\}$$

where $p \in \mathbb{R}^N$ *and* $X \in S(N)$*.* H *is said to be the Hamiltonian.* Tr *is the trace operator,* $B(y,\alpha) = b(y,\alpha)b(y,\alpha)^t$ *and "* $\cdot$ *" is the Euclidean product in* $\mathbb{R}^N$*.*

In the case of discounting to time 0 the HJB would look slightly different. The term $-\delta t$ vanishes then. In the case of an infinite horizon and discounting to time t time does not appear in the HJB at all.

We also use the following notation for the differential operator $\mathfrak{L}$ with control vector α

$$\mathfrak{L}^\alpha v = -\delta v(y) + \frac{1}{2} b^2(y,\alpha) D^2 v(y) + a(y,\alpha) D v(y) \ . \tag{2.7}$$

Therefore (2.6) can be written as

$$0 = -v_t(y) - \sup_{\alpha\in\mathfrak{A}} \{\mathfrak{L}^\alpha v + g(y,\alpha)\} \ .$$

2.3.2 Remark. *At time* $t = T$ *no stochastic influence takes place and the value function is equal to the utility* $\bar{g}$ *of leaving wealth to heirs, i.e.* $v(T,\alpha(T)) = \bar{g}(\alpha(T))$*.*

A usual choice for the function $\bar{g}$ in economic applications is g itself when discounting to time t and $e^{-\delta T} g$ when discounting to time 0.

Equation (2.6) together with 2.3.2 forms an initial value problem, which is to solve backwards for $t < T$.

The optimal controls can be derived from the HJB via differentiation with regard to the control. Of course one has to check that the control is admissible, i.e. that it is an element of A defined above. If that is not the case we have to find the optimal admissible control; hence one has to solve the resulting constraint optimisation problem.

In financial applications constraints often corresponds to no borrowing constraints and minimal and maximal investment bounds. The examples we introduced in chapter 1 all seek to maximise utility. In the context of asset allocation the minimisation of the portfolio risk by postulation minimum return constraints, or vice versa maximising returns and postulation maximum risk constraints is a classical problem. The dynamic programming and HJB framework present an alternative to the well established Markovitz optimisation of the value at risk, which is in contrast to the dynamic programming framework commonplace in financial institutions[3].

[3] For an application of the dynamic programming framework with discussion of the closure to reality cf. LONG-TERM STRATEGIC ASSET ALLOCATION: AN OUT-OF-SAMPLE EVALUATION?, BART DIRISA and FRANZ PALAMA and PETER SCHOTMAMA, July 14, 2009.

Chapter 3

The viscosity solution

In the last chapter we derived a PDE, namely the HJB, from the stochastic control problem. The central question in this chapter is: Is the solution of this HJB also the solution to the stochastic control problem? This issue is not straight forward since neither the existence of solutions to HJBs is clear yet, nor are solutions to HJBs in general unique[1]. Moreover the HJBs are nonlinear and thus have no classical solution in general. We have to seek a "weak" form of a solution. Amongst the solutions we will find the relevant one for economic applications and the answer to our question will be *yes*.

We first define a weak form of solutions, namely viscosity solutions, and then show that the solution to the stochastic control problem, which is the optimal value function (3.3.1), is the unique viscosity solution for the HJB (2.6).

3.1 Notations and Definitions

A key property in the theory of viscosity solutions is a ellipticity condition:

3.1.1 Definition (degenerate elliptic)**.** $\Omega \subseteq \mathbb{R}^N$. *Any equation of the form*

$$F(x, u, Du, D^2u) = 0 \qquad \textit{in } \Omega$$

where $F : \Omega \times \mathbb{R} \times \mathbb{R}^N \times S(N) \to \mathbb{R}$ *is called a degenerate elliptic equation if for all* X, $Y \in S(N)$ *with* $X \geq Y$ *the ellipticity condition*

$$F(x, u, p, X) \leq F(x, u, p, Y)$$

[1] For an example with several solutions of which none is a classical solution see Kohn [12].

is satisfied for all $x \in \Omega$, $u \in \mathbb{R}$ *and* $p \in \mathbb{R}^N$.

The HJB we derived and all the examples from 1 are time dependent, parabolic equations. In fact, parabolic equations are a particular case of degenerate elliptic equations. In order to see this, we have to merge the time variable t and the space variable y into one variable $x = (t, y)$. Next we need to multiply the HJB (2.6) by -1 and then we can write it as $F(x, u, Du, D^2u) = 0$ which fits the conditions of 3.1.1 since $D^2_{tt}u$ does not explicitly appear in the equation. Du and D^2u now denote the gradient and the Hessian with respect to $x = (t, y)$ and not only with respect to the space variable y.

3.1.2 Remark. *Let* $f \in C(\Omega)$ *and* $F : \mathbb{R} \times \mathbb{R}^N \times S(N) \to \mathbb{R}$. *Any equation of the form*

$$F(u, p, X) - f(x) + \gamma u = 0$$

is degenerate elliptic if $F(u, p, X) = 0$ *is degenerate elliptic.*

A proof is given in Crandall et al. [6]. Later on in the proofs of this chapter we will neither use remark 3.1.2 nor the ellipticity property of 3.1.1. Instead we will use the concrete form of the HJB (2.6) which is only a technical difference. Clearly $F = -H$, where H is the Hamiltonian in (2.6) satisfies the ellipticity condition in 3.1.1. We claim that (2.6) now fits the condition of remark 3.1.2 if condition 2.3 is satisfied. More precisely, the condition that insures degenerate ellipticity is 2.3 (iii).

We present some notations. $B_r(x)$ denotes the ball with radius r around x; accordingly B_r denotes the ball with radius r around 0. As defined earlier $S(N)$ denominates the symmetric $N \times N$ matrices. $USC(\Omega)$ (resp. $LSC(\Omega)$) are the upper semi-continuous (resp. lower semi-continuous) functions on Ω into $(\mathbb{R} \cup \{-\infty\})$ (resp. $(\mathbb{R} \cup \{+\infty\})$. Often $\pm\infty$ are not included in the image sets in the definition of USC and LSC but it is needed later on. It enables us to expand the functions from Ω to $\mathbb{R}$. $N(x)$ always denotes a neighbourhood of x.

3.1.3 Definition (sub- and superjet)**.** *Let* $u : \Omega \to \mathbb{R}$ *and* $\hat{x} \in \Omega$.
We define the second-order superjet $J^{2,+}_\Omega u(\hat{x})$ *of* u *at* $\hat{x}$ *as the set*

$$J^{2,+}_\Omega u(\hat{x}) :=$$
$$\left\{ (p, X) \;\middle|\; \lim_{x \to \hat{x}} \frac{u(\hat{x}) - u(x) + < p, x - \hat{x} > + \frac{1}{2} < X(x - \hat{x}), x - \hat{x} >}{|x - \hat{x}|^2} \geq 0 \right\} ,$$

where $(p, X) \in (\mathbb{R}^N \times S(N))$. *Accordingly, we define the second-order subjet*

$J_\Omega^{2,-}u(\hat{x})$ of u at $\hat{x}$ as the set

$$J_\Omega^{2,-}u(\hat{x}) := \left\{(p,X) \;\middle|\; \lim_{x\to\hat{x}} \frac{u(\hat{x})-u(x)+<p,x-\hat{x}>+\frac{1}{2}<X(x-\hat{x}),x-\hat{x}>}{|x-\hat{x}|^2} \leq 0\right\} .$$

Furthermore we define the closures of the above sets:

$$\overline{J}_\Omega^{2,+}u(\hat{x}) := \Big\{(p,X) \;\Big|\; \exists(x_n,p_n,X_n)\in(\Omega\times\mathbb{R}^N\times S(N)),$$
$$(p_n,X_n)\in J_\Omega^{2,+}u(\hat{x}_n) \text{ and } (x_n,u(x_n),p_n,X_n)\to(x,u(x),p,X)\Big\} ,$$

respectively

$$\overline{J}_\Omega^{2,-}u(\hat{x}) := \Big\{(p,X) \;\Big|\; \exists(x_n,p_n,X_n)\in(\Omega\times\mathbb{R}^N\times S(N)),$$
$$(p_n,X_n)\in J_\Omega^{2,-}u(\hat{x}_n) \text{ and } (x_n,u(x_n),p_n,X_n)\to(x,u(x),p,X)\Big\} .$$

The definition of the closures is not straightforward because $J_\Omega^{\pm,2}u$ does not record the values of u while the closures have the extra condition $u(x_n)\to u(x)$. Now we are able to define the term viscosity solution.

3.1.4 Definition (viscosity solution)**.** *A function $u \in USC(\Omega)$ is called a viscosity subsolution of* (2.6) *if*

$$-\Big(u_t(x)-\delta u(x)+H(x,p,X)\Big)\leq 0 ,$$

for all $x\in\Omega$ and $(p,X)\in J_\Omega^{2,+}u(x)$. A function $u\in LSC(\Omega)$ is called a viscosity supersolution of (2.6) *if*

$$-\Big(u_t(x)-\delta u(x)+H(x,p,X)\Big)\geq 0 ,$$

for all $x\in\Omega$ and $(p,X)\in J_\Omega^{2,-}u(x)$. A function u is called a viscosity solution of (2.6) *if u is a viscosity subsolution and a viscosity supersolution of* (2.6).

A viscosity solution u with $u\in C^2(\Omega)$ is called a classical solution.

This definition is not intuitive. The notion of viscosity solution was introduced by Crandall and Lions [5] for first order HJB equations. The motivation was to enable a global analysis in terms of uniqueness and existence of solutions of HJB equations that do not need to be differentiable anywhere. As mentioned above, most of the relevant equations do not have a classical solution. More about the properties of sub- and superjets, as well as viscosity solutions can be found in Crandall et al. [6], Lions [14], Bardi and Capuzzo-Dolcetta [1].

The odd-looking minus sign outside the parenthesis is due to the fact that it has to fit the direction of the inequality sign of the usual definitions of viscosity solutions since one always presumes degenerate elliptic equations.

3.2 About viscosity solutions

We point out some additional facts about sub- and superjets. We use most of these properties later on. If not mentioned differently, the proofs can be found in Crandall et al. [6], Lions [14], Evans [7], Bardi and Capuzzo-Dolcetta [1].

3.2.1 Remark.

$$J_\Omega^{2,+}u(\hat{x}) = -J_\Omega^{2,-}(-u)(\hat{x})$$
$$\overline{J}_\Omega^{2,+}u(\hat{x}) = -\overline{J}_\Omega^{2,-}(-u)(\hat{x})$$

Respectively we get equations for $J_\Omega^{2,-}u(\hat{x})$ and $\overline{J}_\Omega^{2,-}u(\hat{x})$).

3.2.2 Remark. *$J_\Omega^{2,+}u(\hat{x})$ (resp. $\overline{J}_\Omega^{2,-}u(\hat{x})$) is a convex subset in $\mathbb{R}^N \times S(N)$. It may be empty.*

3.2.3 Remark. *For all sets Ω where $\hat{x}$ is an interior point the superjets $J_\Omega^{2,+}u(\hat{x})$ are equal and we denote this set with $J^{2,+}u(\hat{x})$. Respectively we get results for the subjet $J_\Omega^{2,-}u(\hat{x})$ and their closures.*

3.2.4 Lemma. *Let $\varphi \in C^2(\Omega)$ and $(p, X) \in J_\Omega^{2,+}u(\hat{x})$. Then we have*

$$J_\Omega^{2,+}(u-\varphi)(\hat{x}) = \left\{ \Big(p - D\varphi(\hat{x}), X - D^2\varphi(\hat{x})\Big) \;\Big|\; (p, X) \in J_\Omega^{2,+}u(\hat{x}) \right\} .$$

The same holds for $J_\Omega^{2,-}u(\hat{x})$, $\overline{J}_\Omega^{2,+}u(\hat{x})$ and $\overline{J}_\Omega^{2,+}u(\hat{x})$.

Proof. Let $(q, Y) \in J_\Omega^{2,+}(u-\varphi)(\hat{x})$, i.e.

$$u(x) - \varphi(x) \le u(\hat{x}) - \varphi(\hat{x}) + \langle q, x - \hat{x}\rangle + \frac{1}{2}\langle Y(x-\hat{x}), x - \hat{x}\rangle + o(|x-\hat{x}|^2) ,$$

as $x \to \hat{x}$. Further since $\varphi \in C^2(\Omega)$ we obtain

$$\varphi(x) \le \varphi(\hat{x}) + \langle D\varphi(\hat{x}), x - \hat{x}\rangle + \frac{1}{2}\left\langle D^2(\hat{x})(x-\hat{x}), x - \hat{x}\right\rangle + o(|x-\hat{x}|^2) , \quad (3.1)$$

as $x \to \hat{x}$. Adding both inequalities yields

$$u(x) \leq u(\hat{x})+\langle p + D\varphi(\hat{x}), x-\hat{x}\rangle+\frac{1}{2}\left\langle Y + D^2(\hat{x})(x-\hat{x}), x-\hat{x}\right\rangle+o(|x-\hat{x}|^2) \ ,$$

as $x \to \hat{x}$ and therefore $(q + D\varphi(\hat{x}), Y + D^2\varphi(\hat{x})) \in J^{2,+}_\Omega u(\hat{x})$. That implies

$$J^{2,+}_\Omega(u-\varphi)(\hat{x}) \subseteq \left\{ \left(p - D\varphi(\hat{x}), X - D^2\varphi(\hat{x})\right) \,\middle|\, (p,X) \in J^{2,+}_\Omega u(\hat{x}) \right\} .$$

On the other hand $(p, X) \in J^{2,+}_\Omega u(\hat{x})$, hence

$$u(x) \leq u(\hat{x}) + \langle p, x-\hat{x}\rangle + \frac{1}{2}\langle X(x-\hat{x}), x-\hat{x}\rangle + o(|x-\hat{x}|^2) \ ,$$

and by substracting (3.1) we conclude

$$J^{2,+}_\Omega(u-\varphi)(\hat{x}) \supseteq \left\{ \left(p - D\varphi(\hat{x}), X - D^2\varphi(\hat{x})\right) \,\middle|\, (p,X) \in J^{2,+}_\Omega u(\hat{x}) \right\} .$$

□

The following Lemma follows the basic idea of Evans [7], only modified to fit the notion of superjets and subjets respectively. We formulate the result for both superjets and subjets but only give the proof in case of superjets since the modification concerning subjets are obvious.

3.2.5 Lemma. *Let $u : \Omega \to \mathbb{R}$ and $\hat{x} \in \Omega$. If there is a $\Theta > 0$ and a neighbourhood $N(\hat{x})$ of $\hat{x}$ such that*

(i)	$(0, X) \in J^{2,+}_\Omega u(\hat{x})$	*and*
(ii)	$\langle X(x-\hat{x}), x-\hat{x}\rangle \leq -\Theta\|x-\hat{x}\|^2$	*for all $x \in N(\hat{x})$*

hold, u has a strict local maximum at $\hat{x}$.

Proof. Without loss of generality we assume $\hat{x} = u(\hat{x}) = 0$. We proof the remark by contradiction: If u did not have a strict local maximum at $\hat{x}$, there would be a sequence $z_n \in N(\hat{x})$ with $\lim_{n\to\infty} z_n = 0$, $z_n \neq 0$ and $u(z_n) \geq 0$. Condition (i) yields

$$u(x) \leq u(\hat{x}) + \langle 0, x-\hat{x}\rangle + \frac{1}{2}\langle X(x-\hat{x}), (x-\hat{x})\rangle + o(|x-\hat{x}|^2) \ .$$

Using $\hat{x} = u(\hat{x}) = 0$ and inserting condition (ii) we derive for $x \in N(\hat{x})$

$$\begin{aligned} u(x) &\leq \frac{1}{2}\langle Xx, x\rangle + o(|x|^2) \\ &\leq -\frac{1}{2}\Theta|x|^2 + o(|x|^2) \ . \end{aligned}$$

Now we plug in z_n and then by dividing by $|z_n|^2$ we deduce

$$0 \leq \frac{u(z_n)}{|z_n|^2} \leq -\frac{1}{2}\Theta + \frac{o(|z_n|^2)}{|z_n|^2}$$

and by taking the limit $n \to \infty$ we get the contradiction.

□

3.2.6 Lemma. *If $\hat{x} \in \Omega$ the following equation holds:*

$$\begin{aligned} & J_\Omega^{2,+} u(\hat{x}) \\ &= \left\{ (D\varphi(\hat{x}), D^2\varphi(\hat{x})) \;\middle|\; \varphi \in C^2(\Omega) \text{ and } u - \varphi \text{ has a local maximum at } \hat{x} \right\} \end{aligned}$$

Proof. Let $\varphi \in C^2(\Omega)$ be any function such that $u - \varphi$ has a local maximum at $\hat{x}$. We have to show that $(D\varphi(\hat{x}), D^2\varphi(\hat{x})) \in J_\Omega^{2,+} u(\hat{x})$. Since $u - \varphi$ has a maximum at $\hat{x}$, the inequality $u(x) - \varphi(x) \leq u(\hat{x}) - \varphi(\hat{x})$ holds in some neighbourhood $N(\hat{x})$ of $\hat{x}$. As $\varphi \in C^2$, we can develop φ in a Taylor series around $\hat{x}$:

$$\varphi(x) = \varphi(\hat{x}) + \langle D\varphi(\hat{x}), x - \hat{x} \rangle + \frac{1}{2} \left\langle D^2\varphi(\hat{x})(x - \hat{x}), x - \hat{x} \right\rangle + o(|x - \hat{x}|^2) \ .$$

In $N(\hat{x})$ we derive

$$\begin{aligned} u(x) &\leq u(\hat{x}) + \langle D\varphi(\hat{x}), x - \hat{x} \rangle + \frac{1}{2} \left\langle D^2\varphi(\hat{x})(x - \hat{x}), x - \hat{x} \right\rangle + o(|x - \hat{x}|^2) \\ o(|x - \hat{x}|^2) &\geq u(\hat{x}) - u(x) + \langle D\varphi(\hat{x}), x - \hat{x} \rangle + \frac{1}{2} \left\langle D^2\varphi(\hat{x})(x - \hat{x}), x - \hat{x} \right\rangle \ . \end{aligned}$$

Finally, by dividing by $|x - \hat{x}|^2$ we immediately get the result as $x \to \hat{x}$.

