Numerics for partial differential equations

Introduction

- Overview of 2. order PDEs
  (classification, classical model problems)
- Overview of most important discretization concepts
  (finite differences, finite elements, finite volumes)

Finite Differences

- Exm. heat equation
  (stability concepts, convergence)
- Exm. transport equation
  (stability, convergence, dissipation)

Finite Elements

- Exm. Poisson equation
  (matrix assembly, a priori error estimates)
- Adaptivity
  (grid refinement, a posteriori error estimates)

Detailed table of contents

Partial differential equation

Def: Let $\Omega \subset \mathbb{R}^d$ open, $d \geq 2$. A partial differential equation (PDE) is an equation that connects a function $u : \Omega \to \mathbb{R}$ and its partial derivatives,

$$F(D^k u(x), D^{k-1} u(x), \ldots, Du(x), u(x), x) = 0 \quad \forall x \in \Omega,$$

$F : \mathbb{R}^{d^k} \times \mathbb{R}^{d^{k-1}} \times \ldots \times \mathbb{R} \times \Omega \to \mathbb{R}$. k is the order of the PDE, u the sought function

Exm: Laplace-equation $\quad \partial^2 u / \partial x_1^2 + \partial^2 u / \partial x_2^2 = 0$

Def: A PDE problem is a PDE together with boundary conditions on $\Gamma \subset \partial\Omega$

- Dirichlet bc : $u = g$ on Γ (g given)
- Neumann bc : $n^T A D u = g$ on Γ ($A \in \mathbb{R}^{d \times d}$ regular, n = unit normal to Γ)
- Robin bc : $u + n^T A \nabla u = g$ on Γ

The PDE problem is to be solved for $u : \Omega \to \mathbb{R}$

Exm: Laplace equation with $u(x) = x_1$ on $\partial\Omega$ has the solution $u(x) = x_1$.

Classification of 2. order PDEs

Notation: · $u_{x_1} = \partial u / \partial x_1$, $u_{x_1 x_2} = \partial^2 u / \partial x_1 \partial x_2$, ...

· $Du = (u_{x_1}, \dots, u_{x_d})$, $\nabla u = Du^T$

· $\Delta u = \operatorname{div} \nabla u = u_{x_1 x_1} + \dots + u_{x_d x_d}$ "Laplace operator"

· $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}_0^d$ is a multiindex of order $|\alpha| = \alpha_1 + \dots + \alpha_d$.

$D^\alpha u = \partial^{|\alpha|} u / \partial x_1^{\alpha_1} \dots \partial x_d^{\alpha_d}$

· Ω & $x_1, \dots, x_d$ usually denote only spatial variables; if one variable represents time we will call it t and take $[0,T] \times \Omega$ as the domain of the PDE; additionally, $\dot{u} = u_t$.

We will only consider semilinear PDEs: $\sum_{i,j=1}^{d} a_{ij}(x) u_{x_i x_j}(x) + c(\nabla u(x), u(x), x) = 0$

Def: The PDE is called · elliptic, if $A = (a_{ij})_{ij}$ is positive definite

· parabolic, if A is pos. semidefinite with a 1D eigenspace for 0

· hyperbolic, if A has one positive & $d-1$ negative eigenvalues

Elliptic PDEs: energy minimization

Elliptic PDEs often derive from the physical principle of energy minimization:

A physical system in equilibrium attains the state of minimal energy.

Let u be the state, $E[u]$ the energy $\Rightarrow$ $E(u+\varphi) \geq E(u) \quad \forall \varphi: \Omega \to \mathbb{R}$

Exm: The potential energy of an electric charge distribution $u: \Omega \to \mathbb{R}$ (or of an elastic membrane with transversal displacement u or of a temperature distribution u) in an electric field f is

$E(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\, dx + \int_\Omega f u\, dx$. On $\partial\Omega$, $u = g$ is fixed.

$$E(u+\varphi) - E(u) = \int_\Omega \nabla u \cdot \nabla\varphi + \frac{1}{2}|\nabla\varphi|^2\, dx + \int_\Omega f\varphi\, dx$$

Stokes thm./ integration by parts

$$= \int_{\partial\Omega} \varphi \nabla u \cdot n\, dx + \int_\Omega \frac{1}{2}|\nabla\varphi|^2 - \varphi \operatorname{div}\nabla u\, dx + \int_\Omega f\varphi\, dx \overset{!}{\geq} 0 \quad \forall\varphi \text{ with } \varphi = 0 \text{ on } \partial\Omega$$

$\Rightarrow$ Poisson equation $\operatorname{div}\nabla u = \Delta u = f$ with Dirichlet bc $u = g$ on $\partial\Omega$

Parabolic PDEs: gradient flows

A physical system first has to equilibrate, to get to the energy minimum.

Parabolic PDEs derive from the physical principle of gradient flows:

At any time point, the state moves into the direction which allows the fastest energy decrease (at same effort/cost).

Let $\tau D(\frac{\varphi}{\tau})$ be the cost of achieving the change $u \to u+\varphi$ in time τ

$$\Longrightarrow \quad \dot{u} = \lim_{\tau\to 0}\Big(\operatorname*{argmin}_{\varphi}\ \tau D(\tfrac{\varphi}{\tau}) + E(u+\varphi)\Big)\Big/\tau$$

Exm: A change of the charge distribution (displacement/temperature) costs $D(\varphi)=\frac{1}{2}\int_\Omega \varphi^2\,dx$.

$$\tau D(\tfrac{\varphi+\tilde\varphi}{\tau}) + E(u+\varphi+\tilde\varphi) - \big[\tau D(\tfrac{\varphi}{\tau}) + E(u+\varphi)\big] = \int_\Omega \frac{\varphi\tilde\varphi}{\tau} + \frac{\tilde\varphi^2}{2\tau} + \frac{|\nabla\tilde\varphi|^2}{2} + \nabla(u+\varphi)\cdot\nabla\tilde\varphi + f\tilde\varphi\,dx$$

$$= \int_\Omega \tilde\varphi\Big(\frac{\varphi}{\tau} - \Delta(u+\varphi) + f\Big)\,dx + \int_{\partial\Omega}\tilde\varphi\,\nabla(u+\varphi)\cdot n\,dx + O(\tilde\varphi^2) \geq 0 \quad \forall\tilde\varphi \text{ with } \tilde\varphi=0 \text{ on } \partial\Omega$$

$$\Longrightarrow \frac{\varphi}{\tau} = \Delta(u+\varphi) - f \quad \Longrightarrow \varphi\to 0 \text{ for } \tau\to 0$$

$\Longrightarrow$ heat equation $\quad \dot u - \Delta u = -f$ with Dirichlet bc $u=g$ on $\partial\Omega$, $u=u_0$ for $t=0$

Hyperbolic PDEs: conservation laws

Hyperbolic PDEs often derive from the physical principle of conservation:

The change of an extensive quantity (e.g. mass, momentum, energy) in a volume V is only possible via transport through ðV.

Let $q(x,t)$ be the flux of the (mass, energy etc.) density ρ $\Rightarrow \frac{d}{dt}\int_V \rho(x,t)\,dx = -\int_{\partial V} q(x,t)\cdot n\,dx$

$\Rightarrow 0 = \int_V \dot\rho + \operatorname{div} q \, dx \quad \forall V \subset \Omega \quad \Rightarrow \quad \dot\rho + \operatorname{div} q = 0$

gauß theorem

Exm: A material with mass distribution ρ moves at velocity $v(x,t)$ (e.g. water in clouds is transported by wind) $\Rightarrow q = v\rho$

$\Rightarrow$ transport equation $\quad \dot\rho + \operatorname{div}(v\rho) = 0$ with bc $\rho = \rho_0$ for $t = 0$.

Exm: The kinetic & potential energy density within a membrane under tension with displacement u is $\rho(x,t) = \frac{m}{2}|\dot u|^2 + E\frac{|\nabla u|^2}{2}$ (m = membrane mass density, E = elastic modulus), the energy flux is $q = -E\dot u\nabla u \Rightarrow \dot\rho = m\dot u\ddot u + E\nabla u\cdot\nabla\dot u = E\operatorname{div}(\dot u\nabla u)$

$\Rightarrow$ wave equation $\quad \ddot u = \frac{E}{m}\Delta u \quad$ with bc $u = g$ on $\partial\Omega$, $u = u_0$ for $t = 0$

Reduction of hyperbolic PDEs from 2. to 1. order in 2D

$$\ddot{u} - \partial^2 u/\partial x^2 = f(u, \dot{u}, \partial u/\partial x, x, t)$$

$$\Longleftrightarrow \begin{cases} \dot{v} + \partial v/\partial x = f(u, \dot{u}, \partial u/\partial x, x, t) \\ \dot{u} - \partial u/\partial x = v \end{cases}$$

= system of coupled transport equations of 1. order

„ $\partial_t^2 - \partial_x^2 = (\partial_t - \partial_x)(\partial_t + \partial_x)$ "

If f only depends on x&t, one can first solve

$$\dot{v} + \partial v/\partial x = f(x, t)$$

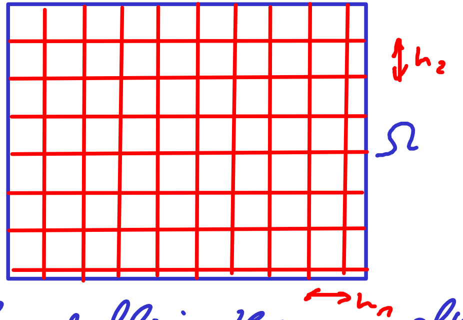
and then

$$\dot{u} - \partial u/\partial x = v$$

=> we will only consider hyperbolic PDEs of 1. order!

Finite Differences

If one has a regular grid on Ω with grid width $h_1, \dots, h_d$ in x_1- ... x_d-direction,



then the derivatives of a sufficiently often differentiable function u can be approximated by difference quotients.

Exm: $\partial u/\partial x_1 \approx \dfrac{u(x_1+h_1,x_2)-u(x_1,x_2)}{h_1}$, $\partial^2 u/\partial x_2^2 \approx \dfrac{u(x_1,x_2+h_2)-2u(x_1,x_2)+u(x_1,x_2-h_2)}{h_2^2}$

We seek an approximation $u_{i_1\dots i_d}$ to $u(\underbrace{(i_1h_1,\dots,i_dh_d)}_{\text{grid points}})$

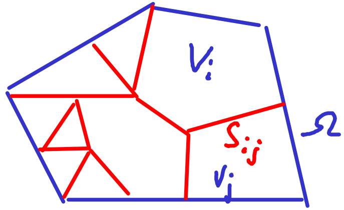
$\Rightarrow$ In the PDE we replace all derivatives by difference quotients and obtain a system of equations in the $u_{i_1\dots i_d}$.

Exm: $\dot u - \partial^2 u/\partial x^2 = 0$, $u(t=0,x)=e^{-x^2}$ $\rightsquigarrow$ $\dfrac{u_{i+1,j}-u_{ij}}{h_1} - \dfrac{u_{i,j+1}-2u_{ij}+u_{i,j-1}}{h_2^2} = 0$, $u_{0j}=e^{-j^2h_2^2}$ $\forall i \geq 0, j$

Introduction: discretization concepts

Finite Volumes

On a mesh of volume elements V_i and their sides S_{ij}



one can apply the principle of conservation on each element: If ρ_{ik} denotes the mean value of ρ on V_i at time $t_k = k\Delta t$ and $\tilde{q}(\rho_{ik}, \rho_{jk}, x, t)$ an approximation to the flux through S_{ij}, then

$$\frac{d}{dt}\int_{V_i} \rho(x,t)\,dx = -\int_{\partial V_i} q(x,t)\cdot n\,dx$$

normal from V_i to V_j

$$\rightsquigarrow \rho_{i,k+1} - \rho_{ik} = \frac{-\Delta t}{|V_i|}\sum_j |S_{ij}|\,\tilde{q}(\rho_{ik}, \rho_{jk}, x_{ij}, t_k)\cdot n_{ij} \qquad \forall k, i$$

midpoint of S_{ij}

which is a set of equations for the ρ_{ik}.

Exm: $\dot{\rho} + \operatorname{div}(v\rho) = 0$, $\rho(t=0,x) = e^{-x^2}$

$$\rightsquigarrow \rho_{i,k+1} - \rho_{ik} = \frac{-\Delta t}{|V_i|}\sum_j |S_{ij}|\, v(x_{ij}, k\Delta t)\cdot n_{ij}\,\frac{\rho_{ik}+\rho_{jk}}{2}, \quad \rho_{i0} = \int_{V_i} e^{-x^2}dx \qquad \forall k>0, i$$

Galerkin methods/Finite Elements

Let V denote a finite-dimensional subspace of the space F of functions $u: \Omega \to \mathbb{R}$ with basis $\phi_1, \dots, \phi_K$. The energy minimization principle can also be applied to V instead of to F:

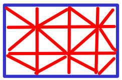
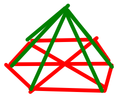
$$u = \operatorname*{argmin}_{v \in F} E(v) = \text{solution of the associated elliptic PDE}$$

$$\rightsquigarrow \hat{u} = \sum_{i=1}^{K} u_i \phi_i = \operatorname*{argmin}_{v \in V} E(v) = \text{approximation of the PDE solution}$$

$$\Rightarrow \quad 0 = \frac{\partial}{\partial u_i} E\Big(\sum_{i=1}^{K} u_i \phi_i\Big) \quad \forall\, i = 1, \dots, K$$

$\hat{u}$ is called <u>Galerkin approximation</u>. If the ϕ_i are local (i.e. $\operatorname{supp}\phi_i \cap \operatorname{supp}\phi_j = \emptyset$ for most i,j), one obtains the special case of <u>finite elements</u>.

<u>Exm:</u> $\Delta u = 0$ on Ω, $u(x) = g(x)$ on $\partial\Omega$ $\qquad E(u) = \int_\Omega \frac{|\nabla u|^2}{2}\, dx$

triangle mesh , V spanned by  $\phi_i(x_j) = \begin{cases} 1 & , j = i \\ 0 & , j \neq i \end{cases}$

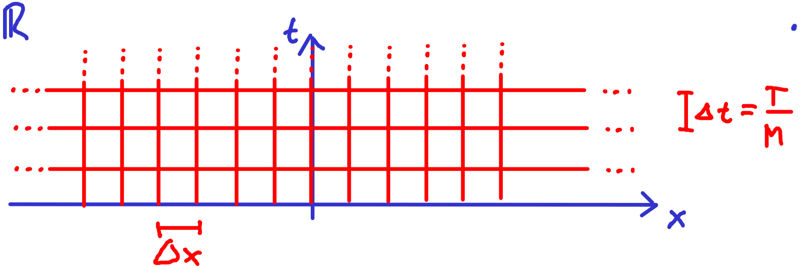
$$\rightsquigarrow \sum_{j=1}^{K} \int_\Omega \nabla\phi_i \cdot \nabla\phi_j \, dx \; u_j = 0 \quad \forall\, i \text{ with } x_i \notin \partial\Omega, \quad u_i = g(x_i) \text{ else}$$

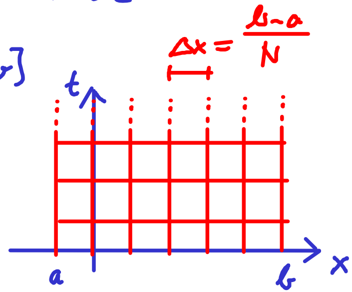
Finite Differences: parabolic PDEs (1D heat equation)

theta-method

$u_t = u_{xx}$ on $[0,T]\times\Omega$ + bc + ic

mesh: • $\Omega = \mathbb{R}$ • $\Omega = [a,b]$





explicit Euler-method: $$\frac{U_j^{m+1} - U_j^m}{\Delta t} = \frac{U_{j-1}^m - 2U_j^m + U_{j+1}^m}{\Delta x^2}$$

$$\Longleftrightarrow \quad U_j^{m+1} = U_j^m + \frac{\Delta t}{\Delta x^2}\left(U_{j-1}^m - 2U_j^m + U_{j+1}^m\right)$$

implicit Euler-method: $$\frac{U_j^{m+1} - U_j^m}{\Delta t} = \frac{U_{j-1}^{m+1} - 2U_j^{m+1} + U_{j+1}^{m+1}}{\Delta x^2}$$

$$\Longleftrightarrow \quad U_j^{m+1} - \frac{\Delta t}{\Delta x^2}\left(U_{j-1}^{m+1} - 2U_j^{m+1} + U_{j+1}^{m+1}\right) = U_j^m$$

Θ-method: $$\frac{U_j^{m+1} - U_j^m}{\Delta t} = (1-\Theta)\frac{U_{j-1}^m - 2U_j^m + U_{j+1}^m}{\Delta x^2} + \Theta\frac{U_{j-1}^{m+1} - 2U_j^{m+1} + U_{j+1}^{m+1}}{\Delta x^2}$$

$\Theta = 0$: expl. Euler ; $\Theta = 1$: impl. Euler ; $\Theta = \frac{1}{2}$: Crank-Nicolson

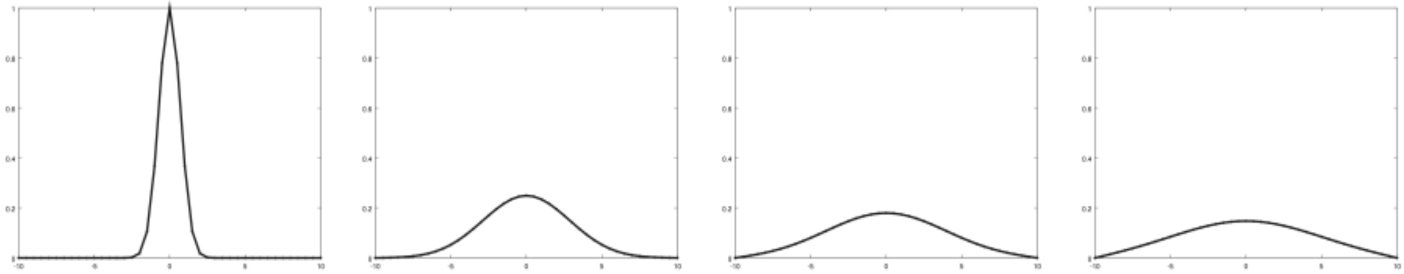
initial condition (ic): $U_j^0 = u(t=0, x_j)$; Dirichlet-bc: $U_0^m = u(t^m, a)$, $U_N^m = u(t^m, b)$

Finite Differences: parabolic PDEs (1D heat equation)

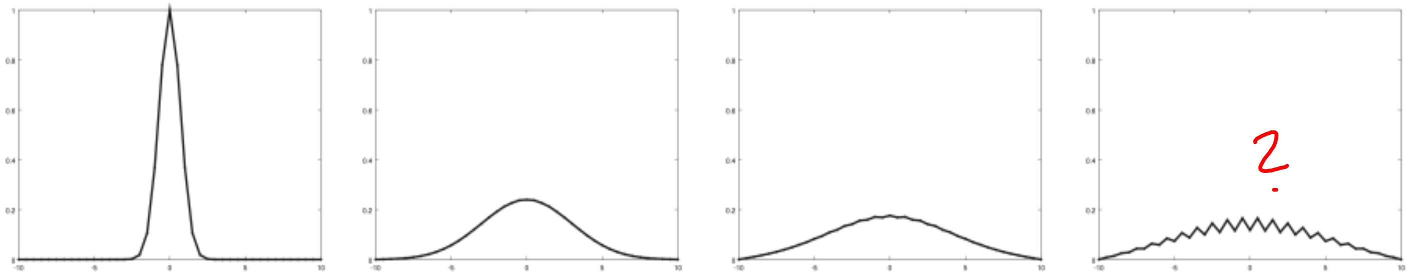
Example simulation

$u_t = u_{xx}$ on $\Omega = [-10, 10]$, $u(t=0,x) = e^{-x^2}$, $u(t,\partial\Omega) = u(0,\partial\Omega) = 0$

expl. Euler, $\Delta x = \frac{1}{2}$, $\Delta t = 0{,}12$



expl. Euler, $\Delta x = \frac{1}{2}$, $\Delta t = 0{,}13$



Fourier-Transform & Plancherel's/Parseval's Theorem

Def: Fourier transform (FT) of $u: \mathbb{R} \to \mathbb{C}$: $\quad \hat{u}(\xi) = F[u](\underset{\hat{\mathbb{R}}}{\xi}) = \int_{\mathbb{R}} u(x)\, e^{-ix\xi}\, dx$

Def: L^2-norm of $u: \Omega \to \mathbb{C}$: $\quad \|u\|_{L^2} = \left(\int_{\Omega} |u(x)|^2\, dx \right)^{1/2}$

$L^2(\Omega) = \{ u: \Omega \to \mathbb{C} \mid u \text{ measurable}, \|u\|_{L^2} < \infty \}$

corresponding L^2 inner product $(u,v)_{L^2} = \int_{\Omega} (u(x), v(x))_{\mathbb{C}}\, dx = \int_{\Omega} \mathrm{Re}\left(\overline{u(x)}\, v(x)\right) dx$

Thm: Inverse FT (IFT) of $\hat{u}: \mathbb{R} \to \mathbb{C}$: $\quad u(x) = F^{-1}[\hat{u}](x) = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{u}(\xi)\, e^{ix\xi}\, d\xi$

Pf: "$\frac{1}{2\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} u(\tilde{x})\, e^{-i\tilde{x}\xi}\, d\tilde{x}\, e^{ix\xi}\, d\xi = \frac{1}{2\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} e^{i(x-\tilde{x})\xi}\, d\xi\, u(\tilde{x})\, d\tilde{x} = \int_{\mathbb{R}} u(\tilde{x})\, \delta(\tilde{x}-x)\, d\tilde{x}$"

$\Rightarrow$ make rigorous via smoothing & approximation argument $\square$

Thm: (Plancherel's/Parseval's Theorem) $\|\hat{u}\|_{L^2} = \sqrt{2\pi}\, \|u\|_{L^2}$

$\Rightarrow \frac{F}{\sqrt{2\pi}}$ is isometry from $L^2(\mathbb{R})$ to $L^2(\mathbb{R})$

Pf: · $\int_{\mathbb{R}} \hat{u}(\xi)\, v(\xi)\, d\xi = \int_{\mathbb{R}} \int_{\mathbb{R}} u(x)\, e^{-ix\xi}\, dx\, v(\xi)\, d\xi$

$$= \int_{\mathbb{R}} \int_{\mathbb{R}} v(\xi)\, e^{-ix\xi}\, d\xi\, u(x)\, dx = \int_{\mathbb{R}} u(x)\, \hat{v}(x)\, dx$$

· choose $v(\xi) = \overline{\hat{u}(\xi)} = 2\pi F^{-1}[\bar{u}](\xi)$ $\square$

L2 stability

Stability $\hat{=}$ small perturbation in the data (e.g. bc or rhs of PDE) results only in small change of the solution

measurement of perturbation strength possible with different norms
$\Rightarrow$ stability wrt. different norms

$$\left.\begin{array}{ll} u_t - u_{xx} = f, & u(t=0) = u_0 \\ \tilde{u}_t - \tilde{u}_{xx} = \tilde{f}, & \tilde{u}(t=0) = \tilde{u}_0 \end{array}\right\} \Longrightarrow \quad v = \tilde{u} - u \text{ solves } v_t - v_{xx} = \tilde{f} - f, \quad v(t=0) = \tilde{u}_0 - u_0$$

for Neumann bc see homework

Thm: Let u solve $u_t - u_{xx} = f$ on $[0,\infty) \times \Omega$ ($\Omega = \mathbb{R}$ or $\Omega = [a,b]$ with 0-Dirichlet-bc).

$$\|u(t)\|_{L^2} \leq \|u(t=0)\|_{L^2} + \int_0^t \|f(s)\|_{L^2}\, ds$$

Cauchy-Schwarz

Pf: · for $\Omega = [a,b]$, $2\|u\|_{L^2} \frac{d}{dt}\|u\|_{L^2} = 2\int_\Omega \dot{u} u \, dx = 2\int_\Omega u(\Delta u + f)\, dx = 2\int_\Omega -|\nabla u|^2 + uf\, dx \leq 2\|u\|_{L^2}\|f\|_{L^2}$

· for $\Omega = \mathbb{R}$, $F[u_t - u_{xx} - f](\xi) = \hat{u}_t(\xi,t) - (i\xi)^2\hat{u}(\xi,t) - \hat{f}(\xi,t)$

$\Rightarrow \forall \xi: \quad \frac{d}{dt}\hat{u} = -\xi^2\hat{u} + \hat{f} \quad \Longrightarrow 2\|\hat{u}\|_{L^2}\frac{d}{dt}\|\hat{u}\|_{L^2} = 2\int_{\mathbb{R}} \mathrm{Re}\left(-\xi^2|\hat{u}|^2 + \bar{\hat{u}}\hat{f}\right) d\xi \leq 2\|\hat{u}\|_{L^2}\|\hat{f}\|_{L^2}$

· in both cases $\quad \frac{d}{dt}\|u\|_{L^2} \leq \|f\|_{L^2}$ $\quad\square$

Finite Differences: parabolic PDEs (1D heat equation)

Semidiscrete Fourier-Transform

Def: The semidiscrete FT of a function U defined on the infinite grid $x_j = j\Delta x$, $j \in \mathbb{Z}$, is

$$\hat{U} : \left[-\frac{\pi}{\Delta x}, \frac{\pi}{\Delta x}\right] \to \mathbb{C}, \qquad \hat{U}(k) = \Delta x \sum_{j=-\infty}^{\infty} U_j e^{-ikx_j}.$$

Def: ℓ_2-norm of $U : Z < \Delta x \mathbb{Z} \longrightarrow \mathbb{C}$:

$$\|U\|_{\ell^2} = \left(\Delta x \sum_{x_j \in Z} |U_j|^2\right)^{\frac{1}{2}}$$

Thm: Inverse semidiscrete FT is

$$U_j = \frac{1}{2\pi} \int_{-\pi/\Delta x}^{\pi/\Delta x} \hat{U}(k) e^{ikj\Delta x} \, dk$$

Thm: (Parseval)

$$\|\hat{U}\|_{\ell^2} = \sqrt{2\pi}\, \|U\|_{\ell^2}$$

Finite Differences: parabolic PDEs (1D heat equation)

l2-Stability

θ-method for $\dot{u} - \Delta u = f$: (*)

$$\boxed{\frac{U_j^{m+1} - U_j^m}{\Delta t} = (1-\theta)\frac{U_{j-1}^m - 2U_j^m + U_{j+1}^m}{\Delta x^2} + \theta\,\frac{U_{j-1}^{m+1} - 2U_j^{m+1} + U_{j+1}^{m+1}}{\Delta x^2} + (1-\theta)F_j^m + \theta F_j^{m+1},\quad U_j^0 = O_j}$$

$$\frac{\tilde{U}_j^{m+1} - \tilde{U}_j^m}{\Delta t} = (1-\theta)\frac{\tilde{U}_{j-1}^m - 2\tilde{U}_j^m + \tilde{U}_{j+1}^m}{\Delta x^2} + \theta\,\frac{\tilde{U}_{j-1}^{m+1} - 2\tilde{U}_j^{m+1} + \tilde{U}_{j+1}^{m+1}}{\Delta x^2} + (1-\theta)\tilde{F}_j^m + \theta \tilde{F}_j^{m+1},\quad \tilde{U}_j^0 = \tilde{O}_j$$

$\Rightarrow$ $\tilde{u} - u$ solves same equations for data $\tilde{F} - F$ & ic $\tilde{O} - O$.

