Universität Münster

Space and Time Localization in Model Order Reduction

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Research Objective

Model order reduction is a powerful tool to efficiently compute solutions of parametrized PDEs. Nevertheless, for **large-scale problems**, the **training** of suitable reduced models is often prohibitively expensive. Therefore, we exploit localization strategies and **employ ideas from multiscale and domain decomposition methods** to efficiently construct localized reduced order models in a **parallel and locally adaptive** manner.

Spectral approximation of transfer operators

For elliptic and parabolic PDEs it is well known that **high frequency modes are damped quickly** over space and time. More precisely, for an elliptic PDE, the **transfer operator**

$$\mathcal{T}_{\Gamma_{out}\to\Omega_{in}} : H^{1/2}(\Gamma_{out}) \to H^1(\Omega_{in}),$$

mapping Dirichlet boundary values on the boundary of an oversampling domain to the solution of the PDE on an inner target domain is **compact** with **rapidly decaying spectrum**.

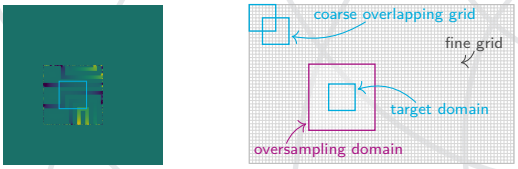
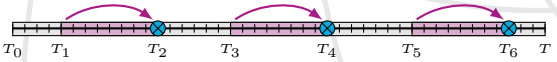


Figure 1: Left: Local solution of elliptic PDE with high-conductivity channels and random Dirichlet boundary values; Right: Overlapping domain decomposition and oversampling domain.

Similar results hold for parabolic PDEs with

$$\mathcal{T}_{T_n\to T_{n+1}} : L^2(\Omega) \to L^2(\Omega),$$

mapping initial values at time T_n to the solution at a later time T_{n+1}.



Idea: Extract modes that still persist in image of $\mathcal{T}$, as these are relevant for approximation. As $\mathcal{T}$ is **compact**, we can approximate its **SVD**

$$\mathcal{T}(u) \approx \sum_{k=1}^{K} \sigma_k(\mathcal{T})\eta_k(u,\xi_k),$$

where the left-singular vectors η_n span **optimal approximation spaces** in the sense of Kolmogorov. Use **randomized SVDs**, which only require applying $\mathcal{T}$ to a few random boundary/initial values.

Localized Reduced Basis Additive Schwarz Methods

Accelerate additive Schwarz methods for elliptic PDEs in multi-query scenarios with local changes using local reduced approximation spaces:

- Compute solutions via **Galerkin projection** onto a search space localized by an **overlapping domain decomposition**.
- Initialize search space as a **spectral coarse space** (e.g. GenEO) and/or use left-singular vectors η_k of $\mathcal{T}_{\Gamma_{out}\to\Omega_{in}}$.
- **Enrich search space** by local additive Schwarz updates in the subdomains selected by a **localized error indicator**.
- High-dimensional computation/communication is only required near enriched subdomains.

Apply **abstract Schwarz theory** to bound the number of required enrichment steps. The approach can be interpreted as a ***locally-adaptive*** **additive Schwarz multi-preconditioned CG method**.

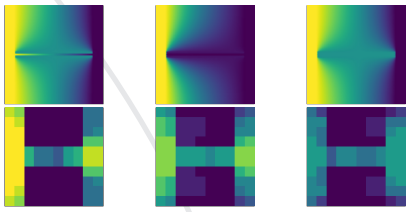


Figure 2: Top: Solutions of elliptic high-contrast benchmark problem with subsequent local changes in diffusivity; Bottom: Number of localized enrichments required for solution update.

Time-parallel construction of reduced basis functions

A well known method to compress solution trajectories of time-dependent PDEs is POD ($\approx$ SVD on time trajectory). Exploit **transfer operator** to avoid high-dimensional global computations in time and obtain **local and parallel in time construction** of problem-adapted basis functions:

- Select time points T_i that are relevant for approximation in a **data-driven** manner via tools from randomized numerical linear algebra.
- Apply $\mathcal{T}_{T_{i-1}\to T_i}$ to random initial values **in parallel** for each i.
- Compute solution via **Galerkin projection** onto reduced space spanned by generated basis functions in each step of time-stepping scheme.

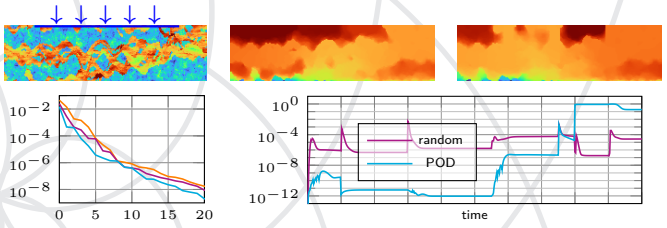


Figure 3: Top: Left: Time-dependent permeability and inflow; Right: Solution of PDE evaluated at two time points; Bottom: Left: Singular value decay of transfer operators; Right: Relative L^2-error over time for POD versus randomized approach based on same computational budget.

Parareal with spectral coarse propagator

The Parareal algorithm **increases parallelism** in the solution of **parabolic PDEs** by replacing sequential time-stepping $u(T) \approx F(u_0, T_0, T)$ by parallel, iterative time-stepping on time slices $[T_n, T_{n+1}]$, $T_0 < \ldots < T_N$, coupled by a fast-to-compute coarse stepper G:

$$U_{n+1}^{k+1} = F(U_n^k, T_n, T_{n+1}) + G(U_n^{k+1}, T_n, T_{n+1}) - G(U_n^k, T_n, T_{n+1}).$$

Standard approach: $G :=$ 'time-stepper with large time-step size'.

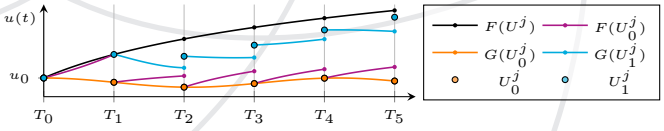


Figure 4: Illustration of first iteration of Parareal algorithm.

- Choose $G(\cdot, T_n, T_{n+1}) := \sum_{k=1}^{K} \sigma_k(\mathcal{T}_{T_n\to T_{n+1}})\eta_k(\cdot,\xi_k)$.
- $N \cdot (K+p)^q$ **parallel local problems** via randomized SVD $(p, q \approx 1)$.
- For K large enough, only a **single global iteration** is needed. Increasing K increases parallelism at the expanse of additional local work.
- Only badly damped modes need to be transferred by G. For Dirichlet boundary conditions, the scheme even converges for $G = 0$!

Outlook: Space-time localized RB methods

- Transfer operator mapping Dirichlet boundary **and** initial conditions on space-time oversampling domain to solution of parabolic PDE on inner space-time target domain can be proven to be **compact**.
- Construct local reduced approximation spaces **in parallel in both space and time** to fully exploit parallelism on modern supercomputers.
- Global coupling via, e.g., partition of unity or DG approaches.

References

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