Now we have to show the reverse direction. The goal is to find a series of functions φ_k that converge uniformly to u in $\hat{x}$, such that $D\varphi_k(\hat{x}) \to p$ and $D^2\varphi_k(\hat{x}) \to X$ for every $(p, X) \in J_\Omega^{2,+} u(\hat{x})$. The limit function will satisfy the right-hand side of the equation in 3.2.6. We start by assuming that $(p, X) \in J_\Omega^{2,+} u(\hat{x})$, i.e.

$$u(x) \leq u(\hat{x}) + \langle p, x - \hat{x} \rangle + \frac{1}{2} \langle X(x - \hat{x}), x - \hat{x} \rangle + o(|x - \hat{x}|^2) \ .$$

We can find a sequence $v_k \in C^2(\Omega)$ which converges uniformly to u and that satisfies

$$\begin{aligned} & \|v_k - u\|_{C(\bar{\Omega})} \leq \frac{1}{k} \qquad \text{for } k = 1, 2, \ldots \qquad \text{and} \\ & v_k(\hat{x}) = u(\hat{x}) \ , \qquad Dv_k(\hat{x}) = p \ , \qquad D^2 v_k(\hat{x}) = X \ . \end{aligned}$$

For example on a neighbourhood $N(\hat{x})$ of $\hat{x}$ we can set

$$v_k(x)\Big|_{N(\hat{x})} = u(\hat{x}) + p(x-\hat{x}) + \frac{1}{2}\langle X(x-\hat{x}), x-\hat{x}\rangle \ .$$

Next we choose a $w \in C^2(\Omega)$ with $0 \le w \le 1$, $w(\hat{x}) = 1$, $Dw(\hat{x}) = 0$ and $\big\langle D^2w(\hat{x})(x-\hat{x}), x-\hat{x}\big\rangle \le -|x-\hat{x}|^2, \quad \forall\, x \in \mathbb{R}^n$. We define ψ_k as

$$\psi_k(x) := u(x) - v_k(x) + \frac{1}{k}w(x) \ .$$

With 3.2.4 we check that ψ_k fulfils the premises of 3.2.5 in $\hat{x}$ and we conclude that $\hat{x}$ is a strict local maximum of ψ_k. Now we choose a sequence $z_k \in C^2(\Omega)$ satisfying

$$0 \le z_k \le \frac{2}{k} \ , \qquad \operatorname{supp}(z_k) \subset B_{r_k}(\hat{x}) \ ,$$
$$z_k(\hat{x}) = \frac{2}{k} \qquad \text{and} \qquad Dz_k(\hat{x}) = D^2 z_k(\hat{x}) = 0 \ .$$

Further we define $\varphi_k \in C^2(\Omega)$ as

$$\varphi_k(x) := v_k(x) - \frac{1}{k}w(x) - z_k(x) \ .$$

We take the limit $k \to \infty$ to obtain

$$\begin{aligned}
\lim_{k\to\infty} \varphi_k(\hat{x}) &= \lim_{k\to\infty} \Big(v_k(\hat{x}) - \frac{1}{k}w(\hat{x}) - z_k(\hat{x})\Big) \\
&= \lim_{k\to\infty} \Big(v_k(\hat{x}) - \frac{3}{k}\Big) \\
&= \lim_{k\to\infty} \Big(u(\hat{x}) - \frac{3}{k}\Big) \\
&= u(\hat{x}) \ ,
\end{aligned}$$

consecutively

$$\begin{aligned}
\lim_{k\to\infty} D\varphi_k(\hat{x}) &= \lim_{k\to\infty} \Big(Dv_k(\hat{x}) - \frac{1}{k}Dw(\hat{x}) - Dz_k(\hat{x})\Big) \\
&= \lim_{k\to\infty} \Big(p - \frac{1}{k}Dw(\hat{x})\Big) \\
&= p \ ,
\end{aligned}$$

and finally

$$\begin{aligned}
\lim_{k\to\infty} D^2\varphi_k(\hat{x}) &= \lim_{k\to\infty} \Big(D^2v_k(\hat{x}) - \frac{1}{k}D^2w(\hat{x}) - D^2z_k(\hat{x})\Big) \\
&= \lim_{k\to\infty} \Big(X - \frac{1}{k}D^2w(\hat{x})\Big) \\
&= X \ .
\end{aligned}$$

In addition we derive for all $x \in \Omega \setminus B_{r_k}(\hat{x})$

$$\begin{aligned} |u(x) - \varphi_k(x)| &\leq |u(x) - v_k(x)| + \frac{1}{k}|w(x)| + z_k(x) \\ &\leq \frac{2}{k} < \frac{3}{k} = u(\hat{x}) - \varphi(\hat{x}) \ , \end{aligned}$$

since z_k vanishes outside $B_{r_k}(\hat{x})$. Contrariwise, for all $x \in B_{r_k}(\hat{x})$ we deduce

$$\begin{aligned} u(x) - \varphi_k(x) &= \underbrace{u(x) - v_k(x) + \frac{1}{k}w(x)}_{=\psi_k(x)<\psi_k(\hat{x})} + \underbrace{z_k(x)}_{\leq z_k(\hat{x})} \\ &< u(\hat{x}) - v_k(\hat{x}) + \frac{1}{k}w(\hat{x}) + z_k(\hat{x}) \\ &= u(\hat{x}) - \varphi_k(\hat{x}) \ . \end{aligned}$$

Hence $u(x) - \varphi_k(x) < u(\hat{x}) - \varphi_k(\hat{x})$ holds in whole $\Omega \setminus \{\hat{x}\}$, and we conclude that $\hat{x}$ is a strict maximum of $u - \varphi_k$. Moreover we obtain

$$u(x) - \varphi_k(x) = u(x) - v_k(x) + \frac{1}{k}w(x) + z_k(x) \geq u(x) - v_k(x) \geq -\frac{1}{k} \ .$$

Therefore $u(x) - \varphi_k(x) < \|u - \varphi_k\|$, for $x \neq \hat{x}$ and $u(\hat{x}) - \varphi_k(\hat{x}) = \|u - \varphi_k\|$ hold in Ω. Due to the construction of φ_k, the limit $\varphi = \lim_{k\to\infty} \varphi_k$ is meaningful and still in $C^2(\Omega)$. We found the desired testfunction φ to complete the proof.

□

Lemma 3.2.6 permits an alternative definition of a viscosity sub- and super solution.

3.2.7 Definition (viscosity solution - alternative to 3.1.4)**.** *A function $\nu \in USC(\Omega)$ is called a viscosity subsolution of* (2.6) *if for all $\varphi \in \mathcal{C}^2$ the following holds:*

$$\begin{aligned} &\nu - \varphi \text{ has a loc. max. at } \hat{y} \in \mathbb{R}^d \\ \Rightarrow &-\Big(\nu_t(\hat{y}) - \delta\nu(\hat{y}) + H(\hat{y}, D\varphi(\hat{y}), D^2\varphi(\hat{y}))\Big) \leq 0 \end{aligned}$$

A function $\nu \in LSC(\Omega)$ is called a viscosity subsolution of (2.6) *if for all $\varphi \in \mathcal{C}^2$ the following holds:*

$$\begin{aligned} &\nu - \varphi \text{ has a loc. min. at } \hat{y} \in \mathbb{R}^d \\ \Rightarrow &-\Big(\nu_t(\hat{y}) - \delta\nu(\hat{y}) + H(\hat{y}, D\varphi(\hat{y}), D^2\varphi(\hat{y}))\Big) \geq 0 \end{aligned}$$

A function ν is called a viscosity solution of (2.6) *if ν is a viscosity subsolution and a viscosity supersolution of* (2.6)*.*

3.3 Existence and uniqueness of the viscosity solution

Now that we have gathered all the tools we are able to give the main results of this chapter. First we will show that the solution to the optimal control problem is a viscosity solution of the HJB that results from it. Then we will show uniqueness and existance of the viscosity solution.

3.3.1 Theorem. *With conditions* (2.3) *the solution to the optimal control problem is a viscosity solution of the HJB* (2.6).

Proof. Let v be the solution to the optimal control problem. We first show that v is a viscosity subsolution and accordingly that v is a viscosity supersolution.

Let $\varphi \in \mathcal{C}^2(\mathbb{R}^N \times \mathbb{R}^+)$ and $\hat{y} \in \mathbb{R}^N$ be chosen so that $v - \varphi$ has a local minimum at $\hat{y}$. We have to show that $v_t(\hat{y},t) - \delta v(\hat{y},t) + H(\hat{y}, D\varphi(\hat{y},t), D^2\varphi(\hat{y},t)) \leq 0$ holds.

With no loss of generality let $v(\hat{y},t) = \varphi(\hat{y},t)$ or else replace φ with $\tilde{\varphi}(\tilde{y},t) = \varphi(\tilde{y},t) + v(\hat{y},t) - \varphi(\hat{y},t)$. We again define $t' := t + \Delta t$. For every control α we set $\phi(t,y,\alpha) = \phi(t,y,\alpha,\omega)$. Now we choose an arbitrary control $\hat{\alpha}$ and get with the dynamic programming principle 2.2.3 (i) for every $t > 0$

$$v(y,t) \geq \int_t^{t'} e^{-\delta(\tau - t)} g(\phi(\tau, y, \hat{\alpha}), \alpha) d\tau + v(\phi(t', y, \hat{\alpha}), t') \ .$$

Since $v(\hat{y},t) - \varphi(\hat{y},t) = 0$ is a local minimum we know that in a neighbourhood $N(\hat{y})$ of $\hat{y}$ the inequality $v(y,t) \geq \varphi(y,t)$ for all $y \in N(\hat{y})$ holds. Further, since a and b are bounded, we conclude

$$\exists t_1 \geq t \text{ with } \phi(t, \hat{y}, \hat{\alpha}) \in N(\hat{y}) \text{ for all } t \in (t, t_1) \ .$$

Hence we obtain for all $t \in (t, t_1)$ according to the dynamic programming principle 2.2.2

$$\begin{aligned}
\varphi(\hat{y},t) &= v(\hat{y},t) \\
&= \sup_{\alpha \in \mathfrak{A}} \mathbb{E}\left[\int_t^{t'} e^{-\delta(\tau - t)} g(\phi(\tau, \hat{y}, \alpha), \alpha) d\tau + v(\phi(t', \hat{y}, \alpha), t')\right] \\
&\geq \mathbb{E}\left[\int_t^{t'} e^{-\delta(\tau - t)} g(\phi(\tau, \hat{y}, \hat{\alpha}), \hat{\alpha})) d\tau + v(\phi(t', \hat{y}, \hat{\alpha}), t')\right] \\
&\geq \mathbb{E}\left[\int_t^{t'} e^{-\delta(\tau - t)} g(\phi(\tau, \hat{y}, \hat{\alpha}), \hat{\alpha}) d\tau\right] + \mathbb{E}\left[\varphi(\phi(t', \hat{y}, \hat{\alpha}), t')\right] \ .
\end{aligned}$$

Now we use Itō's lemma. φ is twice continuously differentiable and ϕ follows an Itō process. According to equation (2.5) (Itō's lemma) we obtain for $\varphi(\phi, t)$ with surpressing the time argument t

$$
\begin{aligned}
&\varphi(\phi(t+\Delta t)) - \varphi(\phi(t)) \\
= &\int_t^{t+\Delta t} \Big[\Big(\varphi_t(\phi(s)) + \nabla\varphi(\phi(s)) \cdot a + \frac{1}{2} b^2 \Delta\varphi(\phi(s))\Big) - \delta\varphi(\phi(s))\Big] \text{ ds} \\
&+ \int_t^{t+\Delta t} b \nabla\varphi(\phi(s)) \text{ d}\omega \ .
\end{aligned}
$$

The expected value of the second integral vanishes. Therefore taking the expected value of the above equation yields

$$
\begin{aligned}
&\mathbb{E}\Big[\varphi(\phi(t+\Delta t))\Big] - \varphi(\phi(t)) \\
= &\mathbb{E}\Big[\int_t^{t+\Delta t} \Big(\varphi_t(\phi(s)) + \nabla\varphi(\phi(s)) \cdot a + \frac{1}{2} b^2 \Delta\varphi(\phi(s))\Big) - \delta\varphi(\phi(s)) \text{ ds } \Big] \\
\approx &\Big[\varphi_t(\phi(t)) + \nabla\varphi(\phi(t)) \cdot a + \frac{1}{2} b^2 \Delta\varphi(\phi(t))\Big] \Delta t - \delta\varphi(\phi(t))\Delta t \ ,
\end{aligned}
$$

whereas the last approximation is possible due to the fact that $\varphi \in C^2$ and a and b fulfil condition (2.3) and by presuming Δt is small enough.

Substituting the latter approximation in the last inequality we stated above, we finally arrive at

$$
\begin{aligned}
&\varphi(\hat{y}, t) \\
\geq &\mathbb{E} \int_t^{t'} e^{-\delta(\tau - t)} g(\phi(\tau, \hat{y}, \hat{\alpha}), \hat{\alpha}) \ d\tau + \varphi(\hat{y}, t) - \Delta t \ \delta\varphi(\phi(t, \hat{y}, \hat{\alpha}), t) \\
&\Delta t \left(\varphi_t(\phi(t, \hat{y}, \hat{\alpha}), t) + \nabla\varphi(\phi(t, \hat{y}, \hat{\alpha}), t) \cdot a + \frac{1}{2} b^2 \Delta\varphi(\phi(t, \hat{y}, \hat{\alpha}), t)\right) \ .
\end{aligned}
$$

And finally, by taking the limit $\Delta t \to 0$ (i.e. $t \to t'$) and approximating the integral with assuming that g is smooth enough, we deduce

$$
\varphi_t(\hat{y}, t) - \delta\varphi(\hat{y}, t) + \Big[g(\hat{y}, \alpha) + \nabla\varphi(\hat{y}, t) \cdot a(\hat{y}, \alpha) + \frac{1}{2} b^2(\hat{y}, \alpha) \Delta\varphi(\hat{y}, t)\Big] \leq 0 \ .
$$

Since that holds for every control $\hat{\alpha}$ and every $\hat{y}$ with $\varphi(\hat{y}, t) = v(\hat{y}, t)$ we conclude

$$
\begin{aligned}
&\varphi_t(\hat{y}, t) - \delta v(\hat{y}, t) \\
&\quad + \sup_{\alpha \in \mathfrak{A}} \Big[g(\hat{y}, \alpha) + \nabla\varphi(\hat{y}, t) \cdot a(\hat{y}, \alpha) + \frac{1}{2} b^2(\hat{y}, \alpha) \Delta\varphi(\hat{y}, t)\Big] \\
&= \varphi_t(\hat{y}, t) - \delta v(\hat{y}, t) + H(\hat{y}, \nabla\varphi(\hat{y}, t), \Delta\varphi(\hat{y}, t)) \\
&\leq 0 \ .
\end{aligned}
$$

Therefore v is a viscosity supersolution.

Now we show the first inequality of definition (3.2.7): Choose $\varphi \in \mathcal{C}^2(\mathbb{R}^N)$ and $\hat{y} \in \mathbb{R}^N$ so that $v - \varphi$ has a local maximum at $\hat{y}$. We have to show that $\varphi_t(\hat{y}, t) - \delta v(y, t) + H(\hat{y}, D\phi(\hat{y}, t), D^2\phi(\hat{y}, t)) \geq 0$. As above we assume $v(\hat{y}, t) = \varphi(\hat{y}, t)$.

Since $\hat{y}$ is a maximum, we can find a neighbourhood $N(\hat{y})$ so that $v(y, t) \leq \varphi(y, t)$ for all $y \in N(\hat{y})$. We are going to construct an $\alpha \in \mathfrak{A}$ with

$$v_t(\hat{y}, t) - \delta v(\hat{y}, t) + \left[g(\hat{y}, \alpha) + a \cdot D\varphi(\hat{y}, t) + \frac{1}{2} b^2 D^2 \varphi(\hat{y}, t) \right] \geq 0 \ . \qquad (3.2)$$

From the dynamic programming principle in the form 2.2.3 part (ii) we deduce that for every $n \in \mathbb{N}$ we can find a control α_n so that the supremum in $v(\hat{y})$ is approximated with an error less than $\frac{1}{n^2}$. Hence we conclude that there exists a sequence $(\alpha_n)_{n \in \mathbb{N}} \subset \mathfrak{A}$ such that

$$\begin{aligned} & v(\hat{y}, t) - \frac{1}{n^2} \\ \leq\ & \mathbb{E} \int_t^{t+\frac{1}{n}} e^{-\delta(\tau - t)} g(\phi(\tau, \hat{y}, \alpha_n), \alpha_n(\tau)) \, \mathrm{d}\tau \ + \mathbb{E} \left[v(\phi(t + \frac{1}{n}, \hat{y}, \alpha_n), t + \frac{1}{n}) \right] \end{aligned}$$

holds. Let n be sufficiently large. Then with condition (2.3) we claim that $\phi(t + \frac{1}{n}, \hat{y}, \alpha_n) \in N(\hat{y})$ and therefore $v(\phi(t + \frac{1}{n}, \hat{y}, \alpha_n), t + \frac{1}{n}) \leq \varphi(\phi(t + \frac{1}{n}, \hat{y}, \alpha_n), t + \frac{1}{n})$ while on the other hand $v(\hat{y}, t) = \varphi(\hat{y}, t)$ holds. Hence:

$$\begin{aligned} \varphi(\hat{y}, t) - \frac{1}{n^2} \leq \mathbb{E} & \int_t^{t+\frac{1}{n}} e^{-\delta(\tau - t)} g(\phi(\tau, \hat{y}, \alpha_n), \alpha_n(\tau)) \, \mathrm{d}\tau \\ & - \mathbb{E} \left[\varphi(\phi(t + \frac{1}{n}, \hat{y}, \alpha_n), t + \frac{1}{n}) \right] \ , \end{aligned}$$

which we rearrange and obtain

$$\begin{aligned} \frac{1}{n^2} \geq \varphi(\hat{y}, t) - \mathbb{E} & \int_t^{t+\frac{1}{n}} e^{-\delta(\tau - t)} g(\phi(\tau, \hat{y}, \alpha_n), \alpha_n(\tau)) \, \mathrm{d}\tau \\ - \mathbb{E} & \left[\varphi(\phi(t + \frac{1}{n}, \hat{y}, \alpha_n), t + \frac{1}{n}) \right] \ . \end{aligned}$$

Now we have to use Itō's lemma again. Although we are already familiar with that, we make the effort to write it down

$$\begin{aligned}
&\varphi(\phi(t+\frac{1}{n},\hat{y},\alpha_n),t+\frac{1}{n})-\varphi(\hat{y},t)\\
&=\int_t^{t+\frac{1}{n}}\Big[\varphi_t(\phi(\tau,\hat{y},\alpha_n),\tau)-\delta\varphi(\phi(\tau,\hat{y},\alpha_n),\tau)+a\cdot\nabla\varphi(\phi(\tau,\hat{y},\alpha_n),\tau)\\
&\quad+\frac{1}{2}b^2\Delta\varphi(\phi(\tau,\hat{y},\alpha_n),\tau)\Big]\ d\tau+\int_t^{t+\frac{1}{n}} b\cdot\nabla\varphi(\phi(\tau,\hat{y},\alpha_n),\tau)\ d\omega\ .
\end{aligned}$$

As, we take the expected value and plug it in the latter inequality to deduce

$$\begin{aligned}
\frac{1}{n^2}\geq -\mathbb{E}\int_t^{t+\frac{1}{n}}&\Big[\varphi_t(\phi(\tau,\hat{y},\alpha_n),\tau)-\delta\varphi(\phi(\tau,\hat{y},\alpha_n),\tau)\\
&+a\cdot\nabla\varphi(\phi(\tau,\hat{y},\alpha_n),\tau)+\frac{1}{2}b^2\Delta\varphi(\phi(\tau,\hat{y},\alpha_n),\tau)\\
&+g(\phi(\tau,\hat{y},\alpha_n),\alpha_n)e^{-\delta\tau-t}\Big]d\tau\ .
\end{aligned}$$

Next we want to proceed as usual, i.e. approximate the integral for $n\to\infty$, but there is some subtleness in the terms a, b and g since they depend on α_n. They might not converge to something meaningful so we have to make some effort before we can take the limit.

The integrand $\varphi_t+a\cdot\nabla\varphi+\frac{1}{2}b^2\Delta\varphi+ge^{-\delta(\tau-t)}:=I$ is Lipschitz continuous in ϕ and uniformly continuous in $\alpha_n(\tau)$. Let L be the Lipschitz constant.

$$\begin{aligned}
0&\leq\frac{1}{n^2}+\mathbb{E}\int_t^{t+\frac{1}{n}} I(\phi(\tau))\ d\tau\\
-\mathbb{E}\int_t^{t+\frac{1}{n}} I(\hat{y})\ d\tau&\leq\frac{1}{n^2}+\mathbb{E}\int_t^{t+\frac{1}{n}} I(\phi(\tau))-I(\hat{y})\ d\tau\\
-\mathbb{E}\int_t^{t+\frac{1}{n}} I(\hat{y})\ d\tau&\leq\frac{1}{n^2}+\mathbb{E}\int_t^{t+\frac{1}{n}} \|\hat{y}-\phi(\tau)\|L\ d\tau\\
-\mathbb{E}\int_t^{t+\frac{1}{n}} I(\hat{y})\ d\tau&\leq\frac{1}{n^2}+\frac{1}{n}\mathbb{E}\|\hat{y}-\phi(\tau)\|L\ .
\end{aligned}$$

Since a is bounded and $\mathbb{E}[b\ d\omega]=0$, we can estimate $\mathbb{E}\,\|\hat{y}-\phi(\tau,\hat{y},\alpha_n)\|=\|a\|\,\tau\leq M_a\tau$ for all $\alpha_n\in\mathfrak{A}$. We resubstitute I and get

$$\begin{aligned}
&-\mathbb{E}\int_t^{t+\frac{1}{n}}\Bigg[\varphi_t(\hat{y},\tau)+a(\hat{y},\alpha_n)\cdot\nabla\varphi(\hat{y},\tau)-\delta\varphi(\hat{y},\tau)\\
&+\frac{1}{2}b^2(\hat{y},\alpha_n)\Delta\varphi(\hat{y},\tau)+g(\hat{y},\alpha_n(\tau))e^{-\delta(\tau-t)}\Bigg]\ d\tau\leq\frac{1}{n^2}+\frac{LM_a}{n^2}\ .
\end{aligned}\tag{3.3}$$

Furthermore we note that

$$\begin{aligned}&\mathbb{E}\int_t^{t+\frac{1}{n}} g(\hat{y})e^{-\delta(\tau-t)}\ d\tau \\ &= \mathbb{E}\int_t^{t+\frac{1}{n}} g(\hat{y})d\tau - \mathbb{E}\int_t^{t+\frac{1}{n}} g(\hat{y})(1-e^{-\delta(\tau-t)})\ d\tau\ .\end{aligned}$$