$\Rightarrow$ for stability consider (as before) norm of the soln. as fcn. of the data & i.c.

<u>Thm:</u> Let $U = (U_j^m)_{j\in Z}^{m\geq 0}$ solve (*) (on $Z = \mathbb{Z}$ or $Z = \{0,1,\dots,N\}$ with 0-Dirichlet-bc).

If $(1-2\theta)\frac{\Delta t}{\Delta x^2} \leq \frac{1}{2}$, $\quad \|U^m\|_{\ell^2} \leq \|O\|_{\ell^2} + \underbrace{\Delta t \sum_{k=1}^{m} \|\theta F^k + (1-\theta) F^{k-1}\|_{\ell^2}}$.

"method is <u>ℓ^2-stable</u>".

$\leq \Delta t \left(\sum_{k=1}^{m-1} \|F^k\|_{\ell^2} + (1-\theta)\|F^0\|_{\ell^2} + \theta\|F^m\|_{\ell^2}\right) =$ quadrature for $\int_0^{m\Delta t} \|F\|_{\ell^2}\,dt$

<u>Cor:</u> Impl. Euler- & Crank-Nicolson-method are unconditionally ℓ^2-stable,
expl. Euler-method is conditionally ℓ^2-stable (for $\Delta t \leq \frac{\Delta x^2}{2}$).

Pf: Abbreviate $\mu = \Delta t / \Delta x^2$

- $\mathcal{Z} = \mathbb{Z}$: Plug in $U_j^m = \frac{1}{2\pi} \int_{-\pi/\Delta x}^{\pi/\Delta x} e^{ikj\Delta x} \hat{u}^m(k)\, dk$

$$\Rightarrow \frac{1}{2\pi} \int_{-\pi/\Delta x}^{\pi/\Delta x} e^{ikj\Delta x} \frac{\hat{u}^{m+1} - \hat{u}^m}{\Delta t}\, dk = \frac{1}{2\pi} \int_{-\pi/\Delta x}^{\pi/\Delta x} (1-\theta) \frac{e^{ik(j-1)\Delta x} - 2e^{ikj\Delta x} + e^{ik(j+1)\Delta x}}{\Delta x^2} \hat{u}^m$$

$$+ \theta \frac{e^{ik(j-1)\Delta x} - 2e^{ikj\Delta x} + e^{ik(j+1)\Delta x}}{\Delta x^2} \hat{u}^{m+1} + e^{ikj\Delta x}\left[(1-\theta)\hat{F}^m + \theta \hat{F}^{m+1}\right] dk$$

$$\Rightarrow 0 = \frac{1}{2\pi} \int_{-\pi/\Delta x}^{\pi/\Delta x} e^{ikj\Delta x} \Big[\hat{u}^{m+1}\left(1 - \theta\mu\left(e^{-ik\Delta x} - 2 + e^{ik\Delta x}\right)\right) - \hat{u}^m\left(1 + (1-\theta)\mu\left(e^{-ik\Delta x} - 2 + e^{ik\Delta x}\right)\right)$$

$$+ \Delta t \left[(1-\theta)\hat{F}^m + \theta \hat{F}^{m+1}\right]\Big]\, dk$$

$$\Rightarrow \hat{u}^{m+1}(k) = \frac{1 + (1-\theta)\mu\left(e^{-ik\Delta x} - 2 + e^{ik\Delta x}\right)}{1 - \theta\mu\underbrace{\left(e^{-ik\Delta x} - 2 + e^{ik\Delta x}\right)}_{-4\sin^2(k\Delta x/2)}} \hat{u}^m(k) - \frac{\Delta t\left[(1-\theta)\hat{F}^m(k) + \theta \hat{F}^{m+1}(k)\right]}{1 - \theta\mu\left(e^{-ik\Delta x} - 2 + e^{ik\Delta x}\right)}, \quad k \in \left[\frac{-\pi}{\Delta x}, \frac{\pi}{\Delta x}\right]$$

$$\Rightarrow \|\hat{u}^{m+1}\|_{L^2} \le \underbrace{\max_k \left| \frac{1 - (1-\theta)4\mu\sin^2\left(\frac{k\Delta x}{2}\right)}{1 + \theta 4\mu \sin^2\left(\frac{k\Delta x}{2}\right)} \right|}_{\le 1 \iff (1-2\theta)\mu \le \frac{1}{2}} \|\hat{u}^m\|_{L^2} + \Delta t\left[\|\theta \hat{F}^{m+1} + (1-\theta)\hat{F}^m\|_{L^2}\right]$$

$\cdot\ \mathcal{Z} = \{0, 1, \dots, N\}$: $U^{m+1} = (I + \theta \mu L)^{-1} \left[(I - (1-\theta)\mu L) U^m + \theta \Delta t F^{m+1} + (1-\theta) \Delta t F^m \right]$

$$U^m = \begin{pmatrix} u_1^m \\ \vdots \\ u_{N-1}^m \end{pmatrix}, \quad L = \begin{pmatrix} 2 & -1 & & \\ -1 & 2 & -1 & \\ & \ddots & \ddots & \\ & -1 & 2 & -1 \\ & & -1 & 2 \end{pmatrix} \in \mathbb{R}^{(N-1)\times(N-1)}$$

By Gerschgorin's Theorem L is pos. semidefinite with eigenvalues $\in [0,4]$.

$$\Rightarrow \|U^{m+1}\|_{\ell^2} \le \underbrace{\|(I + \theta \mu L)^{-1}(I - (1-\theta)\mu L)\|}_{\le \max\limits_{\text{eigenvalues}} \left| \frac{1-(1-\theta)\mu\lambda}{1+\theta\mu\lambda} \right| \le 1 \text{ if } (1-2\theta)\mu \le \frac{1}{2}} \|U^m\|_{\ell^2} + \underbrace{\|(I + \theta\mu L)^{-1}\|}_{\le 1} \Delta t \left[\|\theta F^{m+1} + (1-\theta) F^m\|_{\ell^2} \right]$$

$\cdot$ By induction we deduce in both cases

$$\|U^m\|_{\ell^2} \le \|U^0\|_{\ell^2} + \Delta t \sum_{k=1}^{m} \|\theta F^k + (1-\theta) F^{k-1}\|_{\ell^2} \qquad \square$$

ℓ^2-stability for Neumann boundary conditions: see homework

l2-von Neumann-Stability

Conditions for stability are typically strongest for $\mathcal{Z} = \mathbb{Z}$, since then all frequencies k occur, while on a finite grid there are only finitely many frequencies / eigenmodes. Formally, we can check ℓ^2-stability by inserting $u_j^m = [\lambda(k)]^m e^{ijk\Delta x}$ into the discrete method with $F_j^m = 0$ and by then requiring $|\lambda(k)| \leq 1$.

Def: The method is called ℓ^2-von Neumann-stable, if there exists $C > 0$ independent of Δt such that $\|u^m\|_{\ell^2} \leq C\left[\|u^0\|_{\ell^2} + \Delta t \sum_{k=0}^{m} \|F^k\|_{\ell^2}\right]$, $m = 1, \dots, M = T/\Delta t$.

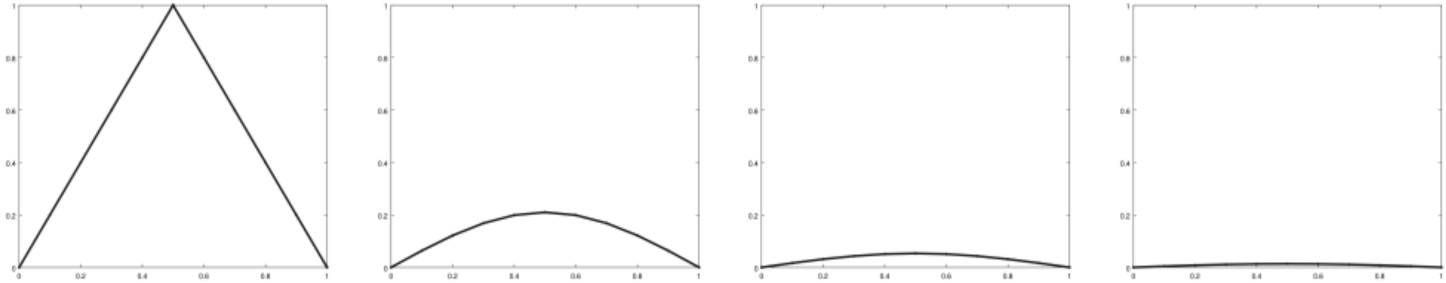
Thm: The θ-method is ℓ^2-von Neumann-stable if $(1-2\theta)\frac{\Delta t}{\Delta x^2} \leq \frac{1}{2} + c\Delta t$.

Pf: homework (use $(1 + c\Delta t)^m \leq (1 + \frac{cT}{M})^M \leq e^{cT}$) $\square$

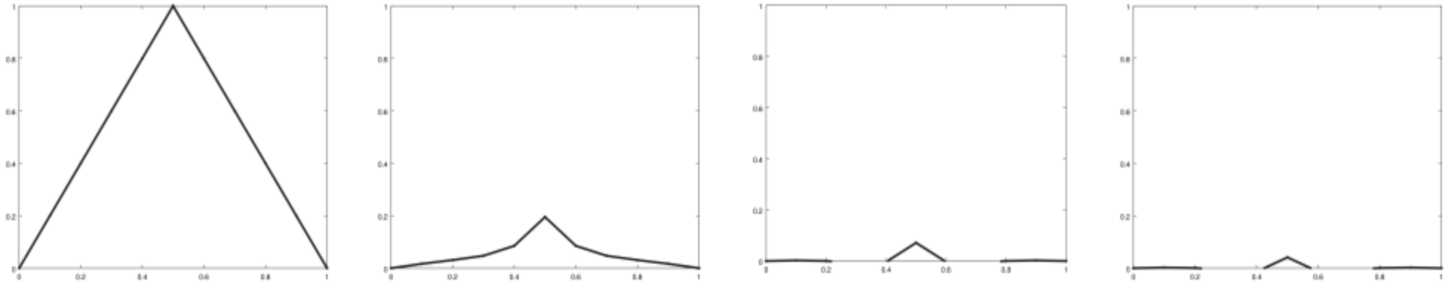
Example simulation

$\dot{u} = u_{xx}$ on $[0,T] \times [0,1]$, $u(0,x) = 1 - 2|x - \frac{1}{2}|$, $u(t,\{0,1\}) = 0$

impl. Euler $\quad \Delta t = \Delta x = \frac{1}{10}$



Crank-Nicolson $\quad \Delta t = \Delta x = \frac{1}{10}$



L^∞-stability: maximum principle

$\Omega = (a,b)$; $\left.\begin{matrix} \dot u - u_{xx} = f, \ u|_{t=0} = u_0, \ u(\cdot,a) = u_a, \ u(\cdot,b) = u_b \\ \dot{\tilde u} - \tilde u_{xx} = f, \ \tilde u|_{t=0} = \tilde u_0, \ \tilde u(\cdot,a) = \tilde u_a, \ \tilde u(\cdot,b) = \tilde u_b \end{matrix}\right\} \Rightarrow \begin{cases} v = \tilde u - u \text{ solves } \dot v - v_{xx} = 0, \\ v|_{t=0} = \tilde u_0 - u_0, \ v(\cdot,a) = \tilde u_a - u_a, \ v(\cdot,b) = \tilde u_b - u_b \end{cases}$

Thm: (Maximum principle) Let $\dot u - \Delta u = f \overset{\geq}{\leq} 0$, then $\displaystyle\overset{\min}{\max_{(t,x)\in[0,T]\times\overline{\Omega}}} u(t,x) = \overset{\min}{\max_{t=0 \vee x\in\partial\Omega}} u(t,x)$.

Pf: · Let $u(t,x)$, $t\in(0,T]$, $x\in\Omega$ be a loc. maximum $\Rightarrow$ $u_x = 0$, $u_t \geq 0$, $u_{xx} \leq 0$

· if $f(t,x) < 0$, we have $0 > f = u_t - u_{xx} \geq 0$ ↯

· if $f(t,x) \leq 0$, set $v(t,x) = u(t,x) + \frac{\varepsilon}{2}x^2 \Rightarrow \dot v - v_{xx} = f - \varepsilon < 0$

$\Rightarrow$ by previous argument, v attains maximum on $\underbrace{\{0\}\times\Omega \cup (0,T]\times\partial\Omega}_{=:R}$

$\Rightarrow u(t,x) \leq v(t,x) \leq \max\limits_{(t,x)\in R} v(t,x) \leq \max\limits_{(t,x)\in R} u(t,x) + \varepsilon \max(|a|,|b|)^2 \xrightarrow[\varepsilon\to 0]{} \max\limits_{(t,x)\in R} u(t,x)$

· analogous for minima □

Cor: (L^∞-stability) $\|\tilde u - u\|_{L^\infty} = \sup\limits_{x,t} |\tilde u(t,x) - u(t,x)| \leq \max\left(\|\tilde u_0 - u_0\|_{L^\infty}, \|\tilde u_a - u_a\|_{L^\infty}, \|\tilde u_b - u_b\|_{L^\infty}\right)$

$\sup\limits_{x\in\Omega} |\tilde u_0(x) - u_0(x)|$ $\qquad$ $\sup\limits_{t\in(0,T)} |\tilde u_a(t) - u_a(t)|$

l^∞-stability: discrete maximum principle

θ-method for $\dot u - \Delta u = f$: $\left(\mu = \frac{\Delta t}{\Delta x^2}\right)$ (*)

$$\boxed{\frac{U_j^{m+1} - U_j^m}{\Delta t} = (1-\theta)\frac{U_{j-1}^m - 2U_j^m + U_{j+1}^m}{\Delta x^2} + \theta\frac{U_{j-1}^{m+1} - 2U_j^{m+1} + U_{j+1}^{m+1}}{\Delta x^2} + (1-\theta)F_j^m + \theta F_j^{m+1}, \quad \begin{cases} U_0^m = A^m, \\ U_N^m = B^m, \\ U_j^0 = O_j \end{cases}}$$

If $\tilde U_j^m$ is soln. for data F & i.c. $\tilde O$ & b.c. $\tilde A, \tilde B$,

then $\tilde U - U$ solves the eq. for data 0, i.c. $\tilde O - O$ & b.c. $\tilde A - A$, $\tilde B - B$

$\Rightarrow$ for stability w.r.t. i.c. & b.c. examine, how $\|\tilde U - U\|$ depends on $\|\tilde A - A\|, \|\tilde B - B\|, \|\tilde O - O\|$

Thm: (Discrete maximum principle) Let U solve (*) with $F = 0$ for $\theta \in [0,1]$ & $\mu(1-\theta) \le \frac{1}{2}$. Then

$$\min\{U_j^m \mid m=0 \vee j=0 \vee j=N\} \le U_j^m \le \max\{U_j^m \mid m=0 \vee j=0 \vee j=N\} \quad \forall j,m$$

Pf:
$$(1+2\theta\mu)U_j^{m+1} = \overbrace{\theta\mu}^{\ge 0}(U_{j+1}^{m+1} + U_{j-1}^{m+1}) + \overbrace{(1-\theta)\mu}^{\ge 0}(U_{j+1}^m + U_{j-1}^m) + \overbrace{(1-2(1-\theta)\mu)}^{\ge 0}U_j^m$$
$$\le 2\theta\mu \underbrace{U^*}_{=\max(U_{j+1}^{m+1}, U_{j-1}^{m+1}, U_{j+1}^m, U_{j-1}^m, U_j^m)} + 2(1-\theta)\mu U^* + (1-2(1-\theta)\mu)U^* \le (1+2\theta\mu)U^*$$

$\Rightarrow$ there cannot be a strict maximum for $m \ne 0$ & $j \notin \{0,N\}$ (analogous for minima) $\square$

Cor: (l^∞-stability) $\theta \in [0,1]$, $\mu(1-\theta) \le \frac{1}{2}$. $\|U - \tilde U\|_{l^\infty} = \max_{j,m}|U_j^m - \tilde U_j^m| \le \max(\|\tilde A - A\|_{l^\infty}, \|\tilde B - B\|_{l^\infty}, \|\tilde O - O\|_{l^\infty})$.

l^∞-stability wrt. data term

Note : ℓ^∞-stability has stricter conditions on μ than ℓ^2-stability.

Crank-Nicolson-method : ℓ^2-stable $\forall \mu$, ℓ^∞-stable for $\mu \leq 1$.

<u>Thm:</u> Let U solve $(*)$ with $A = B = 0 = g$ for $\theta \in [0,1]$, $\mu(1-\theta) \leq \frac{1}{2}$, then $\|U\|_{\ell^\infty} \leq T\,\|F\|_{\ell^\infty}$

$$\|U\|_{\ell^\infty} \leq \Delta t \sum_{k=1}^{m} \|(1-\theta)F^{k-1} + \theta F^k\|_{\ell^\infty} \leq T\,\|F\|_{\ell^\infty}.$$

Pf : $(1+2\theta\mu)U_j^{m+1} = \theta\mu\left(U_{j+1}^{m+1} + U_{j-1}^{m+1}\right) + (1-\theta)\mu\left(U_{j+1}^{m} + U_{j-1}^{m}\right) + \left(1-2(1-\theta)\mu\right)U_j^m + \Delta t\left[(1-\theta)F_j^m + \theta F_j^{m+1}\right]$

$\leq 2\theta\mu\|U^{m+1}\|_{\ell^\infty} + 2(1-\theta)\mu\|U^m\|_{\ell^\infty} + \left(1-2(1-\theta)\mu\right)\|U^m\|_{\ell^\infty} + \Delta t\,\|(1-\theta)F_j^m + \theta F_j^{m+1}\|_{\ell^\infty}$

$\|G\|_{\ell^\infty} = \max\{|G_1|, \dots, |G_{N-1}|\}$

Take maximum over $j \in \{1, \dots, N-1\}$ $\Rightarrow$ $\|U^{m+1}\|_{\ell^\infty} \leq \|U^m\|_{\ell^\infty} + \Delta t\|(1-\theta)F^m + \theta F^{m+1}\|_{\ell^\infty}$

$\Rightarrow \|U^m\|_{\ell^\infty} \leq \underbrace{\|U^0\|_{\ell^\infty}}_{=0} + \Delta t \sum_{k=1}^{m} \|(1-\theta)F^{k-1} + \theta F^k\|_{\ell^\infty}$, $\quad m = 1, \dots, M$ $\qquad \square$

Consistency: truncation error

Def: Let $u_j^m = u(t_m, x_j)$ for a smooth soln. u. The truncation error of the θ-method is

$$T_j^m = \frac{u_j^{m+1} - u_j^m}{\Delta t} - (1-\theta)\frac{u_{j-1}^m - 2u_j^m + u_{j+1}^m}{\Delta x^2} - \theta\frac{u_{j-1}^{m+1} - 2u_j^{m+1} + u_{j+1}^{m+1}}{\Delta x^2} - (1-\theta)f_j^m - \theta f_j^{m+1}$$

How big is the truncation error? Suppose, u solves $\dot{u} - u_{xx} = f$ and is sufficiently often differentiable. Then we perform a Taylor expansion:

$$u_j^{m+1} = u_j^m + u_t(t_m, x_j)\Delta t + O(\Delta t^2) \quad \text{(analogous for } f\text{)}$$

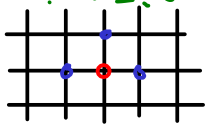
$$u_{j\pm1}^m = u_j^m \pm u_x(t_m, x_j)\Delta x + \tfrac{1}{2}u_{xx}(t_m, x_j)\Delta x^2 \pm \tfrac{1}{6}u_{xxx}(t_m, x_j)\Delta x^3 + O(\Delta x^4)$$

$\Rightarrow$ Expl. Euler-method: $T_j^m = u_t + O(\Delta t) - u_{xx} + O(\Delta x^2) - f_j^m = O(\Delta t + \Delta x^2)$

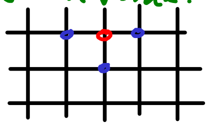
Thm: (Order of consistency) The θ-method satisfies $T_j^m = \begin{cases} O(\Delta t + \Delta x^2), & \theta \neq \frac{1}{2} \\ O(\Delta t^2 + \Delta x^2), & \theta = \frac{1}{2} \end{cases}$,

i.e. it is consistent of 2. order in space and 1. or 2. order in time.

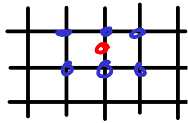
Pf: homework; note: Taylor exp. easiest around



expl. Euler



imp. Euler



CN $\square$

$\Rightarrow$ Only C-N-method is 2. order consistent in time!

Convergence (consistency + stability = convergence!)

$$\frac{U_j^{m+1}-U_j^m}{\Delta t} = (1-\theta)\frac{U_{j-1}^m - 2U_j^m + U_{j+1}^m}{\Delta x^2} + \theta\frac{U_{j-1}^{m+1}-2U_j^{m+1}+U_{j+1}^{m+1}}{\Delta x^2} + (1-\theta)f_j^m + \theta f_j^{m+1}$$

$$\frac{u_j^{m+1}-u_j^m}{\Delta t} = (1-\theta)\frac{u_{j-1}^m - 2u_j^m + u_{j+1}^m}{\Delta x^2} + \theta\frac{u_{j-1}^{m+1}-2u_j^{m+1}+u_{j+1}^{m+1}}{\Delta x^2} + (1-\theta)f_j^m + \theta f_j^{m+1} + T_j^m$$

Set $e_j^m = U_j^m - u_j^m \Rightarrow \dfrac{e_j^{m+1}-e_j^m}{\Delta t} = (1-\theta)\dfrac{e_{j-1}^m - 2e_j^m + e_{j+1}^m}{\Delta x^2} + \theta\dfrac{e_{j-1}^{m+1}-2e_j^{m+1}+e_{j+1}^{m+1}}{\Delta x^2} - T_j^m$

$\Rightarrow$ e solves θ-method with 0-i.c. & 0-b.c & data F_j^m so that $(1-\theta)F_j^m + \theta F_j^{m+1} = -T_j^m$

Thm: Let u (smooth) solve $\dot u - u_{xx} = f$ with ic & bc and U_j^m solve the associated θ-method with $\theta\in[0,1]$

- $\|U^m - u^m\|_{\ell^2} = \begin{cases} O(\Delta t + \Delta x^2), & \theta \neq \frac12 \\ O(\Delta t^2 + \Delta x^2), & \theta = \frac12 \end{cases}$ if $(1-2\theta)\dfrac{\Delta t}{\Delta x^2} \le \dfrac12$, $m = 1,\dots,M$

- $\|U^m - u^m\|_{\ell^\infty} = \begin{cases} O(\Delta t + \Delta x^2), & \theta \neq \frac12 \\ O(\Delta t^2 + \Delta x^2), & \theta = \frac12 \end{cases}$ if $(1-\theta)\dfrac{\Delta t}{\Delta x^2} \le \dfrac12$, $m = 1,\dots,M$

Pf: $\overset{\text{stability}}{\Rightarrow} \|e^m\|_{\ell^2} \le \Delta t \sum_{k=1}^{m} \|\theta F^k + (1-\theta)F^{k-1}\|_{\ell^2} \le T \max_{k=1,\dots,M} \|\theta F^k + (1-\theta)F^{k-1}\|_{\ell^2}$

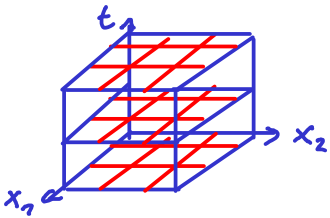
$\|e^m\|_{\ell^\infty} \le T \max_{k=1,\dots,M} \|\theta F^k + (1-\theta)F^{k-1}\|_{\ell^\infty}$

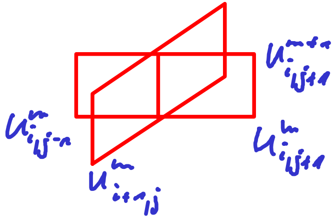
$\overset{\text{Consistency}}{\Rightarrow} \|e^m\|_{\ell^2}, \|e^m\|_{\ell^\infty} \le O(\Delta t^{1 \text{ or } 2} + \Delta x^2)$ $\square$

Summary 2D domain

$\dot{u} - \Delta u = f$ on $\Omega = [a_1, b_1] \times [a_2, b_2]$

with i.c. & b.c.





