Seminar on the Calderón Problem and Applications

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1 Description

For a smooth, bounded domain $\overline{\Omega} \subset \mathbb{R}^n$ and $q \in C^\infty(\overline{\Omega})$, we have the Dirichlet problem

$$\begin{cases} (-\Delta + q)u & = 0 \text{ on } \Omega\,; \\ u|_{\partial\Omega} & = f\,. \end{cases} \tag{$*$}$$

The standard PDE "forward" problem addresses existence, uniqueness, and stability of solutions u to $(*)$.

We look instead at the "inverse" problem for $(*)$, also called the *Calderón problem*: Can we determine q from knowledge of the Dirichlet-to-Neumann map $\Lambda_q : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)$ that sends boundary Dirichlet data $f \in H^{1/2}(\partial\Omega)$ to the Neumann derivative $\partial_\nu u|_{\partial\Omega} \in H^{-1/2}(\partial\Omega)$? By "determination", one is typically concerned with uniqueness, reconstruction, and stability. In this seminar, we concern ourselves mostly with uniqueness: Does $\Lambda_{q_1} = \Lambda_{q_2}$ imply $q_1 = q_2$?

Beyond its motivation from applications such as Electrical Impedance Tomography, the Calderón problem also features in geometric scenarios concerning the determination of a background metric. Most immediately, when Ω above is replaced with a compact Riemannian manifold-with-boundary (M, g) and the first line of $(*)$ is replaced by $-\Delta_g u = 0$, the analogous *Riemannian* Calderón problem for Λ_g remains an open problem generally in dimensions $n \geq 3$. Its resolution in dimension $n = 2$ (up to natural conformal invariance) features prominently in the proof of boundary distance rigidity in two dimensions; this latter problem asks whether knowledge of distances between all points $p, q \in \partial M$ determines the Riemannian metric g in M uniquely. When knowledge of boundary distances is instead replaced by knowledge of areas of minimal surfaces $S \subset M$ with $\partial S \subset \partial M$, the corresponding family of inverse problems of metric determination is actively being researched, motivated by the holographic principle in theoretical physics. Known results in this direction rely on techniques developed for the standard Calderón problem.

The goal of this seminar will be to first understand the technique of constructing Complex Geometric Optics solutions (CGOs) to $(*)$ and its role in the Calderón problem. We then look at various geometric inverse problems mentioned above that involve similar ideas.

2 Topics

1. **Overview**
   We start with the physical motivation of the Calderón problem $(*)$, first posed in the form of a conductivity equation by Calderón in [Cal80], as well as its natural generalization as a Riemannian Calderón problem. We discuss CGO approach ([SU87]), as well as the role of the Calderón problem and CGOs in other geometric inverse problems.

2. **CGOs from scratch**
   Following Chapter 3 of [Sal08], we directly construct the CGOs in dimension $n \geq 3$ needed to recover a smooth potential in the Calderón problem.

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3. **Carleman estimate from scratch**
   Following Chapter 5.1 of [Sal08], we reframe the construction of CGOs in terms of obtaining a suitable Carleman estimate.

4. **Analytic Interlude: The (semi-)classical calculus**
   We digress to review/rapidly introduce the (semi-)classical symbol calculus and their associated (pseudo-)differential operators (cf. Chapters 3–4 of [Zwo12]). Much of this can be black-boxed and presented as fact, although it would be nice to work through basic stationary phase (Chapter 3.4, [Zwo12]) and Gårding inequalities (Chapter 4.7, [Zwo12]).

5. **Carleman estimates using the symbol calculus**
   We apply the semiclassical calculus to obtain the Carleman estimates we had previously calculated directly (cf. Section 3, [LRL12]).

6. **Boundary determination using the symbol calculus**
   We follow [LU89], which calculates the classical symbol of the Dirichlet-to-Neumann map associated with Δ_g on a Riemannian manifold-with-boundary. One can then recover the full Taylor series expansion of g at the boundary ∂M, from which one gets metric rigidity in special cases such as strictly convex, real-analytic manifolds of dimension $n \geq 3$.

7. **The Riemannian Calderón problem**
   We follow [LU01], which resolves the Riemannian Calderón problem in dimension $n = 2$ (up to a conformal class). The proof uses in part the result of [LU89] to build an extension of the initial manifold M upon which one can consider Green's functions. After showing pairs of Green's functions (each based in the exterior of M in the extension) can be viewed as coordinates for points within M, one observes after composing with isothermal coordinates that all other Green's functions are real analytic with respect to this original "Green's" coordinate system. One then uses sheaf theory and the maximal analytic continuation of these Green's functions to reconstruct the original unknown manifold.

8. **Application: Boundary distance rigidity**
   Following the outline of Chapter 11 [PSU23], we briefly go over how the two dimensional Riemannian Calderón problem relates to the resolution of boundary rigidity in two dimensions in [PU05].

9. **Application: Inverse problems for minimal surfaces**
   After providing the necessary background on minimal surfaces, we look at work by [ABN20] and [CLT24] concerning metric rigidity from knowledge of areas of codimension one minimal surfaces. Time permitting, we also discuss the corresponding problem in higher codimension, which is relevant to physics.

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