We substitute the latter into (3.3) and also consider on the one hand that

$$\lim_{n\to\infty} \mathbb{E}\int_t^{t+\frac{1}{n}} (1-e^{-\delta\tau})\ d\tau = 0\ ,$$

and on the other hand that $g(\hat{y},\alpha_n)$ is equally bounded on $N(\hat{y})\times\mathfrak{A}$, to deduce that there is a sequence $(d_n)_{n\in\mathbb{N}}$ with $\lim_{n\to\infty} d_n = 0$ such that

$$\begin{aligned} d_n \geq -n\mathbb{E}\int_t^{t+\frac{1}{n}} \Big[&\varphi_t(\hat{y},\tau) - \delta\varphi(\hat{y},\tau) + a\cdot\nabla\varphi(\hat{y},\tau) \\ &+ \frac{1}{2}b^2\Delta\varphi(\hat{y},\tau) + g(\hat{y},\alpha_n(\tau))\Big]\ d\tau\ . \end{aligned} \tag{3.4}$$

Next look at

$$a_n := n\int_t^{t+\frac{1}{n}} a(\hat{y},\alpha_n(\tau))\ d\tau\ , \qquad b_n := n\int_t^{t+\frac{1}{n}} b^2(\hat{y},\alpha_n(\tau))\ d\tau \quad \text{and}$$

$$g_n := n\int_t^{t+\frac{1}{n}} g(\hat{y},\alpha_n(\tau))\ d\tau\ .$$

Let $co(X)$ be the convex envelope of X. We take a look at

$$(a_n, b_n, g_n) \in co\,\{(a(\hat{y},\alpha), b(\hat{y},\alpha), g(\hat{y},\alpha))\|\alpha\in\mathfrak{A}\}\ .$$

Since $\mathfrak{A}$ is compact and a, b and g are continuous, this convex envelope is compact as well. Thus we can find a subsequence $(a_{n_k}, b_{n_k}, g_{n_k})$ with

$$\lim_{n_k\to\infty}(a_{n_k}, b_{n_k}, g_{n_k}) = (\tilde{a},\tilde{b},\tilde{g}) \in co\,\{(a(x,\alpha), b(x,\alpha), g(x,\alpha))\mid \alpha\in\mathfrak{A}\}\ .$$

Now we are ready to pass to the limit in equation (3.4). With $n_k\to\infty$ we conclude

$$-\left(\varphi_t(\hat{y},t) - \delta\varphi(\hat{y},t) + \tilde{a}\cdot\nabla\varphi(\hat{y},t) + \frac{1}{2}\tilde{b}\Delta\varphi(\hat{y},t) + \tilde{g}\right) \leq 0\ .$$

Since $(\tilde{a},\tilde{b},\tilde{g}) \in co\,\{(a(x,\alpha), b(x,\alpha), g(x,\alpha))\|\alpha\in A\}$, we can find $c_i, i = 1,2,\ldots,l$ with

$$0\leq c_i\leq 1\ , \qquad \sum_{i=1}^{l} c_i = 1\ ,$$

$$\tilde{g} = \sum_{i=1}^{l} c_i g(\hat{y},\alpha_i)\ , \qquad \tilde{a} = \sum_{i=1}^{l} c_i a(\hat{y},\alpha_i) \quad \text{and} \quad \tilde{b} = \sum_{i=1}^{l} c_i b^2(\hat{y},\alpha_i)\ .$$

We derive

$$\begin{aligned}
0 &\geq -\Big(\varphi_t(\hat y,t)+\tilde a\cdot\nabla\varphi(\hat y,t)+\frac{1}{2}\tilde b\Delta\varphi(\hat y,t)+\tilde g-\delta\varphi(\hat y,t)\Big)\\
&= -\Big(\varphi_t(\hat y,t)-\sum_{i=1}^{l}c_i a(\hat y,\alpha_i)\cdot\nabla\varphi(\hat y,t)+\frac{1}{2}\sum_{i=1}^{l}c_i b^2(\hat y,\alpha_i)\Delta\varphi(\hat y,t)\\
&\quad+\sum_{i=1}^{l}c_i g(\hat y,\alpha_i)-\delta\varphi(\hat y,t)\Big)\\
&= \sum_{i=1}^{l}c_i\Big\{-\Big(\varphi_t(\hat y,t)-\delta\varphi(\hat y,t)+\cdot a(\hat y,\alpha_i)\nabla\varphi(\hat y,t)\\
&\quad+\frac{1}{2}b^2(\hat y,\alpha_i)\Delta\varphi(\hat y,t)+g(\hat y,\alpha_i)\Big)\Big\}\\
&\geq \min_{i=1,\dots,l}c_i\Big\{-\Big(\varphi_t(\hat y,t)-\delta\varphi(\hat y,t)+a(\hat y,\alpha_i)\cdot\nabla\varphi(\hat y,t)\\
&\quad+\frac{1}{2}b^2(\hat y,\alpha)\Delta\varphi(\hat y,t)+g(\hat y,\alpha_i)\Big)\Big\}\ .
\end{aligned}$$

Now we choose an $i\in\{1,\dots,l\}$ which satisfies the minimum and set $\alpha=\alpha_i$ and finally multiply by -1 to deduce

$$\Big(\varphi_t(\hat y,t)-\delta\varphi(\hat y,t)+a(\hat y,\alpha)\cdot\nabla\varphi(\hat y,t)+\frac{1}{2}b^2(\hat y,\alpha)\Delta\varphi(\hat y,t)+g(\hat y,\alpha)\Big)\geq 0$$

which is exactly the second inequality of (3.2.7). Since it holds for every $\hat y$ with $v(\hat y)=\varphi(\hat y)$, we conclude that v is a viscosity subsolution.

□

Now we know that the solution to the optimal control problem is a viscosity solution but we still need the uniqueness. In order to achieve that, we show a comparison principle which states that a viscosity supersolution is always greater than or equal to any viscosity subsolution. We basically follow the line of reasoning of Crandall et al. [6]. With the comparison principle we will be able to proof the uniqueness result.

First we give two results on semiconvex functions which we will need later on to maintain the comparison principle.

3.3.2 Lemma (Jensen's Lemma)**.** *Let $\varphi:\mathbb{R}^N\to\mathbb{R}$ be semiconvex and $\hat x$ be a strict local maximum of φ. Then for any $r,\ \delta>0$ the set*

$$K=\Big\{x\in B_r(\hat x)\ \Big|\ \exists\ p\in B_\delta(0)\ \textit{such that}\ \varphi(x)+\langle p,x\rangle\ \textit{has a local maximum at}\ x\Big\}$$

has positive measure.

Proof. We presume that r is small enough so that $\hat{x}$ is the only maximum of φ in $B_r(\hat{x})$ and φ is twice differentiable. Let $p \in B_\delta(0)$ and $\bar{x}$ be a maximum of $\varphi(x) + \langle p, x \rangle$. Then

$$\varphi(\hat{x}) + \langle p, \hat{x} \rangle \leq \varphi(\bar{x}) + \langle p, \bar{x} \rangle$$

holds. Furthermore we derive

$$\big|\varphi(\hat{x}) - \varphi(\bar{x})\big| \leq \big| - \langle p, \hat{x} - \bar{x} \rangle \big| \leq \delta \big|\hat{x} - \bar{x}\big| \leq \delta r \ .$$

From the latter inequality and the continuity of φ we can conclude that if δ is sufficiently small, every maximum of $\varphi(x) + \langle p, x \rangle$ with respect to $B_r(\hat{x})$ lies in the interior of $B_r(\hat{x})$. Further, for every $p \in B_\delta(0)$ there exists a $\bar{x} \in K$ such that $\bar{x}$ is a maximum of $\varphi(x) + \langle p, x \rangle$. Moreover, since $D\varphi + p = 0$ holds at maximum points of φ_p, we get $D\varphi(K) \supset B_\delta(0)$.

Let $\lambda \geq 0$. Therefore $\varphi(x) + \frac{\lambda}{2}|x|^2$ is convex and we deduce

$$\begin{aligned} 0 &\leq D^2\varphi(x) + \lambda I \\ -\lambda I &\leq D^2\varphi(x) \qquad \leq 0 \\ |\det(D^2\varphi(x))| &\leq \lambda^N \end{aligned}$$

since $D^2\varphi(x) \leq 0$ on K. Finally we obtain

$$\begin{aligned} 0 &\leq \mathrm{meas}(B_\delta) \\ &\leq \mathrm{meas}(D\varphi(K)) \\ &\leq \int_K |\det(D^2\varphi(x))| \ \mathrm{dx} \\ &\leq \mathrm{meas}(K)|\lambda^N| \ . \end{aligned}$$

We found a lower bound on the measure of K depending only on the semi-convexity constant λ.

In the more general case, where φ is not necessarily smooth, we approximate it via mollification with smooth testfunctions φ_ϵ with the same semi-convexity constant λ and which converge uniformly to φ on $B_r(\hat{x})$. The corresponding sets K_ϵ obey the above estimate for small ϵ. Furthermore the set

$$K \supseteq \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} K_{1/m}$$

satisfies the above estimate.

□

Next we recall a result called Alexandrov's theorem.

3.3.3 Theorem (Alexandrov's Theorem)**.** *Let $\phi : \mathbb{R}^N \to \mathbb{R}$ be semiconvex. Then ϕ is twice differentiable almost everywhere on $\mathbb{R}^N$.*

The proof is based on Rademachers theorem which states that Lipschitz functions are differentiable almost everywhere. An accurate proof can be found in Crandall et al. [6].

We are heading towards the comparison principle. In order to give a structured proof we show one more lemma in advance.

3.3.4 Lemma. *Let $\Omega_u \subseteq \mathbb{R}^{N_u}$ and $\Omega_v \subseteq \mathbb{R}^{N_v}$ be locally compact, $\Omega := \Omega_u \times \Omega_v$, $u \in USC(\Omega_u)$ and $v \in LSC(\Omega_v)$. $\varphi \in C^2(N(\Omega))$. Let $\hat{\chi} = (\hat{x}, \hat{y}) \in \Omega$ be a local maximum of $u(x) - v(y) - \varphi(x, y)$. Then*

$$\forall\ \epsilon \geq 0\ \exists\ X \in S(n_u), Y \in S(n_v)\ :$$

$$(D_x\phi(\hat{x}, \hat{y}), X) \in \overline{J}_\Omega^{2,+} u(\hat{x}) \text{ and } (D_y\varphi(\hat{x}, \hat{y}), Y) \in \overline{J}_\Omega^{2,-} v(\hat{y})$$

holds. Furthermore X and Y satisfy the estimate

$$\begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq A + \epsilon A^2\ , \tag{3.5}$$

where $A = D^2\varphi(\hat{x}, \hat{y}) \in S(N_u + N_v)$.

In order to proof this theorem we make two reductions and reformulate the theorem as 3.3.5:

The first reduction is $\Omega_u = \mathbb{R}^{N_u}$ (respectively $\Omega_v = \mathbb{R}^{N_v}$). If this is not the case, we restrict u to a neighbourhood $N(\hat{x})$ of $\hat{x}$ and extend the restriction by setting $u(x) = -\infty$ for $x \notin N(\hat{x})$ (respectively we restrict v to the neighbourhood $N(\hat{y})$ of $\hat{y}$ and extend it by $v(y) = \infty$ for $y \notin N(\hat{x})$). Since we included $\pm\infty$ in the definitions of USC and LSC, $u \in USC(\Omega_u)$ and $v \in LSC(\Omega_v)$ will still hold. Moreover we have

$$\overline{J}_\Omega^{2,-} u(\hat{x}) = \overline{J}^{2,-} u(\hat{x})\ , \qquad \overline{J}_\Omega^{2,-} v(\hat{y}) = \overline{J}^{2,-} v(\hat{y})$$

and $(\hat{x}, \hat{y})$ is still a maximum of $u - v + \varphi$ relative to $\mathbb{R}^{N_u + N_v}$.

As the second reduction we presume that $\hat{x} = \hat{y} = 0$, $u(0) = v(0) = 0$, $\varphi(\cdot) = \frac{1}{2}\langle A\cdot, \cdot\rangle$ for some quadratic $A \in S(N_u + N_v)$ and that $(0, 0)$ is a strict global maximum of $u - v - \varphi$. With a translation we can move the origin to $(\hat{x}, \hat{y})$. Then we replace φ, u and v by

$$\begin{aligned} \tilde{\varphi}(x, y) &= \varphi(x, y) - \Big(\varphi(0, 0) + \langle D\varphi(0, 0), (x, y)\rangle\Big) \\ \tilde{u}(x) &= u(x) - \Big(u(0) + \langle D_x\varphi(0, 0), (x)\rangle\Big) \\ \tilde{v}(y) &= v(y) - \Big(v(0) + \langle D_y\varphi(0, 0), (y)\rangle\Big)\ . \end{aligned}$$

Now we already reduced to $D\tilde{\varphi}(0) = 0 = \hat{x} = \hat{y}$. Using the Taylor series expansion of φ we derive

$$\begin{aligned} \varphi(x,y) &= \varphi(0,0) + \langle D\varphi(0,0),(x,y)\rangle \\ &\qquad + \frac{1}{2}\left\langle D^2\varphi(0,0)(x,y),(x,y)\right\rangle + O(|(x,y)|^3) \\ \Rightarrow \quad \tilde{\varphi}(x,y) &= \frac{1}{2}\left\langle D^2\varphi(0,0)(x,y),(x,y)\right\rangle + O(|(x,y)|^3) \end{aligned}$$

We state that

$$u(x) - v(y) - \tilde{\varphi}(x,y) \le u(0) - v(0) - \tilde{\varphi}(0,0) = 0$$

holds. Hence if $|(x,y)| \neq 0$ is small enough and $\eta > 0$, we can replace $\tilde{\varphi}$ to obtain

$$u(x) - v(y) - \frac{1}{2}\langle D\tilde{\varphi}(0,0)(x,y),(x,y)\rangle - \frac{\eta}{2}\langle I(x,y),(x,y)\rangle < 0 \ .$$

The local maximum $(0,0)$ can be extended to a global maximum by restricting $u - v - \varphi$ to a neighborhood of $(0,0)$ and then via extending $u - v - \varphi$ exactly as described in the first reduction above. Next we take the limit $\eta \to 0$ to obtain that $(0,0)$ is a strict local maximum of $u(x) - v(y) - \frac{1}{2}\langle A(x,y),(x,y)\rangle$, where we have set $A := D\varphi(0,0)$.

The reformulation of 3.3.4 now reads:

3.3.5 Lemma. *Let $u \in USC(\mathbb{R}^{N_u})$, $v \in LSC(\mathbb{R}^{N_v})$ and $u(0) = v(0) = 0$. Suppose that for $A \in S(N_u + N_v)$ and all $(x,y) \in \mathbb{R}^{N_u} \times \mathbb{R}^{N_v} =: \Omega$*

$$u(x) - v(y) - \frac{1}{2}\langle A(x,y),(x,y)\rangle \le 0 \tag{3.6}$$

holds. Then for each $\epsilon \ge 0$ there exists an $X \in S(N_u)$ and a $Y \in S(N_v)$, such that

$$(0,X) \in \overline{J}^{2,+}_{\Omega} u(0) \quad \textit{and} \quad (0,Y) \in \overline{J}^{2,-}_{\Omega} v(0) \tag{3.7}$$

and the following inequality holds

$$\begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \le A + \epsilon A^2 \ . \tag{3.8}$$

Proof. Let $\epsilon \ge 0$. By applying the Cauchy-Schwarz inequality we deduce

$$\langle A(x,y),(x,y)\rangle \le \left\langle (A + \epsilon A^2)\psi, \psi\right\rangle + \left(\frac{1}{\epsilon} + \|A\|\right)|(x,y) - \psi|^2$$

for (x,y),$\psi \in \mathbb{R}^{N_u + N_v}$. Fix

$$\lambda := \frac{1}{\epsilon} + \|A\|$$

and with assumption (3.6) we obtain

$$\left(u(x) - \frac{\lambda}{2}|x - \psi_1|^2\right) + \left(-v(y) - \frac{\lambda}{2}|y - \psi_2|^2\right) \leq \left\langle (A + \epsilon A^2)\psi, \psi \right\rangle \ . \quad (3.9)$$

Next we define

$$\hat{u}(\psi_1) = \sup_{x \in \mathbb{R}^N} \left(u(x) - \frac{\lambda}{2}|x - \psi_1|^2\right)$$
$$\hat{v}(\psi_2) = \sup_{y \in \mathbb{R}^N} \left(-v(y) - \frac{\lambda}{2}|y - \psi_2|^2\right) \ .$$

On the one hand we deduce $\hat{u}(0) \geq u(0) = 0$ and $\hat{v}(0) \geq v(0) = 0$. But on the other hand inequality (3.9) yields $\hat{u}(0) + \hat{v}(0) \leq 0$. Therefore we conclude $\hat{u}(0) = \hat{v}(0) = 0$.

Since $\hat{u}(\psi_1) + \hat{v}(\psi_2) + \frac{\lambda}{2}|\psi|^2$ is convex, we can apply Alexandrov's theorem 3.3.3 and receive that $\hat{u}(\psi_1) + \hat{v}(\psi_2)$ is differentiable almost everywhere. Set

$$\omega(\psi) := \hat{u}(\psi_1) + \hat{v}(\psi_2) - \frac{1}{2}\left\langle (A + \epsilon A^2)\psi, \psi \right\rangle - |\psi|^4 \ .$$

We note that $\omega(0) = 0$ is a strict maximum of ω since $u(\psi_1) - v(\psi_2) - \frac{1}{2}\langle (A)\psi, \psi\rangle \leq 0$ and $-|\psi|^4 < 0$ for $\psi \neq 0$. We now apply Jensen's Lemma 3.3.2 which states that the set

$$K = \Big\{(x, y) \in B_r \ \Big| \ \exists\, p \in B_\delta : (\hat{u} + \hat{v})(x, y) + \langle q, (x, y)\rangle$$
$$\text{has a local maximum at } (x, y)\Big\}$$

has a positive measure. Applying Alexandrov's theorem 3.3.3 we deduce that for every $\delta > 0$ there exists a $q_\delta \in \mathbb{R}^N$ with $|q_\delta| \leq \delta$ such that $\omega(\psi) + \langle q_\delta, \psi\rangle$ has a maximum point $\psi_\delta \in K$ with $|\psi_\delta| \leq \delta$ where $\omega(\psi) + \langle q_\delta, \psi\rangle$ is twice differentiable at ψ_δ. Hence we attain

$$\big(D(\hat{u} + \hat{v})(\psi_\delta), D^2(\hat{u} + \hat{v})(\psi_\delta)\big) \in J^2(\hat{u} + \hat{v})((\psi_\delta)) \ .$$

As $\delta \to 0$ we can ensure that $D^2(\hat{u}(\psi_1^\delta) + \hat{v}(\psi_2^\delta))$ is convergent by chosing an appropriate subsequence of δ. Thus, as $\delta \to 0$ and by noting that $D^2_{\psi_1\psi_2}(\hat{u} + \hat{v}) = 0$ we obtain

$$\big(D(\hat{u} + \hat{v})(\psi), D^2(\hat{u} + \hat{v})(\psi)\big) \in \overline{J}^2(\hat{u} + \hat{v})((\psi))$$
$$\left(0, \begin{pmatrix} X & 0 \\ 0 & -Z \end{pmatrix}\right) \in \overline{J}^2(\hat{u} + \hat{v})((0, 0))$$

for some $X \in S(N_u)$ and $y \in S(N_v)$. Since $u - v - \varphi$ is twice differentiable at $(0,0)$, we claim that $\hat{u}(x)$ and $\hat{v}(y)$ are twice differentiable at $x = y = 0$, hence $(0, X) \in \overline{J}^2 \hat{u}(0)$ and $(0, -Y) \in \overline{J}^2 \hat{v}(0)$. Since ω is convex, we know $D^2\omega(\psi_\delta) \leq 0$ and derive

$$D^2(\hat{u}(\psi_1^\delta) + \hat{v}(\psi_2^\delta)) - (A + \epsilon A^2) - 12|(\psi^\delta)^2| \leq 0$$
$$D^2(\hat{u}(\psi_1^\delta) + \hat{v}(\psi_2^\delta)) \leq A + \epsilon A^2 + O(\delta^2) \ .$$

And as above by taking the limit $\delta \to 0$ we conclude

$$D^2(\hat{u} + \hat{v})(0,0) = \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq A + \epsilon A^2 \ ,$$

which completes the proof of inequality (3.8). We still have to show that (3.7) holds. We recall

$$\hat{u}(0) = u(0) = \hat{v}(0) = v(0) = 0 \ . \tag{3.10}$$

For any x, $\psi \in \mathbb{R}^{n_u}$ we deduce

$$\begin{aligned} & u(x) - \frac{\lambda}{2}|\psi - x|^2 \\ \leq\ & \hat{u}(\psi) \\ =\ & \hat{u}(0) + \langle 0, \psi - 0 \rangle + \frac{1}{2}\langle X(\psi - 0), \psi - 0 \rangle + O(|\psi - 0|^2) \\ =\ & u(0) + \langle 0, \psi - 0 \rangle + \frac{1}{2}\langle X(\psi - 0), \psi - 0 \rangle + O(|\psi - 0|^2) \ . \end{aligned}$$

By putting $\psi = x$ we obtain

$$u(x) = u(0) + \langle 0, x - 0 \rangle + \frac{1}{2}\langle X(x - 0), x - 0 \rangle + O(|x - 0|^2)$$

and conclude that $(0, X) \in \overline{J}^{2,+} u(0)$. Starting over from equation (3.10) with $y, \psi \in \mathbb{R}^{n_v}$ and replacing $u(x)$ by $-v(y)$ we derive $(0, -Y) \in \overline{J}^{2,+}(-v(0))$ which according to remark 3.2.1 is the same as $(0, Y) \in \overline{J}^{2,-} v(0)$.