Thm: 1) The θ-method for $\Omega = [a_1, b_1] \times [a_2, b_2]$ with grid widths $\Delta t, \Delta x_1, \Delta x_2$ and $\mu_1 = \frac{\Delta t}{\Delta x_1^2}$, $\mu_2 = \frac{\Delta t}{\Delta x_2^2}$, $\theta \in [0,1]$, is

- ℓ^2-stable for $(1-2\theta)(\mu_1 + \mu_2) \leq \frac{1}{2}$,
- ℓ^∞-stable for $(1-\theta)(\mu_1 + \mu_2) \leq \frac{1}{2}$,

2) it is consistent with truncation error

- $T^m_{ij} = O(\Delta t + \Delta x_1^2 + \Delta x_2^2)$ for $\theta \neq \frac{1}{2}$
- $T^m_{ij} = O(\Delta t^2 + \Delta x_1^2 + \Delta x_2^2)$ for $\theta = \frac{1}{2}$

3) it is convergent of same order in the ℓ^2/ℓ^∞-norm, if it is stable in that norm.

Pf: homework $\square$

Alternating Direction Implicit (ADI) method

$\dot{u} - \Delta u = 0$

Write $\delta_1^2 U_{ij}^m = U_{i-1,j}^m - 2U_{ij}^m + U_{i+1,j}^m$; analogously δ_2^2

Crank-Nicolson : $\left(1 - \frac{1}{2}\mu_1\delta_1^2 - \frac{1}{2}\mu_2\delta_2^2\right) U_{ij}^{m+1} = \left(1 + \frac{1}{2}\mu_1\delta_1^2 + \frac{1}{2}\mu_2\delta_2^2\right) U_{ij}^m$

<u>Def:</u> <u>ADI-meth:</u> $\left(1 - \frac{1}{2}\mu_1\delta_1^2\right)\left(1 - \frac{1}{2}\mu_2\delta_2^2\right) U_{ij}^{m+1} = \left(1 + \frac{1}{2}\mu_1\delta_1^2\right)\left(1 + \frac{1}{2}\mu_2\delta_2^2\right) U_{ij}^m$

$$\Leftrightarrow \left.\begin{cases} \left(1 - \frac{1}{2}\mu_1\delta_1^2\right) U_{ij}^{m+1/2} = \left(1 + \frac{1}{2}\mu_2\delta_2^2\right) U_{ij}^m & \Big| \left(1 + \frac{1}{2}\mu_1\delta_1^2\right)\cdot \ldots \\ \left(1 - \frac{1}{2}\mu_2\delta_2^2\right) U_{ij}^{m+1} = \left(1 + \frac{1}{2}\mu_1\delta_1^2\right) U_{ij}^{m+\frac{1}{2}} & \Big| \left(1 - \frac{1}{2}\mu_1\delta_1^2\right)\cdot \ldots \end{cases}\right\} (*)$$

ADI differs from CN only in the terms $\frac{1}{4}\mu_1\mu_2\delta_1^2\delta_2^2$. It has the advantage that the resulting linear systems of equations (with matrices $1 - \frac{1}{2}\mu_1\delta_1^2$ and $1 - \frac{1}{2}\mu_2\delta_2^2$) are tridiagonal and can therefore be readily inverted by the Thomas-algorithm (the matrices of CN are much more complicated)!

<u>Thm:</u> The ADI method is ℓ^2-stable and for $\mu_1, \mu_2 \leq 1$ also ℓ^∞-stable. It is consistent & convergent of 2. order in space & time.

Pf: homework; use $(*)$ to derive stability. $\square$

Perspective: nonlinear PDEs

For nonlinear PDEs the stability of the method is often difficult to show.

Ex: $\dot{u} = \Delta u + 2u(1-u^2) + f$ can be discretized as

$$U_{ij}^{m+1} - U_{ij}^{m} = \mu_1 \delta_1^2 U_{ij}^{m+1} + \mu_2 \delta_2^2 U_{ij}^{m+1} - 2\Delta t \left(U_{ij}^{m+1}\right)^3 + 2\Delta t\, U_{ij}^{m} + \Delta t\, F_{ij}^{m}$$

U does not satisfy the maximum principle (u does not either), but one has:

Thm: $\|U^m\|_{\ell^\infty} \leq \max\left(1, \|U^0\|_{\ell^\infty}, \max_{x_{ij}\in\partial\Omega, n\leq m} |U_{ij}^n|\right) + T\|F\|_{\ell^\infty}$

Pf: Let $|U_{ij}^{m+1}| > 1$ for some $x_{ij} \in \text{int}\,\Omega$,

$$\begin{aligned}
|U_{ij}^{m+1}|(1+2\mu_1+2\mu_2+2\Delta t) &\leq \left|U_{ij}^{m+1}(1+2\mu_1+2\mu_2) + 2\Delta t\left(U_{ij}^{m+1}\right)^3\right| \\
&= \left|U_{ij}^{m} + \mu_1\left(U_{i-1,j}^{m+1} + U_{i+1,j}^{m+1}\right) + \mu_2\left(U_{i,j-1}^{m+1} + U_{i,j+1}^{m+1}\right) + 2\Delta t\, U_{ij}^{m} + \Delta t\, F_{ij}^{m}\right| \\
&\leq (1+2\Delta t)\|U^m\|_{\ell^\infty} + \Delta t\|F\|_{\ell^\infty} + 2(\mu_1+\mu_2)\|U^{m+1}\|_{\ell^\infty}
\end{aligned}$$

$\Rightarrow \|U^{m+1}\|_{\ell^\infty} \leq \max\left(1, \max_{x_{ij}\in\partial\Omega} |U_{ij}^{m+1}|\right)$ or $\|U^{m+1}\|_{\ell^\infty} \leq \|U^m\|_{\ell^\infty} + \Delta t\|F\|_{\ell^\infty}$

$\Rightarrow \|U^{m+1}\|_{\ell^\infty} \leq \max\left(1, \max_{x_{ij}\in\partial\Omega} |U_{ij}^{m+1}|, \|U^m\|_{\ell^\infty}\right) + \Delta t\|F\|_{\ell^\infty}$

$\Rightarrow$ result follows by induction $\square$

Perspective: nonlinear PDEs

Such boundedness results allow to reduce everything to the linear case.

<u>Thm:</u> (stability) Let $\tilde{u}$ solve the discrete system for $\tilde{F}$, but same i.c./b.c. Then there exists $C>0$ s.t. for Δt small enough (both depends on $\|F\|_{\ell^\infty}$, $\|\tilde{F}\|_{\ell^\infty}$, ic/bc): $\|\tilde{u}-u\|_{\ell^\infty} \le C\|\tilde{F}-F\|_{\ell^\infty}$

Pf: $V = \tilde{u} - u$

$$V_{ij}^{m+1} - V_{ij}^{m} = \mu_1\delta_1^2 V_{ij}^{m+1} + \mu_2\delta_2^2 V_{ij}^{m+1} - 2\Delta t\underbrace{\left[\left(\tilde{u}_{ij}^{m+1}\right)^3 - \left(u_{ij}^{m+1}\right)^3\right]} + 2\Delta t\, V_{ij}^m + \Delta t\left(\tilde{F}_{ij}^m - F_{ij}^m\right)$$

$$= V_{ij}^{m+1}\left(\left(\tilde{u}_{ij}^{m+1}\right)^2 + \tilde{u}_{ij}^{m+1}u_{ij}^{m+1} + \left(u_{ij}^{m+1}\right)^2\right) =: V_{ij}^{m+1} B_{ij}^{m+1}$$

Now standard argument for linear case:

$$V_{ij}^{m+1}\left(1+2\mu_1+2\mu_2+2\Delta t\, B_{ij}^{m+1}\right) = V_{ij}^m + \mu_1\left(V_{i-1,j}^{m+1} + V_{i+1,j}^{m+1}\right) + \mu_2\left(V_{i,j-1}^{m+1} + V_{i,j+1}^{m+1}\right) + 2\Delta t\, V_{ij}^m + \Delta t\left(\tilde{F}_{ij}^m - F_{ij}^m\right)$$

$$\Rightarrow \|V^{m+1}\|_{\ell^\infty}\left(1-2\Delta t\|B\|_{\ell^\infty}\right) \le \|V^m\|_{\ell^\infty}(1+2\Delta t) + \Delta t\|\tilde{F}-F\|_{\ell^\infty}$$

$\|B\|_{\ell^\infty} \le \left(\|\tilde{u}\|_{\ell^\infty} + \|u\|_{\ell^\infty}\right)^2$ depends only on $F, \tilde{F}$, ic, bc. If $\Delta t \le \min\left\{\frac{1}{2}, \frac{1}{8\|B\|_{\ell^\infty}}\right\}$, then

$$\frac{1+2\Delta t}{1-2\|B\|_{\ell^\infty}\Delta t} \le 1+2\Delta t+4\|B\|_{\ell^\infty}\Delta t, \quad \frac{1}{1-2\|B\|_{\ell^\infty}\Delta t} \le \frac{4}{3}$$ and thus

$$\|V^{m+1}\|_{\ell^\infty} \le \|V^m\|_{\ell^\infty}\overbrace{\left(1+2\Delta t+4\|B\|_{\ell^\infty}\Delta t\right)}^{=:q} + 2\Delta t\|\tilde{F}-F\|_{\ell^\infty}$$

$$\le \|V^0\|_{\ell^\infty} q^{m+1} + 2\Delta t\|\tilde{F}-F\|_{\ell^\infty}\sum_{i=0}^{m-1} q^i$$

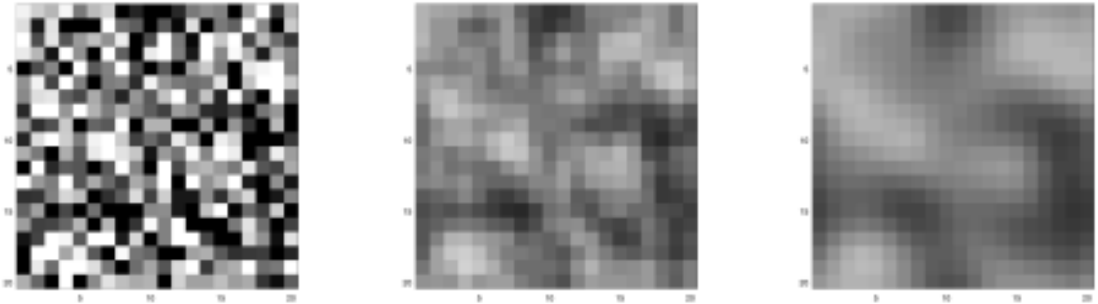
$$\sum_{i=0}^{m-1} q^i = \frac{q^m-1}{q-1} \le \frac{q^M}{\Delta t} \le \frac{\left(1+\left(2+4\|B\|_{\ell^\infty}\right)\frac{T}{M}\right)^M}{\Delta t} \le \frac{e^{\left(2+4\|B\|_{\ell^\infty}\right)T}}{\Delta t} \qquad \square$$

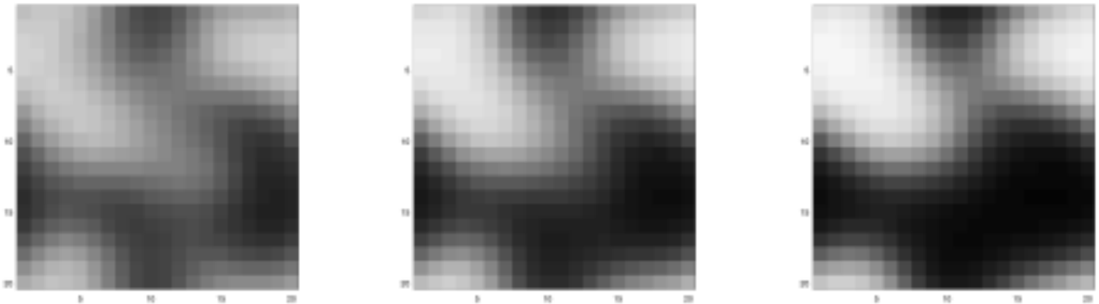
Convergence now follows from consistency and stability as before.

Example nonlinear PDE

$\dot{u} = \Delta u + 2u(1-u^2)$ is gradient flow of $E(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 + (u^2-1)^2 \, dx$

$\Rightarrow$ $u = 1$ (white) or -1 (black) are assumed in the long-run.





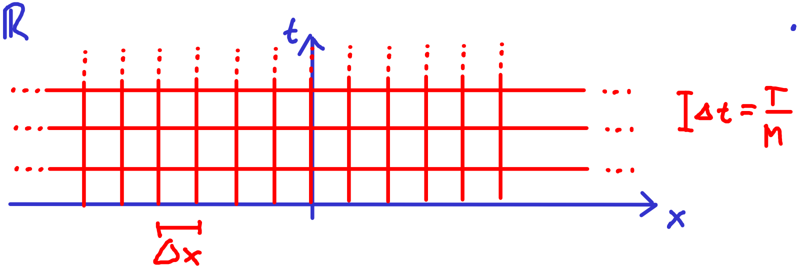
Upwind-method

what is interpretation?

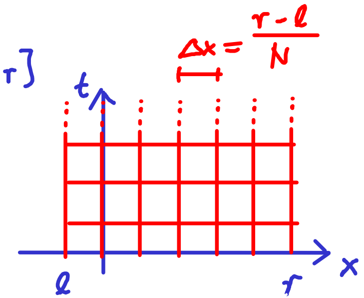
$$u_t + a(t,x)\,u_x + c(t,x)\,u = f(t,x) \quad \text{on } [0,T]\times\Omega, \quad a \geq 0$$

ic : $u(t=0,x) = u_0(x)$, bc : $u(t,l) = u_l(t)$

grid : • $\Omega = \mathbb{R}$



• $\Omega = [l, r]$



upwind-method :

$$\frac{U_j^{m+1} - U_j^m}{\Delta t} + A_j^m \frac{U_j^m - U_{j-1}^m}{\Delta x} + C_j^m U_j^m = F_j^m$$

$$\Longleftrightarrow U_j^{m+1} = U_j^m - \frac{\Delta t}{\Delta x} A_j^m \left(U_j^m - U_{j-1}^m\right) - \Delta t\, C_j^m U_j^m + \Delta t\, F_j^m$$

ic : $U_j^0 = u(t=0, x_j)$; bc : $U_0^m = u(t^m, l)$

If $a \leq 0$ choose :

$$\frac{U_j^{m+1} - U_j^m}{\Delta t} + A_j^m \frac{U_{j+1}^m - U_j^m}{\Delta x} + C_j^m U_j^m = F_j^m$$

ic : $U_j^0 = u(t=0, x_j)$; bc : $U_N^m = u(t^m, r)$

Example simulation

$$\dot{u} + u_x = 0, \quad u(0,x) = e^{-200(x-\frac{1}{2})^2} + 0.8\, e^{-150(x-0.8)^2} =: u_0(x)$$

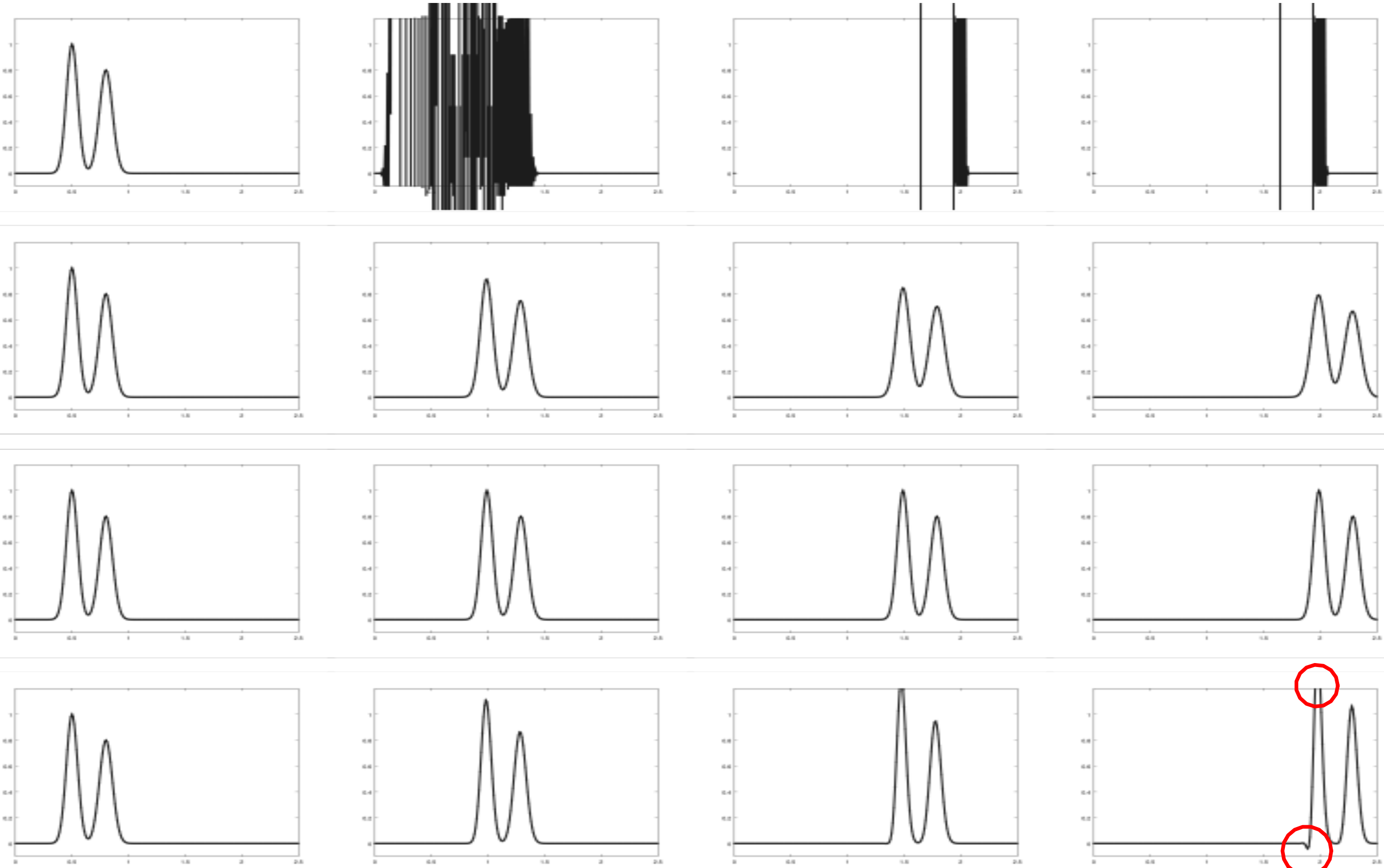
$$\Rightarrow \quad u(t,x) = u_0(x-t)$$

forward differences

$\frac{\Delta t}{\Delta x} = 0.9$

$\frac{\Delta t}{\Delta x} = 1$

$\frac{\Delta t}{\Delta x} = 1.1$



Lebesgue-spaces

Def: Let $\Omega \subset \mathbb{R}^n$ open, $p \in [1,\infty]$. $\|f\|_{L^p} = \begin{cases} \left(\int_\Omega |f|^p dx\right)^{\frac{1}{p}}, & p \neq \infty \\ \operatorname{esssup}_{x\in\Omega} |f(x)| = \inf_{\tilde{f}=f \text{ a.e.}} \sup_{x\in\Omega} |\tilde{f}(x)|, & p=\infty \end{cases}$

$$L^p(\Omega) = \{ f:\Omega \to \mathbb{R} \text{ measurable} \mid \|f\|_{L^p} < \infty \}$$

Thm: (Hölder-inequality) For $p,q \in [1,\infty]$, $\frac{1}{p}+\frac{1}{q}=1$, $f \in L^p(\Omega)$, $g \in L^q(\Omega)$ we have

$$\int_\Omega |fg|\,dx \leq \|f\|_{L^p} \|g\|_{L^q}$$

Pf: · For $a,b \geq 0$, $\lambda \in (0,1)$ one has

$$a^\lambda b^{(1-\lambda)} \leq \lambda a + (1-\lambda) b$$ (geometric $\leq$ arithmetic mean),

since $f(t) = t^\lambda - \lambda t + \lambda - 1$ satisfies $f(t) \leq f(1) = 0 \implies t^\lambda \leq \lambda t + 1 - \lambda$; now pick $t = \frac{a}{b}$

· choose $a = \left(\frac{|f(x)|}{\|f\|_{L^p}}\right)^p$, $b = \left(\frac{|g(x)|}{\|g\|_{L^q}}\right)^q$, $\lambda = \frac{1}{p}$, $1-\lambda = \frac{1}{q}$

$$\implies \frac{|f(x)g(x)|}{\|f\|_{L^p}\|g\|_{L^q}} \leq \frac{1}{p}\left(\frac{|f(x)|}{\|f\|_{L^p}}\right)^p + \frac{1}{q}\left(\frac{|g(x)|}{\|g\|_{L^q}}\right)^q$$; now integrate on Ω. $\square$

Lp-stability

$u_t + a(t,x)u_x + c(t,x)u = f(t,x)$ on $[0,T]\times\Omega$, $p\in[1,\infty)$

Thm: Let $u_0(x), f(x) = 0$ for $|x|$ big enough, $\Omega=\mathbb{R}$, sol. $u\geq 0$, then

$$\|u(t,\cdot)\|_{L^p} \leq \|u_0\|_{L^p} \exp\left(\int_0^t \|c(s,\cdot)\|_{L^\infty} + \|a_x(s,\cdot)\|_{L^\infty} ds\right)$$
$$+ \int_0^t \|f(\tau,\cdot)\|_{L^p} \exp\left(\int_\tau^t \|c(s,\cdot)\|_{L^\infty} + \|a_x(s,\cdot)\|_{L^\infty} ds\right) d\tau$$

Pf: $p\|u\|_{L^p}^{p-1} \frac{d}{dt}\|u\|_{L^p} = \frac{d}{dt}\|u\|_{L^p}^p = p\int_\Omega u^{p-1}(f - au_x - cu)\,dx$

$$\leq p\left[\|u\|_{L^p}^p \|c\|_{L^\infty} + \|u\|_{L^p}^{p-1}\|f\|_{L^p}\right] - \underbrace{\int_\Omega a\,(u^p)_x\,dx}_{= -\int_\Omega a_x u^p dx} \leq p\left[\|u\|_{L^p}^p\|c\|_{L^\infty} + \|u\|_{L^p}^{p-1}\|f\|_{L^p} + \|a_x\|_{L^\infty}\|u\|_{L^p}^p\right]$$

$\Rightarrow \frac{d}{dt}\|u\|_{L^p} \leq \left(\|c\|_{L^\infty} + \|a_x\|_{L^\infty}\right)\|u\|_{L^p} + \|f\|_{L^p}$; now use Gronwall $\square$

Thm: For a, c constant, $\Omega=\mathbb{R}$, one has $u(t,x) = e^{-ct}u_0(x-at) + \int_0^t e^{-c(t-s)} f(s, x-a(t-s))\,ds$

$\Rightarrow \|u(t,\cdot)\|_{L^p} \leq e^{-ct}\|u_0\|_{L^p} + \int_0^t e^{-c(t-s)}\|f(s,\cdot)\|_{L^p}\,ds \quad \forall p\in[1,\infty]$

Finite differences: hyperbolic PDEs (1D transport equation)

l2-stability

$\dot u + a u_x + c u = f$, $\quad a, c$ constant, $a \geq 0$, $\Omega = \mathbb{R}$

$$\frac{U_j^{n+1} - U_j^n}{\Delta t} + a \frac{U_j^n - U_{j-1}^n}{\Delta x} + c U_j^n = F_j^n$$

__Thm:__ (ℓ^2-stability) Let U_j^n be the soln. of the upwind method, $\mu = a\frac{\Delta t}{\Delta x}$. One has

$$\|U^m\|_{\ell^2} \leq \begin{cases} e^{-ct^m}\|U^0\|_{\ell^2} + \Delta t \sum_{n=0}^{m-1} \|F^n\|_{\ell^2} e^{-c(t^m - t^n)} & \text{if } \mu \leq 1 - c\Delta t \text{ „}\ell^2\text{-stable"} \\ e^{(c+2k)t^m}\|U^0\|_{\ell^2} + \Delta t \sum_{n=0}^{m-1} \|F^n\|_{\ell^2} e^{(c+2k)(t^m - t^n)} & \text{if } \mu \leq 1 + \underset{\substack{\vee\\0}}{k}\Delta t \text{ „}\ell^2\text{-von Neumann-stable"} \end{cases}$$

Pf: Set $U_j^m = \frac{1}{2\pi}\int_{-\pi/\Delta x}^{\pi/\Delta x} \hat{u}^m(k) e^{ikj\Delta x}\, dk$

$$\Rightarrow \frac{\hat{u}^{m+1}(k) - \hat{u}^m(k)}{\Delta t} + a\frac{\hat{u}^m(k) - e^{-ik\Delta x}\hat{u}^m(k)}{\Delta x} + c\hat{u}^m(k) = \hat{F}^m(k) \quad \forall k \in [-\pi/\Delta x, \pi/\Delta x]$$

$$\Rightarrow \hat{u}^{m+1}(k) = \left(1 - \mu(1 - e^{-ik\Delta x}) - c\Delta t\right)\hat{u}^m(k) + \Delta t \hat{F}^m(k) =: \lambda(k)\hat{u}^m(k) + \Delta t \hat{F}^m(k)$$

- Let $\Lambda = \max_k |\lambda(k)| \Rightarrow \|U^{m+1}\|_{\ell^2} = \frac{1}{\sqrt{2\pi}}\|\hat{u}^{m+1}\|_{L^2} \leq \Lambda \frac{1}{\sqrt{2\pi}}\|\hat{u}^m\|_{L^2} + \Delta t \frac{1}{\sqrt{2\pi}}\|\hat{F}^m\|_{L^2}$

for $a<0$ this would be $\approx 1 + \tilde{c}|\mu|$!!

$$= \Lambda\|U^m\|_{\ell^2} + \Delta t\|F^m\|_{\ell^2} \leq \Lambda^{m+1}\|U^0\|_{\ell^2} + \sum_{n=0}^{m} \Delta t \Lambda^{m-n}\|F^n\|_{\ell^2}$$

- $|\lambda(k)|^2 = (1 - c\Delta t)^2 + 4\mu\left(\sin^2\frac{k\Delta x}{2}\right)(\mu - (1 - c\Delta t)) \Rightarrow \Lambda \leq \begin{cases} 1 - c\Delta t \leq e^{-c\Delta t} & \text{if } \mu \leq 1 - c\Delta t \\ 1 + (c+2k)\Delta t \leq e^{(c+2k)\Delta t} & \text{if } \mu \leq 1 + k\Delta t \end{cases}$ □

Finite differences: hyperbolic PDEs (1D transport equation)

l^∞-stability

$\dot{u} + a u_x + c u = f, \quad a \geq 0, \quad \Omega = \mathbb{R}$

$$\frac{U_j^{m+1} - U_j^m}{\Delta t} + A_j^m \frac{U_j^m - U_{j-1}^m}{\Delta x} + C_j^m U_j^m = F_j^m$$

<u>Thm</u>: (ℓ^∞-stability) Let U_j^m be the soln. of the upwind method, $m \geq 0$, $j \in \mathbb{Z}$, $\mu = \|A\|_{\ell^\infty} \frac{\Delta t}{\Delta x}$.

