The localized reduced basis multi-scale method with online enrichment

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(joint work with Mario Ohlberger)

We are interested in the efficient and reliable numerical solution of parametric multi-scale problems, the multi-scale (parametric) character of which is indicated by ε (μ) if expressed in the general notation of (1). It is well known that solving parametric multi-scale problems accurately can be challenging and computationally costly for small scales ε and for a strong dependency of the solution on μ.

Two traditional approaches exist to reduce this computational complexity: numerical multi-scale methods and model order reduction techniques. Numerical multi-scale methods reduce the complexity of multi-scale problems with respect to ε, while model order reduction techniques reduce the complexity of parametric problems with respect to μ (for both see [3] and references therein).

The localized reduced basis multiscale (LRBMS) method is a combination of both to reduce the complexity of parametric multi-scale problems with respect to ε and μ simultaneously. It performs well, for instance in the context of two-phase flow problems (see [1]), but still requires solving (1) on the ε scale for several parameters μ, just like classical RB methods. Therefore, we propose an extension to the LRBMS method which requires a smaller number of full solutions of (1) by further incorporating localization ideas from numerical multi-scale methods.

Following the notation of [3], we consider solutions ${}^{\varepsilon}_{\mu}u_h \in U_h$ of the parameterized variational multi-scale problem

$$R^{\varepsilon}_{\mu}[{}^{\varepsilon}_{\mu}u_h](v_h) = 0 \qquad \forall v_h \in V_h, \tag{1}$$

with trial and test function spaces $U_h, V_h : \Omega \subset \mathbb{R}^d \to \mathbb{R}$, $d = 1, 2, 3$, and an ε- and μ-dependent mapping $R^{\varepsilon}_{\mu} : U_h \to V_h'$. The approximation spaces U_h and V_h are associated with a fine triangulation τ_h of Ω resolving the ε scale.

In general, numerical multi-scale methods capture the macroscopic behavior of the solution in coarse approximation spaces, e.g., $V_H \subset V_h$, usually associated with a coarse triangulation $\mathcal{T}_H$ of Ω, and recover the microscopic behavior of the solution by local fine-scale corrections. Inserting this additive decomposition into (1) yields a coupled system of a fine- and a coarse-scale variational problem. By appropriately selecting trial and test spaces and defining the localization operators to decouple this system, a variety of numerical multi-scale methods can be recovered, e.g., the multi-scale finite element method, the variational multi-scale method and the heterogeneous multi-scale method (see [3] and references therein).

Model order reduction using reduced basis (RB) methods, on the other hand, is based on the idea to introduce a reduced space $V_{\text{red}} \subset V_h$, spanned by solutions of (1) for a limited number of parameters μ. These training parameters are iteratively selected by an adaptive greedy procedure. Depending on the choice of the training parameters and the nature of the problem V_{red} is expected to be of a significantly smaller dimension than V_h. Additionally, if R^{ε}_{μ} allows for an affine decomposition with respect to μ, its components can be projected onto V_{red}, which can then be used to effectively split the computation into an offline and online part (see [1, 3]). In the offline phase all parameter-independent quantities are precomputed, such that the online phase's complexity only depends on V_{red}.

The idea of the combined LRBMS approach, for $U_h = V_h$, is to generate a local reduced space $V_{\text{red}}^T \subset V_h^T$ for each coarse element of $\mathcal{T}_H$, given a tensor product type decomposition of the fine approximation spaces, $V_h = \oplus_{T\in\mathcal{T}_H} V_h^T$. The coarse reduced space is then given as $V_{H,\text{red}} := \oplus_{T\in\mathcal{T}_H} V_{\text{red}}^T \subset V_h$, resulting in a multiplicative decomposition of the solution into ${}^{\varepsilon}_{\mu}u_{H,\text{red}}(x) = \sum_{i=1}^{\dim(V_{H,\text{red}})} u_i^{\mu}(x)\varphi_i^{\varepsilon}(x)$, where the reduced basis functions φ_i^{ε} capture the microscopic behaviour of the solution and the coefficient functions u_i^{μ} only vary on the coarse triangulation.

We detail the LRBMS method in the context of linear elliptic parametric multiscale problems, which arise for instance as the pressure equation in the two-phase flow context: : find ${}^{\varepsilon}_{\mu}u_h \in V_h$, such that $-\nabla\cdot(a_{\mu}^{\varepsilon}\nabla\, {}^{\varepsilon}_{\mu}u_h) = f_u$ holds in a weak sense with homgeneous dirichlet boundary conditions. In this context, the residual in (1) is given as $R_{\mu}^{\varepsilon}[\cdot] := {}^{\varepsilon}_{\mu}A[\cdot] - F_{\mu}$, where ${}^{\varepsilon}_{\mu}A$ and F_{μ} can be expressed as

$$ {}^{\varepsilon}_{\mu}A[\cdot] = \sum_{T\in\mathcal{T}_H} {}^{\varepsilon}_{\mu}A^T[\cdot] + \sum_{T,S\in\mathcal{T}_H} {}^{\varepsilon}_{\mu}A^{T,S}[\cdot], \qquad\qquad F_{\mu} = \sum_{T\in\mathcal{T}_H} F_{\mu}^T, $$

where the coupling operators ${}^{\varepsilon}_{\mu}A^{T,S}$ are given as in the SWIP discontinuous galerkin context for any nontrivial combination of $T, S \in \mathcal{T}_H$ (see [1] for details). The local opearators and functionals ${}^{\varepsilon}_{\mu}A^T$ and F_{μ}^T can be given by any suitable discretization inside the coarse element T, for instance by a continuous finite element discretization in a local fine space V_h^T of piecewise linear polynomials on the fine triangulation inside the coarse element T. The local reduced spaces $V_{\text{red}}^T :=< \Phi^T >$ are then spanned by local reduced bases Φ^T which are computed by restricting and compressing global solution snapshots.