□

Finally we gathered all ingredients to proof a comparison principle:

3.3.6 Theorem. *Let $\Omega \subseteq \mathbb{R}^N$ be bounded and open. Let $u \in USC(\bar{\Omega})$ be a subsolution of* (2.6) *in Ω and $v \in LSC(\bar{\Omega})$ be a supersolution of* (2.6) *in Ω and $u \leq v$ on $\delta\Omega$. Then:*

$$u \leq v \text{ in } \bar{\Omega}$$

Proof. We proof $u \leq v$ by contradiction. We assume $v(k) < u(k)$ for a $k \in \Omega$. Therefore we can set $0 < \delta = u(k) - v(k)$ and further define $M_\eta := \sup_{\bar{\Omega}\times\bar{\Omega}} u(x) - v(y) - \frac{\eta}{2}|x-y|^2$. Suppose M_η takes its maximum at (x_η, y_η). The maximum is achieved because $(u(x) - v(y)) \in USC(\Omega)$ and $\bar{\Omega}\times\bar{\Omega}$ is compact, thus $M_\eta < \infty$. Moreover $(x_\eta, y_\eta) \in \mathring{\Omega}\times\mathring{\Omega}$ since $u \leq v$ on $\partial\Omega$. That yields for $\eta > 0$

$$
\begin{aligned}
u(x_\eta) - v(y_\eta) &\geq u(x_\eta) - v(y_\eta) - \frac{\eta}{2}|x_\eta - y_\eta|^2 = M_\eta \\
&\geq u(k) - v(k) - \frac{\eta}{2}|k-k|^2 \\
&= \delta > 0 \ .
\end{aligned}
$$

Furthermore, since M_η decreases as η increases and $M_\eta \geq 0$, the limit $\lim_{\eta\to\infty} M_\eta$ exists and is finite. We deduce

$$
\begin{aligned}
M_{\frac{\eta}{2}} &= u(x_{\frac{\eta}{2}}) - v(y_{\frac{\eta}{2}}) - \frac{\eta}{2}\frac{1}{2}|x_{\frac{\eta}{2}} - y_{\frac{\eta}{2}}|^2 \\
&\geq u(x_\eta) - v(y_\eta) - \frac{\eta}{4}|x_\eta - y_\eta|^2 \\
&= u(x_\eta) - v(y_\eta) - \frac{\eta}{2}|x_\eta - y_\eta|^2 + \frac{\eta}{4}|x_\eta - y_\eta|^2 \\
&= M_\eta + \frac{\eta}{4}|x_\eta - y_\eta|^2 \ ,
\end{aligned}
$$

which can be rearranged to

$$
4(M_{\frac{\eta}{2}} - M_\eta) \geq \eta|x_\eta - y_\eta|^2 \geq 0 \ .
$$

Taking the limit we conclude

$$
\lim_{\eta\to\infty} \eta|x_\eta - y_\eta|^2 = 0 \ . \tag{3.11}
$$

Let now $(\hat{x}, \hat{y}) := (x_\alpha, y_\alpha)$. We apply lemma 3.3.4 with $\varphi = \frac{\eta}{2}|x-y|^2 \in C^2$ to obtain

$$
(\eta(\hat{x},\hat{y}), X) \in \overline{J}_\Omega^{2,+} u(\hat{x}), \quad (\eta(\hat{x},\hat{y}), Y) \in \overline{J}_\Omega^{2,-} v(\hat{y}) \quad \text{and}
$$

$$
\begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq A + \epsilon A^2 = \eta \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} + \epsilon 2\eta^2 \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} \ .
$$

Multiplying the latter with $(1, \ \ldots, \ 1)^t$ we deduce $X \leq Y$.

Since u and v are sub- and supersolutions we denote in addition

$$
\begin{aligned}
-D_t\varphi_1(\hat{x}) + \delta u(\hat{x}) - H(\hat{x}, \eta(\hat{x}-\hat{y}), X) \leq 0 \quad \text{and} \\
-D_t\varphi_2(\hat{y}) + \delta v(\hat{y}) - H(\hat{y}, \eta(\hat{x}-\hat{y}), Y) \geq 0 \ .
\end{aligned}
$$

Rearranging and adding the two inequalities yields

$$\delta\left(u(\hat{x})-v(\hat{y})\right) \\ \leq H(\hat{x},\eta(\hat{x}-\hat{y}),X)-H(\hat{y},\eta(\hat{x}-\hat{y}),Y)+\left(D_t\varphi_1(\hat{x})-D_t\varphi_2(\hat{y})\right) .$$

We recall the general inequality

$$\sup_\eta f_1(\eta)-\sup_\eta f_2(\eta) \\ =\sup_\eta\Big(f_1(\eta)-\sup_\beta\{f_2(\beta)\}\Big)\leq\sup_\eta\Big(f_1(\eta)-f_2(\eta)\Big) . \tag{3.12}$$

Now we take a closer look at the Hamiltonians on the right-hand side and derive with the above inequality (3.12)

$$H(\hat{x},\eta(\hat{x}-\hat{y}),X)-H(\hat{y},\eta(\hat{x}-\hat{y}),Y) \\ =\sup_{\eta\in\mathfrak{A}}\left\{\frac{1}{2}Tr(B(\hat{x},\alpha)X)+a(\hat{x},\alpha)\cdot\eta(\hat{x}-\hat{y})+g(\hat{x},\alpha)\right\} \\ -\sup_{\alpha\in\mathfrak{A}}\left\{\frac{1}{2}Tr(B(\hat{y},\alpha)Y)+a(\hat{y},\alpha)\cdot\eta(\hat{x}-\hat{y})+g(\hat{y},\alpha)\right\} \\ \leq\sup_{\alpha\in\mathfrak{A}}\Big\{\frac{1}{2}Tr\Big(B(\hat{x},\alpha)X-B(\hat{y},\alpha)Y\Big)+ \\ a(\hat{y},\alpha)\cdot\eta(\hat{x}-\hat{y})-a(\hat{x},\alpha)\cdot\eta(\hat{x}-\hat{y})+g(\hat{y},\alpha)-g(\hat{x},\alpha)\Big\} .$$

Now we estimate each summand seperately: The first one is $\frac{1}{2}Tr(B(\hat{x},\alpha)X-B(\hat{y},\alpha)Y)$. Since B is symmetric and positive definite, we can write it as $B=\Sigma\Sigma^t$ for some matrix Σ. Starting over from (3.8) and choosing $\epsilon=\frac{1}{\eta}$ we deduce

$$\begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix}\leq A+\epsilon A^2 \\ \Rightarrow\begin{pmatrix} {}^t\Sigma(x)\Sigma(x) & {}^t\Sigma(y)\Sigma(x) \\ {}^t\Sigma(x)\Sigma(y) & {}^t\Sigma(y)\Sigma(y) \end{pmatrix}\begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \\ \leq\eta(1+2\epsilon\eta)\begin{pmatrix} {}^t\Sigma(x)\Sigma(x) & {}^t\Sigma(y)\Sigma(x) \\ {}^t\Sigma(x)\Sigma(y) & {}^t\Sigma(y)\Sigma(y) \end{pmatrix}\begin{pmatrix} I & -I \\ -I & I \end{pmatrix} \\ \Rightarrow\begin{pmatrix} ({}^t\Sigma(x)\Sigma(x))X & -({}^t\Sigma(y)\Sigma(x))Y \\ ({}^t\Sigma(x)\Sigma(y))X & -({}^t\Sigma(y)\Sigma(y))Y \end{pmatrix} \\ \leq 3\eta\begin{pmatrix} {}^t\Sigma(x)\Sigma(x)-{}^t\Sigma(y)\Sigma(x) & {}^t\Sigma(y)\Sigma(x)-{}^t\Sigma(x)\Sigma(x) \\ {}^t\Sigma(x)\Sigma(y)-{}^t\Sigma(y)\Sigma(y) & {}^t\Sigma(y)\Sigma(y)-{}^t\Sigma(x)\Sigma(y) \end{pmatrix} .$$

The trace operator preserves the inequality, hence

$$Tr\left(({}^t\Sigma(x)\Sigma(x))X-({}^t\Sigma(y)\Sigma(y))Y\right) \\ \leq 3\eta Tr\left(({}^t\Sigma(x)-{}^t\Sigma(y))(\Sigma(x)-\Sigma(y))\right) .$$

Since by assumption (2.3) B satisfies a Lipschitz condition, Σ as well as ${}^t\Sigma$ satisfy a Lipschitz condition. Therefore we can estimate

$$Tr\,(B(\hat{x},\alpha)X - B(\hat{y},\alpha)Y) \leq 3\eta L_b^2 \|\hat{x}-\hat{y}\|^2 \ .$$

For the second summand we derive

$$\begin{aligned} & a(\hat{x},\alpha)\cdot\eta(\hat{x}-\hat{y}) - a(\hat{y},\alpha)\cdot\eta(\hat{x}-\hat{y}) \\ &= \eta(\hat{x}-\hat{y})(a(\hat{x},\alpha)-a(\hat{y},\alpha)) \\ &\leq \eta\|\hat{x}-\hat{y}\|L_a\|\hat{x}-\hat{y}\| \\ &= \eta L_a \|\hat{x}-\hat{y}\|^2 \ . \end{aligned}$$

Moreover the last summand can be estimated by

$$g(\hat{x},\alpha) - g(\hat{y},\alpha) \leq L_g\|\hat{x}-\hat{y}\| \ .$$

Putting all ingredients together we conclude

$$\begin{aligned} 0 < \delta\gamma &\leq \delta\,(u(\hat{x}) - v(\hat{y})) \\ &\leq H(\hat{x},\eta(\hat{x}-\hat{y}),X) - H(\hat{y},\eta(\hat{x}-\hat{y}),Y) \\ &\leq (L_a + 3L_b^2)\eta\|\hat{x}-\hat{y}\|^2 + L_g\|\hat{x}-\hat{y}\| \ . \end{aligned}$$

Now we take the limit $\eta \to \infty$. The right-hand side tends to zero by (3.11) which leads to the contradiction since $\delta\gamma$ does not depend on η.

□

Theorem 3.3.7 does not only hold true for the HJB (2.6) but for any equation $F(x,u,Du,D^2u) = 0$ where F is proper, i.e. F is degenerate elliptic and fulfils a monotonicity condition in u ($F(x,u,Du,D^2u) \leq F(x,v,Du,D^2u)$ whenever $u \leq v$). The last part of the proof is simply the application of the fact that the HJB (2.6) can be written as a degenerate elliptic equation. Now we get to the result of this chapter:

3.3.7 Theorem. *With condition* (2.3) *the optimal value function v of the optimal control problem* (3.3.1) *is the unique solution of the Hamilton-Jacobi-Bellmann equation* (2.6).

Proof. With theorem (3.3.1) we know the optimal value function v is a viscosity solution of (2.6).

Let u be another viscosity solution of (2.6). By definition v is a viscosity subsolution and u a viscosity supersolution. Hence with the comparison principle $v \leq u$ holds. Vice versa: v is also a viscosity supersolution and u a viscosity subsolution. Hence with the comparison principle also $u \leq v$ holds. That yields $v = u$ and therefore we obtain the desired uniqueness result.

□

Chapter 4

A finate difference scheme

Now we develop an unconditionally stable finate difference scheme and show its convergence to the viscosity solution.

4.1 Deriving the scheme

Let $i \in I \in \mathbb{N}^N$ be a multi-index with a minimum i^0 and a maximum i^p so that we have $p+1$ different indices. We define a grid $\{y_i\}$ with $y_{i^0} = \min\{y(t)\}$ and $y_{i^p} = \max\{y(t)\}$ assuming that we can put the gridpoints in some kind of order. We write $h = |y_i - y_j|$ for the distance between two contiguous gridpoints and set $\Delta y = \max\{h\}$. This is a simplified notation to avoid additional indices and does not necessarily imply an equidistant grid. For the time discretisation we write t_n with $n = 0 \ldots n_{max}$, $t_0 = 0$ and $t_{n_{max}} = T$. $\Delta\tau$ denotes the maximal distance between two timesteps $\max_n\{|t_n - t_{n+1}|\}$.

We introduce some notations.

4.1.1 Notations.

- *Let v_i^n be an approximation of $v(y_i, \alpha^n)$ at node y_i and time t_n.*
- $v^n := [v_{i^0}^n, \ldots, v_{i^p}^n]^t$
- *We will write $(y)_i$ for the i-th component of y not to be confused with the grid node y_i defined above.*
- *We write $v_{(i+1)_k}^n$ (resp. $v_{(i-1)_k}^n$) with $k \in \{1 \ \ldots \ N\}$ for v_j^n with $j_k = i_k + 1$ (resp. $j_k = i_k - 1$) and $j_l = i_l$ for $l = 1 \ldots N$ and $l \neq k$.*

- $v^n_{i+1} \in \mathbb{R}^N$ *(respectively* $v^n_{i-1} \in \mathbb{R}^N$*) denotes the vector of all* $v^n_{(i+1)_k}$ *(resp. of all* $v^n_{(i-1)_k}$*) for* $k = 1 \dots N$.

- *We write* $v^n_{(i+1,+1)_{k,l}}$ *with* $k, l \in \{1 \ \dots \ N\}$ *and* $k < l$ *for* v^n_j *with* $j_k = i_k + 1$, $j_l = i_l + 1$ *and* $j_m = i_m$ *for* $m = 1 \dots N$ *and* $m \neq k, l$; *analogous* $v^n_{(i-1,-1)_{k,l}}$, $v^n_{(i+1,-1)_{k,l}}$ *and* $v^n_{(i-1,+1)_{k,l}}$.

- $v^n_{i\pm1,\pm1} \in \mathbb{R}^{2\binom{N}{2}}$ *(respectively* $v^n_{i\pm1,\mp1} \in \mathbb{R}^{2\binom{N}{2}}$*) denotes the vector of all a* $v^n_{(i+1,+1)_{k,l}}$ *and* $v^n_{(i-1,-1)_{k,l}}$ *(respectively of all* $v^n_{(i+1,-1)_{k,l}}$ *and* $v^n_{(i-1,+1)_{k,l}}$*) for* $j = 1, \dots (N-1)$ *and* $k = (j+1), \dots N$. *The specific order is not important as long as there is an order. For eample:*

$$\begin{cases} (v^n_{i\pm1,\pm1})_{(j-1)N+k-\frac{1}{2}j(j+1)} & = v^n_{(i+1,+1)_{k,l}} \\ (v^n_{i\pm1,\pm1})_{\frac{1}{2}N(N-1)+(j-1)N+k-\frac{1}{2}j(j+1)} & = v^n_{(i-1,-1)_{k,l}} \end{cases}$$

  each with $j = 1, \dots (N-1)$ *and* $k = (j+1), \dots N$.

We define a discrete form of the differential operator (2.7) at v^n by replacing the first and second derivatives with appropriate difference quotients:

$$(\mathfrak{L}^\alpha_{\Delta y} v^n)_i = c^n_i v^n_{i+1} + d^n_i v^n_i + e^n_i v^n_{i-1} + f^n_i(+) v^n_{i\pm1,\pm1} + f^n_i(-) v^n_{i\pm1,\mp1} \ . \quad (4.1)$$

This notation is more intuitive rather than precise. The coefficients are transposed vectors of matching sizes so that the expression $(\mathfrak{L}^\alpha_{\Delta y} v^n)_i \in \mathbb{R}$ defined above is meaningful. The number of approximated values of v used in this expression is $4\binom{N}{2} + 2N + 1$. This corresponds to an approximation with nine grid nodes if $N = 2$ and with 19 grid nodes if $N = 3$.

The coefficients c_i, d_i and e_i are determined by the choice of forward, backward or central differencing for the first derivative and by the choice of difference quotient for the second derivative. $f_i(+)$ and $f_i(-)$ only depend on the second order difference quotient. All of them also depend on y^n_i and α^n.

In this approach we will use the following difference quotients: The choices for the first derivatives are

forward $\quad D^+_{(y)_j} v(y_i, \alpha^n) = \dfrac{v^n_{(i+1)_j} - v^n_i}{\Delta y}$,

backward $\quad D^-_{(y)_j} v(y_i, \alpha^n) = \dfrac{v^n_i - v^n_{(i-1)_j}}{\Delta y}$ and

central $\quad D^c_{(y)_j} v(y_i, \alpha^n) = \dfrac{v^n_{(i+1)_j} - v^n_{(i-1)_j}}{2\Delta y}$.

For the second derivative in one direction the usual difference quotient is

$$D^2_{(y)_j} v(y_i, \alpha^n) = \frac{v^n_{(i+1)_j} - 2v^n_i + v^n_{(i-1)_j}}{(\Delta y)^2} \ .$$

The choices for the mixed second derivatives are

$$D^{2+}_{(y)_j,(y)_k} v(y_i, \alpha^n) = \frac{\big(2v_i + v_{(i+1,+1)_{j,k}} + v_{(i-1,-1)_{j,k}} - v_{(i+1)_j} - v_{(i-1)_j} - v_{(i+1)_k} - v_{(i-1)_k}\big)}{2\Delta y \Delta \tau} \text{ and}$$

$$D^{2-}_{(y)_j,(y)_k} v(y_i, \alpha^n) = \frac{-\big(2v_i + v_{(i+1,-1)_{j,k}} + v_{(i-1+1)_{j,k}} - v_{(i+1)_j} - v_{(i-1)_j} - v_{(i+1)_k} - v_{(i-1)_k}\big)}{2\Delta y \Delta \tau} \ .$$

We will use fully implicit ($\Theta = 0$) or Crank-Nicolson ($\Theta = 1/2$) discretisation of the HJB (2.6). By using the discrete operator (4.1) we can write the finite difference scheme as

$$\begin{aligned} 0 = \frac{v^{n+1}_i - v^n_i}{\Delta \tau} &+ (1-\Theta) \sup_{\alpha^{n+1} \in \mathfrak{A}} \Big\{ (\mathfrak{L}^{\alpha^{n+1}}_{\Delta y} v^{n+1})_i + g^{n+1}_i \Big\} \\ &+ \Theta \sup_{\alpha^n \in \mathfrak{A}} \big\{ (\mathfrak{L}^{\alpha^n}_{\Delta y} v^n)_i + g^n_i \big\} \qquad \text{for } i = 0 \dots p \end{aligned} \tag{4.2}$$

where n denotes the timestep and $g^n_i := g(y_i, \alpha^n)$. For notational reasons we abbreviate the right side of (4.4.2) with $G^{n+1}_i(v^{n+1}, v^n, \Delta y, \Delta \tau)$ and also write $G^{n+1}_i(\varphi, \Delta)$ if φ is a function and Δy, $\Delta \tau$ are understood. Hence $G^{n+1}_i = 0$ is the equation for the node y_i at time t_n. We denote the solution of the scheme resulting from the equations for all gridpoints by v^Δ.

Hence we have one equation at each node y_i with p unknowns to calculate at each timestep. We can gather these equations in a matrix equation for each timestep

$$Mv = b \tag{4.3}$$

with

$$M_{kk} = \frac{1}{\tau}(\Theta - 1) d^{n+1}_i \ ,$$

$M_{kl} = 0$, for most l. Otherwise it depends on the node order. Moreover

$$b_k = \frac{1}{\Delta \tau} - \Big((1-\Theta) g^{n+1}_i + \Theta \big((\mathfrak{L}^{\alpha^n}_h v^n)_i + g^n_i \big) \Big) \ .$$

M is a sparse $(p \times p)$-matrix with at most $(4\binom{N}{2} + 2N + 1)p$ non-zero entries.

We need the corresponding matrix M to be a M-matrix, i.e. all off-diagonal entries must be nonpositive and all diagonal entries must be positive. We implement an upwind scheme to ensure that this is the case. That means we choose the forward or backward difference quotient for the first and mixed second derivatives depending on the signum of their coefficient.