If $\mu \leq 1$, $\quad \|U^m\|_{\ell^\infty} \leq e^{t^m \|C\|_{\ell^\infty}} \|U^0\|_{\ell^\infty} + \Delta t \sum_{k=0}^{m-1} e^{(t^{m-1} - t^k)\|C\|_{\ell^\infty}} \|F^k\|_{\ell^\infty}$.

Pf: $U_j^{m+1} = U_j^m \left(1 - A_j^m \frac{\Delta t}{\Delta x}\right) + A_j^m \frac{\Delta t}{\Delta x} U_{j-1}^m - \Delta t\, C_j^m U_j^m + \Delta t\, F_j^m$

$\Rightarrow |U_j^{m+1}| \leq \|U^m\|_{\ell^\infty} \left(1 - A_j^m \frac{\Delta t}{\Delta x}\right) + A_j^m \frac{\Delta t}{\Delta x} \|U^m\|_{\ell^\infty} + \Delta t \|C\|_{\ell^\infty} \|U^m\|_{\ell^\infty} + \Delta t \|F^m\|_{\ell^\infty}$

$\Rightarrow \|U^{m+1}\|_{\ell^\infty} \leq (1 + \|C\|_{\ell^\infty} \Delta t) \|U^m\|_{\ell^\infty} + \Delta t \|F^m\|_{\ell^\infty}$

$\Rightarrow$ now induction in m $\square$

<u>Thm</u>: For c = constant & $\mu + c\Delta t \leq 1$ even $\|U^m\|_{\ell^\infty} \leq e^{-c t^m} \|U^0\|_{\ell^\infty} + \Delta t \sum_{k=0}^{m-1} e^{-c(t^{m-1} - t^k)} \|F^k\|_{\ell^\infty}$

Pf: homework $\square$

How about $\Omega = [\ell, r]$ and Dirichlet b.c. in ℓ?

Finite differences: hyperbolic PDEs (1D transport equation)

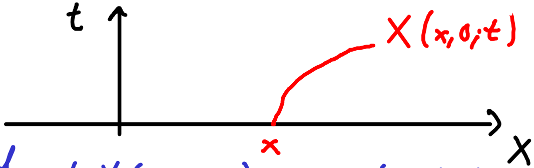
Domain of Dependence

$$u_t(t,x) + a(t,x)\,u_x(t,x) + c(t,x)\,u(t,x) = f(t,x)$$

<u>Def</u>: The curve defined by $\frac{d}{dt}X(x,s;t) = a(t,X(x,s;t))$, $\quad X(x,s;s) = x$

is called <u>characteristic curve</u> of the PDE.

<u>Ex</u>: $\dot u + t\,u_x = 0 \;\Rightarrow\; \frac{d}{dt}X(x,s;t) = t \;\Rightarrow\; X(x,s;t) = x + \frac{t^2-s^2}{2}$



We have $\frac{d}{dt}u(t,X(x,s;t)) = u_t(t,X(x,s;t)) + u_x(t,X(x,s;t))\,\overbrace{\frac{d}{dt}X(x,s;t)}^{=a(t,X)}$

$$= -c(t,X(x,s;t))\,u(t,X(x,s;t)) + f(t,X(x,s;t))$$

With $\tilde g(t) := g(t,X(x,s;t))$ we obtain $\dot{\tilde u} = -\tilde c\,\tilde u + \tilde f$ $\Rightarrow$ Along char. curve one has an ODE

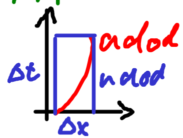
$\Rightarrow \tilde u(t) = \tilde u(s)\,e^{-\int_s^t \tilde c(r)\,dr} + \int_s^t \tilde f(\tau)\,e^{-\int_\tau^t \tilde c(r)\,dr}\,d\tau$ only depends on c, f, u along $(t, X(x,s;t))$!

<u>Ex</u>: $\dot u + t\,u_x = 0 \;\Rightarrow\; u\left(t, x + \frac{t^2-s^2}{2}\right) = u(s,x) \;\Rightarrow\; u(t,x) = u\left(0, x - \frac{t^2}{2}\right)$

<u>Def</u>: The analytical <u>domain of dependence</u> (dod) of (s,x) is $\{(X(x,s;t),t) \,|\, 0 \le t \le s\}$.

<u>Courant-Friedrichs-Lewy-condition</u>: analytical $\subset$ numerical dod

$\Rightarrow a\frac{\Delta t}{\Delta x} \le 1$



Finite differences: hyperbolic PDEs (1D transport equation)

Convergence (concistency + stability)

<u>Def</u>: Let u solve $\dot{u} + a u_x + c u = f$. The <u>truncation error</u> of the upwind method is

$$T_j^m = \frac{u_j^{m+1} - u_j^m}{\Delta t} + a_j^m \frac{u_j^m - u_{j-1}^m}{\Delta x} + c_j^m u_j^m - f_j^m \qquad \left(\text{with } g_j^m = g(t^m, x_j)\right)$$

<u>Thm</u>: (consistency) The upwind method is first order consistent, i.e. a smooth soln u satisfies $T_j^m = O(\Delta t + \Delta x)$.

Pf : homework (via Taylor expansion) □

Just like for the heat eq the error solves $e_j^m = U_j^m - u_j^m$: $\frac{e_j^{m+1} - e_j^m}{\Delta t} + a_j^m \frac{e_j^m - e_{j-1}^m}{\Delta x} + c_j^m e_j^m = -T_j^m$

we only prove ℓ^2-convergence for $a, c = \text{const}$

<u>Thm</u>: (Convergence) Let U_j^m, $j \in \mathbb{Z}$, $m = 0, \dots, M = \frac{T}{\Delta t}$ solve the upwind-method for $\dot{u} + a u_x + c u = f$ on $[0,T] \times \Omega$, $\Omega = \mathbb{R}$, $a \geq 0$ + i.c., u smooth, $\mu \leq 1$. Then $\|e^m\|_{\ell^2}, \|e^m\|_{\ell^\infty} = O(\Delta t + \Delta x)\ \forall m$

Pf : Stability $\Longrightarrow$ $\|e^m\| \leq \Delta t \sum_{k=0}^{m-1} e^{\text{const.}T} \|T^m\| \leq T e^{\text{const}\,T}\, O(\Delta t + \Delta x)$. □

only on that boundary where characteristics enter

<u>Thm</u>: Now let $\Omega = [l, r]$ & use b.c. $u(t, l) \overset{!}{=} g(t)$. $\|e^m\|_{\ell^\infty} = O(\Delta t + \Delta x)$, $m = 1, \dots, M$.

Pf : homework (first derive ℓ^∞-stability for 0-b.c.) □

Methods of higher order consistency

$u_t + a u_x = 0$

sign of a plays no role any more!

Def: · method of central differences $\quad \frac{U_j^{m+1} - U_j^m}{\Delta t} + A_j^m \frac{U_{j+1}^m - U_{j-1}^m}{2\Delta x} = 0$

· Lax-Friedrichs-method $\quad \frac{U_j^{m+1} - \frac{1}{2}(U_{j-1}^m + U_{j+1}^m)}{\Delta t} + A_j^m \frac{U_{j+1}^m - U_{j-1}^m}{2\Delta x} = 0$

· Lax-Wendroff-method $\quad \frac{U_j^{m+1} - U_j^m}{\Delta t} + a \frac{U_{j+1}^m - U_{j-1}^m}{2\Delta x} = \frac{1}{2}\Delta t a^2 \frac{U_{j+1}^m - 2U_j^m + U_{j-1}^m}{\Delta x^2}$

for $a = $ const.

What is the Lax-Wendroff-method based on?

$u_t = -a u_x \quad \Rightarrow u_{tt} = -a_t u_x - a u_{xt} = -a_t u_x - a(-a u_x)_x$

Taylor expansion $\Rightarrow \quad u_j^{m+1} = u_j^m + \Delta t \underbrace{[u_t]_j^m}_{= -[a u_x]_j^m} + \frac{\Delta t^2}{2} \underbrace{[u_{tt}]_j^m}_{= -[a_t u_x - a(a u_x)_x]_j^m} + O(\Delta t^3)$

$$\Rightarrow \frac{u_j^{m+1} - u_j^m}{\Delta t} + \left[\left(a + \frac{1}{2}\Delta t a_t\right) u_x\right]_j^m = \frac{1}{2}\Delta t \left[a (a u_x)_x\right]_j^m + O(\Delta t^2)$$

Thm: (Consistency) The truncation error of the three methods for a smooth soln u is $O(\Delta t + \Delta x^2)$, resp. $O(\Delta t^2 + \Delta x^2)$ for LW.

Pf: homework □

l2-stability

$u_t + a u_x = 0$ on $\mathbb{R}$ with $a = \text{const.}$

If we insert $u_j^m = \frac{1}{2\pi}\int_{-\pi/\Delta x}^{\pi/\Delta x} \hat{u}^m(k) e^{ikx_j}\,dk$ into the methods, we obtain

centr. diff: $$\frac{\hat{u}^{m+1}(k) - \hat{u}^m(k)}{\Delta t} + a\frac{e^{ik\Delta x}\hat{u}^m(k) - e^{-ik\Delta x}\hat{u}^m(k)}{2\Delta x} = 0$$

Lax-Friedrichs: $$\frac{\hat{u}^{m+1}(k) - \frac{1}{2}\left(e^{ik\Delta x}\hat{u}^m(k) + e^{-ik\Delta x}\hat{u}^m(k)\right)}{\Delta t} + a\frac{e^{ik\Delta x}\hat{u}^m(k) - e^{-ik\Delta x}\hat{u}^m(k)}{2\Delta x} = 0$$

Lax-Wendroff: $$\frac{\hat{u}^{m+1}(k) - \hat{u}^m(k)}{\Delta t} + a\frac{e^{ik\Delta x}\hat{u}^m(k) - e^{-ik\Delta x}\hat{u}^m(k)}{2\Delta x} = \frac{1}{2}\Delta t a^2 \frac{\hat{u}^m(k)\left(e^{ik\Delta x} - 2 + e^{-ik\Delta x}\right)}{\Delta x^2}$$

$\Rightarrow \hat{u}^{m+1}(k) = \lambda(k)\hat{u}^m(k)$ for all $k \in \left[-\frac{\pi}{\Delta x}, \frac{\pi}{\Delta x}\right]$ with

$$\lambda(k) = \begin{cases} 1 - \frac{\mu}{2}\left(e^{ik\Delta x} - e^{-ik\Delta x}\right) = 1 - \mu i \sin k\Delta x & , \text{CD} \\ \frac{1}{2}\left[e^{ik\Delta x} + e^{-ik\Delta x} - \mu\left(e^{ik\Delta x} - e^{-ik\Delta x}\right)\right] = \cos k\Delta x - \mu i \sin k\Delta x & , \text{LF} \\ 1 - \frac{\mu}{2}\left(e^{ik\Delta x} - e^{-ik\Delta x}\right) + \frac{\mu^2}{2}\left(e^{ik\Delta x} - 2 + e^{-ik\Delta x}\right) = 1 - \mu i \sin k\Delta x - 2\mu^2 \sin^2\frac{k\Delta x}{2} & , \text{LW} \end{cases}$$

Thm: (ℓ^2-stability) LF & LW are ℓ^2-stable for $|\mu| \leq 1$, CD is never ℓ^2-stable.

Pf: LF: $|\lambda(k)|^2 = \cos^2 k\Delta x + \mu^2 \sin^2 k\Delta x \leq 1$; LW: $|\lambda(k)|^2 = \left(1 - 2\mu^2\sin^2\frac{k\Delta x}{2}\right)^2 + \mu^2 \sin^2 k\Delta x = 1 - 4\mu^2(1-\mu^2)\sin^4\frac{k\Delta x}{2} \leq 1$

CD: $|\lambda(k)|^2 = 1 + \mu^2\sin^2 k\Delta x > 1 \Rightarrow \left|\hat{u}^m(k)\right| \geq (1+c)^{\frac{m}{2}}\left|\hat{u}^0(k)\right|$ for $\forall c \leq \mu^2\sin^2(k\Delta x)$

$\Rightarrow \Delta t \xrightarrow[\mu\,\text{const.}]{} 0$ implies $\sqrt{2\pi}\,\|u^M\|_{\ell^2} = \|\hat{u}^M\|_{L^2} \geq \left\|\hat{u}^M|_{\left[\frac{\pi}{4\Delta x}, \frac{3\pi}{4\Delta x}\right]}\right\|_{L^2} \geq \left(1 + \frac{\mu^2}{2}\right)^{\frac{M}{2}} \left\|\hat{u}^0|_{\left[\frac{\pi}{4\Delta x}, \frac{3\pi}{4\Delta x}\right]}\right\|_{L^2} \to \infty$ $\square$

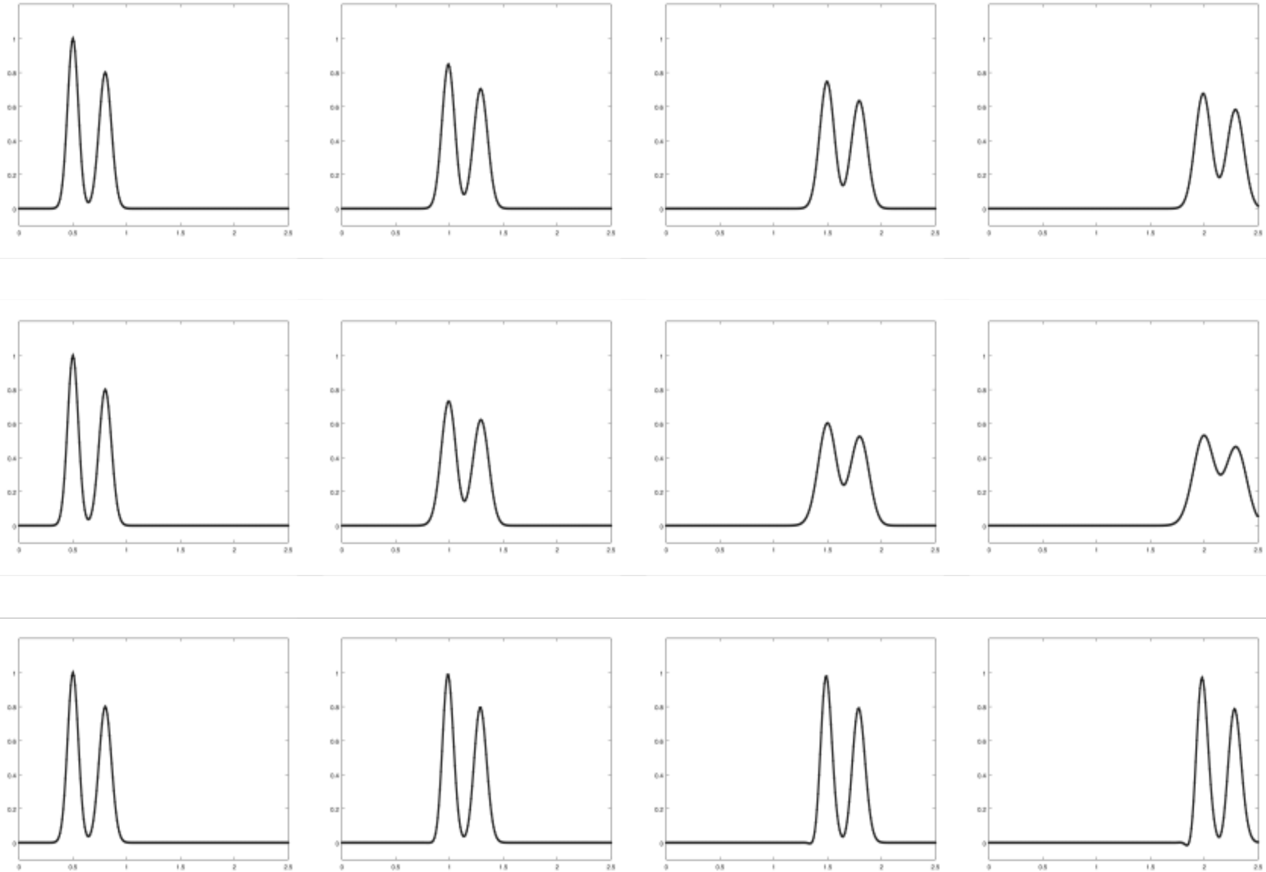
Example simulation

$u_t + u_x = 0 \quad , \quad \mu = \frac{\Delta t}{\Delta x} = 0.7$

upwind

LF

LW



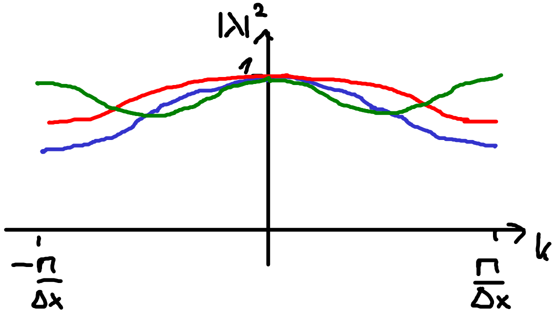
All methods are convergent in ℓ^2 (follows from stability + consistency), but ...

Numerical dissipation

$u_t + u_x = 0$ has soln. $u(t,x) = u(0, x-t)$ $\Rightarrow$ $\|u\|_{L^2} = \text{const.}$

$\sqrt{2\pi}\,\|U^m\|_{\ell^2} = \|\hat{U}^m\|_{L^2}$ with $|\hat{U}^m(k)| \leq |\lambda(k)|^m |\hat{U}^0(k)|$

for $$|\lambda(k)|^2 = \begin{cases} 1 - 4\mu(1-\mu)\sin^2\frac{k\Delta x}{2} & \text{upwind} \\ 1 - (1-\mu^2)\sin^2 k\Delta x & \text{LF} \\ 1 - 4\mu^2(1-\mu^2)\sin^4\frac{k\Delta x}{2} & \text{LW} \end{cases}$$



$\Rightarrow$ frequency components at $k=0$ stay unchanged due to $|\lambda(0)|=1$, but frequency components further away from $k=0$ are damped $\Rightarrow$ "numerical dissipation"

Def: A scheme is called dissipative of order $r \in \mathbb{N}$ if there is $\delta > 0$ such that
$$|\lambda(k)|^2 \leq 1 - \delta \left|\sin\frac{k\Delta x}{2}\right|^{2r} \text{ near } k=0.$$
(upwind & LF: $r=1$, LW: $r=2$)

Perspective: 2D and nonlinearity

Conservation principle: $u_t + \mathrm{div}\,[q(x,u)] = 0 \quad \Rightarrow \quad u_t + q_u \cdot u_x + \mathrm{tr}\, q_x = 0$, $\Omega \subset \mathbb{R}^2$

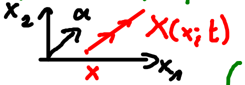
$\Rightarrow$ general form: $u_t(t,x) + a(t,x,u(t,x)) \cdot \nabla u(t,x) = f(t,x,u(t,x))$

soln. satisfies $u(t, X(x;t)) = U(x;t)$ ODE, i.e. concept of dod also applies here

for characteristic equations
$$\begin{cases} \dot X(x;t) = a(t, X(x;t), U(x;t)), & X(x;0) = x \\ \dot U(x;t) = f(t, X(x;t), U(x;t)), & U(x;0) = u(0,x) \end{cases}$$

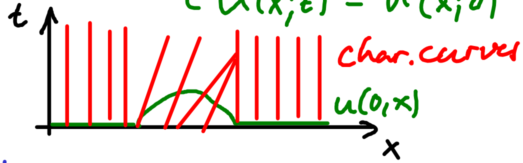
Ex: $u_t + a \cdot \nabla u = 0$, $a \in \mathbb{R}^2 \Rightarrow X(x;t) = at + x$, $u(t, X(x;t)) = U(x;t) = U(x;0)$, d.h. $u(t,y) = u(0, y - at)$

char. curve



Ex: (Burgers equation) $u_t + \left(\frac{u^2}{2}\right)_x = 0$, i.e. $a(t,x,u) = u$ $\Rightarrow$ $\begin{cases} X(x;t) = U(x;0)\,t + x \\ U(x;t) = U(x;0) \end{cases}$

$\Rightarrow$ u is constant along char. curves



The upwind method now reads

$$\frac{U_{ij}^{m+1} - U_{ij}^m}{\Delta t} + \frac{1}{\Delta x}\left[a(t^m, x_{ij}, U_{ij}^m)^+ \cdot \begin{pmatrix} U_{ij}^m - U_{i-1,j}^m \\ U_{ij}^m - U_{i,j-1}^m \end{pmatrix} + a(t^m, x_{ij}, U_{ij}^m)^- \cdot \begin{pmatrix} U_{i+1,j}^m - U_{ij}^m \\ U_{i,j+1}^m - U_{ij}^m \end{pmatrix} \right] = f(t^m, x_{ij}, U_{ij}^m)$$

neg. part

+ ic + bc on that part of $\partial\Omega$ where char. curves enter

If $\left| a(t,x,u) \cdot \begin{pmatrix} \Delta t / \Delta x_1 \\ \Delta t / \Delta x_2 \end{pmatrix} \right| \le 1 \ \forall t, x, u$, then ℓ^2-/ℓ^∞-stability follows as before.

Elliptic PDE theory: introduction to Euler-Lagrange equations

Poisson equation with homogeneous Dirichlet boundary conditions: $\begin{cases} -\Delta u = f & \text{on } \Omega \subset \mathbb{R}^d \\ u = 0 & \text{on } \partial\Omega \end{cases}$

We had interpreted this as necessary condition for the fact that u minimizes

$$E(u) = \int_\Omega \frac{1}{2}|\nabla u(x)|^2 - f(x)\,u(x)\,dx \qquad (*)$$

among all functions $\Omega \to \mathbb{R}$ with 0-b.c. In detail:

<u>Def</u>: (Gâteaux derivative) Let X be a vector space, $E: X \to \mathbb{R}$. The <u>Gâteaux derivative</u> of E in $u \in X$ in direction $v \in X$ is $\partial_u E(u)(v) = \frac{d}{dt} E(u+tv)\Big|_{t=0}$.

<u>Thm</u>: (necessary optimality condition) If u minimizes the functional E and if $\partial_u E(u)(v)$ exists, then $\partial_u E(u)(v) = 0$. why?

<u>Ex</u>: For $(*)$ we have $\partial_u E(u)(v) = \int_\Omega \nabla u \cdot \nabla v - f v \, dx$. If u is sufficiently often differentiable, using integration by parts we obtain $\partial_u E(u)(v) = \int_\Omega v\,(-\Delta u - f)\,dx$.

If u is a minimum, then $\partial_u E(u)(v) = 0$ for "all" v $\Rightarrow -\Delta u - f = 0$.

Actually, only the so-called <u>weak form</u> of the PDE holds

$$0 = \partial_u E(u)(v) = \int_\Omega \nabla u \cdot \nabla v - f v\, dx \qquad \forall v: \Omega \to \mathbb{R},\ v|_{\partial\Omega} = 0.$$

How often must u be differentiable to interpret this as $-\Delta u = f$?

Elliptic PDE theory: weak derivative

$\Omega \subset \mathbb{R}^d$

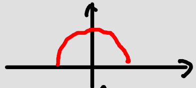
Def: · $C^m(\overline{\Omega})$ is the space of <u>m-times continuously differentiable functions</u> $f:\overline{\Omega}\to\mathbb{R}$ with

$$\|f\|_{C^m} := \sup_{x\in\Omega,\ |\alpha|\le m} |D^\alpha f(x)| < \infty$$

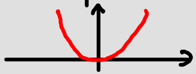
· $C_0^\infty(\Omega)$ is the space of <u>infinitely often continuously differentiable functions</u> $f:\Omega\to\mathbb{R}$ with compact support $\mathrm{supp}(f) := \overline{\{x\in\Omega \mid f(x)\neq 0\}} \subset \Omega$.

· $L^1_{loc}(\Omega) = \{f:\Omega\to\mathbb{R} \mid f\in L^1(\omega)\ \forall\, \overline{\omega}\subset\Omega \text{ compakt}\}$ = <u>locally integrable functions</u>

Ex: $x \mapsto \sqrt{1-x^2} \in C^0([-1,1])\setminus C^1([-1,1])$

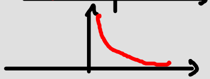


$x \mapsto |x|^3 \in C^2([-1,1])\setminus C^3([-1,1])$



$x \mapsto \begin{cases} \exp\left(-\frac{1}{1-x^2}\right), & x\in(-1,1) \\ 0 & \text{else} \end{cases} \in C_0^\infty(\mathbb{R})$

$x \mapsto \frac{1}{x} \in L^1_{loc}((0,\infty))$



Def: Let $u\in L^1_{loc}(\Omega)$, α a multiindex. $w\in L^1_{loc}$ is called <u>weak α-derivative</u> of u, if

$$\int_\Omega w(x)v(x)\,dx = (-1)^{|\alpha|}\int_\Omega u(x)\,D^\alpha v(x)\,dx \quad \forall\, v\in C_0^\infty(\Omega).$$

Ex: · $u\in C^m(\overline{\Omega})$, $v\in C_0^\infty(\Omega) \Rightarrow \int_\Omega v\,\partial_{x_i}u\,dx \overset{\text{int. by parts}}{=} -\int_\Omega u\,\partial_{x_i}v\,dx \Rightarrow \partial_{x_i}u$ is weak derivative with respect to x_i.

analogously, $D^\alpha u$ is weak α-derivative for $|\alpha|\le m$

· $x \mapsto |x|$ has weak derivative $x \mapsto \begin{cases} -1, & x<0 \\ 1, & \text{else} \end{cases}$ ⟵ check!