Here we propose an online enrichment step as an addition to the LRBMS method to reduce the need for global solution snapshots. While it is not feasible in the RB framework to compute solution snapshots during the online phase, the LRBMS frameworks allows us to carry out local computations in the online phase to enrich the local reduced bases. The idea of the LRBMS method with online enrichment is as follows: the initial construction of the local reduced bases is carried out as described above but using fewer training parameters and thus less global snapshots. Given local error indicators $|||\, {}^{\varepsilon}_{\mu}u_h|_T - {}^{\varepsilon}_{\mu}u_{H,\text{red}}|_T |||_T \leq \eta_{\text{red}}^T({}^{\varepsilon}_{\mu}u_{H,\text{red}})$ we efficiently asses the quality of the reduced solution ${}^{\varepsilon}_{\mu}u_{H,\text{red}} \in V_{H,\text{red}}$ with respect to the reference solution ${}^{\varepsilon}_{\mu}u_h \in V_h$ during the online phase and select coarse elements $\hat{\mathcal{T}}_H \subseteq \mathcal{T}_H$ where the local reduced bases are insufficient for the current parameter μ. In a local offline phase we compute a local correction function $\varphi_{\text{cor}}^{T^\delta} \in V_h^{T^\delta}$ for each $T \in \hat{\mathcal{T}}_H$ on an oversampled domain $T \subset T^\delta$ by solving

$$ {}^{\varepsilon}_{\mu}A^{T^\delta}[\varphi_{\text{cor}}^{T^\delta}](v_h) = F_{\mu}^{T^\delta}(v_h) - {}^{\varepsilon}_{\mu}A^{T^\delta}[{}^{\varepsilon}_{\mu}u_{H,\text{red}}|_{T^\delta}](v_h) \qquad \forall v_h \in V_h^{T^\delta}. $$

We then restrict this correction function to T and enrich the existing local reduced basis on T by adding $\varphi_{\text{cor}}^{T^\delta}|_T$ after orthonormalization. This process is repeated until the quality of the reduced solution meets the prescribed torelance again. After this local offline phase all quantities are made available in a reduced fashion again and the online phase continues. We repeat this process for each parameter, which is not yet captured by the local reduced bases.

We exemplify the local offline phase by a 2d thermalblock problem, illustrated in figure 1, where the local reduced bases have been trained in the offline phase with one global solution snapshot to μ_{train}, computed on a fine triangulation τ_h with 2500 elements utilizing the discretiation framework `Dune-Fem` [2]. During the online phase we solve for a test parameter μ_{test} which only differs from μ_{train} locally in T_0 and T_2. Since the reduced basis is insufficient for μ_{test}, a local offline phase is started to enrich the local bases until the indicated errors fall below $5e^{-4}$ (figure 1, right). As indicators η^T_{red} we use the true relative error in the local energy norm. For a local oversampling size of ten for example (red line, circular markers), the error torelance is reached after 14 iterations. The resulting sizes of the local reduced bases, $(|\Phi^0|, |\Phi^1|, |\Phi^2|, |\Phi^3|) = (14, 6, 14, 6)$, show the local influence of the parameter component μ_2 and the symmetry of the problem (figure 1, left).

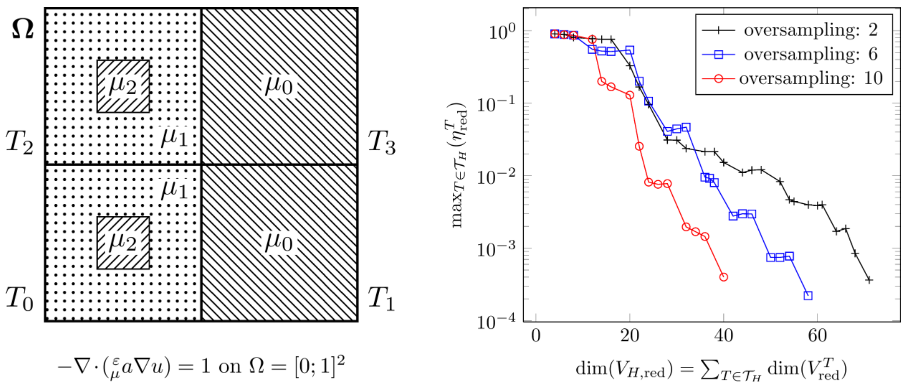

FIGURE 1. Thermalblock example with the values of the piecewise constant parametric diffusion $^{\varepsilon}_{\mu}a$ given by $\mu = (\mu_0, \mu_1, \mu_2)'$ and the coarse triangulation $\mathcal{T}_H = \cup_{i=0}^{3} T_i$ (left). Error evolution during the local offline phase (right): maximum local relative error for $\mu_{\text{test}} = (0.1, 1, 0.01)'$ against size of the coarse reduced space with the local reduced bases trained only with $\mu_{\text{train}} = (0.1, 1, 1)'$ for several sizes of oversampling layers (colored).

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