Since all edge points $v^n_{i\pm1,\pm1}$ and $v^n_{i\pm1,\mp1}$ only appear in the second mixed derivatives and have to be nonpositive this choice is straightforward: If the coefficient of that derivative in equation (2.6) is negative, i.e. $b(y_i,\alpha) < 0$ (resp. positive $b(y_i,\alpha) > 0$), then choose difference quotient $D^{2+}_{(y)_j,(y)_k}v(y_i,\alpha^n)$ (resp. $D^{2-}_{(y)_j,(y)_k}v(y_i,\alpha^n)$), i.e. the coefficient of $v^n_{i\pm1,\pm1}$ (resp. $v^n_{i\pm1,\mp1}$) will be nonpositive. Precisely

$$\begin{cases} (f^n_i(+))_{(j-1)N+k-\frac{1}{2}j(j+1)} = \dfrac{1}{\Delta y\Delta\tau}\max\{0;(b(y_i,\alpha))_j(b(y_i,\alpha))_k\} \\ \qquad \text{with } j = 1\ldots(n-1) \text{ and } k = j\ldots N \\ (f^n_i(+))_{\frac{1}{2}N(N-1)+l} = (f^n_i(+))_l \\ \qquad \text{with } l = 1\ldots\dfrac{1}{2}N(N-1)+l \end{cases}$$

$$\begin{cases} (f^n_i(-))_{(j-1)N+k-\frac{1}{2}j(j+1)} = \dfrac{1}{\Delta y\Delta\tau}\min\{0;(b(y_i,\alpha))_j(b(y_i,\alpha))_k\} \\ \qquad \text{with } j = 1\ldots(n-1) \text{ and } k = j\ldots N \\ (f^n_i(-))_{\frac{1}{2}N(N-1)+l} = (f^n_i(-))_l \\ \qquad \text{with } l = 1\ldots\dfrac{1}{2}N(N-1)+l\ . \end{cases}$$

The points $v^n_{i\pm1}$ appear in the second and first derivatives. So c^n_i and e^n_i are determined by $a(y_i,\alpha^n)$ and $b(y_i,\alpha^n)$.

$$(c^n_i)_j = \frac{1}{\Delta y\Delta\tau}\Big(b_{jj} - \frac{1}{2}\sum_k b_{jk}\Big) + \frac{1}{\Delta y}\min\{0,a_j\}$$
$$(e^n_i)_j = \frac{1}{\Delta y\Delta\tau}\Big(b_{jj} - \frac{1}{2}\sum_k b_{jk}\Big) + \frac{1}{\Delta y}\max\{0,a_j\}$$

with $j = 1\ldots N$. The coefficient d_i for the center point v_i is determined by the first and second derivatives as well as $-\delta$ and reads

$$d^n_i = -\delta + \frac{1}{\Delta y\Delta\tau}\sum_j\Big(-2b_{jj} + \sum_k b_{jk}\Big) + \frac{1}{\Delta y}\sum_k |a_k|\ .$$

4.1.2 Remark (coefficient condition)**.** *The coefficients c, e, $f(+)$ and $f(-)$ defined above have to be nonpositive. d has to be positive.*

We can use the central difference quotient for the first derivative if the coefficient condition 4.1.2 is not violated. With $\mathbb{1} = (1...1)^t$ beeing a vector with all elements 1 and of matching size, we note that

$$c_i^n \mathbb{1} + d_i^n + e_i^n \mathbb{1} + f_i^n(+)\mathbb{1} + f_i^n(-)\mathbb{1} = -\delta \ . \tag{4.4}$$

4.1.3 Definition (M-matrix)**.** *A square matrix M with entries m_{ij} is said to be a M-marix if it satisfies the following conditions (ii), (iii) and (iv):*

(i) all diagonal entries are strictly positive

(ii) all off-diagonal entries are nonpositive

(iii) M is nonsingular

(iv) $M^{-1} \geq 0$

Condition (i) of definition 4.1.3 is a direct consequence of conditions (ii), (iii) and (iv).

4.1.4 Remark. *The matrix M defined by* (4.3) *is a M-matrix if the coefficients condtion 4.1.2 holds.*

4.2 Boundary conditions for financial applications

Unknown boundary conditions often pose problems for the computation. In a financial context if the boundary conditions are unknown, a typical feature is that $b(y, \alpha)$ vanishes as the spatial variable y approaches the lower (zero) boundary. Moreover the assumption $D^2 v \approx 0$ at the upper boundaries is commonplace. Hence the equation reduces at the boundaries. In addition to this, one can make extra assumptions to make sure a classical solution exists. We refer to Forsyth and Labahn [8] and the references therein for more about this obstacle. With assuming that the controls are time independent one can deduce an ODE whose solution provides boundary conditions. Another possibility is simply to "guess" boundary conditions for the upper boundary. The error with that will be small if the upper boundary is far away from the domain of financial interest.

One example of boundary conditions are the Dirichlet type. They read $v = f^{BC}$ on $\partial\Omega$ where we assume that f^{BC} is a continuous function. A classical idea is to add the term $-\epsilon\Delta v$ to the HJB and look for a solution of the problem

$$\begin{cases} \tilde{v}_t(y) - \delta\tilde{v}(y) + H(y, D\tilde{v}(y), D^2\tilde{v}(y)) - \epsilon\Delta\tilde{v} = 0 & \text{in } \Omega \\ \tilde{v} = f^{BC} & \text{on } \partial\Omega \end{cases} .$$

The solution $\tilde{v}$ can be expected to be more regular than v itself[1]. This idea of "vanishing viscosity" motivated the term viscosity solution in the first place. As regularity one can assume $\tilde{v}$ to be in $C^2(\Omega) \cap C(\Omega)$, thus to be a classical solution in Ω.

In general boundary conditions do not have to be continuous. We briefly look at this issue later in 4.4. For now we make the assumption that either $b = 0$ or $D^2 v = 0$ is satisfied at the spatial boundary for $t \leq T$.

4.3 Convergence of the scheme

Convergence to the viscosity solution follows from stability, consistency and monotonicity.

4.3.1 Definition (consistency)**.** *Let $\mathfrak{L} : C^k(\omega) \to C^0(\omega)$ be a differential operator of order k and $\tilde{\mathfrak{L}}_h : C^k(\omega)- > C^0(\omega)$ a discrete approximation. The approximation is consistent according to a norm $\|\cdot\|$ if*

$$\lim_{h \to 0} \|\tilde{\mathfrak{L}}_h \varphi - \mathfrak{L}\varphi\| = 0$$

and consistence of order m if

$$\|\tilde{\mathfrak{L}}_h \varphi - \mathfrak{L}\varphi\| \leq C(h^m)$$

for all $\varphi \in C^{k+m}$.

This definition only covers the case in which no problems occur at the boundary. For a general degenerate elliptic problem we need a more complex definition of consistency. We give the required definition in section 4.4 because we have not introduced discontinuous viscosity solutions yet.

4.3.2 Theorem. *The approximation* (4.4.2) *of the HJB* (2.6) *is consistent according to the $\|\cdot\|_\infty$-norm.*

Proof. Let $\varphi(x) \in C^3(\Omega)$ be a test function. We have to show that

$$\begin{aligned} \lim_{\Delta y, \Delta\tau \to 0} \Bigg\| &- \varphi_t - \sup_\alpha \left\{ \mathfrak{L}^\alpha \varphi(y_i, \alpha^{n+1}) + g(y_i, \alpha^{n+1}) \right\} \\ &- G_i^{n+1}(\varphi^{n+1}, \varphi^n, \Delta y, \Delta\tau) \Bigg\| &&= 0 \end{aligned} \tag{4.5}$$

[1]cf. Continuous Dependence Estimates for Viscosity Solutions of Fully Nonlinear Degenerate Elliptic Equations, ESPEN R. JAKOBSEN and KENNETH H. KARLSEN, Electronic Journal of Differential Equations, Vol. 39 (2002), pp. 1–10.

holds.

First we show by using Taylor series expansion and with the discretisation described above that

$$\|\tilde{\mathfrak{L}}_{\Delta y}\varphi - \mathfrak{L}\varphi\| = O(\Delta x) \tag{4.6}$$

holds. We show that the second order mixed derivative $D^{2+}_{(y)_j,(y)_k} v(y_i, \alpha^n)$ has an error of size $O(h_j) + O(h_k)$ where h_k, and h_j are the distances to the next nodes denoted with Δy above to indicate the changes for non-equidistant grids. The Taylor series in multiple dimension with the multi-index κ is

$$T(x_0) = \sum_{|\kappa|\geq 0} \frac{1}{\kappa!}\frac{\partial^\kappa f}{\partial x^\kappa}(x_0 - x)^\kappa \ . \tag{4.7}$$

We develop φ in a Taylor series around $\tilde{x}$, $\hat{x}$ and $\bar{x}$ where

$$\tilde{x} = \begin{pmatrix} x_1 \\ \dots \\ x_j + h_j \\ \dots \\ x_k + h_k \\ \dots \\ x_d \end{pmatrix} \qquad \hat{x} = \begin{pmatrix} x_1 \\ \dots \\ x_j + h_j \\ \dots \\ \dots \\ \dots \\ x_d \end{pmatrix} \qquad \bar{x} = \begin{pmatrix} x_1 \\ \dots \\ \dots \\ \dots \\ x_k + h_k \\ \dots \\ x_d \end{pmatrix} .$$

We derive

$$\begin{aligned}
\varphi(\tilde{x}) = \varphi(x) &+ \frac{\partial\varphi}{\partial x_j}(x)h_j + \frac{\partial\varphi}{\partial x_k}(x)h_k + \frac{\partial^2\varphi}{\partial x_j^2}(x)\frac{h_j^2}{2} + \frac{\partial^2\varphi}{\partial x_k^2}(x)\frac{h_k^2}{2} + \\
&\frac{\partial^2\varphi}{\partial x_j x_k}(x)h_j h_k + \frac{\partial^3\varphi}{\partial x_j^3}(x)\frac{h_j^3}{6} + \frac{\partial^3\varphi}{\partial x_k^3}(x)\frac{h_k^3}{6} + \\
&\frac{\partial^3\varphi}{\partial x_j^2 x_k}(x)\frac{h_j^2 h_k}{2} + \frac{\partial^3\varphi}{\partial x_j x_k^2}(x)\frac{h_j h_k^2}{2} + \sum_{|\kappa|=4}\frac{\partial^4\varphi}{\partial x^\kappa}(\tilde{X})\frac{1}{\kappa!}(\tilde{x} - x)^\kappa \\
\varphi(\hat{x}) = \varphi(x) &+ \frac{\partial\varphi}{\partial x_j}(x)h_j + \frac{\partial^2\varphi}{\partial x_j^2}(x)\frac{h_j^2}{2} + \frac{\partial^3\varphi}{\partial x_j^3}(x)\frac{h_j^3}{6} + \frac{\partial^4\varphi}{\partial x_j^4}(\hat{X})\frac{h_j^4}{24} \\
\varphi(\bar{x}) = \varphi(x) &+ \frac{\partial\varphi}{\partial x_k}(x)h_k + \frac{\partial^2\varphi}{\partial x_k^2}(x)\frac{h_k^2}{2} + \frac{\partial^3\varphi}{\partial x_k^3}(x)\frac{h_k^3}{6} + \frac{\partial^4\varphi}{\partial x_k^4}(\bar{X})\frac{h_k^4}{24} \ .
\end{aligned}$$

That yields

$$\begin{aligned}
&\frac{\varphi(\tilde{x}) - \varphi(\hat{x}) - \varphi(\bar{x}) + \varphi(x)}{h_j h_k} \\
&= \frac{\partial^2\varphi}{\partial x_j x_k}(x) + \frac{\partial^3\varphi}{\partial x_j^2 x_k}(x)\frac{h_j}{2} + \frac{\partial^3\varphi}{\partial x_j x_k^2}(x)\frac{h_k}{2} \\
&+ \sum_{|\kappa|=4}\frac{\partial^4\varphi}{\partial x^\kappa}(\tilde{x})\frac{1}{\kappa!}\frac{(\tilde{x} - x)^\kappa}{h_j h_k} \ .
\end{aligned} \tag{4.8}$$

By doing the exact same calculation with altering the (+)-signs in $\tilde{x}$, $\hat{x}$ and $\bar{x}$ to (−)-signs, we get the following analogous result

$$
\begin{aligned}
&\frac{\varphi(\tilde{x}^-) - \varphi(\hat{x}^-) - \varphi(\bar{x}^-) + \varphi(x)}{h_j h_k} \\
&= \frac{\partial^2 \varphi}{\partial x_j x_k}(x) - \frac{\partial^3 \varphi}{\partial x_j^2 x_k}(x)\frac{h_j}{2} - \frac{\partial^3 \varphi}{\partial x_j x_k^2}(x)\frac{h_k}{2} \\
&\quad + \sum_{|\kappa|=4} \frac{\partial^4 \varphi}{\partial x^\kappa}(\tilde{x})\frac{1}{\kappa!}\frac{(\tilde{x}-x)^\kappa}{h_j h_k} \ .
\end{aligned} \tag{4.9}
$$

Adding (4.8) and (4.9) and by setting $x = y_i$, $\tilde{x} = y_{(i+1,+1)_{jk}}$, $\hat{x} = y_{(i+1)_j}$, $\bar{x} = y_{(i+1)_k}$ and $h_{max} = \Delta y$, we obtain that $D^2_{(y)_j,(y)_k}\varphi(y_i, \alpha^n)$ is of order $O((\Delta y)^2)$. The calculations for the other difference quotients are analogous and can be found in many books. We conclude that (4.6) holds.

Next we look at the term

$$\|\tilde{\mathfrak{L}}_h^{n+1}\varphi - \mathfrak{L}^n\varphi\| = O(\Delta x) + O(\Delta\tau) \ .$$

This follows since the coefficients of the PDE a, b and also g are continuous functions of time.

Using Taylor series expansion, we deduce for the time derivative

$$
\begin{aligned}
&\varphi_i^{n+1} = \varphi_i^n + (\varphi_t)_i^n \Delta\tau + O((\Delta\tau)^2) \\
\Leftrightarrow \quad &(\varphi_t)_i^n = \frac{\varphi_i^{n+1} - \varphi_i^n}{\Delta\tau} + O((\Delta\tau)) \ .
\end{aligned}
$$

That yields

$$\left\|\frac{\varphi_i^{n+1} - \varphi_i^n}{\Delta\tau} - (\varphi_t)_i^n\right\| = O(\Delta\tau) \ . \tag{4.10}$$

Now we put all pieces together to derive

$$
\begin{aligned}
&\|\tilde{\mathfrak{L}}_{\Delta y}\varphi - \mathfrak{L}\varphi\| \\
&= \Big|(\varphi_t)_i^{n+1} - \max_{\alpha^{n+1}\in\mathfrak{A}} \{(\mathfrak{L}^\alpha\varphi) + g\}_i^{n+1} - \Big(\frac{\varphi_i^{n+1} - \varphi_i^n}{\Delta\tau} \\
&\quad + (1-\Theta) \sup_{\alpha^{n+1}\in\mathfrak{A}} \Big\{(\mathfrak{L}_{\Delta y}^{\alpha^{n+1}}\varphi^{n+1})_i + g_i^{n+1}\Big\} + \Theta \sup_{\alpha^n\in\mathfrak{A}} \{(\mathfrak{L}_{\Delta y}^{\alpha^n}\varphi^n)_i + g_i^n\}\Big)\Big| \\
&\le \Big|(\varphi_t)_i^{n+1} - \frac{\varphi_i^{n+1} - \varphi_i^n}{\Delta\tau}\Big| \\
&\quad + (1-\Theta)\,\Big| \max_{\alpha^{n+1}\in\mathfrak{A}} \{(\mathfrak{L}^\alpha\varphi) + g\}_i^{n+1} - \sup_{\alpha^{n+1}\in\mathfrak{A}} \Big\{(\mathfrak{L}_{\Delta y}^{\alpha^{n+1}}\varphi^{n+1})_i + g_i^{n+1}\Big\}\Big| \\
&\quad + \Theta\,\Big| \max_{\alpha^{n+1}\in\mathfrak{A}} \{(\mathfrak{L}^\alpha\varphi) + g\}_i^{n+1} - \sup_{\alpha^n\in\mathfrak{A}} \{(\mathfrak{L}_{\Delta y}^{\alpha^n}\varphi^n)_i + g_i^n\}\Big| \\
&= O(\Delta\tau) + O(\Delta y) + \Theta\,\Big| \max_{\alpha^{n+1}\in\mathfrak{A}} \{(\mathfrak{L}^\alpha\varphi) + g\}_i^{n+1} - \sup_{\alpha^n\in\mathfrak{A}} \{(\mathfrak{L}_{\Delta y}^{\alpha^n}\varphi^n)_i + g_i^n\}\Big| \\
&= O(\Delta\tau) + O(\Delta y)\ .
\end{aligned}
$$

We have shown that (4.5) holds. □

4.3.3 Definition (stability)**.** *Dicretisation* (4.4.2) *is stable if* $\|v^{n+1}\|_\infty \le C$ *for* $n = 0, \ldots N$*, for* $\Delta\tau \to 0$ *and* $\Delta y \to 0$ *and* C *is some constant independent of* $\Delta\tau$ *and* Δy*.*

4.3.4 Theorem. *The approximation* (4.4.2) *of the HJB* (2.6) *is stable if coefficient condition 4.1.2 is satisfied and appropriate boundary condition (e.g. as described in 4.2) are imposed.*

Proof. The discrete equation reads according to (4.4.2) and (4.1)

$$
\begin{aligned}
v_i^{n+1} = v_i^n &- \Delta\tau(1-\Theta)\Big\{c_i^{n+1}v_{i+1}^{n+1} + d_i^{n+1}v_i^{n+1} + e_i^{n+1}v_{i-1}^{n+1} \\
&+ f_i^{n+1}(+)v_{i\pm1\pm1}^{n+1} + f_i^{n+1}(-)v_{i\pm1\mp1}^{n+1} + g_i^{n+1}\Big\} \\
&- \Delta\tau\Theta\Big\{c_i^n v_{i+1}^n + d_i^n v_i^n + e_i^n v_{i-1}^n \\
&+ f_i^n(+)v_{i\pm1\pm1}^n + f_i^n(-)v_{i\pm1\mp1}^n + g_i^n\Big\}\ .
\end{aligned}
$$

We surpressed the α dependence to simplify the notation and deduce

$$|v_i^{n+1}| \le |v_i^n| + \Delta\tau(1-\Theta)\Big| c_i^{n+1} v_{i+1}^{n+1} + d_i^{n+1} v_i^{n+1} + e_i^{n+1} v_{i-1}^{n+1} + f_i^{n+1}(+) v_{i\pm1\pm1}^{n+1} + f_i^{n+1}(-) v_{i\pm1\mp1}^{n+1} + g_i^{n+1}\Big| + \Delta\tau\Theta\Big| c_i^n v_{i+1}^n + d_i^n v_i^n + e_i^n v_{i-1}^n + f_i^n(+) v_{i\pm1\pm1}^n + f_i^n(-) v_{i\pm1\mp1}^n + g_i^n\Big|$$

$$|v_i^{n+1}| \le \Delta\tau(1-\Theta)\Big(\|c_i^{n+1}\|_\infty + |d_i^{n+1}| + \|e_i^{n+1}\|_\infty + \|f_i^{n+1}(+)\|_\infty + \|f_i^{n+1}(-)\|_\infty\Big)\|v^{n+1}\|_\infty + \Big(1 + \Delta\tau\Theta(\|c_i^n\|_\infty + |d_i^n| + \|e_i^n\|_\infty + \|f_i^n(+)\|_\infty + \|f_i^n(-)\|_\infty)\Big)\|v^n\|_\infty + \Delta\tau \max\Big\{|g_i^n|, |g_i^{n+1}|\Big\}$$

$$|v_i^{n+1}| \le \Delta\tau C_1 \|v^{n+1}\|_\infty + (1+\Delta\tau C_2)\|v^n\|_\infty + \Delta\tau \max\Big\{|g_i^n|, |g_i^{n+1}|\Big\} .$$

That yields

$$\|v^{n+1}\|_\infty \le \frac{1+\Delta\tau C_2}{1-\Delta\tau C_1}\|v^n\|_\infty + \frac{\Delta\tau}{1-\Delta\tau C_1}\max\Big\{\|g^n\|_\infty, \|g^{n+1}\|_\infty\Big\} .$$

For $\Delta\tau \to \infty$ the coefficient converges to 1. So we conclude that there is an $\epsilon > 0$ and a constant $\hat{C}_\epsilon$ independent of τ so that for all $\Delta\tau \le \epsilon$ the inequality $\|v^{n+1}\|_\infty \le \hat{C}_\epsilon \|v^n\|_\infty$ holds. From that we conclude that there are constants C_ϵ^n for all $n = 1, \dots T$ independent of τ so that for all $\Delta\tau \le \epsilon$ a stability estimate is given by

$$\|v^{n+1}\|_\infty \le C_\epsilon^{n+1}\|v^0\|_\infty .$$

□

Recall the definition of G_i^{n+1} as the right-hand side of equation (4.4.2).