Elliptic PDE theory: Sobolev spaces

Def: · A metric space X is called complete, if every Cauchy sequence in X converges.

· A complete normed space X with inner product $\langle\cdot,\cdot\rangle_X$ is called Hilbertspace, if

$\|x\|_X = \sqrt{\langle x,x\rangle} \quad \forall x \in X$

· Let $m \in \mathbb{N}$, $p \in [1,\infty]$. The (m,p)-Sobolev space is

$$W^{m,p}(\Omega) = \{ f \in L^p(\Omega) \mid f \text{ has weak } \alpha\text{-derivative } D^\alpha f \in L^p(\Omega) \ \forall |\alpha| \leq m \}$$

with norm $\|f\|_{W^{m,p}} = \left(\sum_{|\alpha|\leq m} \|D^\alpha f\|_{L^p}^p \right)^{\frac{1}{p}}$

· Let $m \in \mathbb{N}$. The m-Hilbert space is $H^m(\Omega) = W^{m,2}(\Omega)$

with inner product $\quad (f,g)_{H^m} = \sum_{|\alpha|\leq m} \int_\Omega D^\alpha f \, D^\alpha g \, dx.$

Thm: $W^{m,p}(\Omega)$ is complete.

Ex: · $x \mapsto \frac{1}{x^2} \quad \in W^{m,p}((1,\infty)) \quad \forall p \in [1,\infty], m \geq 0$

· $x \mapsto \sqrt{|x|} \quad \in \left(W^{1,p}((-1,1)) \cap L^q((-1,1)) \right) \setminus W^{2,r}((-1,1)) \quad \forall p \in [1,2), q,r \in [1,\infty]$

· $x \mapsto \begin{cases} -1, & x \leq 0 \\ 1, & \text{else} \end{cases} \quad \notin W^{1,p}((-1,1)) \ \forall p$

Thm: (trace theorem) Let Ω have Lipschitz boundary. $T: W^{1,p}(\Omega) \to L^p(\partial\Omega)$, $u \mapsto u|_{\partial\Omega}$ is linear & bounded.

Elliptic PDE theory: linear elliptic boundary value problems

Apparently, the minimization of $E(u)=\int_\Omega \frac{|\nabla u|^2}{2}-fu\,dx$ already makes sense on $H^1(\Omega)\supset C^2(\bar\Omega)$.

<u>Def</u>: $u\in H_0^1(\Omega)=\{v\in H^1(\Omega)\,|\,v=0 \text{ on } \partial\Omega\}$ is called <u>weak solution</u> of $-\Delta u=f$ in Ω, $u=0$ on $\partial\Omega$, if
$$0=\partial_u E(u)(v)=\int_\Omega \nabla u\cdot\nabla v-fv\,dx \quad \forall v\in H_0^1(\Omega).$$

To obtain more general elliptic PDEs and boundary conditions, we now consider
$$E(u)=\int_\Omega \tfrac12\nabla u(x)^T A(x)\nabla u(x)+\frac{q(x)}{2}u(x)^2-f(x)u(x)\,dx+\int_{\Gamma_1}\frac{p(x)}{2}u(x)^2-g(x)u(x)\,dx$$
for $A(x)\in\mathbb{R}^{d\times d}$ continuously differentiable symmetric positive definite, $q\in L^\infty(\Omega)$, $q\geq 0$, $\Gamma_1\subset\partial\Omega$ with positive measure, $p\in L^\infty(\Gamma_1)$, $p\geq 0$, $f\in L^2(\Omega)$, $g\in L^2(\Gamma_1)$, to be minimized for $u\in H^1(\Omega)$ (why does this make sense?) with $u=h\in L^2(\Gamma_2)$ on $\Gamma_2=\partial\Omega\setminus\Gamma_1$.

$\Rightarrow$ For all $v\in H^1(\Omega)$ with $v|_{\Gamma_2}=0$ we have
$$0=\partial_u E(u)(v)=\int_\Omega \nabla v^T A\nabla u+quv-fv\,dx+\int_{\Gamma_1}puv-gv\,dx \qquad \text{"weak form"}$$
if u smooth
$$=\int_\Omega v[-\operatorname{div}(A\nabla u)+qu-f]\,dx+\int_{\Gamma_1}(n^TA\nabla u+pu-g)v\,dx$$

$\Rightarrow$ $-\operatorname{div}(A\nabla u)+qu=f$ in Ω with
$$\begin{cases} u=h & \text{on } \Gamma_2 & \text{Dirichlet bc}\\ n^TA\nabla u=g & \text{on } \Gamma_1\cap\{p=0\} & \text{Neumann bc}\\ n^TA\nabla u+pu=g & \text{on } \Gamma_1 & \text{Robin bc}\end{cases}$$

$\Rightarrow$ exactly one bc everywhere on $\partial\Omega$

Elliptic PDE theory: weak solution

The weak form of the PDE is obtained from the energy E as $0=\partial_u E(u)(v)$ for all allowed v; from the PDE itself it is obtained by multiplication of the PDE with v & integration by parts.

Ex: $\Delta u = f$ in Ω, $u=0$ on $\partial\Omega$ $\Rightarrow$ allowed v are $v \in H_0^1(\Omega)$

$\Rightarrow -\int_\Omega v \Delta u \, dx = \int_\Omega v f \, dx \Rightarrow \int_\Omega \nabla u \cdot \nabla v \, dx = \int_\Omega v f \, dx \;\; \forall v \in H_0^1(\Omega)$ is weak form

Def: The weak form of
$$\begin{cases} -\mathrm{div}(A(x)\nabla u(x)) + b(x)\cdot\nabla u(x) + q(x)u(x) = f(x) & \text{in } \Omega \\ u(x) = h(x) & \text{on } \Gamma_2 = \partial\Omega\setminus\Gamma_1 \\ (A(x)\nabla u(x))\cdot n + p(x)u(x) = g(x) & \text{on } \Gamma_1 \end{cases}$$
is $0 = \int_\Omega \nabla v^T A \nabla u + v b\cdot\nabla u + quv - fv \, dx + \int_{\Gamma_1} puv - gv \, dx \quad \forall v \in H^1_{\Gamma_2}(\Omega) = \{v \in H^1(\Omega) \,|\, v|_{\Gamma_2} = 0\}$
& $u = h$ on Γ_2

- $u \in H^1(\Omega)$ is called a weak solution of the PDE if it satisfies its weak form.

Elliptic PDE theory: associated bilinear form

By setting $u(x) = \tilde{u}(x) + h(x)$ $\forall x \in \Omega$ one can reduce the b.c. $u|_{\Gamma_2} = h$ to $\tilde{u}|_{\Gamma_2} = 0$.

Ex: $-\Delta u = 0$ in Ω, $u = h$ on $\partial\Omega$ $\Rightarrow$ Find $u \in H^1(\Omega)$ with $\int_\Omega \nabla u \cdot \nabla v \, dx = 0$ $\forall v \in H_0^1(\Omega)$ & $u|_{\partial\Omega} = h$

$\Rightarrow$ Find $\tilde{u} \in H_0^1(\Omega)$ with $\int_\Omega \nabla \tilde{u} \cdot \nabla v + \nabla h \cdot \nabla v \, dx = 0$ $\forall v \in H_0^1(\Omega)$

Therefore we only consider 0-boundary values. Hence, let u be a weak soln., i.e.

$u \in H^1_{\Gamma_2}$ with $0 = \int_\Omega \nabla v^T A \nabla u + v b \cdot \nabla u + q u v - f v \, dx + \int_{\Gamma_1} p u v - g v \, dx$ $\forall v \in H^1_{\Gamma_2}(\Omega)$.

Set
$$a(u,v) = \int_\Omega \nabla v^T A \nabla u + v b \cdot \nabla u + q u v \, dx + \int_{\Gamma_1} p u v \, dx \quad \text{"associated bilinear form"}$$
$$\ell(v) = \int_\Omega f v \, dx + \int_{\Gamma_1} g v \, dx$$

a is a bilinear form, i.e. linear in u & v $\left(a(u + \lambda \tilde{u}, v) = a(u,v) + \lambda a(\tilde{u}, v)\right.$, analogous in $v)$

ℓ is a linear form $\left(\ell(v + \lambda \tilde{v}) = \ell(v) + \lambda \ell(\tilde{v})\right)$

u is weak soln. of PDE $\iff$ $\boxed{u \in H^1_{\Gamma_2} \ \& \ a(u,v) = \ell(v) \ \forall v \in H^1_{\Gamma_2}}$

Elliptic PDE theory: existence, uniqueness, equivalence to minim.

Def: Let X be a Hilbert space, $a: X \times X \to \mathbb{R}$ a bilinear form, $\ell: X \to \mathbb{R}$ linear.

We require the following conditions: $\exists\, c, C > 0$ such that for all $u, v \in X$ we have

coercivity/ellipticity: $\quad a(v,v) \geq c\|v\|_X^2$

boundedness: $\quad |a(u,v)| \leq C\|u\|_X \|v\|_X$

boundedness: $\quad |\ell(v)| \leq C\|v\|_X$

Thm: (Lax-Milgram-lemma) Let X be a Hilbert space, a a coercive bounded bilinear form, ℓ linear and bounded. Then there exists a unique solution $u \in X$ of

$$a(u,v) = \ell(v) \quad \forall v \in X.$$

Thm: Let in addition a be symmetric, i.e., $a(u,v) = a(v,u)\ \forall u,v \in X$. Then

$$a(u,v) = \ell(v) \quad \forall v \in X \qquad \Longleftrightarrow \qquad u = \underset{\forall \tilde{u} \in X}{\operatorname{argmin}}\ \tfrac{1}{2}a(\tilde{u},\tilde{u}) - \ell(\tilde{u})$$

Pf: „$\Rightarrow$" Let $v \in X \Rightarrow \frac{1}{2}a(v,v) - \ell(v) - \left(\frac{1}{2}a(u,u) - \ell(u)\right) = \frac{1}{2}a(v-u, v-u) + a(v-u, u) - \ell(v-u)$

$$= \tfrac{1}{2}a(v-u, v-u) \geq \tfrac{c}{2}\|v-u\|_X^2$$

„$\Leftarrow$" Set $E(u) = \frac{1}{2}a(u,u) - \ell(u)$, then $0 = \partial_u E(u)(v) = a(u,v) - \ell(v)\ \forall v \in X$ $\square$

Elliptic PDE theory: weak solution to Poisson-problem

$\Omega \subset \mathbb{R}^d$ bounded, Lipschitz

Thm: $a(v,u) = \int_\Omega \nabla u \cdot \nabla v \, dx$ is a coercive, bounded bilinear form on $H_0^1(\Omega)$,
$\ell(v) = \int_\Omega f v \, dx$ with $f \in L^2(\Omega)$ is a bounded linear form on $H_0^1(\Omega)$.

Cor: There exists a unique weak solution u of $-\Delta u = f$ in Ω, $u=0$ on $\partial\Omega$.

Pf (Thm):
- $|\ell(v)| \le \int_\Omega |f v| \, dx \le \|f\|_{L^2} \|v\|_{L^2} \le \|f\|_{L^2} \sqrt{\|v\|_{L^2}^2 + \|\nabla v\|_{L^2}^2} = \|f\|_{L^2} \|v\|_{H^1}$
- $|a(u,v)| \le \sum_{i=1}^d \int_\Omega |\partial_{x_i} u \, \partial_{x_i} v| \, dx \le \sum_{i=1}^d \|\partial_{x_i} u\|_{L^2} \|\partial_{x_i} v\|_{L^2}$
  $\le \left(\|u\|_{L^2}, \|\partial_{x_1} u\|_{L^2}, \dots, \|\partial_{x_d} u\|_{L^2} \right)^T \cdot \left(\|v\|_{L^2}, \|\partial_{x_1} v\|_{L^2}, \dots, \|\partial_{x_d} v\|_{L^2} \right)^T$
  $\le \sqrt{\|u\|_{L^2}^2 + \|\partial_{x_1} u\|_{L^2}^2 + \dots + \|\partial_{x_d} u\|_{L^2}^2} \cdot \sqrt{\|v\|_{L^2}^2 + \|\partial_{x_1} v\|_{L^2}^2 + \dots + \|\partial_{x_d} v\|_{L^2}^2} = \|u\|_{H^1} \|v\|_{H^1}$
- $a(v,v) = \int_\Omega |\nabla v|^2 \, dx = \frac{1}{2} \|\nabla v\|_{L^2}^2 + \frac{1}{2} \|\nabla v\|_{L^2}^2 \ge \frac{1}{2} \|\nabla v\|_{L^2}^2 + C \|v\|_{L^2}^2 \ge \min\left(\frac{1}{2}, C\right) \|v\|_{H^1}^2$

  $= \|\partial_{x_1} v\|^2 + \dots + \|\partial_{x_d} v\|^2$ Poincaré-inequality!
- bilinearity & linearity obvious (check!) $\square$

Elliptic PDE theory: Poincaré inequality

Thm: (Poincaré-(Friedrichs-)inequality) Let $\Omega \subset \mathbb{R}^d$ be bounded, connected with Lipschitz body.

Let $\Gamma \subset \partial\Omega$ be simply connected with $|\Gamma| > 0$. There exists $C > 0$ such that

$$\|v\|_{L^2} \le C \|\nabla v\|_{L^2} \quad \text{for all} \quad v \in H^1_\Gamma = \{ v \in H^1(\Omega) \mid v = 0 \text{ on } \Gamma \}.$$

Pf · in 1D, wlog. $\Omega = [a,b]$, $v(a) = 0$:

$$\|v\|_{L^2}^2 = \int_a^b v(x)^2\,dx = \int_a^b \left(\int_a^x v'(y)\,dy \right)^2 dx \le \int_a^b \left(\int_a^b |v'(y)|\,dy \right)^2 dx = (b-a) \left(\int_a^b |v'(x)|\,dx \right)^2$$

$$\le (b-a) \left(\|1\|_{L^2} \|v'\|_{L^2} \right)^2 = (b-a)^2 \|v'\|_{L^2}^2$$

· in 2D with $\Omega = [a,b] \times [c,d]$, $v(a, x_2) = 0$:

$$\|v\|_{L^2}^2 = \int_c^d \int_a^b v^2\,dx = \int_c^d \|v(\cdot, x_2)\|_{L^2}^2\,dx_2 \le (b-a)^2 \int_c^d \|\partial_{x_1} v(\cdot, x_2)\|_{L^2}^2\,dx_2$$

$$\le (b-a)^2 \int_c^d \int_a^b (\partial_{x_1} v)^2 + (\partial_{x_2} v)^2\,dx = (b-a)^2 \|\nabla v\|_{L^2}^2$$

· in 2D, Ω simply connected: $\exists \phi : [a,b] \times [c,d] \to \Omega$ Lipschitz with ϕ^{-1} Lipschitz,

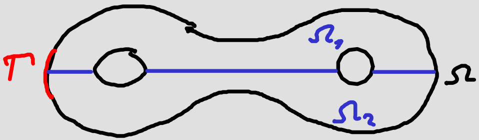
$\phi^{-1}(\Gamma) = \{a\} \times [c,d]$; let L be the Lipschitz constant of ϕ, ϕ^{-1}.

$$\|v\|_{L^2}^2 = \int_\Omega v^2\,dx = \int_a^b \int_c^d (v \circ \phi)^2 |\det D\phi|\,dx \le 2L^2 \|v \circ \phi\|_{L^2}^2 \le 2L^2 (b-a)^2 \|\nabla(v \circ \phi)\|_{L^2}^2 = C \|\nabla\phi\, (\nabla v) \circ \phi\|_{L^2}^2$$

$$\le C L^2 \|(\nabla v) \circ \phi\|_{L^2}^2 = C L^2 \int_a^b \int_c^d (\nabla v)^2 \circ \phi\,dx = C L^2 \int_\Omega \nabla v^2 \, |\det D\phi^{-1}|\,dx \le 2CL^4 \|\nabla v\|_{L^2}^2$$

Elliptic PDE theory: Poincaré inequality II

- analogous proof in d dimensions for simply connected domain Ω Lipschitz-boundary, $v=0$ on $\Gamma \subset \partial\Omega$ simply connected with $|\Gamma|>0$.
- For arbitrary connected Ω with Lipschitz boundary, $v=0$ on $\Gamma \subset \partial\Omega$ connected with $|\Gamma|>0$ decompose Ω into Lipschitz domains $\Omega_1,\dots,\Omega_N$ with $v=0$ on part of the boundary



and use $\|v\|_{L^2}^2 = \|v|_{\Omega_1}\|_{L^2}^2 + \dots + \|v|_{\Omega_N}\|_{L^2}^2$,

$\|\nabla v\|_{L^2}^2 \geq \|\nabla v|_{\Omega_1}\|_{L^2}^2 + \dots + \|\nabla v|_{\Omega_N}\|_{L^2}^2$ $\square$

Thm: The following, non-improvable inequalities hold for all $v \in H_0^1(\Omega)$.

1) $\|v\|_{L^2}^2 \leq \left(\frac{b-a}{\pi}\right)^2 \|\nabla v\|_{L^2}^2$ if $\Omega=[a,b]$

2) $\|v\|_{L^2}^2 \leq \frac{1}{2\pi^2} \|\nabla v\|_{L^2}^2$ if $\Omega=[0,1]^2$

Pf: 1) homework: with semidiscrete Fourier-transform

2) reduce to case 1) $\square$

Elliptic PDE theory: weak solution to linear elliptic bvp

Thm: Let Ω be Lipschitz, $\Gamma_1, \Gamma_2 = \partial\Omega \setminus \Gamma_1$ with $|\Gamma_1|, |\Gamma_2| > 0$, $\cdot\ A \in L^\infty(\Omega; \mathbb{R}^{d\times d})$ with $\xi^T A \xi \geq c|\xi|^2 \ \forall \xi \in \mathbb{R}^d$

$\cdot\ b \in L^\infty(\Omega; \mathbb{R}^d)$ with $|b(x)|^2 \leq 4\tilde{c}\, q(x) \quad \forall x \in \Omega$ — $\tilde{c} < c$

also other conditions possible, see homework

$\cdot\ q \in L^\infty(\Omega)$ with $q \geq 0$

$\cdot\ p \in L^\infty(\Gamma_1)$ with $p \geq 0$

$\cdot\ f \in L^2(\Omega)$, $g \in L^2(\Gamma_1)$

Then $a(u,v) = \int_\Omega \nabla v^T A \nabla u + v\, b\cdot\nabla u + q u v \, dx + \int_{\Gamma_1} p u v \, dx$ is a coercive, bounded bilinear form on $H^1_{\Gamma_2}(\Omega)$, and $l(v) = \int_\Omega f v \, dx + \int_{\Gamma_1} g v \, dx$ is bounded & linear.

Cor: $\exists!$ weak solution of

$$\begin{cases} -\operatorname{div}(A(x)\nabla u(x)) + b(x)\cdot\nabla u(x) + q(x)u(x) = f(x) & \text{in } \Omega \\ u(x) = 0 & \text{on } \Gamma_2 \\ (A(x)\nabla u(x))\cdot n + p(x)u(x) = g(x) & \text{on } \Gamma_1 \end{cases}$$

Pf (Thm): $\cdot\ |l(v)| \leq \int_\Omega |fv|\,dx + \int_{\Gamma_1} |gv|\,dx \leq \|f\|_{L^2}\|v\|_{L^2} + \|g\|_{L^2(\Gamma_1)}\|v\|_{L^2(\Gamma_1)} \underset{\text{trace theorem}}{\leq} \left(\|f\|_{L^2} + C\|g\|_{L^2(\Gamma_1)}\right)\|v\|_{H^1}$

$\cdot\ |a(u,v)| \leq \|A\|_{L^\infty}\|\nabla v\|_{L^2}\|\nabla u\|_{L^2} + \|b\|_{L^\infty}\|v\|_{L^2}\|\nabla u\|_{L^2} + \|q\|_{L^\infty}\|u\|_{L^2}\|v\|_{L^2} + \|p\|_{L^\infty}\|u\|_{L^2(\Gamma_1)}\|v\|_{L^2(\Gamma_1)}$

$\leq \text{const.} \cdot \|v\|_{H^1}\|u\|_{H^1}$

$\cdot\ a(v,v) \geq \int_\Omega c|\nabla v|^2 - |b||v\nabla v| + q v^2 \geq \int_\Omega (c-\tilde{c})|\nabla v|^2 + \tilde{c}\left(|\nabla v| - \frac{|b|}{2\tilde{c}}|v|\right)^2 dx \geq (c-\tilde{c})\|\nabla v\|^2_{L^2} \geq \text{const}\,\|v\|^2_{H^1}$ $\square$

Ritz-Galerkin-method: existence, uniqueness, minimization

Thm: Let $a: X \times X \to \mathbb{R}$ be a coercive, bounded bilinear form on the Hilbert space X, $\ell: X \to \mathbb{R}$ linear and bounded. Let $X_h \subset X$ be a closed subspace. Then there exists a unique solution $u_h \in X_h$ of $a(u_h, v_h) = \ell(v_h)\ \forall v_h \in X_h$.

Def: u_h is called Ritz-Galerkin-approximation of the solution $u \in X$ to $a(u,v) = \ell(v)\, \forall v \in X$.

Pf: a is coercive & bounded bilinear form on X_h, ℓ is bounded linear form on X_h (check!)

$\Rightarrow$ Lax-Milgram yields unique u_h. $\square$

Thm: If additionally a is symmetric, then $u_h = \underset{u \in X_h}{\arg\min}\ \frac{1}{2} a(u,u) - \ell(u)$.

Pf: analogous to proof in X. $\square$

The Ritz-Galerkin-approximation u_h thus possesses the same properties as u!

If $\dim X_h < \infty$, then there is a basis $\phi_1, \dots, \phi_N \in X_h$. Each $u_h \in X_h$ can be written as $u_h = \sum_{i=1}^{N} U_i \phi_i$.

$$\Rightarrow \quad a(u_h, v_h) = (U_1, \dots, U_N)\, A \begin{pmatrix} V_1 \\ \vdots \\ V_N \end{pmatrix}, \qquad \ell(v_h) = (B_1, \dots, B_N) \begin{pmatrix} V_1 \\ \vdots \\ V_N \end{pmatrix}$$

with $A_{ij} = a(\phi_i, \phi_j)$, $B_i = \ell(\phi_i)$

$\Rightarrow$ Ritz-Galerkin-approximation solves $A^T \begin{pmatrix} U_1 \\ \vdots \\ U_N \end{pmatrix} = \begin{pmatrix} B_1 \\ \vdots \\ B_N \end{pmatrix}$

Ritz-Galerkin-method: basics of error estimates

$a: X \times X \to \mathbb{R}$ with $a(v,v) \geq c\|v\|_X^2$ & $|a(u,v)| \leq C\|u\|_X\|v\|_X$, $\quad \ell: X \to \mathbb{R}$ with $|\ell(v)| \leq \tilde{C}\|v\|_X$

$X_h \subset X$, $\quad a(u,v) = \ell(v) \;\forall v \in X$, $\quad a(u_h, v_h) = \ell(v_h) \quad \forall v_h \in X_h$

<u>Thm</u>: (Galerkin-orthogonality) $\quad a(u_h - u, v_h) = 0 \quad \forall v_h \in X_h$

Pf: Subtract $a(u, v_h) = \ell(v_h) \;\forall v_h \in X_h$ from $a(u_h, v_h) = \ell(v_h) \;\forall v_h \in X_h$ $\square$

<u>Def</u>: If a is symmetric, then $\langle u, v\rangle_a = a(u,v)$ is an inner product on X, the <u>energy inner product</u>. $\quad \|u\|_a = \sqrt{\langle u,u\rangle_a}$ is called <u>energy norm</u>.

Galerkin-orthogonality says that the approximation error $u_h - u$ is orthogonal to X_h wrt. $\langle\cdot,\cdot\rangle_a$.

<u>Cor</u>: $\cdot$ (Céa-lemma) $\quad \|u_h - u\|_X \leq \frac{C}{c} \min_{v_h \in X_h} \|v_h - u\|_X \quad$ „almost-best approximation"

$\cdot$ (best approximation) If a is symmetric, then $\quad \|u_h - u\|_a \leq \min_{v_h \in X_h} \|v_h - u\|_a$

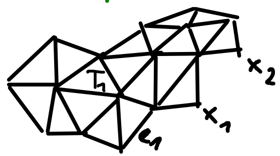
Pf: $\cdot$ $c\|u_h - u\|_X^2 \leq a(u_h - u, u_h - u) \underset{\text{Galerkin-orth.}}{=} a(u_h - u, v_h - u) \leq C\|u_h - u\|_X \|v_h - u\|_X \quad \forall v_h \in X_h$

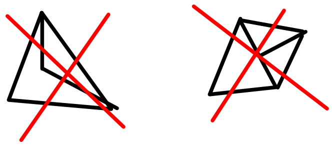
$\cdot$ a symmetric $\Rightarrow$ $\|u_h - u\|_a^2 = a(u_h - u, u_h - u) = a(u_h - u, v_h - u) \underset{\text{Cauchy-Schwarz}}{\leq} \|u_h - u\|_a \|v_h - u\|_a \quad \forall v_h \in X_h$ $\square$

u_h is (almost) the best possible approximation to u. It only remains to estimate $\min_{v_h \in X_h} \|v_h - u\|_{X_h}$.