4.3.5 Definition (monotonicity)**.** *The discrete scheme* (4.4.2) *is monotone if for all* $\epsilon^l \in \mathbb{R}^p$, $\epsilon_j^l \ge 0$ *for* $l \in \{n, (n+1)\}$ *and all* $j = 1 \dots N$ *with* $\epsilon_i^{n+1} = 0$*:*

$$G_i^{n+1}\Big(v^{n+1} + \epsilon^{n+1}, v^n + \epsilon^n\Big) \le G_i^{n+1}\Big(v^{n+1}, v^n\Big) .$$

The monotonicity of the FDM scheme is the discrete equivalent of the ellipticity condition 3.1.1.

4.3.6 Theorem (monotonicity)**.** *If the coefficient condition 4.1.2 and appropriate boundary conditions are imposed, the discretisation* (4.4.2) *is monotone.*

Proof. For all $\epsilon^{n+1}, \epsilon^n \geq 0$ with $\epsilon_i^{n+1} = 0$ we obtain

$$G_i^{n+1}\Big(v^{n+1} + \epsilon^{n+1}, v^n + \epsilon^n\Big) - G_i^{n+1}\Big(v^{n+1}, v^n\Big)$$

$$\begin{aligned}
0 = &\frac{v_i^{n+1} - (v_i^n + \epsilon_i^n)}{\Delta\tau} + (1-\Theta) \sup_{\alpha^{n+1}\in\mathfrak{A}} \Big\{(\mathfrak{L}_{\Delta y}^{\alpha^{n+1}} v^{n+1} + \epsilon^{n+1})_i + g_i^{n+1}\Big\} \\
&+ \Theta \sup_{\alpha^n\in\mathfrak{A}} \big\{(\mathfrak{L}_{\Delta y}^{\alpha^n} v^n + \epsilon^n)_i + g_i^n\big\} \\
&- \frac{v_i^{n+1} - (v_i^n)}{\Delta\tau} - (1-\Theta) \sup_{\alpha^{n+1}\in\mathfrak{A}} \Big\{(\mathfrak{L}_{\Delta y}^{\alpha^{n+1}} v^{n+1})_i + g_i^{n+1}\Big\} \\
&- \Theta \sup_{\alpha^n\in\mathfrak{A}} \big\{(\mathfrak{L}_{\Delta y}^{\alpha^n} v^n)_i + g_i^n\big\} \\
\leq &-\frac{\epsilon_i^n}{\Delta\tau} + (1-\Theta) \sup_{\alpha^{n+1}\in\mathfrak{A}} \Big\{(\mathfrak{L}_{\Delta y}^{\alpha^{n+1}} \epsilon^{n+1})_i\Big\} + \Theta \sup_{\alpha^n\in\mathfrak{A}} \big\{(\mathfrak{L}_{\Delta y}^{\alpha^n} \epsilon^n)_i\big\} \\
\leq &-\frac{\epsilon_i^n}{\Delta\tau} + (1-\Theta)\|\epsilon^{n+1}\|_\infty \sup_{\alpha^n\in\mathfrak{A}} \Big\{c_i^{n+1}\mathbb{1} + e_i^{n+1}\mathbb{1} + f_i^{n+1}(+)\mathbb{1} + f_i^{n+1}(-)\mathbb{1}\Big\} \\
&+ \Theta\|\epsilon^n\|_\infty \sup_{\alpha^n\in\mathfrak{A}} \Big\{c_i^n\mathbb{1} + d_i^n + e_i^n\mathbb{1} + f_i^n(+)\mathbb{1} + f_i^n(-)\mathbb{1}\Big\} \\
\leq &\, 0
\end{aligned}$$

where we used inequalities (3.12) and (4.4). □

Now we can can conclude the desired convergence result in the following theorem. The proof is based on the presentation of Barles and Souganidis [3].

4.3.7 Theorem (convergence of FDM)**.** *If the coefficient condition 4.1.2 and appropriate boundary conditions are imposed, the solution of the discretisation* (4.4.2) *converges to the viscosity solution of the HJB* (2.6) *in* Ω*.*

Proof. We recall that v^Δ is the solution to the numerical scheme resulting from (4.4.2). We define

$$\overline{u} := \limsup_{\substack{\Delta y\to 0\\ \Delta\tau\to 0\\ y\to y_i}} v^\Delta \quad \text{and} \qquad \underline{u} := \liminf_{\substack{\Delta y\to 0\\ \Delta\tau\to 0\\ y\to y_i}} v^\Delta\ .$$

Hence we have $\overline{u} \in USC$ and $\underline{u} \in LSC$. The definitions of $\overline{u}$ and $\underline{u}$ yield $\underline{u} \leq \overline{u}$. If we can show that $\overline{u}$ is a supersolution and $\underline{u}$ a subsolution of (2.3.1), we can apply the comparison principle 3.3.7 and immediately get $\underline{u} \geq \overline{u}$. Therefore we conclude $u \equiv \underline{u} = \overline{u}$ is the unique solution of (2.3.1).

Furthermore by the definition of $\overline{u}$ and $\underline{u}$, v^Δ is locally uniform convergent to u.

It is left to show that $\overline{u}$ is a supersolution and $\underline{u}$ a subsolution. Let $\varphi \in C^\infty(\Omega)$ and $\hat{x} \in \Omega$ such that $\hat{x}$ is a local maximum of $\overline{u}(\hat{x}) - \varphi(\hat{x})$. Without loss of generality we assume that $\overline{u}(\hat{x}) - \varphi(\hat{x}) = 0$ is a strict local maximum. We note that φ fulfils the condition

$$\begin{cases} \overline{u}(\hat{x}) = \varphi(\hat{x}) & \text{and} \\ \varphi(x) > \overline{u}(x) & \text{for } x \in B_r(\hat{x}) \setminus \{\hat{x}\} \end{cases}$$

for some $r > 0$.

We observe that $v^\Delta \leq \overline{u}$ holds by definition and in addition $\overline{u} \leq \varphi$ holds in $B_r(\hat{x})$. So we conclude $v^\Delta \leq \varphi$ in $B_r(\hat{x})$. Hence the monotonicity of G 4.3.6 yields

$$G_i^{n+1}(\varphi, \Delta y, \Delta\tau) \leq G_i^{n+1}(v^\Delta, \Delta y, \Delta\tau) = 0 \ . \tag{4.11}$$

Thus we deduce

$$\begin{aligned} -G_i^{n+1}(\varphi, \Delta y, \Delta\tau) &\geq 0 \\ -\Big(\varphi_t(\hat{y}) - \delta\varphi(\hat{x}) + H(\hat{y}, D\varphi(\hat{x}), D^2\varphi(\hat{x}))\Big) - G_i^{n+1}(\varphi, \Delta y, \Delta\tau) &\geq \\ -\Big(\varphi_t(\hat{x}) - \delta\varphi(\hat{x}) + H(\hat{y}, D\varphi(\hat{x}), D^2\varphi(\hat{x}))\Big) \end{aligned}$$

for all Δ. According to 4.3.2 G is a consistent scheme. Using that, we derive from the above inequality

$$\begin{aligned} &\lim_{\Delta \to 0} \| - \Big(\varphi_t(\hat{y}) - \delta\varphi(\hat{y}) + H(\hat{x}, D\varphi(\hat{x}), D^2\varphi(\hat{x}))\Big) - G_i^{n+1}(\varphi, \Delta y, \Delta, \tau)\| \\ &\geq -\Big(\varphi_t(\hat{x}) - \delta\varphi(\hat{y}) + H(\hat{x}, D\varphi(\hat{x}), D^2\varphi(\hat{x}))\Big) \\ 0 &\geq -\Big(\varphi_t(\hat{x}) - \delta\varphi(\hat{y}) + H(\hat{x}, D\varphi(\hat{x}), D^2\varphi(\hat{x}))\Big) \ . \end{aligned}$$

Since $\varphi(\hat{x}) = \overline{u}(\hat{x})$ for all t we obtain

$$0 \geq -\Big(\varphi_t(\hat{x}) - \delta\overline{u}(\hat{x}) + H(\hat{x}, D\varphi(\hat{x}), D^2\varphi(\hat{x}))\Big) \ .$$

A similar series of steps shows that if $\hat{x} \in \Omega$ is a minimum of $\underline{u}(\hat{x}) - \varphi(\hat{x})$ the inequality

$$0 \leq -\Big(\varphi_t(\hat{x}) - \delta\underline{u}(\hat{x}) + H(\hat{x}, D\varphi(\hat{x}), D^2\varphi(\hat{x}))$$

holds and thus that $\underline{u}$ is a supersolution. □

We always needed "appropriate" boundary conditions. But even in less complicated financial examples one can have discontinuous boundary conditions, e.g. problems including barrier options. We will briefly discuss general boundary conditions in the following section.

4.4 Discontinuous boundary conditions

In the case of more general boundary conditions we need to make some extra effort. The definition of consistency 4.3.1 covers the case of boundary conditions as in section 4.2 (cf. Forsyth and Labahn [8]). In the case of general boundary conditions we need a more complicated definition of viscosity solutions and consistency (cf. Barles and Souganidis [3], Barles [2]). The idea is to write the HJB and the boundary condition in one equation and obtain a solution in $\bar{\Omega}$ instead of Ω.

$$F(x,u,p,X) := \begin{cases} H(x,u,p,X) & \text{in } \Omega \\ H^{BC}(x,u,p) & \text{on } \partial\Omega \end{cases} \tag{4.12}$$

where $H(x,u,p,X) = 0$ is degenerate elliptic. This way one thinks of the boundary condition as a discontinuity of the equation. By F^* we denote the upper semicontinuous envelope of F defined as

$$F^*(x,u,Du,D^2u) = \limsup_{\substack{y \in \bar{\Omega} \\ y \to x}} \left[F(y,u,Du,D^2u) \right] .$$

Respectively by F_* with

$$F^*(x,u,Du,D^2u) = \liminf_{\substack{y \in \bar{\Omega} \\ y \to x}} \left[F(y,u,Du,D^2u) \right]$$

we denote the lower semicontinuous envelope of F.

4.4.1 Definition (discontinuous viscosity solution)**.** *The function u is a viscosity solution of $F(x,u,Du,D^2u) = 0$ on $\bar{\Omega}$ if (and only if) it is a viscosity solution of $HJB = 0$ in Ω and if $F^* \le 0$ on $\partial\Omega$ and $F_* \ge 0$ on $\partial\Omega$.*

4.4.2 Definition (consistency on $\bar{\Omega}$)**.** *Define $\Delta = \max(\Delta y, \Delta t)$. The scheme with boundary condition is called consistent if any smooth function φ*

$$\limsup_{\substack{\Delta \to 0 \\ \xi \to 0 \\ y \to y_i}} \frac{G(\varphi + \xi, \Delta y, \Delta t)}{\Delta} \le F^*(x,u,Du,D^2u) \ ,$$

as well as

$$\liminf_{\substack{\Delta \to 0 \\ \xi \to 0 \\ y \to y_i}} \frac{G(\varphi + \xi, \Delta y, \Delta t)}{\Delta} \ge F_*(x,u,Du,D^2u)$$

hold for all $x \in \bar{\Omega}$.

Now we have the definitions at hand to proof a general convergence result; in particular theorem 4.3.7 holds true for general boundary conditions and on $\bar{\Omega}$.

4.4.3 Remark. *If the coefficient condition 4.1.2 and the discretisation* (4.4.2) *is consistent according to definition 4.4.2, the solution of the discretisation* (4.4.2) *converges to the viscosity solution of the HJB 4.4.2 in* $\bar{\Omega}$.

Proof. We only have to show that $\overline{u}$ (respectively $\underline{u}$) defined in the proof of 4.3.7 is a supersolution (respectively subsolution) as defined in 4.4.1. The rest of the proof stays the same.

We set $\Delta = max \Delta y, \Delta t$. Let $\varphi \in C^\infty(\Omega)$ and $\hat{x} \in \Omega$ such that $\hat{x}$ is a local maximum of $\overline{u}(\hat{x}) - \varphi(\hat{x})$. Without loss of generality we assume that $\overline{u}(\hat{x}) - \varphi(\hat{x}) = 0$ is a strict local maximum and that φ fulfils the condition

$$\begin{cases} \varphi(x) \geq 2 \sup_\Delta \|v^\Delta\|_\infty & \text{for } x \in \bar{\Omega} \setminus B_r(\hat{x}) \quad \text{and} \\ \varphi(x) > \overline{u}(x) & \text{for } x \in B_r(\hat{x}) \end{cases}$$

Then there exist sequences $\Delta y_n \in \mathbb{R}^+$, $\Delta \tau_n \in \mathbb{R}^+$ and $x_n \in \bar{\Omega}$ with

$$\Delta y_n \to 0 \ , \qquad \Delta \tau_n \to 0 \qquad \text{and} \qquad x_n \to \hat{x}$$

as $n \to 0$ such that $v^{\Delta_n}(x_n) \to \overline{u}(\hat{x})$ and x_n is a global maximum of $v^{\Delta_n} - \varphi$. Now set $\xi_n := v^{\Delta_n}(x_n) - \varphi(x_n)$. Since φ is continuous, we know that $\varphi(x_n) \to \varphi(\hat{x})$. Moreover $v^{\Delta_n}(x_n) \to \overline{u}(\hat{x}) = \varphi(\hat{x})$. We conclude that $\xi_n \to 0$.

Next we note that since $v^{\Delta_n}(x) - \varphi(x) \leq v^{\Delta(x)_n}(y_n) - \varphi(y_n)$, we obtain

$$v^{\Delta_n}(x) \leq \varphi(x) + \xi_n$$

for all $x \in \bar{\Omega}$. Now we can apply the monotonicity of G. That yields

$$G(\varphi(x_n) + \xi_n, \Delta_n y, \Delta_n t) \leq G(v^{\Delta_n}, \Delta_n y, \Delta_n t) = 0$$

for all Δ_n. Taking the limit $n \to \infty$ and the consistency of G as defined in 4.4.2 we deduce

$$\begin{aligned} 0 &\geq \liminf_{n\to\infty} \frac{G(\varphi(x_n) + \xi_n, \Delta_n y, \Delta_n t)}{\Delta} \\ &\geq \liminf_{\substack{\Delta\to 0 \\ x\to\hat{x} \\ \xi\to 0}} G(\varphi(\hat{x}) + \xi, \Delta y, \Delta t) \\ &\geq F_*(\varphi(\hat{x})) \\ &= F_*(\overline{u}(\hat{x})) \ . \end{aligned}$$

In the last step we used the fact that $\overline{u}(\hat{x}) = \varphi(\hat{x})$. We conclude that $\overline{u}$ is a subsolution according to definition 4.4.1. An analogous derivation shows that $\underline{u}$ is a supersolution.

$\square$

Chapter 5

Example 3: Asset allocation with stochastic interest and income

In section 1.3 we introduced a complex financial example (example 3) we now want to return to. In this chapter we will fit the economic model in the notation we used in definition 2.1.1 and 2.3.1 and derive the HJB. Then we will present a special case where we can give a closed form solution. Further we will discuss problems of the implementation. Numerical results will be presented in the next chapter.

5.1 The model setup

The first item of the problem is the state equation which we didn't explicitly name in section 1.3. It is given by the dynamics of wealth, interest rate and income

$$\begin{pmatrix} dW \\ dr \\ dy \end{pmatrix} = a\ dt + b \begin{pmatrix} d\omega_r \\ d\omega_S \\ d\omega_y \end{pmatrix}$$

with $a \in \mathbb{R}^3$

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} rW + \lambda_r(\theta_B\sigma_B + \theta_S\sigma_S\rho_{SB}) - \lambda_S\theta_S\sigma_S\sqrt{1-\rho_{SB}^2} - c + y \\ \kappa(\bar{r} - r) \\ y(\xi_0(s) - \xi_1 r) \end{pmatrix}$$

and $b \in \mathbb{R}^{3\times 3}$

$$b = \begin{pmatrix} \theta_B \sigma_B(r,s) + \theta_S \sigma_S \rho_S B & \theta_S \sigma_S \sqrt{1-\rho_{SB}^2} & 0 \\ -\sigma_r & 0 & 0 \\ -y\sigma_y(t)\rho_{yr} & y\sigma_y(t)\hat{\rho}_{ys} & y\sigma_y(s) \end{pmatrix} .$$

The control vector is

$$\alpha = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} c \\ \theta_B \\ \theta_S \end{pmatrix} .$$

A benchmark for the parameters is listed in the appendix A.3. For further discussion of the economic adjustment of the parameters we refer to Munk and Sørensen [16].

Next we need the utility function, which is given by (1.9) in chapter 1. The problem now reads: Find the control vector α that maximises the indirect utility function (1.10). Further we are interested in the value of the indirect utility function.

Following the line of reasoning of chapter 2 we get the HJB via the dynamic programming principle. Skipping right to the results the relevant HJB for this problem is given by equation (2.6) of definition 2.3.1

$$\begin{aligned} 0 &= J_t(y) - \delta J(y) + H(y, \Delta J(y), \nabla J(y)) \\ &= J_t(y,t) - \delta J(y) + \sup_{\alpha} \left\{ U(\alpha) + a\nabla J(y,\alpha) + \frac{1}{2} Tr(bb^T \Delta J(y,\alpha)) \right\} \\ &= J_t(y) - \delta J(y) \\ &\qquad + \sup_{\theta,c} \Big\{ J_W(rW + \theta^T \Sigma\lambda - c + y) + \frac{1}{2} J_{WW} \theta^T \Sigma \Sigma^T \theta \\ &\qquad\qquad + J_r \kappa(\bar{r} - r) + \frac{1}{2} J_{rr} \sigma_r^2 + J_y y(\xi_0 + \xi_1 r) + \frac{1}{2} J_{yy} y^2 \sigma_y^2 \\ &\qquad\qquad - J_{yr} \theta^T \Sigma \begin{pmatrix} 1 \\ 0 \end{pmatrix} \sigma_r + J_{Wy} y \sigma_y \theta^T \Sigma \rho_{yP} + J_{ry} y \rho_{yr} \sigma_y \sigma_r \Big\} . \end{aligned}$$

From the results of chapter 3, namely theorem 3.3.7, we know that the viscosity solution to this equation is the solution of the economic stochastic control problem if the problem fulfils condition (2.3). Moreover with 3.1.2 we know that the HJB has a degenerate elliptic formulation. We verify condition (2.3).

5.1.1 Remark. *The benchmark parameters listed in A.3 fulfil condition (2.3).*

Proof. Clearly every closed interval in $\mathbb{R}$ fulfils condition (i) for the set of control values. Moreover the coefficients satisfy a Lipschitz condition on a compact domain in $\mathbb{R} \times \mathbb{R} \times \mathbb{R}$. Hence condition (ii) is satisfied. The proof of (iii) is a straightforward calculation of the coefficient vector a and the matrix B on the domain using the definition of a and B above.

□

5.2 Analytic solution for a special case

If the income is fully spanned, i.e. the income stream can be replicated by a portfolio of stocks and bonds, and additionaly there are no portfolio constraints, then the income can be replicated by the traded assets and we can provide a closed form solution. We present the findings of Munk and Sørensen [16].

To be able to replicate the income stream by bonds and stocks, the correlations ρ_{yS} and ρ_{yB} have to satisfy

$$\rho_{yS} + \rho_{SB}\rho_{yr} = \pm\sqrt{(1-\rho_{yr}^2)(1-\rho_{SB}^2)}\ .$$

We define the function H, the human wealth, as

$$H(y,r,t) = y\int_t^T h(t,s)(B^S(r,t))^{1-\xi_1}ds \tag{5.1}$$

with h given by

$$\begin{aligned}&\ln h(t,s)\\ =& \int_t^s \Big(\xi_0(u) - \sigma_y(u)(\rho_{yB}\lambda_r + \rho_{yS}\lambda_S) - (\xi_1 - 1)\rho_{yB}\sigma_r\sigma_y(u)b(s-u)\Big)du\\ &+ \xi_1(\xi_1 - 1)\frac{\sigma_r^2}{2\kappa^2}\Big(s - t - b(s-t) - \frac{\kappa}{2}b(s-t)^2\Big)\ .\end{aligned}$$

The bond volatility b is

$$b(\tau) = \frac{1}{\kappa}(1 - e^{-\kappa\tau})\ .$$

The value function J is determined by

$$J(W,r,y,t) = \frac{1}{1-\gamma}g(r,t)^\gamma(W + H(y,r,t))^{1-\gamma} \tag{5.2}$$

where g is defined as

$$g(r,t) = \int_t^T f(s-t)(B^S(r,t))^{\frac{\gamma-1}{\gamma}}ds + \epsilon f(T-t)(B^S(r,t))^{\frac{\gamma-1}{\gamma}}$$

and f is given by

$$lnf(\tau) = \Big(-\frac{\delta}{\gamma} + \frac{1-\gamma}{2\gamma^2}\|\lambda\|^2\Big)\tau + \frac{1-\gamma}{\gamma^2}\Big((\bar{r} - y_\infty)(\tau - b(\tau)) - \frac{\sigma_r^2}{4\kappa}b(\tau)^2\Big) .$$

The optimal controls can be derived by the first order condition of a maximum. Hence they are

$$\begin{aligned} c(t) &= \frac{W + H(y(t), r(t), t)}{g(r(t), t)} , \\ \theta_B(t) &= \frac{1}{\gamma\sigma_B}(W+H)\left(\lambda_r - \rho_{SB}\frac{\lambda_S}{\sqrt{1-\rho_{SB}}}\right) - H\frac{\sigma_y(t)}{\sigma_B}\frac{\rho_{yB} - \rho_{yS}\rho_{SB}}{1-\rho_{SB}^2} \\ &\quad + \left(1 - \frac{1}{\gamma}\right)\frac{\sigma_r}{\sigma_B}(W+H)G(r,t) \\ &\quad + (\xi_1 - 1)\frac{\sigma_r}{\sigma_B}y\int_t^T b(s-t)h(t,s)(B^S)^{1-\xi_1}ds \text{ and} \\ \theta_S(t) &= \frac{1}{\gamma}(W+H)\frac{\lambda_S}{\sigma_S\sqrt{1-\rho_{SB}^2}} - H\sigma_y(t)\frac{\rho_{yS} - \rho_{SB}\rho_{yB}}{\sigma_S(1-\rho_{SB}^2)} \end{aligned}$$

where G is given by

$$\begin{aligned} & G(r,t) \\ = & \frac{\gamma}{1-\gamma}\frac{g_r(r,t)}{g(r,t)} \\ = & \frac{\int_t^T b(s-t)f(s-t)(B^S(r,t))^{\frac{\gamma-1}{\gamma}}ds + \epsilon b(T-t)f(T-t)(B^T(r,t))^{\frac{\gamma-1}{\gamma}}}{\int_t^T f(s-t)(B^S(r,t))^{\frac{\gamma-1}{\gamma}}ds + \epsilon f(T-t)(B^T(r,t))^{\frac{\gamma-1}{\gamma}}} . \end{aligned}$$

5.2.1 Proposition. *With the above assumptions, i.e. fully spanned income and no portfolio constraints, the human wealth H, the value function J and the controls c, θ_B and θ_S are given by respectively* (5.1), (5.2) *and* $c(t)$, $\theta_B(t)$ *and* $\theta_S(t)$ *defined above.*

A proof of this proposition is given in Munk and Sørensen [16]. The key idea for the derivation is on the one hand the fact that the income stream can be fully replicated by a portfolio of which consists of bonds and stocks since $\sqrt{\rho_{yB}^2 + \rho_{yS}^2} = 1$. On the other hand it is commonplace to make the assumption that the indirect utility function has the form

$$\tilde{J}(W, r, t) = \frac{1}{1-\gamma}g(r,t)^\gamma W^{1-\gamma}$$

when the income y equals zero. With income the indirect utility function is given by

$$\hat{J}(W, r, y, t) = \tilde{J}(W + H(y, r, t), r, t) .$$

The idea for the justification is similar to the idea of example 1 given by Kohn [12]. We have derived the value function by making some assumptions. To proof that value function $\hat{J}$, which was derived under assumptions, is the correct value function J, one can make use of the verification theorem. That states in our case that, if function $\hat{J}$ solves the HJB, it is an upper bound for the value function, i.e. $J \leq \hat{J}$. On the other hand every control provides a lower bound for the maximised indirect utility. One can use the control implicated by $\hat{J}$ to derive the lower bound and conclude that $\hat{J} = J$. We refer to Kohn [12] for more about the verification theorem[1].

5.3 Reduction to a second order PDE

Next we make the assumption that the value function J is homogeneous of degree $1-\gamma$ in (W, y), i.e.

$$J(kW, r, ky, t) = k^{1-\gamma} J(W, r, y, t)$$

and with defining F with $F(x, r, t) = J(x, r, 1, t)$ we conclude that

$$J(W, r, y, t) = y^{1-\gamma} F(x, r, t) \ ,$$

where $x = W/y$ is the wealth to income ratio. For a justification of this assumption see Munk and Sørensen [16].

Now we substitute F into the equation for J in 5.1 and by excessive use of the chain rule we deduce

$$\begin{aligned} 0 = F_t - \hat{\delta}(r,t)F + \sup_{\alpha} \Big\{ & U(c) + F_r(\kappa[r - \bar{r}] + (1-\gamma)\rho_{yr}\sigma_y(t)\sigma_r) \\ & + F_x\Big(1 - \hat{c} + x\Big[(1-\xi_1)r - \xi_0(t) + \gamma\sigma_y(t)^2 + \pi_B\sigma_B(t)(\lambda_r - \gamma\sigma_y(t)\rho_{yB}) \\ & + \pi_S(\psi - \gamma\rho_{ys}\sigma_y(t)\sigma_S)\Big]\Big) + F_{rr}\frac{1}{2}\sigma_r^2 \\ & + F_{xx}\frac{1}{2}x^2\Big((\pi_B\sigma_B)^2 + (\pi_S\sigma_S)^2 - 2\pi_B\pi_S\sigma_B\sigma_S\rho_{SB} + \sigma_y(t)^2 \\ & - 2\sigma_y(t)(\pi_B\sigma_B\rho_{yB} + \pi_S\sigma_S\rho_{yS})\Big) \\ & - F_{xr}x\sigma_r\Big(\pi_B\sigma_B + \pi_S\sigma_S\rho_{SB} + \rho_{yr}\sigma_y(t)\Big)\Big\} \ . \end{aligned}$$

[1] Moreover, as stated in chapter 1, more about verification theorems in the context of viscosity solutions can be found in [Verification Theorems within the Framework of Viscosity Solutions, XUN YU ZHOU, Journal of Mathematical Analysis and Applications, 176 (1993)] and [On some recent aspects of stochastic control and their applications, HUYEN PHAM, Probability Surveys Vol. 2 (2005), pp. 506–549].

Now we derive the controls by deriving $D_{\pi_B}F = 0$ and $D_{\pi_S}F = 0$ and solving the resulting equations for π_B and π_S. We obtain

$$\pi_B = \frac{F_x}{xF_{xx}} \frac{1}{\sigma_B(1-\rho_{SB}^2)}\Big(\gamma\sigma_y(t)(\rho_{yB} - \rho_{yS}\rho_{SB}) - \lambda_r + \frac{\psi\rho_{SB}}{\sigma_S}\Big)$$
$$\frac{F_{xr}}{xF_{xx}}\Big(\frac{\sigma_r}{\sigma_B}\Big) + \frac{\sigma_y(t)}{\sigma_B(1+\rho_{SB}^2)}\Big(\rho_{yB} - \rho_{yS}\rho_{SB}\Big)$$

and

$$\pi_S = \frac{F_x}{xF_{xx}}\Big(\frac{1}{\sigma_S(1-\rho_{SB}^2)}\Big(- \frac{\psi}{\sigma_S} + \rho_{SB}\lambda_r + \gamma\sigma_y(t)(\rho_{yS} - \rho_{SB}\rho_{yB})\Big)$$
$$+ \frac{\sigma_y}{\sigma_S(1-\rho_{SB}^2)}\Big(\rho_{yS} - \rho_{SB}\rho_{yB}\Big) .$$

5.4 The boundary conditions

At any of the boundaries we approximate the first order derivatives with a one-sided difference quotient that only involves points on the grid, e.g. at the boundary $x = x_{max}$ we use $D_x^- F$ to approximate the derivative F_x.

At the lower boundary for x ($x_{min} = 0$) the coefficient of the second derivative of F with respect to x vanishes. No other boundary condition is needed at that boundary. At all other boundaries we "guess" that the second order derivative in the outward direction (either F_{rr} or F_{xx}) vanishes.

For the mixed second order derivative it is also necessary to use approximations that only involve points on the grid, i.e. using a difference quotient with four points that results either from 4.8 or from 4.9. This way we can still use an upwind scheme for the mixed second derivatives at the boundaries[2].

These boundary conditions are in accordance with the conditions stated in 4.2. We have to make sure that the boundaries (in particular x_{max}) are sufficiently far from the most economically interesting domain since the guess of the second derivatives will cause inaccuracy there.

Moreover there is no mathematical restriction for the interest rate to stay positive. Of course negative interest rates are economically nonsense. The grid we use is equidistant with $r \in [-0.03; 0.07]$, $x \in [0; 20]$ and $t \in [0; 30]$.

[2]We thank Professor Munk for his considerate comments.

5.5 A concrete FDM scheme

Applying the FDM scheme from chapter 4 is a pure technical issue. When using the scheme (4.4.2) it is necessary to find the equations for the coefficients c, d, e, $f(+)$ and $f(-)$ of 4.1. The subscripts for the spatial domain i are two-dimensional in our implementation, thus we redefine i as (i, j) for the spatial domain, where i corresponds to x and j corresponds to r. By α we denote the three controls (π_b, π_S, c). The main procedure of the implementation is to write these coefficients in a matrix. The algorithm is given in the appendix A.1 , algorithm A.1.1.

The final time condition is the utility function U

$$J(y, T) = \epsilon U(W_T)$$

if $\epsilon = 1$. Further we have neither consumption nor investment at time T, i.e. $c(T) = \theta_B(T) = \theta_S(T) = 0$. Given the value of J at time T we can compute an approximation of J at time $T - \Delta t$ with scheme (4.4.2). We can remove the sup-term in the HJB if we insert the optimal values of α. The initial guess will always be the approximations for the optimal values at $t + \Delta t$ computed one step before. After solving scheme (4.4.2) we compute the new optimal values of α, i.e. the maximum of v via the standard first order condition $D_\alpha J = 0$. The algorithm for the solver is given by 5.5.1.

```
Algorithm 5.5.1: SOLVERE3()

comment: Solves example 1
(F, α) := InitialiseFinalTime(T)
SaveTimeStep(T, F, α)
for t := (T − 1) to 0
  do { comment: time loop
       while no convergence of F
         do { comment: Maximisation loop
              (M, U) := AllocateMU(α)
              F := SolveMatrixEquation(M, U)
              α := DeriveNewControls(F)
       SaveTimeStep(t, F, α)
```

The approximation of the supremum of the HJB

One issue of the computation is the precision with which one wants to approximate the optimal controls. As stated above the algorithm substitutes a guess for the controls into the equation to remove the sup-term, numerically solves the PDE and then derives the optimal control of the new solution via the first order condition for a maximum. With these new controls we solve the PDE again. We continue this loop until the relative change in the value function falls below some threshold. This iteration is proven to converge always (cf. Forsyth and Labahn [8] and the references therein). However we actually need a concrete threshold which we want to be undercut by the error. If we demand the relative change of the value function between two iteration steps to fall below a fixed number of percentage, we will face a problem. If the value function approaches zero, the expected truncation error and the relative change of the value function can be disproportional. To avoid unreasonable levels of accuracy we introduce a parameter *scale*, so the threshold will be

$$\max_i \left(\frac{(v_i^n)^{k+1} - (v_i^n)^k}{\max\{(v_i^n)^{k+1}, \text{scale}\}} \right) .$$

This leads to the algorithm

Algorithm 5.5.2: $\text{OPTIMISATIONLOOP1}(previousF, \Delta x)$

comment: Exit condition: Percentage change in value function

$Scale \longleftarrow= 0.01$
$Threshold \longleftarrow= 0.001$
comment: This refers to 0.1% change in the value function

while $RelChange < Threshold$

- $F_{old} \longleftarrow= F_{new}$
- $(M, U) \longleftarrow= \text{ALLOCATEMU}(previousF, \alpha)$
- $F_{new} \longleftarrow= \text{SOLVEMATRIXEQUATION}(M, U)$
- **comment:** Solves matrix equation MU=B
- $\alpha \longleftarrow= \text{DERIVENEWCONTROLS}(F_{new})$
- $RelChange \longleftarrow= \max_{(i,j)} \Big((F_{new} - F_{old}) / \max\{F_{new}; \text{Scale}\} \Big)$

Another way to determine the threshold is via the local truncation error (LTE). We will first make a definition.

5.5.1 Definition (local truncation error (LTE)). *Let v be the exact solution of HJB. The local truncation error LTE is defined as*

$$LTE(i,n) = -D_t^+ v(y_i,t_n) + \tilde{\mathfrak{L}}^{\alpha}_{\Delta y} v(y_i,t_n) \ .$$

That is the error we make when inserting the exact solution into the discretisation. Of course we do not know the exact solution but we can use a Taylor series expansion of the solution and put it in the numerical scheme, only regarding the Taylor series to a certain order and dropping the terms of higher order. We approximate the LTE of the difference quotients with

$$LTE(D^+_{(y)_j} v(y,t)) \approx \frac{\Delta}{2} v_{y_j y_j}(y,t) \quad , LTE(D^2_{(y)_j} v(y,t)) \approx \frac{\Delta^2}{4!} v_{y_j y_j y_j y_j}(y,t) \ .$$

This follows directly by inserting the Taylor series (4.7) of the exact solution into the difference quotients. We did the calculation earlier in the context of consistency. Now with decreasing Δ, the approximated LTE will be dominated by the errors in the difference quotients of the first derivatives. We therefore approximate the LTE of the discretisation (4.1) with LTE^* by

$$LTE^* \approx \Delta t \frac{\Delta^2}{2} \sum_i \Big(a_i |D_{x_i x_i} F| \Big)$$

(4.1). Since we do not know the exact solution, we have to use the computation of the value function to derive LTE^*. In the computation we approximate the derivatives by difference quotients which enable us to use an upwind scheme. We can use central differencing when deriving LTE^*. The error guess does not claim high accuracy but it is easy to compute since we have to calculate an approximation of the derivatives of the computed solution anyway in order to derive the controls. It gives us a rough idea what kind of error we are looking for. What we actually have to do is to solve the PDE with an old guess of the optimal controls to get the new value function, then compute the new guess of the optimal controls, then calculate a guess of the LTE with the new value function and the new controls and finally compare the guess of the LTE and the absolute difference between the old and the new value function. The algorithm will be

Algorithm 5.5.3: OPTIMISATIONLOOP2($previousF, \Delta x$)

comment: Exit condition: Guess of the LTE

while $\max_{(i,j)} \Big(\big(F_{new}(i,j) - F_{old}(i,j)\big) - LTE^*(i,j) \Big) > 0$

$$\begin{cases} (M,U) \longleftarrow \text{ALLOCATEMU}(previousF, \alpha) \\ F \longleftarrow \text{SOLVEMATRIXEQUATION}(M,U) \\ \textbf{comment: } \text{Solves matrix equation MF=U} \\ \alpha \longleftarrow \text{DERIVENEWCONTROLS}(U) \\ LTE^* \longleftarrow \text{GUESSLTE}(F, \alpha) \end{cases}$$

Implementation obstacles

To obtain an efficient algorithm we have to be careful with the computer memory. When allocating the matrix M to solve the differential equation we only store the non-zero entries of the matrix. Then for solving the matrix equation $MF = U$ we use the iterative generalized minimal residual method (GMRES). Due to the form of the utility function, the solution of the HJB will be very small as the income-to-wealth ratio x increases. However we have to choose x_{max} sufficiently large since the error will be high at the boundary. We have to make sure that the error tolerance of the GMRES method is small enough to ensure a good calculation. If the solution approaches zero, even small ripples can cause the computation to fail. This is due to the fact that the computation of the controls depends on $(D^2F)^{-1}$.

Chapter 6

Numerical results of the example problems

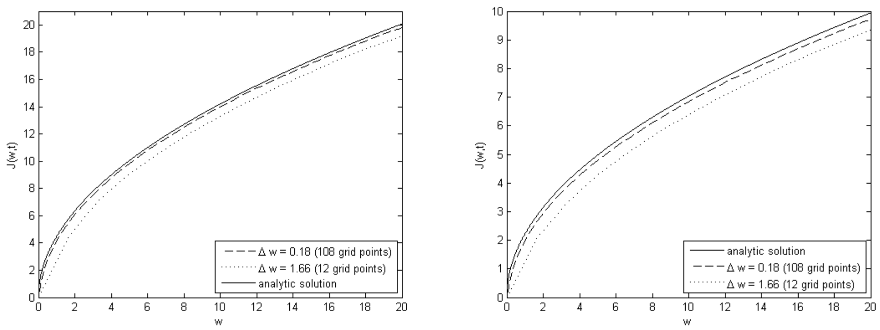
6.1 Results of example 1

Instead of implementing an efficient code for the ODE (1.3) of example 1, we use this example as a test scenario for the implementation of example 3. Only few changes have to be made, most importantly it serves as a test of the matrix allocation algorithm. The algorithm is given by 5.5.1 and A.1.1.

Since we use the code from example 3 we reduce the grid to a $n \times 2$ grid because we only have one variable. By exchanging the variables and using a $2 \times n$ grid we produce exactly the same result. This reaffirms the code for example 3. According to equations (1.4) and (1.5) we have to compute J_w as well as J_{ww} to derive the new controls. Although we use an upwind scheme in the matrix allocation algorithm, we use central differencing in the derivation of the new controls. This appears to be much more accurate.

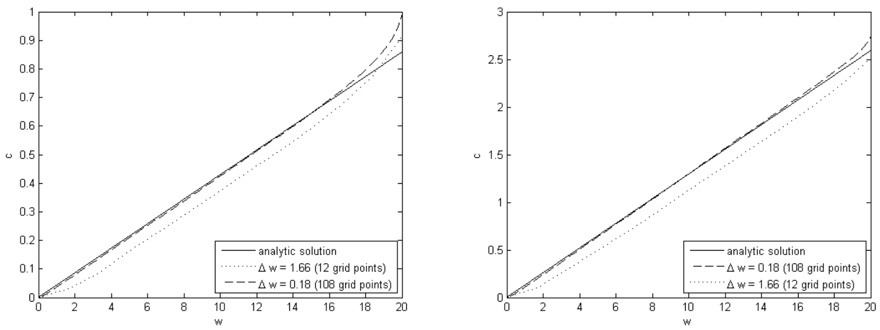
Figure 6.1 shows the analytic solution with two different grid computations. Subfigure 6.1a shows time step 50 of 200, i.e. $t = 7.45$, while 6.1b shows time step 150 of 200, i.e. $t = 22.35$. The implied time step size is $\Delta t = 0.15$. With a relatively coarse space grid (the dotted lines - $\Delta w \approx 1.66$) the computation still looks acceptable if we take the inaccurate boundary conditions into account. By using 108 spatial grid points the computation already is quite accurate.

The resulting computations of the controls are shown in figures 6.2 and 6.3. Though the computation is quite accurate, it corrupts near the upper bound for w. This can be explained through the inaccurate boundary conditions. We assumed the second derivative with respect to w to vanish at the upper bound of w. This is certainly not the correct solution, hence the accuracy at the upper boundary is low. Furthermore, in order to derive the optimal investment I, we need to compute F_{ww}. That is not possible at the upper boundary, thus we cannot derive the control investment here. The boundary condition $F_{ww} = 0$ does not help to compute the optimal investment either since the derivation of I depends on $(F_{ww})^{-1}$. We therefore set $I \equiv 0$ at this boundary. That explains the error in the computation shown in figure 6.3.



(a) $t = 7.45$ (b) $t = 22.35$

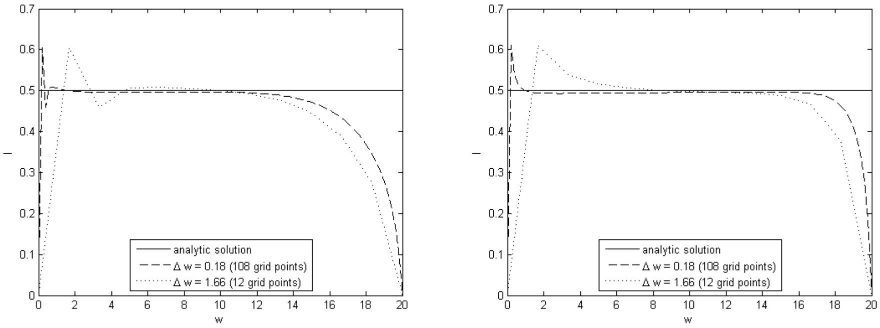
Figure 6.1: Example 1: Analytic value function with two computations.



(a) $t = 7.45$ (b) $t = 22.35$

Figure 6.2: Example 1: Analytic control consumption with two computations.