Linear FE: ansatz

Def: · Let $\Omega \subset \mathbb{R}^2$. A <u>triangulation</u> of Ω is a triple $(\mathcal{V}, \mathcal{E}, \mathcal{T})$ with <u>vertex set</u> $\mathcal{V} = (x_1, \dots, x_N) \in (\mathbb{R}^2)^N$, <u>edge set</u> $\mathcal{E} = (e_1, \dots, e_L)$, <u>triangle set</u> $\mathcal{T} = (T_1, \dots, T_M)$, where every edge / every triangle is the convex hull of two/three vertices, every intersection of two triangles is either a common side/corner or empty, $\bar{\Omega} = \bigcup_{i=1}^{M} T_i$, each edge is a triangle side, and $\mathring{T}_i \neq \emptyset \ \forall i$.



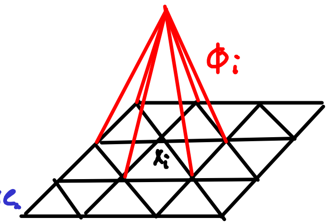


· $h_T = \text{diam } T$, $\quad h = \max_{T \in \mathcal{T}} h_T$ = grid width

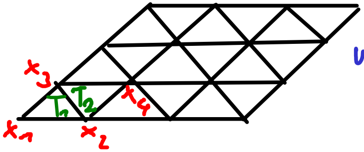
If one chooses $X_h \subset H^1(\Omega)$ as the set of piecewise affine functions, $X_h = \{ v : C(\Omega) \mid v|_T \in \mathcal{P}_1 \ \forall T \in \mathcal{T} \}$ ($\mathcal{P}_n = \{$polynomials of degree $\leq n\}$), a basis is given by the "FE hat functions" $\phi_1, \dots, \phi_N \in X_h$, which are defined by $\phi_i(x_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{else} \end{cases}$.

The elements of the chosen basis are also called <u>FE form functions</u>.

Since $\text{supp}\, \phi_i \cap \text{supp}\, \phi_j = \emptyset$ for most FE form functions, $A_{ij} = a(\phi_i, \phi_j)$ is sparse.



Linear FE: matrix assembly



vertex array $= \begin{bmatrix} -x_1- \\ -x_2- \\ \vdots \\ -x_N- \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0.5 & 0.5 \\ 1.5 & 0.5 \\ & \vdots \end{bmatrix}$

connectivity array $= \begin{array}{c} \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 4 \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ 7 & 8 & 9 \\ \cdots & \cdots & \cdots \end{bmatrix} \end{array} \begin{array}{l} \leftarrow T_1 \\ \\ \\ \\ \leftarrow T_i \\ \\ \end{array}$

The computation of matrix $A = (a(\phi_i, \phi_j))_{ij}$ is called <u>assembly</u>.

Since a is of the form $\quad a(u,v) = \int_\Omega Q(u,v)\,dx = \sum_{\ell=1}^{M} \int_{T_\ell} Q(u,v)\,dx$ for a bilinear operator Q, one iterates over the triangles.

here without $\partial\Omega$-contribution; this works analogously...

Let $T_\ell = co(x_{i_1}, x_{i_2}, x_{i_3})$, then only $\phi_{i_1}, \phi_{i_2}, \phi_{i_3}$ are nonzero on T_ℓ.

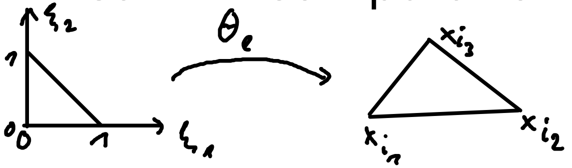
$\Rightarrow \int_{T_\ell} Q(\phi_m, \phi_n)\,dx = 0$ for $m \notin \{i_1, i_2, i_3\}$ or $n \notin \{i_1, i_2, i_3\}$.

$\Rightarrow$ one computes the local system matrix $A^\ell = \left(\int_{T_\ell} Q(\phi_{i_m}, \phi_{i_n})\,dx \right)_{m,n=1,2,3}$ and adds it into A at position (i_m, i_n), $m,n = 1,2,3$, i.e. one computes

$$A = \sum_{\ell=1}^{M} L_\ell^T A^\ell L_\ell \quad \text{for } L_\ell = \begin{array}{c} \begin{pmatrix} 0 & \cdots & 1 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & \cdots & 1 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 1 & 0 & \cdots & 0 \end{pmatrix} \\ \begin{array}{ccccccccccc} & & \uparrow & & & \uparrow & & \uparrow & & & \\ & & i_1 & & & i_2 & & i_3 & & & \end{array} \end{array}$$

<u>Ex</u>: $\quad A = \begin{bmatrix} \cdots & \cdots & \cdots & \cdots & \cdots \\ & \cdots + x & \cdots + x & \cdots + x & \\ & \cdots + x & \cdots + x & \cdots + x & \\ & \cdots + x & \cdots + x & \cdots + x & \\ & \cdots & \cdots & \cdots & \cdots \end{bmatrix} \qquad A^2 = \begin{pmatrix} x & x & x \\ x & x & x \\ x & x & x \end{pmatrix}$

Linear FE: computation of local system matrix



reference triangle T triangle T_e

$\Theta_e(\xi) = x_{i_1} + \overbrace{\left(x_{i_2}-x_{i_1} \,\middle|\, x_{i_3}-x_{i_1}\right)}^{D\Theta_e} \begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix}$ maps T onto T_e.

triangle area $= |T_e| = |(x_{i_2}-x_{i_1}) \times (x_{i_3}-x_{i_1})|/2 = \det D\Theta_e/2$

Set $\psi_1(\xi) = 1-\xi_1-\xi_2$, $\psi_2(\xi) = \xi_1$, $\psi_3(\xi) = \xi_2 \Rightarrow$ on T_e one has $\phi_{i_m}\circ\Theta_e = \psi_m$, $\nabla\phi_{i_m}\circ\Theta_e = D\Theta_e^{-T}\nabla\psi_m$

Now A^e can be computed using the ψ_i and Θ_e!

Ex: • $a(u,v) = \int_\Omega u v \, dx \Rightarrow A^e_{mn} = \int_{T_e} \phi_{i_m}\phi_{i_n}\, dx = \int_T \psi_m \psi_n \, |\det D\Theta_e| \, d\xi = 2|T_e| \int_T \psi_m\psi_n \, d\xi$

$$\Rightarrow A^e = \frac{|T_e|}{12}\begin{pmatrix}2 & 1 & 1\\ 1 & 2 & 1\\ 1 & 1 & 2\end{pmatrix}$$

• $a(u,v) = \int_\Omega \nabla u \cdot \nabla v \, dx \Rightarrow A^e_{mn} = \int_{T_e} \nabla\phi_{i_m}\cdot\nabla\phi_{i_n}\, dx = \int_T \nabla\psi_m^T D\Theta_e^{-1} D\Theta_e^{-T} \nabla\psi_n \, |\det D\Theta_e| \, d\xi$

$$= |T_e| \, \nabla\psi_m^T D\Theta_e^{-1} D\Theta_e^{-T} \nabla\psi_n$$

$$D\Theta_e^{-1} = \frac{1}{2|T_e|}\begin{pmatrix}(x_{i_3}-x_{i_1})_2 & -(x_{i_3}-x_{i_1})_1\\ -(x_{i_2}-x_{i_1})_2 & (x_{i_2}-x_{i_1})_1\end{pmatrix} \Rightarrow D\Theta_e^{-1} D\Theta_e^{-T} = \frac{1}{4|T_e|^2}\underbrace{\begin{pmatrix}|x_{i_3}-x_{i_1}|^2 & -(x_{i_3}-x_{i_1})\cdot(x_{i_2}-x_{i_1})\\ -(x_{i_3}-x_{i_1})\cdot(x_{i_2}-x_{i_1}) & |x_{i_2}-x_{i_1}|^2\end{pmatrix}}_{R}$$

$$\Rightarrow A^e = \frac{1}{4|T_e|}\begin{pmatrix}-1 & -1\\ 1 & 0\\ 0 & 1\end{pmatrix} R \begin{pmatrix}-1 & 1 & 0\\ -1 & 0 & 1\end{pmatrix} = \frac{1}{4|T_e|}\begin{pmatrix}\text{---}\; x_{i_2}-x_{i_3} \;\text{---}\\ \text{---}\; x_{i_3}-x_{i_1} \;\text{---}\\ \text{---}\; x_{i_1}-x_{i_2} \;\text{---}\end{pmatrix}\left(x_{i_2}-x_{i_3} \,\middle|\, x_{i_3}-x_{i_1} \,\middle|\, x_{i_1}-x_{i_2}\right)$$

Linear FE: mass & stiffness matrix for regular grid

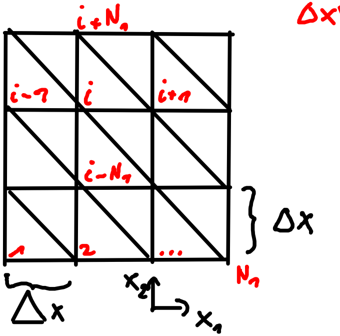
<u>Def</u>: Let $\phi_1, \dots, \phi_N$ be a basis of the FE-space X_h.

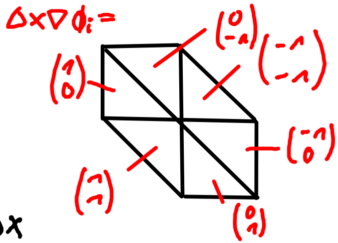
<u>mass matrix</u> $M_h = \left(\int_\Omega \phi_i \phi_j \, dx \right)_{ij}$, <u>stiffness matrix</u> $L_h = \left(\int_\Omega \nabla\phi_i \cdot \nabla\phi_j \, dx \right)_{ij}$

Assembly of $B_i = \ell(\phi_i)$ also works via iteration over all triangles (homework);

if $\ell(v) = \int_\Omega f v \, dx$ for $f \in X_h$ one can also calculate $B = M_h \begin{pmatrix} f(x_1) \\ \vdots \\ f(x_N) \end{pmatrix}$.

On a regular grid we can often calculate the system matrix by hand:





<u>Ex</u>: $\int_\Omega |\nabla\phi_i|^2 dx = \frac{1}{2}\left[\begin{pmatrix} 0 \\ -1 \end{pmatrix}^2 + \begin{pmatrix} 1 \\ 0 \end{pmatrix}^2 + \begin{pmatrix} 1 \\ 1 \end{pmatrix}^2 + \begin{pmatrix} 0 \\ 1 \end{pmatrix}^2 + \begin{pmatrix} -1 \\ 0 \end{pmatrix}^2 + \begin{pmatrix} -1 \\ -1 \end{pmatrix}^2 \right] = 4$

$\int_\Omega \nabla\phi_i \cdot \nabla\phi_{i+1} dx = \frac{1}{2}\left[\begin{pmatrix} -1 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} -1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right] = -1$

$\int_\Omega \nabla\phi_i \cdot \nabla\phi_{i-1} dx = \int_\Omega \nabla\phi_i \cdot \nabla\phi_{i+N_1} dx = \int_\Omega \nabla\phi_i \cdot \nabla\phi_{i-N_1} dx = -1$

$\Rightarrow L_h = \begin{pmatrix} \dots -1 \dots -1 \; \overset{\ddots}{4} \; -1 \dots -1 \dots \end{pmatrix}$

$\Rightarrow L_h \begin{pmatrix} u_1 \\ \vdots \\ u_N \end{pmatrix} = -u_{i-N_1} - u_{i-1} + 4u_i - u_{i+1} - u_{i+N_1}$

finite differences for $-\Delta$: $\dfrac{-u_{i-N_1} - u_{i-1} + 4u_i - u_{i+1} - u_{i+N_1}}{\Delta x^2}$

<u>Ex</u>: $\int_\Omega \phi_i^2 dx = \frac{\Delta x^2}{2}$, $\int_\Omega \phi_i \phi_{i\pm1} dx = \int_\Omega \phi_i \phi_{i\pm N_1} dx = \int_\Omega \phi_i \phi_{i\pm(N_1-1)} dx = \frac{\Delta x^2}{12}$ (homework) $\Rightarrow M_h = \frac{\Delta x^2}{12} \begin{pmatrix} \dots 1 \; 1 \dots 1 \; \overset{\ddots}{6} \; 1 \dots 1 \; 1 \dots \end{pmatrix}$

Linear FE: interpolation error (on reference triangle)

Def: $I_h u = \sum_{i=1}^{N} u(x_i)\,\phi_i \in X_h$ is called FE-interpolant of u

Thm: There exists $C>0$ such that for all $u \in H^2(T)$ we have $\|u - I_h u\|_{L^2} \le C |u|_{H^2} = C \|D^2 u\|_{L^2}$,

$\|\nabla u - \nabla I_h u\|_{L^2} \le C |u|_{H^2}$.

reference triangle

$\|A\|_{L^2}^2 = \int \sum_{i,j} |A_{ij}|^2 dx$

Pf: Let $v \in H^1([0,1]^2)$ with $\int_0^1 v(s,0)\,ds = 0$.

$$\|v\|_{L^2}^2 = \int_0^1\int_0^1 \Big(v(x_1,x_2) - \int_0^1 v(s,0)\,ds\Big)^2 dx_1\,dx_2 = \int_0^1\int_0^1 \Big|\int_0^1 v(x_1,x_2) - v(s,x_2)\,ds + \int_0^1 v(s,x_2) - v(s,0)\,ds\Big|^2 dx_1\,dx_2$$

$$= \int_0^1\int_0^1 \Big|\int_0^1\int_s^{x_1} \frac{\partial v}{\partial x_1}(t,x_2)\,dt\,ds + \int_0^1\int_0^{x_2} \frac{\partial v}{\partial x_2}(s,t)\,dt\,ds\Big|^2 dx_1\,dx_2$$

$$\le \int_0^1\int_0^1 \Big(\int_0^1\int_0^1 \Big|\frac{\partial v}{\partial x_1}(t,x_2)\Big|\,dt\,ds + \int_0^1\int_0^1 \Big|\frac{\partial v}{\partial x_2}(s,t)\Big|\,dt\,ds\Big)^2 dx_1\,dx_2$$

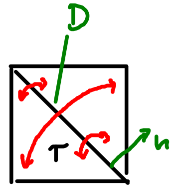
$$\le 2\int_0^1\int_0^1 \Big(\int_0^1\int_0^1 \Big|\frac{\partial v}{\partial x_1}(t,x_2)\Big|\,dt\,ds\Big)^2 + \Big(\int_0^1\int_0^1 \Big|\frac{\partial v}{\partial x_2}(s,t)\Big|\,dt\,ds\Big)^2 dx_1\,dx_2$$

$$\le 2\int_0^1\int_0^1\int_0^1\int_0^1 \Big|\frac{\partial v}{\partial x_1}(t,x_2)\Big|^2 dt\,ds\,dx_1\,dx_2 + 2\int_0^1\int_0^1\int_0^1\int_0^1 \Big|\frac{\partial v}{\partial x_2}(s,t)\Big|^2 ds\,dt\,dx_1\,dx_2$$

$$= 2\Big\|\frac{\partial v}{\partial x_1}\Big\|_{L^2}^2 + 2\Big\|\frac{\partial v}{\partial x_2}\Big\|_{L^2}^2 = 2\|\nabla v\|_{L^2}^2$$

Analogous for v with $\int_0^1 v(0,s)\,ds = 0$.

Linear FE: interpolation error - cont.



- Let $v \in H^1(T)$, $\hat{v} : [0,1]^2 \to \mathbb{R}$, $\hat{v} = v$ on T, $\hat{v}(x_1, x_2) = v(1-x_2, 1-x_1)$ on $[0,1]^2 \setminus T$

  then $\hat{v} \in H^1([0,1]^2)$ with $\|\hat{v}\|_{L^2}^2 = 2\|v\|_{L^2}^2$ (clear), $\|\nabla \hat{v}\|_{L^2}^2 = 2\|\nabla v\|_{L^2}^2$ (since $\nabla \hat{v} = \nabla v$ on T and $\nabla \hat{v}(x_1,x_2) = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \nabla v(1-x_2, 1-x_1)$ on $[0,1]^2 \setminus T$: Let $\varphi \in C_0^\infty([0,1]^2; \mathbb{R}^2)$, then

  $$\int_0^1\int_0^1 \hat{v}\, \mathrm{div}\,\varphi \, dx = \int_T v \,\mathrm{div}\,\varphi\, dx + \int_{T^c} \hat{v}\, \mathrm{div}\,\varphi\, dx = \int_D v \varphi \cdot n \, dx - \int_T \nabla v \cdot \varphi \, dx + \int_D \hat{v} \varphi \cdot (-n)\, dx - \int_{T^c} \nabla \hat{v} \cdot \varphi \, dx = -\int_0^1\int_0^1 \nabla \hat{v} \cdot \varphi \, dx)$$

- Let $v \in H^1(T)$ with $\int_0^1 v(s,0)\, ds = 0$ or $\int_0^1 v(0,s)\, ds = 0$, then $\|v\|_{L^2}^2 = \frac{1}{2}\|\hat{v}\|_{L^2}^2 \le \|\nabla \hat{v}\|_{L^2}^2 = 2\|\nabla v\|_{L^2}^2$

  $\Longrightarrow$ with $v = \nabla u - \nabla I_h u$ we have $\|\nabla u - \nabla I_h u\|_{L^2}^2 \le 2\|D^2 u - D^2 I_h u\|_{L^2}^2 = 2\|D^2 u\|_{L^2}^2$

- Let $v \in H^2(T)$ with $v(0,0) = v(1,0) = 0$.

  $$\|v\|_{L^2}^2 = \int_0^1 \|v(x_1,\cdot)\|_{L^2}^2 \, dx_1 \le 2\int_0^1 v(x_1,0)^2 + \|v(x_1,\cdot) - v(x_1,0)\|_{L^2}^2 \, dx_1 \overset{\text{Poincaré}}{\le} 2\|v(\cdot,0)\|_{L^2}^2 + 2\int_0^1 \left\|\frac{\partial v}{\partial x_2}(x_1,\cdot)\right\|_{L^2}^2 dx_1$$

  $$\underset{\text{Poincaré}}{\le} 2\left\|\frac{\partial v}{\partial x_1}(\cdot,0)\right\|_{L^2}^2 + 2\left\|\frac{\partial v}{\partial x_2}\right\|_{L^2}^2 \underset{\text{trace theorem}}{\le} 2C\left\|\frac{\partial v}{\partial x_1}\right\|_{H^1}^2 + 2\|\nabla v\|_{L^2}^2 \le 2(C+1)\|\nabla v\|_{L^2}^2 + 2C\|D^2 v\|_{L^2}^2$$

  $$\underset{\text{from before}}{\le} (6C+4)\|D^2 v\|_{L^2}^2$$

  $\Longrightarrow$ with $v = u - I_h u$ we have $\|u - I_h u\|_{L^2}^2 \le (6C+4)\|D^2 u - D^2 I_h u\|_{L^2}^2 = (6C+4)\|D^2 u\|_{L^2}^2$ $\square$

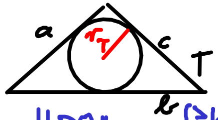
Linear FE: quasiuniform triangulation

Def: A triangulation $(\mathcal{V}, \mathcal{E}, \mathcal{T})$ is called $\frac{1}{\rho}$-quasiuniform, if for all $T_\ell \in \mathcal{T}$ the affine transformation $\theta_\ell : T \to T_\ell$ of the reference triangle T onto T_ℓ satisfies $\operatorname{cond} D\theta_\ell \le \rho$.

condition number e.g. w.r.t. 2-norm

Here's a geometric interpretation:

Thm: $r_T \ge 2h_T/\rho \;\; \forall T \in \mathcal{T} \overset{1)}{\Rightarrow} \frac{1}{\rho}\text{-quasiuniform} \overset{2)}{\Rightarrow} r_T \ge h_T/8\rho \;\; \forall T \in \mathcal{T}$



Pf: Let θ be the reference transformation. Incircle $r_T = \frac{2|T|}{a+b+c}$, $2|T| = |\det D\theta| = \sigma_1\sigma_2$, $\|D\theta\|_2 = \sigma_1 \begin{cases} \ge h_T/2 \\ \le 2h_T \end{cases}$,

$\sigma_1 \le a+b+c \le \sigma_1 + \sigma_1 + \sqrt{2}\,\sigma_1 \le 4\sigma_1$ $\quad$ 1) $\frac{\|D\theta\|_2}{\rho} \le \frac{2h_T}{\rho} \le r_T \le \sigma_2 = \|D\theta^{-1}\|_2^{-1}$ $\quad$ 2) $r_T \ge \frac{\sigma_2}{4} \ge \frac{\sigma_1}{4\rho} \ge \frac{h_T}{8\rho}$ $\square$

Thm: Let T_ℓ be a triangle of a $\frac{1}{\rho}$-quasiuniform triangulation, $e \subset T_\ell$ an edge. Then exists $C \equiv C(\rho)$ such that

- trace theorem: $\quad \|v\|_{L^2(e)} \le C\left(h_{T_\ell}^{-\frac{1}{2}} \|v\|_{L^2(T_\ell)} + h_{T_\ell}^{\frac{1}{2}} \|\nabla v\|_{L^2(T_\ell)}\right) \quad \forall v \in H^1(T_\ell)$
- Poincaré-inequality: $\quad \|v\|_{L^2(T_\ell)} \le C h_{T_\ell} \|\nabla v\|_{L^2(T_\ell)} \quad \forall v \in H^1_0(T_\ell)$
- Poincaré-inequality II: $\|v - \bar{v}\|_{L^2(T_\ell)} \le C h_{T_\ell} \|\nabla v\|_{L^2(T_\ell)}$, $\bar{v} = \frac{1}{|T_\ell|}\int_{T_\ell} v\,dx \quad \forall v \in H^1(T_\ell)$
- interp. error: $\quad \|v - I_h v\|_{L^2(T_\ell)} \le C h_{T_\ell}^2 |v|_{H^2(T_\ell)}$, $\|\nabla v - \nabla I_h v\|_{L^2(T_\ell)} \le C h_{T_\ell} |v|_{H^2(T_\ell)} \quad \forall v \in H^2(T_\ell)$

Pf: homework (show & use $h_{T_\ell}/\sqrt{2} \le \|D\theta_\ell\|_2 \le \sqrt{2}\,h_{T_\ell}$, $\|D\theta_\ell^{-1}\|_2 \le \rho/\|D\theta_\ell\|_2 \le \sqrt{2}\,\rho/h_{T_\ell}$, $|T_\ell| = \frac{|\det D\theta_\ell|}{2} = \frac{\|D\theta_\ell\|_2}{2\|D\theta_\ell^{-1}\|_2}$)

Linear FE: a priori error estimates

$a : X \times X \to \mathbb{R}$ with $a(v,v) \geq c\|v\|_X^2$ & $|a(u,v)| \leq C\|u\|_X\|v\|_X$, $\quad \ell : X \to \mathbb{R}$ with $|\ell(v)| \leq \tilde{C}\|v\|_X$

$X_h \subset X = H^1_\Gamma(\Omega)$, $\quad a(u,v) = \ell(v) \;\forall v \in X$, $\quad a(u_h, v_h) = \ell(v_h) \quad \forall v_h \in X_h$

We already know $\|u - u_h\|_{H^1}^2 \leq C' \min_{v_h \in X_h} \|u - v_h\|_{H^1}^2$ — how big is this?

<u>Thm:</u> On a $\hat{\rho}$-quasiuniform triangulation of grid with h there exists $\hat{C} > 0$, depending only on $\hat{\rho}$, with $\quad \|u - I_h u\|_{L^2} \leq \hat{C} h^2 |u|_{H^2}$, $\quad \|\nabla u - \nabla I_h u\|_{L^2} \leq \hat{C} h |u|_{H^2}$.

Pf: $\cdot\ \|u - I_h u\|_{L^2}^2 = \sum_{\ell=1}^{M} \|u - I_h u\|_{L^2(T_\ell)}^2 \leq \sum_{\ell=1}^{M} \hat{C} h^4 |u|_{H^2(T_\ell)}^2 = \hat{C} h^4 \, |u|_{H^2}^2$

$\cdot$ analogous for $\|\nabla u - \nabla I_h u\|_{L^2}^2$ $\quad\square$

<u>Cor:</u> $\|u - u_h\|_{H^1}^2 \leq C\|u - I_h u\|_{H^1}^2 \leq C\hat{C}^2 h^2 (1 + h^2) |u|_{H^2}^2 \leq \tilde{C} h^2 |u|_{H^2}^2$

Linear FE: Aubin-Nitsche duality trick

$a: X \times X \to \mathbb{R}$ coercive, bounded, bilinear, $\ell: X \to \mathbb{R}$ bounded, linear, $X_h \subset X$ subspace (red: $X = H^1_\Gamma$)

$u \in X$: $a(u,v) = \ell(v)$ $\forall v \in X$, $u_h \in X_h$: $a(u_h, v_h) = \ell(v_h)$ $\forall v_h \in X_h$

We know $\|u_h - u\|_{H^1} \leq Ch|u|_{H^2} \Longrightarrow \|u_h - u\|_{L^2} \leq Ch|u|_{H^2}$; however, one often actually has $\|u_h - u\|_{L^2} \leq Ch^2|u|_{H^2}$

<u>Thm</u>: (Aubin-Nitsche-lemma) Let H (red: L^2) be a Hilbert space with continuous embedding $X \hookrightarrow H$, $w \in X$ with $a(v,w) = (u_h - u, v)_H$ $\forall v \in X$. There exists $C > 0$ depending on a with

adjoint bilinear form

$$\|u_h - u\|_H^2 \leq C\|u_h - u\|_X \inf_{w_h \in X_h} \|w - w_h\|_X$$

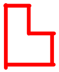
Pf: $\|u_h - u\|_H^2 = (u_h - u, u_h - u)_H = a(u_h - u, w) = a(u_h - u, w - w_h) \leq C\|u_h - u\|_X \|w - w_h\|_X$ $\forall w_h \in X_h$ $\square$

is this well-defined?

<u>Cor</u>: Let $X = H^1_\Gamma(\Omega)$. Assume for a fact the solution w_r of $a(v, w_r) = (r, v)_{L^2}$ $\forall v \in H^1_\Gamma$ satisfies $|w_r|_{H^2} \leq c\|r\|_{L^2}$ $\forall r \in L^2(\Omega)$. Then $\|u_h - u\|_{L^2} \leq \tilde{C}h^2|u|_{H^2}$

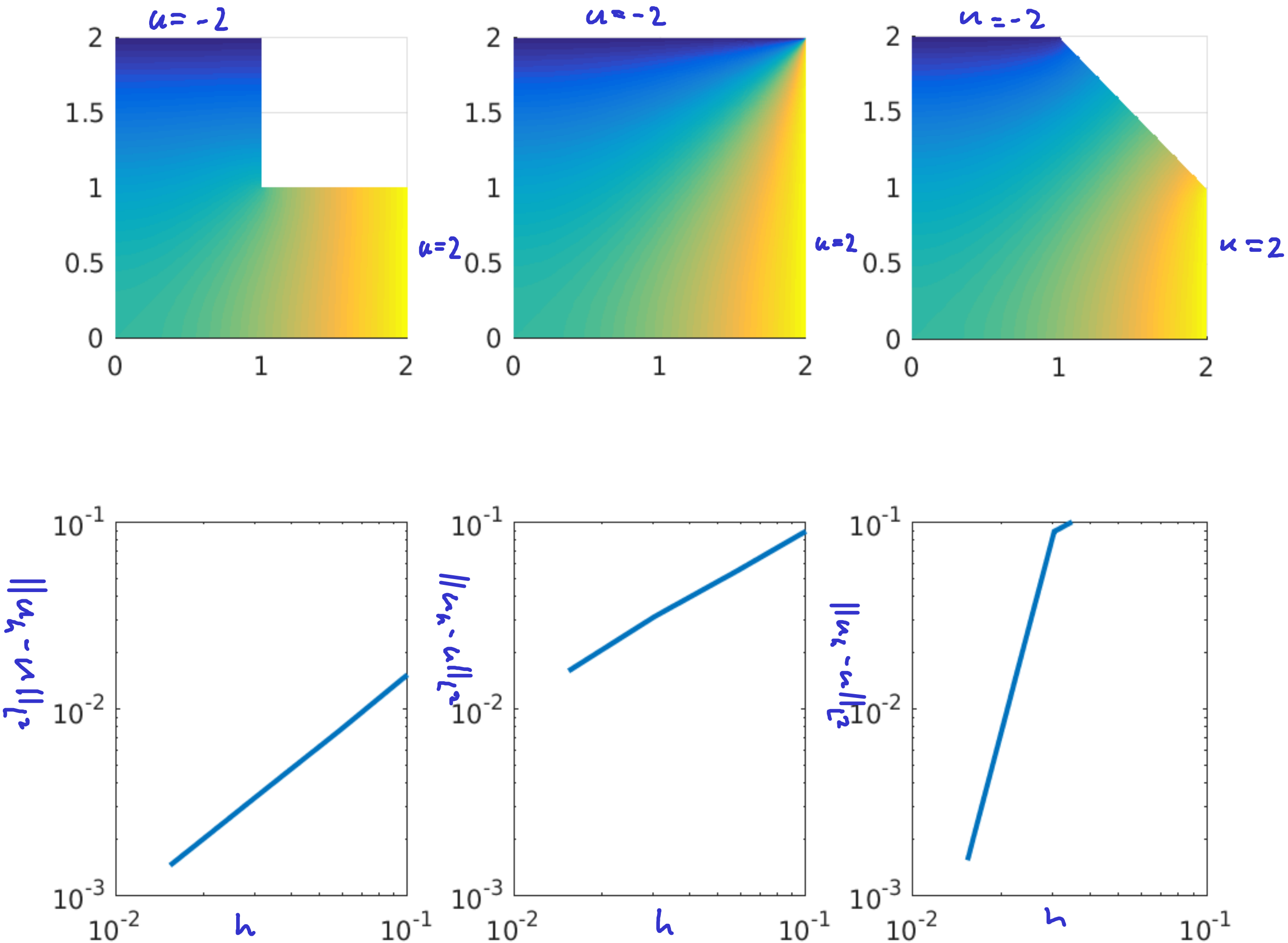
Pf: Take $H = L^2(\Omega)$, then $\|u_h - u\|_{L^2}^2 \leq C\|u_h - u\|_{H^1} \inf_{w_h \in X_h} \|w - w_h\|_{H^1} \leq \hat{C}h|u|_{H^2}\, h|w|_{H^2} \leq \hat{C}ch^2|u|_{H^2}\|u_h - u\|_{L^2}$ $\square$

Note: The estimates require that $u \in H^2$ & $w_r \in H^2$ with $|w_r|_{H^2} \leq C\|r\|_{L^2}$.

There are situations in which $u, w_r \notin H^2$, e.g. for $\Omega =$  $\Rightarrow \|u_h - u\| = O(h^\alpha)$ or smaller

Linear FE: example simulation

$-\Delta u = 0$ on Ω



Linear FE: regularity (Friedrichs' theorem)

We showed $\boxed{\|u_h - u\|_{H^1} \le Ch|u|_{H^2} \quad \& \quad \|u_h - u\|_{L^2} \le Ch^2|u|_{H^2}}$ if $|w_r|_{H^2} \le C\|r\|_{L^2}$

Do $u, w_r \in H^2$ and $|w_r|_{H^2} \le C\|r\|_{L^2}$ hold, and how big is $|u|_{H^2}$?

<u>Ex</u>: $\Omega = [0,1]^2$, $a(u,v) = \int_\Omega \nabla u \cdot \nabla v + uv\,dx$, $\ell(v) = \int_\Omega fv\,dx$,

$$\begin{cases} u \in X = H_0^1(\Omega), & a(u,v) = \ell(v) \ \forall v \in X \\ w_r \in X, & a(v,w_r) = (r,v)_{L^2} \ \forall v \in X \end{cases} \Rightarrow \begin{cases} |u|_{H^2} \le C\|f\|_{L^2} \\ |w_r|_{H^2} \le C\|r\|_{L^2} \end{cases}$$

Pf: (proof for u, analogous for w_r) $\quad \|u\|_{L^2}^2 \le \|u\|_{H^1}^2 \le a(u,u) = \ell(u) \le \|f\|_{L^2}\|u\|_{L^2}$

$$|u|_{H^2}^2 = \int_\Omega \left|\frac{\partial^2 u}{\partial x_1^2}\right|^2 + 2\left|\frac{\partial^2 u}{\partial x_1 \partial x_2}\right|^2 + \left|\frac{\partial^2 u}{\partial x_2^2}\right|^2 dx = \int_\Omega \left|\frac{\partial^2 u}{\partial x_1^2}\right|^2 - 2\frac{\partial}{\partial x_2}\frac{\partial^2 u}{\partial x_1 \partial x_2}\frac{\partial u}{\partial x_1} + \left|\frac{\partial^2 u}{\partial x_2^2}\right|^2 dx = \int_\Omega \left|\frac{\partial^2 u}{\partial x_1^2}\right|^2 + 2\frac{\partial^2 u}{\partial x_1^2}\frac{\partial^2 u}{\partial x_2^2} + \left|\frac{\partial^2 u}{\partial x_2^2}\right|^2 dx$$

$$= \|\Delta u\|_{L^2}^2 = \|f - u\|_{L^2}^2 \le 2\|f\|_{L^2}^2 + 2\|u\|_{L^2}^2 \le 2(1+1)\|f\|_{L^2}^2 \qquad \square$$

<u>Thm</u>: (e.g. Gilbarg & Trudinger Thm. 8.12) Let Ω bounded, $\partial\Omega$ C^2-regular,

$a(u,v) = \int_\Omega \nabla v^T A \nabla u + u\, b \cdot \nabla v - c \cdot \nabla u\, v - d\,uv\,dx$, $\ell(v) = \int_\Omega fv\,dx$,

A, b Lipschitz continuous, $c, d \in L^\infty(\Omega)$, $f \in L^2(\Omega)$, a coercive, $h \in H^2(\Omega)$.

The weak solution $u \in H^1(\Omega)$ of $a(u,v) = \ell(v) \ \forall v \in H^1(\Omega)$, $u = h$ on $\partial\Omega$,

satisfies $\|u\|_{H^2} \le C(\|u\|_{L^2} + \|f\|_{L^2} + \|h\|_{H^2})$ (C depending on Ω, A, b, c, d)

Similar statements hold for piecewise smooth boundary such that no two neighbouring pieces have Neumann boundary con
(see e.g. Bacuta, Mazzucato, Nistor, Zikatanov: Interface and mixed boundary value problems on n-dimensional polyhedral

FEM: general FE spaces

Def: A <u>finite element</u> is a triple $(K, \mathcal{P}, \mathcal{N}) =$ (<u>element domain</u>, <u>space of form functions</u>, <u>set of nodal variables</u>) with

- $K \subset \mathbb{R}^n$ open, bounded, simply connected with piecewise smooth boundary
- $\mathcal{P}$ a k-dimensional space of functions $f: K \to \mathbb{R}$
- $\mathcal{N} = \{N_1, \ldots, N_k\}$ a set of linearly independent linear functionals on $\mathcal{P}$

Def: Let $(K, \mathcal{P}, \mathcal{N})$ be a finite element, and let $\{\psi_1, \ldots, \psi_k\}$ be a basis of $\mathcal{P}$ dual to $\mathcal{N}$, i.e. $N_i(\psi_j) = \delta_{ij}$. Such a basis is called <u>nodal basis</u> for $\mathcal{P}$.

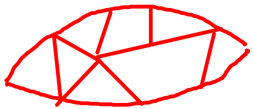
Thm: $\psi, \tilde{\psi} \in \mathcal{P}$ with $N(\psi) = N(\tilde{\psi})\ \forall N \in \mathcal{N} \Rightarrow \psi = \tilde{\psi}$

Pf: $N(\psi - \tilde{\psi}) = N(\psi) - N(\tilde{\psi}) = 0\ \forall N \Rightarrow$ it suffices to show that $N(\phi) = 0\ \forall N \Rightarrow \phi = 0$.

Let $\phi = \sum_{i=1}^{k} \alpha_i \psi_i$, then $\quad 0 = N_j(\phi) = \sum_{i=1}^{k} \alpha_i N_j(\psi_i) = \alpha_j \quad \forall j$. $\square$

Def: A <u>subdivision</u> of a domain $\Omega \subset \mathbb{R}^n$ is a finite set $\{K_1, \ldots, K_M\}$ of open sets with

(1) $K_i \cap K_j = \emptyset$ for $i \neq j$ $\qquad$ (2) $\overline{K}_1 \cup \ldots \cup \overline{K}_M = \overline{\Omega}$



Def: Let $\{K_1, \ldots, K_M\}$ be a subdivision of Ω and $(K_i, \mathcal{P}^i, \mathcal{N}^i)$ be a finite element $\forall i = 1, \ldots, M$.
$X_h = \{v: \Omega \to \mathbb{R} \mid v|_{K_i} \in \mathcal{P}^i,\ N(v|_{K_i}) = N(v)\ \forall N \in \mathcal{N}^i\ \forall i = 1, \ldots, M\}$ is called <u>finite element space</u>.

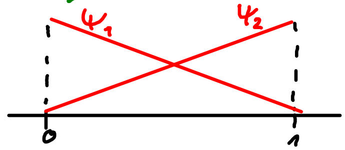
FEM: FE examples

Ex: (1D Lagrange element) $K = (0,1)$, $\mathcal{P} = \mathcal{P}_1 = \{\text{polynomials of degree } 1\}$, $\mathcal{N} = \{N_1, N_2\}$

dim $\mathcal{P}$ = ?

$N_1(v) = v(0)$, $N_2(v) = v(1)$

nodal basis: $\psi_1(x) = 1 - x$, $\psi_2(x) = x$

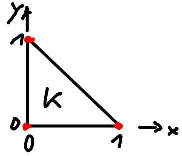


dim $\mathcal{P}$ = ?

Ex: (2D Lagrange element) $K = T := \{(x,y) \in [0,1]^2 \mid y \leq 1 - x\}$, $\mathcal{P} = \mathcal{P}_1$, $\mathcal{N} = \{N_1, N_2, N_3\}$

$N_1(v) = v(0,0)$, $N_2(v) = v(1,0)$, $N_3(v) = v(0,1)$

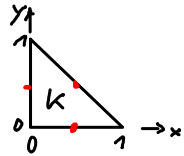
nodal basis: $\psi_1(x,y) = 1 - x - y$, $\psi_2(x,y) = x$, $\psi_3(x,y) = y$



Ex: (2D Crouzeix-Raviart element) $K = T$, $\mathcal{P} = \mathcal{P}_1$, $\mathcal{N} = \{N_1, \dots, N_3\}$

$N_1(v) = v(\frac{1}{2}, \frac{1}{2})$, $N_2(v) = v(0, \frac{1}{2})$, $N_2(v) = v(\frac{1}{2}, 0)$

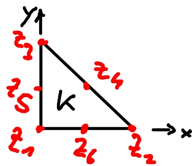
nodal basis: $\psi_1(x,y) = 2(x+y) - 1$, $\psi_2(x,y) = 1 - 2x$, $\psi_3(x,y) = 1 - 2y$



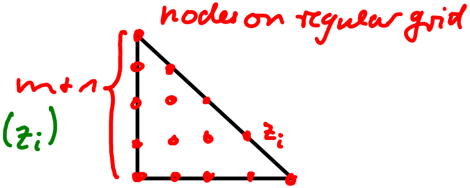
Ex: (2D Lagrange element) $K = T$, $\mathcal{P} = \mathcal{P}_2$, $\mathcal{N} = \{N_1, \dots, N_6\}$; $\quad N_i(v) = v(z_i)$;

$\psi_1(x,y) = (1-2x-2y)(1-x-y)$, $\psi_2(x,y) = -x(1-2x)$, $\psi_3(x,y) = -y(1-2y)$,

$\psi_4(x,y) = 4xy$, $\psi_5(x,y) = 4y(1-x-y)$, $\psi_6(x,y) = 4x(1-x-y)$



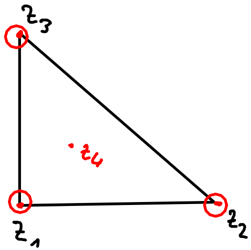
FEM: FE examples



Ex: (2D Lagrange element of order m)

$K = T$, $\mathcal{P} = \mathcal{P}_m$, $\mathcal{N} = \{N_1, \dots, N_k\}$, $k = 1+2+\dots+(m+1)$, $N_i(v) = v(z_i)$

Ex: (2D Hermite element) $K = T$, $\mathcal{P} = \mathcal{P}_3$ dim $\mathcal{P}$ = ?



$N_1(v) = v(z_1)$, $N_2(v) = v(z_2)$, $N_3(v) = v(z_3)$, $N_4(v) = v(z_4)$

$N_5(v) = \frac{\partial v}{\partial x}(z_1)$, $N_6(v) = \frac{\partial v}{\partial x}(z_2)$, $N_7 = \frac{\partial v}{\partial x}(z_3)$

$N_8(v) = \frac{\partial v}{\partial y}(z_1)$, $N_9(v) = \frac{\partial v}{\partial y}(z_2)$, $N_{10} = \frac{\partial v}{\partial y}(z_3)$

nodal basis: determine via polynomial interpolation

Ex: (2D Hermite element of order m)

$K = T$, $\mathcal{P} = \mathcal{P}_m$, $\mathcal{N} = \{N_1, \dots, N_k\}$, $k = \frac{(m+1)(m+2)}{2}$

m	1	2	3	4	5	6	7	8
k	3	6	10	15	21	28	36	45

$N_i(v)$ = function values & directional derivatives up to order ℓ in z_1, z_2, z_3

there are $\ell+1$ independent directional derivatives of order ℓ

$\Longrightarrow 3(1+2+\dots+(\ell+1)) = 3\frac{(\ell+1)(\ell+2)}{2} =: L$ form functions $\to$ pick $\ell = \left\lfloor \frac{-3+\sqrt{1+8k/3}}{2} \right\rfloor$

& complement with $k-L$ form functions in z_4

FEM: interpolant

Def: · Let $(K, \mathcal{P}, \mathcal{N})$ be a finite element with nodal basis $\{\psi_1, \dots, \psi_k\}$. The local interpolant of a function $v: K \to \mathbb{R}$ is $I_K v = \sum_{i=1}^{k} N_i(v)\,\psi_i$.

· Let $\{K_1, \dots, K_M\}$ be a subdivision of Ω and $(K_i, \mathcal{P}_i^i, \mathcal{N}^i)$ a finite element $\forall i = 1, \dots, M$.
The global interpolant $I_h v$ of $v: \Omega \to \mathbb{R}$ is defined by $I_h v|_{K_i} = I_{K_i} v \;\; \forall i$.

The associated finite element space can be represented as $X_h = \{ I_h v \mid v \in C^\infty(\overline{\Omega}) \}$.

Def: If $I_h v \in C^r(\Omega)$ $\forall v \in C^\infty(\overline{\Omega})$, then X_h is called a C^r finite element space.

Thm: For a triangulation with Lagrange or Hermite elements, the associated FE space is a C^0 FE space.

Pf: It suffices to show that on an edge $e = \overline{K_i} \cap \overline{K_j}$ between two elements we have

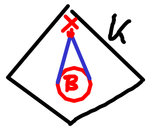
$$I_{K_i} v|_e = I_{K_j} v|_e \quad \forall v \in C^\infty(\overline{\Omega}).$$

For a Lagrange element of order m, $I_{K_i} v|_e$, $I_{K_j} v|_e$ are 1D polynomials of degree m
$\Rightarrow$ the conditions $I_{K_i} v(z_r) = v(z_r) = I_{K_j} v(z_r)$ for the $m+1$ points z_r on e imply $I_{K_i} v|_e = I_{K_j} v|_e$.

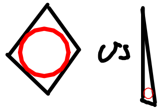
Hermite element: homework $\square$

FEM: Bramble-Hilbert lemma

<u>Def:</u> · $K \subset \mathbb{R}^n$ is called <u>star-shaped</u> wrt. a ball $B \subset K$ if the convex hull of $\{x\} \cup B$ satisfies $co(\{x\} \cup B) \subset K$ for all $x \in K$.



· A subdivision $\{K_1, \dots, K_M\}$ of Ω is called <u>$\frac{1}{\rho}$-quasiuniform</u> or regular if $\forall i = 1, \dots, M: \quad \frac{1}{\rho} \le \frac{r_{K_i}}{h_{K_i}}$, where r_{K_i} = diameter of the largest ball in K_i



Stephen, not David

<u>Thm:</u> (Bramble-Hilbert lemma) Let $\{K_1, \dots, K_M\}$ be a $\frac{1}{\rho}$-quasiuniform subdivision of the bounded, open domain Ω, and let $(K_i, \mathcal{P}^i, \mathcal{N}^i)$ be finite elements with

(1) K_i is star-shaped wrt. a ball,

(2) $\mathcal{P}_{m-1} \subset \mathcal{P}^i \subset C^m(\overline{K_i})$,

(3) $\mathcal{N}^i \subset$ bounded linear forms on $C^\ell(\overline{K_i})$,

} this form of the lemma actually is due to Dupont & Scott

then for $m > \ell + \frac{n}{2}$ and $0 \le j \le m$ we have: $|v - I_{K_i} v|_{H^j(K_i)} \le C(m,n,\rho)\, h_{K_i}^{m-j}\, |v|_{H^m(K_i)}$

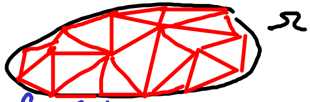
$$\Longrightarrow \quad |v - I_h v|_{H^j(\Omega)} \le C(m,n,\rho)\, h^{m-j}\, |v|_{H^m(\Omega)} \quad \text{if} \quad I_h v \in H^j(\Omega)$$

<u>Cor:</u> $\|u_h - u\|_{H^1} \le C h^{m-1} |u|_{H^m}$ for our elliptic problems & C^0 FE approximations

Variational crimes: deviation from exact methods

So far we only considered FE approximations in which u_h satisfies the same weak equation as u (i.e. $a(u,v) = \ell(v)\ \forall v \in X$), only on a subspace X_h. In practice, however, one often deviates from that, e.g.

- Ω has a curvilinear boundary $\partial\Omega$, but is approximated by triangulations



- the system matrix $A_{mn} = \int_\Omega Q(\phi_m, \phi_n)\, dx$ or right-hand side $B_m = \int_\Omega f \phi_m\, dx$ are only computed approximately, e.g. via numerical quadrature
- the finite element space X_h is not a subspace of the solution space X (e.g. Crouzeix-Raviart-FE-space $\not\subset H^1(\Omega)$)

Such deviations are called „variational crimes".

Ex: $\ell(v) = \int_\Omega f(x) v(x)\, dx$ is replaced by midpoint quadrature on each triangle T_ℓ of the triangulation $\mathcal{T}$, i.e. by

$$\ell_h(v) = \sum_{\ell=1}^{M} |T_\ell|\ f(\bar{x}_\ell)\, v(\bar{x}_\ell) \quad \text{with } \bar{x}_\ell \text{ the midpoint of } T_\ell.$$

Variational crimes: Strang's first lemma

$u \in X$ solves $a(u,v) = \ell(v)\ \forall v \in X$; a bilinear, coercive, bounded; ℓ linear, bounded

$u_h \in X_h$ solves $a_h(u_h, v_h) = \ell_h(v_h)\ \forall v_h \in X_h$; a_h bilinear, uniformly coercive; ℓ_h linear

$X_h \subset X$

i.e. $a_h(v_h, v_h) \geq c\|v_h\|_X^2$ indep. of h

By the way: a_h & ℓ_h only need to be defined on X_h! E.g., $a_h(u_h, v_h)$ could evaluate the gradient of u_h at a quadrature point, which would not be well-defined for an arbitrary function u from $X = H^1(\Omega)$!

Thm: (Strang's first lemma) There exists $K \geq 0$ independent of h such that

$$\|u_h - u\|_X \leq K \left[\inf_{v_h \in X_h} \left(\|u - v_h\|_X + \sup_{w_h \in X_h} \frac{|a(v_h, w_h) - a_h(v_h, w_h)|}{\|w_h\|_X} \right) + \sup_{w_h \in X_h} \frac{|\ell(w_h) - \ell_h(w_h)|}{\|w_h\|_X} \right]$$

Pf: Let $v_h \in X_h$ and $w_h = u_h - v_h$.

$$c\|u_h - v_h\|_X^2 \leq a_h(u_h - v_h, w_h) = a(u - v_h, w_h) + [a(v_h, w_h) - a_h(v_h, w_h)] + [a_h(u_h, w_h) - a(u, w_h)]$$

$$\leq C\|u - v_h\|_X \|w_h\|_X + [a(v_h, w_h) - a_h(v_h, w_h)] + [\ell_h(w_h) - \ell(w_h)]$$

$$\Rightarrow c\|u_h - v_h\|_X \leq C\|u - v_h\|_X + \frac{|a(v_h, w_h) - a_h(v_h, w_h)|}{\|w_h\|_X} + \frac{|\ell(w_h) - \ell_h(w_h)|}{\|w_h\|_X}$$

Now use $\|u_h - u\|_X \leq \|u - v_h\|_X + \|u_h - v_h\|_X$ $\square$

Variational crimes: Strang's first lemma - application

Ex: Solve $-\Delta u + u = f$ on $\Omega = [0,1]^2$, $u = 0$ on $\partial\Omega$

$\Rightarrow X = H_0^1(\Omega)$, $a(u,v) = \int_\Omega \nabla u \cdot \nabla v + uv\,dx$, $\ell(v) = \int_\Omega f v\,dx$ satisfy LM conditions

$\frac{1}{\rho}$-quasiuniform triangulation $(\mathcal{V}, \mathcal{E}, \mathcal{T})$ with grid width h; X_h = 1. order-Lagrange FE-space

$a_h = a$, $\ell_h(v_h) = \sum_{T_\ell \in \mathcal{T}} |T_\ell| \sum_{x \in \mathcal{V} \cap \overline{T_\ell}} \frac{f(x)v(x)}{3}$ "trapezium rule"

$u_h \in X_h$ solves $a_h(u_h, v_h) = \ell_h(v_h)$ $\forall v_h \in X_h$; how big is $\|u_h - u\|_{H^1}$?

There is $K > 0$ with $\|u_h - u\|_{H^1} \leq K\left(h\,|u|_{H^2} + h^2\,\|f\|_{C^2}\right) \leq \tilde{K}\left(h\,\|f\|_{L^2} + h^2\|f\|_{C^2}\right)$

Pf: Strang: $\|u_h - u\|_{H^1} \leq K\left[\inf_{v_h \in X_h} \|u - v_h\|_{H^1} + \sup_{w_h \in X_h} \frac{|\ell(w_h) - \ell_h(w_h)|}{\|w_h\|_{H^1}}\right]$

Bramble-Hilbert: $\inf_{v_h \in X_h} \|u - v_h\|_{H^1} \leq \|u - I_h u\|_{H^1} \leq C(\rho)\, h\, |u|_{H^2}$

$$|\ell(w_h) - \ell_h(w_h)| = \left|\int_\Omega f w_h\,dx - \int_\Omega I_h(f w_h)\,dx\right| \leq \sum_{T \in \mathcal{T}} \int_T |f w_h - I_h(f w_h)|\,dx$$

$$\leq \sum_{T_\ell \in \mathcal{T}} \|f w_h - I_h(f w_h)\|_{L^2(T_\ell)} \sqrt{|T_\ell|} \leq \sum_{T_\ell \in \mathcal{T}} \sqrt{|T_\ell|}\, C(\rho)\, h^2\, |f w_h|_{H^2(T_\ell)}$$

$$\leq \left(\sum_{T_\ell \in \mathcal{T}} |T_\ell|\right)^{\frac{1}{2}} C(\rho) h^2 \left(\sum_{T_\ell \in \mathcal{T}} |f w_h|^2_{H^2(T_\ell)}\right)^{\frac{1}{2}} = C(\rho)\sqrt{|\Omega|}\, h^2 \left(\sum_{T_\ell \in \mathcal{T}} |f w_h|^2_{H^2(T_\ell)}\right)^{\frac{1}{2}}$$

$$|f w_h|^2_{H^2(T)} = \int_T \left(w_h D^2 f + \nabla f \otimes \nabla w_h + \nabla w_h \otimes \nabla f\right)^2 dx \leq 8\left(\max_{x \in \Omega} |D^2 f(x)|^2 + \max_{x \in \Omega} |Df(x)|^2\right) \|w_h\|^2_{H^1(T)}$$

$\square$

Variational crimes: Strang's second lemma

$u \in X$ solves $a(u,v) = \ell(v)$ $\forall v \in X$; a bilinear, coercive, bounded; ℓ linear, bounded

$X_h \not\subset X$, i.e. in particular $\|\cdot\|_X$ is not defined on X_h!