The next figure 6.4 shows the error of the value function (figure 6.4a) and the control consumption (figure 6.4b) for three different computations. We can

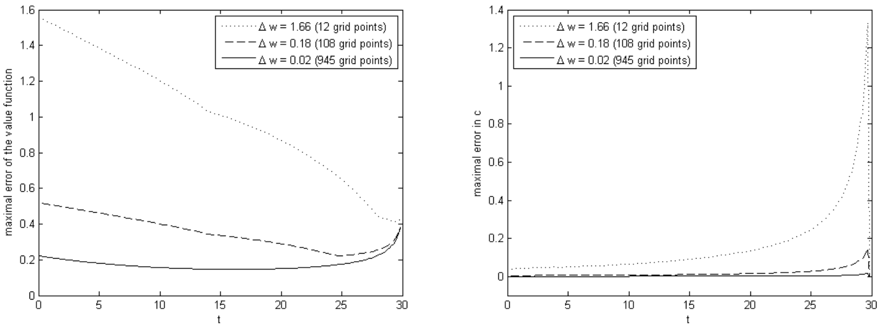


(a) $t = 7.45$

(b) $t = 22.35$

Figure 6.3: Example 1: Analytic control investment with two computations.

observe that the accuracy increases as the grid refines. Again the incorrect boundary data cause inaccuracy at the boundaries. Especially at the upper boundary for w the conditions $J_{ww} = 0$ as well as $I = 0$ cause a massive error. We did not draw the error for the control I since it is dominated by the inaccurate boundary conditions as can be seen in figure 6.3.



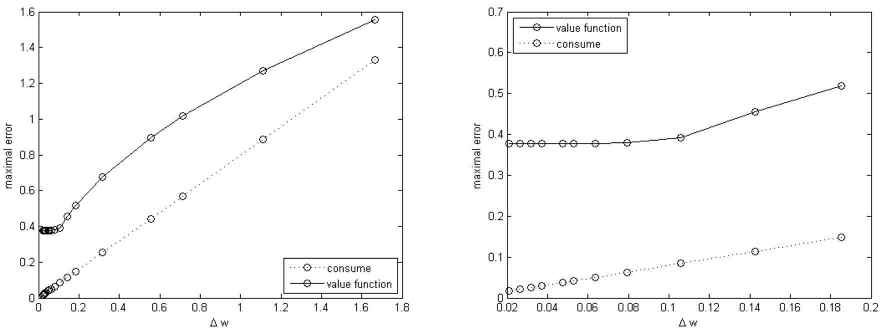
(a) Value function

(b) Control consumption

Figure 6.4: Example 1: Maximal error of J respectively c over the time period. $\Delta t = 0.15$, i.e. 200 time steps.

Figure 6.5 shows the maximal error of the value function and the control consumption for 16 different grids. The subfigure 6.5a shows that the error decreases as the grid refines in the spatial variable. Subfigure 6.5b is a close-up of 6.5a without the 5 coarsest grids. One can see that only increasing the number of spatial grid points while the number of time grid points is unchanged, allows for the error only decrease to a certain point. This is in accordance with 4.3.2 which states that the order of consistency is

$O(\Delta\tau) + O(\Delta y)$. In addition the linear-like decrease of the error reflects the order of consistency. We point out that figure 6.5 shows the maximal error of the whole spatial domain in all timesteps. One may have a higher speed of convergence to the analytic solutions at individual grid points, especially if the usage of central differencing for the first derivatives is possible[1].



(a) Maximal error for 16 grids

(b) Maximal error for 11 grids

Figure 6.5: Example 1: Maximal error of F and C over the space-time domain. $\Delta t = 0.15$, i.e. 200 time steps.

During each time step the computation has to maximise the controls. Figure 6.6 shows the number of iteration it takes the computation to converge, i.e. for the relative change in the value function to be under a specified threshold (0.1% here). The odd peak in the middle could be caused by an increase in the dynamics of the equation. We do not investigate this issue in more detail since the convergence is the desired objective. Most of the time we need few iterations.

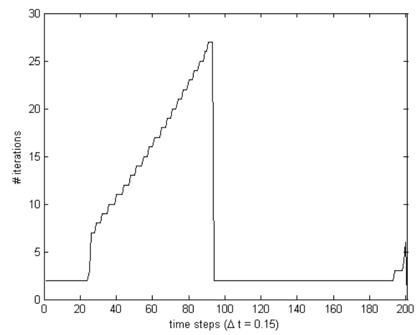


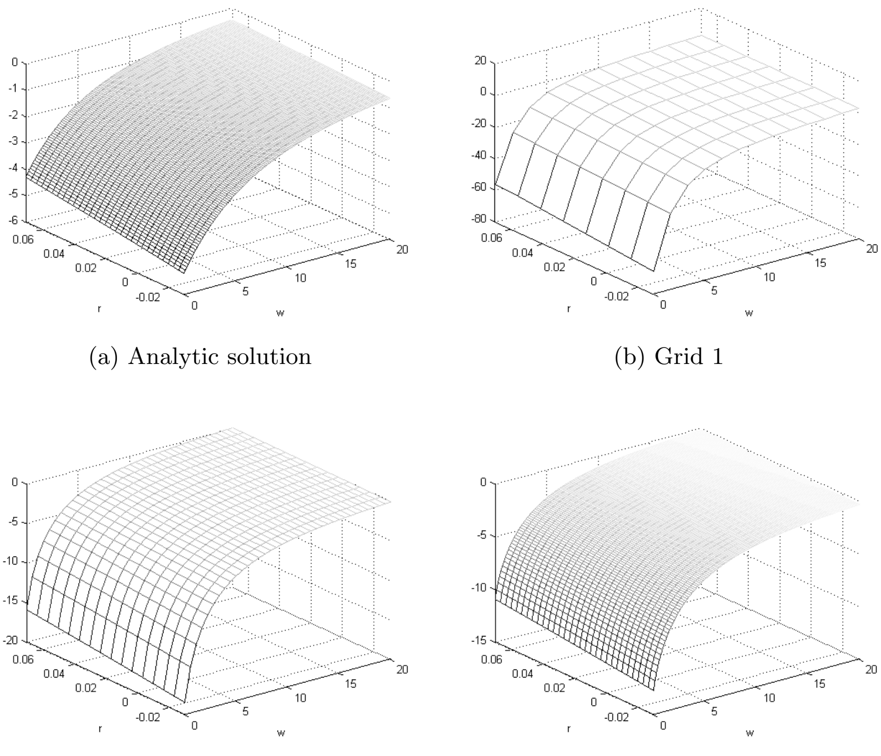
Figure 6.6: Example 1: Number of iterations per time step.

[1]More about the use of central differencing can be found in MAXIMAL USE OF CENTRAL DIFFERENCING FOR HAMILTON-JACOBI-BELLMAN PDEs IN FINANCE, J. WANG and P. A. FORSYTH, SIAM Journal on Numerical Analysis, vol. 46 no. 3, pp. 1580-1601.

6.2 Results of example 3

The technical resources one needs for the computation of example 3 increase fast with refining the grid. A two dimensional PDE has to be solved for each time step several times because of the maximisation loop. For the calculation we used the grids defined in the appendix A.3.

First we take a look at the value function in figure 6.7. It shows the computation of three grids along with the analytical solution. We can see a high imprecision of the computation but at the same time the error decreases drasticly with the refinement of the grid as can be seen in figure 6.7d as well as in figure 6.9b. On the one hand side an error is caused by the incorrect boundary conditions, including the incorrect controls at the boundaries. On the other hand the total error also consists of the inaccuracy of the FDM scheme.



(a) Analytic solution

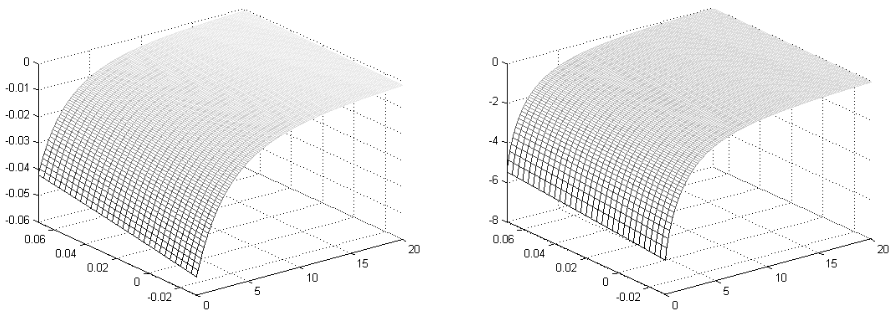
(b) Grid 1

(c) Grid 4

(d) Grid 7

Figure 6.7: Example 3: Value function at time $t = 5$

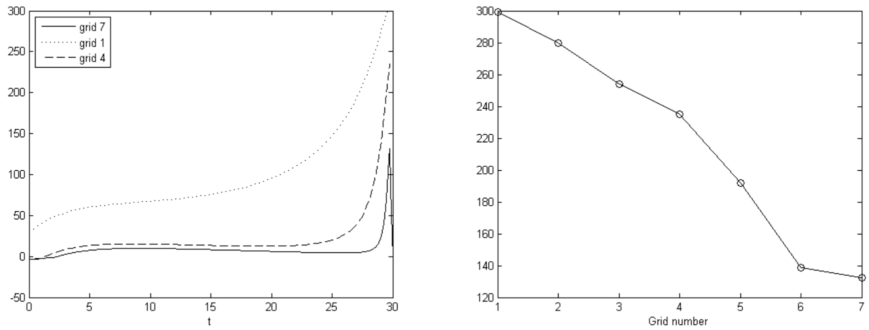
We point out that the error differs in time and becomes much higher near the final time. Figure 6.9a shows the high inaccuracy at near the final time. In figure 6.8 the valufunction is plotted at a later time step than in figure 6.7.



(a) Analytic solution (b) Grid 7

Figure 6.8: Example 3: Value function at time $t = 20$

The overall convergence of the sceme is reflected in figure 6.9b. It also shows that the grids are to coarse for a good computation.

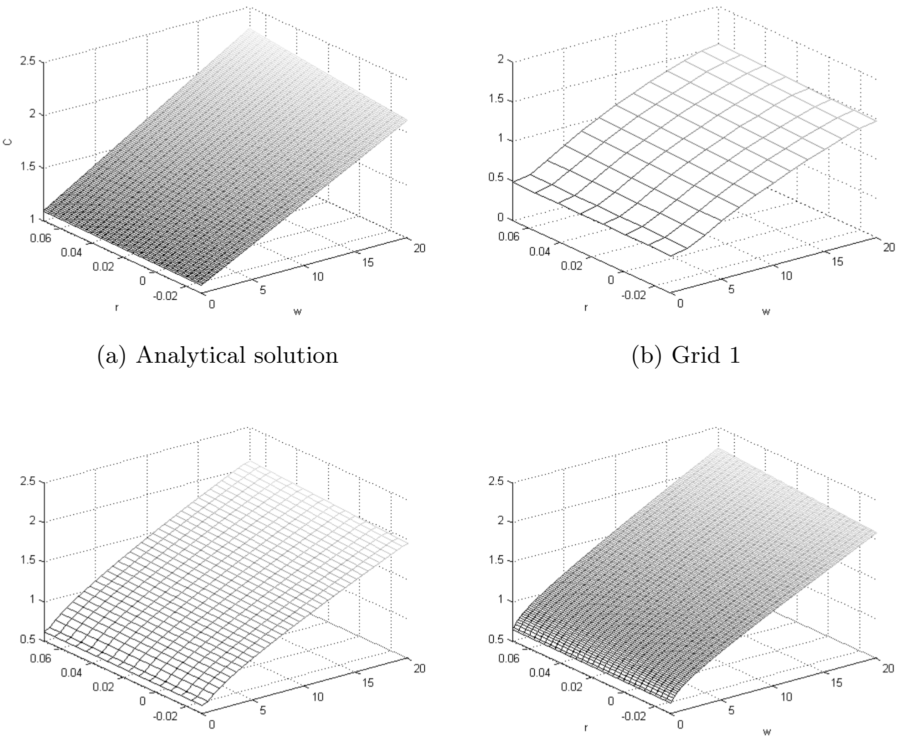


(a) Error per time for grid 1, 4, 7 (b) Error per grid

Figure 6.9: Example 3: Error of the value function

The convergence result stated in chapter 4 is valid for the value function. We did not investigate the corresponding errors in the controls. We can see a relatively robust approximation of the consumption in figure 6.10. The derivation of the optimal consumption only involves the first derivative of the value function with respect to x. At the upper boundary for x the value function approaches zero, hence the derivative with respect to x tends to zero as well. This explains a smaller error than in the value function.

At the lower boundary for x the computation falls much faster than the analytical solution. Therefore the derivative has a higher error which causes an error in the consumption. However the important observation is that the error decreases as the grid refines, i.e. the difference between the solution 6.10a and the computations falls from grid 1 (figure 6.10b) to grid 7 (figure 6.10d).

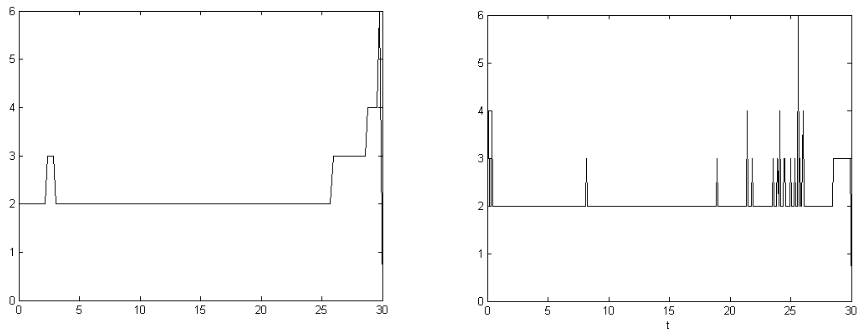


(a) Analytical solution (b) Grid 1

(c) Grid 4 (d) Grid 7

Figure 6.10: Example 3: Consumption

The last figure 6.11 shows the number of iterations it took for the maximisation loop given by algorithm 5.5.2 to stop and hence for the value function to converge. Most of the time only few iterations are needed. Only near the final time the final time a few more iterations are needed.



(a) Grid 4

(b) Grid 7

Figure 6.11: Example 3: Number of iterations per time step

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Appendix A

Details on the numerical implementation

A.1 Algorithms

Algorithm A.1.1: ALLOCATEMU($PreviousF, \alpha$)

comment: Allocates matrix M and load vector U

$\left\{\begin{array}{l}\textbf{comment: } \text{Calculate the HJB coefficients for the derivatives} \\ (a_r; a_x; B_{rr}; B_{xx}; B_{rx}) \longleftarrow= \\ \qquad \text{CALCULATECOEFFICIENTS}(PreviousF, \alpha)\end{array}\right.$

for each *inner Gridpoint* (i, j)

$\left\{\begin{array}{l}M \longleftarrow= \text{CALCULATEM}(PreviousF, \alpha) \\ U \longleftarrow= \text{CALCULATEU}(PreviousF, \alpha)\end{array}\right.$

return $((M, U))$

Algorithm A.1.2: CALCULATEM$(PreviousF, \alpha)$

$$\begin{cases} \textbf{comment: } (i,j) \rightarrow \text{diagonal matrix entries} \\ M_{(i+(j-1)*(i_{max}),i+(j-1)*(i_{max}))} \longleftarrow= ... \\ \quad -\frac{1}{\Delta t} - \hat{\delta} - 2\frac{B_{xx}}{(\Delta x)^2} - 2\frac{B_{rr}}{(\Delta r)^2} - abs(\frac{a_r}{\Delta r})... \\ \quad -abs(\frac{a_x}{\Delta x}) + abs(\frac{B_{rx}}{\Delta x \Delta r}) \end{cases}$$

$$\begin{cases} \textbf{comment: } (i \pm 1, j), (i, j \pm 1) \rightarrow \text{off-diagonal entries} \\ M_{(i+(j-1)*(i_{max}),i-1+(j-1)*(i_{max}))} \longleftarrow= ... \\ \quad \frac{B_{xx}}{(\Delta x)^2} - abs(\frac{B_{rx}}{2\Delta x \Delta r} - \min(0, \frac{a_x}{\Delta x})) \\ M_{(i+(j-1)*(i_{max}),i+(j-2)*(i_{max}))} \longleftarrow= ... \\ \quad \frac{B_{rr}}{(\Delta r)^2} - abs(\frac{B_{rx}}{2\Delta x \Delta r}) - \min(0, \frac{a_r}{\Delta r}) \\ M(_{iI+(iJ-1)*(CapI),iI+1+(iJ-1)*(CapI))} \longleftarrow= ... \\ \quad \frac{B_{xx}}{(\Delta x)^2} - abs(\frac{B_{rx}}{2\Delta x \Delta r}) + \max(0, \frac{a_x}{\Delta x}) \\ M_{(i+(j-1)*(i_{max}),i+(j+1-1)*(i_{max}))} \longleftarrow= ... \\ \quad \frac{B_{rr}}{(\Delta r)^2} - abs(\frac{B_{rx}}{2\Delta x \Delta r}) + \max(0, \frac{a_r}{\Delta r}) \end{cases}$$

$$\begin{cases} \textbf{comment: } (i \pm 1, j \pm 1), (i \pm 1, j \pm 1) \rightarrow \text{off-diagonal entries} \\ M_{(i+(j-1)*(i_{max}),i+1+(j+1-1)*(i_{max}))} \longleftarrow= \max(0, \frac{B_{rx}}{2\Delta x \Delta r}) \\ M_{(i+(j-1)*(i_{max}),i-1+(j-1-1)*(i_{max}))} \longleftarrow= \max(0, \frac{B_{rx}}{2\Delta x \Delta r}) \\ M_{(i+(j-1)*(i_{max}),i+1+(j-1-1)*(i_{max}))} \longleftarrow= -\min(0, \frac{B_{rx}}{2\Delta x \Delta r}) \\ M_{(i+(j-1)*(i_{max}),i-1+(j+1-1)*(i_{max}))} \longleftarrow= -\min(0, \frac{B_{rx}}{2\Delta x \Delta r}) \end{cases}$$

Algorithm A.1.3: CALCULATEU$(PreviousF, \alpha)$

$$U_{(i+(j-1)*(i_{max}))} \longleftarrow== -\frac{PreviousF(j,i)}{\Delta t} - \text{UTILITY}(\alpha)$$

return (U)

A.2 Parameters of example 1

Economic parameters	
δ	0.01
μ	0.03
σ	0.2
γ	0.5
r	0.02

Numeric parameters	
interval for w	$[0;20]$
interval for t	$[0;30]$
maximal iterations in optimisation loop	60
scale as used in 5.5.2	5.e05
threshold as used in 5.5.2	1%

A.3 Parameters of example 3

Economic parameters	
δ	0.03
γ	4
κ	0.5
$\bar{T}$	10
σ_r	0.02
σ_S	0.2
ρ_{yS}	0.05
ρ_{yr}	-0.01
ρ_{yB}	-ρ_{yr}
ρ_{rS}	0.1
ρ_{SB}	-ρ_{rS}
$\bar{r}$	0.02
ξ_1	0.25
ψ	0.04
λ_r	0.1
ϵ	1

If one wants to derive an analytical solution the condition $\rho_{yS} + \rho_{SB}\rho_{yr} = \pm\sqrt{(1-\rho_{yr}^2)(1-\rho_{SB}^2)}$ has to be fulfiled. By choosing $\rho_{SB} = 0$ and $\rho_{yS} = \rho_{yB} = \sqrt{0.5}$ satisfy this condition.

Numeric parameters	
interval for W/y	$[0;20]$
interval for r	$[-0.03;0,07]$
interval for t	$[0;30]$
maximal iterations in optimisation loop	60
scale as used in 5.5.2	5e-05
threshold as used in 5.5.2	1%

Grid parameters							
Grid number	1	2	2	4	3	6	7
Grid points x	13	19	29	37	64	109	141
Grid points r	8	11	15	16	19	22	31
Grid points t	71	91	106	127	141	316	421

A.4 Attachment CD

As an attachment to this thesis we provide a CD with the implementation of example 1 and 3 in MatLab to verify the computations presented in chapter 6. For example 1 the executing files are

- "/diploma T Neugebauer/example i/E1analytic.m"
- "/diploma T Neugebauer/example i/solverE1.m"
- "/diploma T Neugebauer/example i/CreateParNum.m"

The former one will calculate the analytic solution while the second one solves the PDE. In the latter file one can change the parameters stated above.

For example 3 the executing files are

- "/diploma T Neugebauer/example munk/EBConvergenceS9.m"
- "/diploma T Neugebauer/analytic solution/AnalyticSolution.m"

to compute the PDE solution and the analytic solution respectively.

All numerical cumputations used for chapter 6 can be found in the folders "/diploma T Neugebauer/example i" for example 1 and "/diploma T Neugebauer/results example munk" for example 3.

Acknowlegdement

First of all I would like to express my sincerest gratitude to my supervisor Professor Martin Burger for his guidance and patience. His advice was extremely valueable to me.

Next I wish to thank my co-supervisor Professor Nicole Branger for the insightful discussion. Furthermore I would like to thank Professor Claus Munk for his helpful comments.

I wish to thank my family for their encouragement.

Affirmation

I hereby affirm that I have written this diploma independently and that I have not used any other sources than those cited.

I affirm that I have implemented the programs on the attached CD independently.

Tobias Neugebauer