$\Rightarrow$ use (grid-dependent) norm $\|\cdot\|_h$ on $X + X_h$ — u_h well-defined if X_h Hilbert space with norm $\|\cdot\|_h$

$u_h \in X_h$ solves $a_h(u_h, v_h) = \ell_h(v_h)$ $\forall v_h \in X_h$; a_h bilinear; ℓ_h linear

$$a_h(v_h, v_h) \geq c\|v_h\|_h^2 \quad \forall v_h \in X_h \; ; \; |a_h(u, v_h)| \leq C\|u\|_h \|v_h\|_h \quad \forall u \in X + X_h, v_h \in X_h$$

<u>Thm</u>: (Strang's second lemma; Berger-Scott-Strang-lemma) $\exists K > 0$ independent of h such that

$$\|u_h - u\|_h \leq K\Big[\underbrace{\inf_{v_h \in X_h} \|u - v_h\|_h}_{\text{"approximation error"}} + \underbrace{\sup_{w_h \in X_h} \frac{|a_h(u, w_h) - \ell_h(w_h)|}{\|w_h\|_h}}_{\text{"consistency error"}}\Big]$$

Pf: Let $v_h \in X_h$, $w_h = u_h - v_h$. $c\|u_h - v_h\|_h^2 \leq a_h(u_h - v_h, w_h) = a_h(u - v_h, w_h) + [\ell_h(w_h) - a_h(u, w_h)]$

$$\leq C\|u_h - v_h\|_h \|w_h\|_h + |\ell_h(w_h) - a_h(u, w_h)|$$

$$\Rightarrow \|u_h - v_h\|_h \leq \frac{C}{c}\|u - v_h\|_h + \frac{1}{c}\frac{|\ell_h(w_h) - a_h(u, w_h)|}{\|w_h\|_h};$$ now use $\|u - u_h\|_h \leq \|u - v_h\|_h + \|u_h - v_h\|_h$ $\square$

<u>Ex</u>: Examples for the norm $\|\cdot\|_h$ are given by

$$\|v\|_{m,h} = \sqrt{\sum_{T_\ell \in \mathcal{T}} \|v\|_{H^m(T_\ell)}^2} \qquad \left(\neq \|v\|_{H^m(\Omega)}\right)$$

Variational crimes: Strang's second lemma - application

- $\Omega=[0,1]^2$, $X=H_0^1(\Omega)$, $f\in L^2(\Omega)$, $a(u,v)=\int_\Omega \nabla u\cdot\nabla v+uv\,dx$, $\ell(v)=\int_\Omega fv\,dx$ for $u,v\in X$
- Crouzeix-Raviart-elements on quasiuniform triangulation $\mathcal{T}_h$, i.e.

$X_h=\{v\in L^2(\Omega)\,|\,v|_T\in\mathcal{P}_1\ \forall T\in\mathcal{T}_h$ & v is continuous at midpoints of triangle sides & $v=0$ at midpoints of triangle sides on $\partial\Omega\}$

$a_h(u,v)=\sum_{T\in\mathcal{T}_h}\int_T \nabla u\cdot\nabla v+uv\,dx$, $\ell_h(v)=\int_\Omega fv\,dx$ for $u,v\in X_h\not\subset X$!

- $u\in X$ solves $a(u,v)=\ell(v)\ \forall v\in X$, $u_h\in X_h$ solves $a_h(u_h,v_h)=\ell_h(v_h)\ \forall v_h\in X_h$
- we know: $u\in H^2(\Omega)$ with $\|u\|_{H^2}\le C\|f\|_{L^2}$

Thm: $\|u_h-u\|_{1,h}\le ch\|f\|_{L^2}$ for a $c>0$.

Pf: $\cdot\ a_h(u,v_h)-\ell_h(v_h)=\sum_{T_i}\int_{T_i}\nabla u\cdot\nabla v_h+uv_h-fv_h\,dx \overset{\text{int. by parts \& } u\in H^2}{=} \sum_{T_i}\int_{\partial T_i}\nabla u\cdot n\,v_h\,dx-\int_{T_i}v_h\underbrace{(\Delta u-u+f)}_{=0}dx$

$=\sum_{T_i}\sum_{\text{edge } e\in\partial T_i}\int_e \nabla u(x)\cdot n(v_h(x)-v_h(x_e))\,dx$ with x_e = midpoint of e

since $\nabla u|_{T_i}\in H^1(T_i)$, for $e=T_1\cap T_2$ the trace theorem implies $\nabla u|_{T_1}\cdot n_{T_1}=-\nabla u|_{T_2}\cdot n_{T_2}$ on e

$\int_e w_h-w_h(x_e)\,dx=0$

$=\sum_{T_i}\sum_{e\in\partial T_i}\int_e(\nabla u-\nabla I_hu)\cdot n\,(v_h-v_h(x_e))\,dx\le\sum_{T_i}\sum_{e\in\partial T_i}\|\nabla(u-I_hu)\|_{L^2(e)}\,\|v_h-v_h(x_e)\|_{L^2(e)}$

Variational crimes: Strang's second lemma - application (cont.)

- $\|\nabla(u-I_h u)\|_{L^2(e)} \le \check{C}\left(h^{-\frac{1}{2}}\|\nabla(u-I_h u)\|_{L^2(T_i)} + h^{\frac{1}{2}}\|D^2(u-I_h u)\|_{L^2(T_i)}\right)$ trace theorem

$$\le \bar{C}\; h^{\frac{1}{2}}\,\|D^2(u-I_h u)\|_{L^2(T_i)} \quad \text{Bramble-Hilbert}$$

$$= \bar{C}\; h^{\frac{1}{2}}\,|u|_{H^2(T_i)}$$

- $\|v_h - v_h(x_e)\|_{L^2(e)} \le \|v_h - \overline{v_h}(T_i)\|_{L^2(e)}$ $\quad \overline{v_h}(T_i) = \frac{1}{|T_i|}\int_{T_i} v_h\,dx$

$$\le \check{C}\left(h^{-\frac{1}{2}}\|v_h - \overline{v_h}(T_i)\|_{L^2(T_i)} + h^{\frac{1}{2}}\|\nabla(v_h - \overline{v_h}(T_i))\|_{L^2(T_i)}\right) \quad \text{trace theorem}$$

$$\le C h^{\frac{1}{2}}|v_h|_{H^1(T_i)} \quad \text{Poincaré-ineq.}$$

$\Rightarrow |a_h(u,v_h) - \ell_h(v_h)| \le \sum_{T_i} C\,h\,|u|_{H^2(T_i)}\,|v_h|_{H^1(T_i)} \underset{\text{Cauchy-Schwarz}}{\le} C h\,|u|_{H^2(\Omega)}\,\|v_h\|_{1,h}$

- $\inf_{v_h\in X_h}\|u-v_h\|_{1,h}^2 \le \|u-I_h u\|_{1,h}^2 = \sum_{T_i}\|u-I_h u\|_{H^1(T_i)}^2 \le \sum_{T_i} C h^2 |u|_{H^2(T_i)}^2 = C h^2 |u|_{H^2(\Omega)}^2$

- conditions of Strang's second lemma satisfied $\Longrightarrow \|u_h - u\|_h \le \tilde{K}\left(h|u|_{H^2} + h|u|_{H^2}\right)$ □

Variational crimes: duality (extension of Aubin-Nitsche)

- $u \in X$ solves $a(u,v) = \ell(v)\ \forall v \in X$; a bilinear, coercive, bounded; ℓ linear, bounded
- Aubin-Nitsche-cond.: H Hilbert space with continuous embedding $X \hookrightarrow H$
- Strang's 2. lemma-cond.: $a_h(v_h,v_h) \geq c\|v_h\|_h^2\ \forall v_h \in X_h$; $|a_h(u,v)| \leq C\|u\|_h\|v\|_h\ \forall u,v \in X+X_h$

  ℓ_h linear & bounded on X_h, $\|\cdot\|_h$ norm on Hilbert space X_h
- $u_h \in X_h$ solves $a_h(u_h,v_h) = \ell_h(v_h)\quad \forall v_h \in X_h$
- additionally: a_h defined on $X+X_h$, $a_h = a$ on X, $X_h \subset H$, $\|\cdot\|_h$ defined on $X+X_h$

is not used in proof

Thm: $w_r \in X$ solves $a(v,w_r) = (r,v)_H\ \forall v \in X$, $w_h \in X_h$ solves $a_h(v_h,w_h) = (r,v_h)_H\ \forall v_h \in X_h$.

$$\|u_h-u\|_H \leq \sup_{r \in H} \frac{1}{\|r\|_H}\Big(C\|u_h-u\|_h\|w_r-w_h\|_h + \big[(u_h-u,r)_H - a_h(u_h-u,w_r)\big] + \big[\ell_h(w_h) - \ell(w_r) - a_h(u,w_h-w_r)\big]\Big)$$

Pf: Set $r = u_h - u$. $(u_h-u,u_h-u)_H = a_h(u_h-u,w_r-w_h) + \text{rest}$ like Aubin-Nitsche

$$\text{rest} = (u_h-u,r)_H - a_h(u_h-u,w_r-w_h) = \big[(u_h-u,r)_H - a_h(u_h-u,w_r)\big] + a_h(u_h-u,w_h)$$

$$= \big[(u_h-u,r)_H - a_h(u_h-u,w_r)\big] + \big[\ell_h(w_h) - \ell(w_r) - a_h(u,w_h-w_r)\big]$$

$$\Rightarrow \|u_h-u\|_H \leq \frac{1}{\|r\|_H}\Big(C\|u_h-u\|_h\|w_r-w_h\|_h + \big[(u_h-u,r)_H - a_h(u_h-u,w_r)\big] + \big[\ell_h(w_h) - \ell(w_r) - a_h(u,w_h-w_r)\big]\Big)$$

$\square$

Variational crimes: duality - application

Solve $-\Delta u + u = f$ on $[0,1]^2$ with 0-Dirichlet bc with Crouzeix-Raviart elements $\Rightarrow$ setting as before

<u>Thm:</u> $\|u_h - u\|_{L^2(\Omega)} \leq C h^2 \|f\|_{L^2(\Omega)}$ for a $C > 0$.

Pf: $H = L^2(\Omega)$, $\|\cdot\|_h = \|\cdot\|_{1,h}$. We already know $\|u_h - u\|_{1,h} \leq Ch\|f\|_{L^2}$, $\|w_h - w_r\|_{1,h} \leq Ch\|r\|_{L^2}$.

- Furthermore, we have shown two slides ago

$$|(v_h, g)_{L^2} - a_h(v_h, w_g)| = |(v_h, g)_{L^2} - a_h(w_g, v_h)| \leq Ch|w_g|_{H^2}\|v_h\|_{1,h} \quad (*)$$

Note: In previous calculation, test function was $v_h \in X_h$, but also $v_h \in X + X_h$ is possible.

To this end replace $v_h(x_e)$ with $\overline{v_e} = \frac{1}{|e|}\int_e v_h(x)\,dx$ (is the same on X_h).

$$\Rightarrow \begin{cases} (u_h - u, r)_{L^2} - a_h(u_h - u, w_r) \overset{(*)}{\leq} Ch|w_r|_{H^2}\|u_h - u\|_{1,h} \leq \tilde{C}h^2\|r\|_{L^2}\|f\|_{L^2} \\ \ell_h(w_h) - \ell(w_r) - a_h(u, w_h - w_r) \overset{(*)}{\leq} Ch|u|_{H^2}\|w_h - w_r\|_{1,h} \leq \tilde{C}h^2\|f\|_{L^2}\|r\|_{L^2} \end{cases}$$

- conditions from previous slide satisfied

$$\Rightarrow \|u_h - u\|_{L^2} \leq \sup_{r \in L^2(\Omega)} \frac{1}{\|r\|_{L^2}}\left(Ch^2\|f\|_{L^2}\|r\|_{L^2}\right) \qquad \square$$

Duality: dual spaces

Def: Let X be a normed vector space. The dual space X^* is the space of all bounded linear functionals on X. For $f \in X^*$, $x \in X$ we also write $f(x) = \langle f, x \rangle_{X^*, X}$. A norm on X^* is defined by $\|f\|_{X^*} = \sup_{x \in X, x \neq 0} \frac{f(x)}{\|x\|_X}$.

Thm: By construction, $\langle f, x \rangle_{X^*, X} \leq \|f\|_{X^*} \|x\|_X$.

Ex: $L^p(\Omega)^* = L^q(\Omega)$ for $p \in [1, \infty)$ & $\frac{1}{p} + \frac{1}{q} = 1$. ($L^q(\Omega) \subset L^p(\Omega)^*$ by Hölder's inequality.)

Def: · Let $\Omega \subset \mathbb{R}^n$ open. The space of test functions is $C_c^\infty(\Omega) = \{\phi \in C^\infty(\Omega) \mid \operatorname{supp}\phi \text{ compact}\}$.

$\phi_j \xrightarrow[j\to\infty]{} \phi$ in $C_c^\infty(\Omega)$, if there exists a compact $K \subset \Omega$ with $\operatorname{supp}\phi_j \subset K$ $\forall j$ and $\|\phi_j - \phi\|_{C^n} \xrightarrow[j\to\infty]{} 0$ $\forall n$. Ex.: $\phi(x) = e^{-(1-|x|^2)^{-1}}$ for $|x|<1$ & $\phi(x) = 0$ else

· The space of distributions is $\mathcal{D}'(\Omega) = \{f : C_c^\infty(\Omega) \to \mathbb{R} \mid f \text{ linear \& continuous}\}$ (topological dual space to $C_c^\infty(\Omega)$). Any $f \in L^1(\Omega)$ can be interpreted as a distribution by setting $f(\phi) := \int_\Omega f(x)\phi(x)\,dx$. The notation $\int_\Omega f\phi\,dx$ for $f(\phi)$ is even used if $f \notin L^1(\Omega)$. Ex.: $\int_\Omega \delta\phi\,dx = \phi(0)$

· For $f \in \mathcal{D}'(\Omega)$ and multiindex α the distributional derivative $D^\alpha f \in \mathcal{D}'(\Omega)$ is defined as $D^\alpha f(\phi) = (-1)^{|\alpha|} f(D^\alpha \phi)$ (if $f \in C^{|\alpha|} \subset L^1 \subset \mathcal{D}'$, this equals the classical derivative!) Ex.: $H' = \delta$

Finite Elements: elliptic PDEs (2D Poisson equation)

Duality: H^{-1}

Proof possible with Lax-Milgram!

<u>Thm</u>: $C_c^\infty(\Omega)$ is dense in $H_0^1(\Omega)$, i.e. $\forall u \in H_0^1(\Omega), \varepsilon > 0 \; \exists \phi \in C_c^\infty(\Omega) : \|u-\phi\|_{H^1} < \varepsilon$.

<u>Thm</u>: (Riesz representation theorem) A Hilbert space X is isometrically isomorphic to X^*, i.e. for every $f \in X^*$ there exists exactly one $g \in X$ with $\|f\|_{X^*} = \|g\|_X$ and $f(v) = (g,v)_X \; \forall v \in X$.

<u>Ex</u>: Every $f \in L^2(\Omega)^*$ can be written as $f(v) = \int_\Omega g v \, dx$ for a $g \in L^2(\Omega)$.

<u>Def</u>: $H^{-1}(\Omega) := H_0^1(\Omega)^*$, and we write $\langle f,u\rangle_{H^{-1},H_0^1} = \int_\Omega f u \, dx$.

<u>Thm</u>: $H^{-1}(\Omega) = \{ f \in \mathcal{D}'(\Omega) \mid \exists g \in H_0^1(\Omega) : \langle f,u\rangle_{\mathcal{D}',C_c^\infty} = (g,u)_{H^1} \; \forall u \in H_0^1(\Omega)\} =: F$.

Pf: $\subset$: Clearly, $H^{-1} \subset \mathcal{D}'$, and by Riesz, $\forall f \in H^{-1}(\Omega) \subset \mathcal{D}'(\Omega) \; \exists g \in H_0^1(\Omega)$: $\langle f,v\rangle_{\mathcal{D}',C_c^\infty} = \langle f,v\rangle_{H^{-1},H_0^1} = (g,v)_{H^1} \; \forall v \in C_c^\infty(\Omega) \subset H_0^1(\Omega)$

$\supset$: For $f \in F$, $v \in H_0^1(\Omega)$ one can define $f(v) = \lim_{i\to\infty} f(v_i)$ for a sequence $v_i \in C_c^\infty(\Omega)$ with $v_i \xrightarrow[i\to\infty]{} v$ in $H_0^1(\Omega)$.

The limit exists since $f(v_i)$ is Cauchy due to $|f(v_i) - f(v_j)| = (g, v_i - v_j) \le \|g\|_{H^1} \|v_i - v_j\|_{H^1} \xrightarrow[i,j\to\infty]{} 0$, and

it does not depend on v_i (since $v_i, \tilde v_i \longrightarrow v \Rightarrow |f(v_i) - f(\tilde v_i)| \le \|g\|_{H^1} \|v_i - \tilde v_i\|_{H^1} \to 0$);

also, this f is clearly linear with $|f(v)| = \lim_{i\to\infty} |f(v_i)| = \lim_{i\to\infty} (g, v_i)_{H^1} \le \limsup_i \|g\|_{H^1} \|v_i\|_{H^1} = \|g\|_{H^1} \|v\|_{H^1}$. $\square$

<u>Ex</u>:
- $L^2(\Omega) \subset H^{-1}(\Omega)$ with $\langle f,u\rangle_{H^{-1},H_0^1} = \int_\Omega f u \, dx$ for $f \in L^2(\Omega)$
- $\delta \in H^{-1}((-1,1))$:
  - $f_1 : \left(x \mapsto \frac{1-|x|}{2}\right) \in H_0^1((-1,1)) \subset L^2((-1,1)) \subset H^{-1}((-1,1))$
  - $f_2 = \delta + f_1 \in H^{-1}((-1,1))$, since $\forall u \in C_c^\infty((-1,1))$: $f(u) = u(0) + \int_{-1}^1 \frac{1-|x|}{2} u(x) \, dx = (-f_1, u)_{H^1}$

Duality: elliptic differential operators

Thm: $H^{-1}(\Omega)^* = H^1_0(\Omega)$ with $\langle u, f\rangle_{H^{-1}(\Omega)^*, H^{-1}(\Omega)} = \langle f, u\rangle_{H^{-1}, H^1_0}$.

Pf: $\cdot$ $H^1_0(\Omega) \subset H^{-1}(\Omega)^*$ (one always has $X \subset (X^*)^*$)

$\cdot$ Let now $z \in H^{-1}(\Omega)^*$ and $R: H^{-1}(\Omega) \to H^1_0(\Omega)$ be the isomorphism from Riesz' theorem.

$u_z = R(z \circ R^{-1}) \in H^1_0(\Omega)$ satisfies $z(f) = (z\circ R^{-1})(Rf) = \langle z\circ R^{-1}, Rf\rangle_{H^{-1}, H^1_0} = (Rf, R(z\circ R^{-1}))_{H^1} = \langle f, u_z\rangle = f(u_z)\ \forall f \in H^{-1}(\Omega)$. $\square$

Thm: Let $A \in L^\infty(\Omega; \mathbb{R}^{n\times n})$, $b \in L^\infty(\Omega; \mathbb{R}^n)$, $c \in L^\infty(\Omega)$. The differential operator $L: H^1_0(\Omega) \to H^{-1}(\Omega)$,

distributional derivative — weak derivative

$Lu = -\operatorname{div}(A\nabla u) + b\cdot\nabla u + cu$ is bounded

Pf: Let $u \in H^1_0(\Omega)$, $v \in C^\infty_c(\Omega)$. $\langle Lu, v\rangle_{\mathcal{D}', C^\infty_c} = \int_\Omega \nabla v\cdot A\nabla u + v\, b\cdot\nabla u + c u v\, dx \le C_1 \|u\|_{H^1}\|v\|_{H^1}$

Let now $v \in H^1_0(\Omega)$, $v_j \in C^\infty_c(\Omega)$, $v_j \to v$. $\langle Lu, v\rangle_{H^{-1}, H^1_0} := \lim_{j\to\infty} \langle Lu, v_j\rangle_{\mathcal{D}', C^\infty_c}$ exists (why?),

is linear & bounded in $v \in H^1_0(\Omega)$ (why?), and $\|L\| \le C_1$ (why?). Hint: same argument as previous slide $\square$

<u>Def</u>: Let X, Y be normed spaces, $L: X \to Y$ linear and bounded. The <u>adjoint operator</u>

$L^*: Y^* \to X^*$ is defined by $\langle f, Lx\rangle_{Y^*, Y} = \langle L^* f, x\rangle_{X^*, X}$.

<u>Ex</u>: $L: H^1_0(\Omega) \to H^{-1}(\Omega)$, $Lu = -\operatorname{div}(A\nabla u) + b\cdot\nabla u + cu \Rightarrow L^* v = -\operatorname{div}(A^T\nabla v) - \operatorname{div}(b v) + c v$,

since $\int_\Omega (Lu) v\, dx = \int_\Omega u L^* v\, dx$ after integrating by parts twice.

Adaptivity: idea of adaptive FE-solutions

a priori error estimate $\|u - u_h\| \leq \eta_h(u)$

a posteriori error estimate $\|u - u_h\| \leq \eta_h(u_h)$

Def: An error estimator $\eta_h(u_h)$ is called reliable, if $\|u - u_h\| \leq \eta_h(u_h)$.

It is called efficient with constant c, if $c\,\eta_h(u_h) \leq \|u - u_h\|$.

An error estimator can often be written as a sum $\eta_h(u_h)^2 = \sum_{T_\ell \in S} \eta_{T_\ell}(u_h)^2$ over all FE.

Given TOL > 0: $\eta_{T_\ell}(u_h) < \frac{TOL}{\sqrt{M}}$ (→ refinement criterion) $\Rightarrow \|u - u_h\| \leq \eta_h(u_h) < TOL$ (stopping criterion)

Alg. (Adaptive FE)

choose initial subdivision

while stopping criterion not satisfied

  compute u_h & $\eta_h(u_h)$

  for $\ell = 1, \dots, M$

    if refinement criterion satisfied for T_ℓ

      refine T_ℓ into smaller elements

Finite Elements: elliptic PDEs (2D Poisson equation)

Adaptivity: a posteriori error estimate

Def: The residual of an elliptic PDE $Lu = f$ with $L: H_0^1(\Omega) \to H^{-1}(\Omega)$, $f \in L^2(\Omega)$ is

$$R: H_0^1(\Omega) \to H^{-1}(\Omega), \quad R(v) = Lv - f.$$

Let $a(u,v) = \langle Lu, v \rangle_{H^{-1}, H_0^1}$, $\ell(v) = (f, v)_{L^2}$, $a(v,v) \geq c\|v\|_{H^1}^2$, $a(u,v) \leq C\|u\|_{H^1}\|v\|_{H^1}$ $\forall u, v \in H_0^1(\Omega)$

$a(u,v) = \ell(v)$ $\forall v \in H_0^1(\Omega)$, $\quad a(u_h, v_h) = \ell(v_h)$ $\forall v_h \in X_h \subset H_0^1(\Omega)$

e.g. $L = (-\Delta + id)$, $a(u,v) = \int_\Omega \nabla u \cdot \nabla v + uv \, dx$

Thm: $c\|u - u_h\|_{H^1(\Omega)} \leq \|R(u_h)\|_{H^{-1}(\Omega)} \leq C\|u - u_h\|_{H^1(\Omega)}$

Pf: $\cdot$ $\langle R(u_h), v \rangle_{H^{-1}, H_0^1} = \langle Lu_h - f, v \rangle_{H^{-1}, H_0^1} = \langle L(u_h - u), v \rangle_{H^{-1}, H_0^1} = a(u_h - u, v)$ $\quad \forall v \in H_0^1(\Omega)$

$\cdot$ $\|R(u_h)\|_{H^{-1}} = \sup_{v \neq 0} \langle R(u_h), v \rangle_{H^{-1}, H_0^1} / \|v\|_{H^1} \leq C\|u_h - u\|_{H^1}$

$\cdot$ $\|R(u_h)\|_{H^{-1}} \geq \langle R(u_h), u_h - u \rangle_{H^{-1}, H_0^1} / \|u_h - u\|_{H^1} \geq c\|u_h - u\|_{H^1}$ $\quad \square$

Thus, $\frac{1}{c}\|R(u_h)\|_{H^{-1}(\Omega)}$ is a reliable & efficient error estimator.

Adaptivity: a posteriori error estimate in L^2

- $u \in X$ solves $a(u,v) = \ell(v)\ \forall v \in X$; a bilinear, coercive, bounded; ℓ linear, bounded

  $a(u,v) = \langle Lu, v\rangle_{H^{-1},H^1_0} = \langle u, L^* v\rangle_{H^1_0, H^{-1}}$ for a differential operator $L: H^1_0(\Omega) \to H^{-1}(\Omega)$

  a & ℓ such that $w \in H^2$ & $a(v,w) = (f,v)_{L^2}\ \forall v \in H^1_0(\Omega)$ implies $|w|_{H^2} \le C\|f\|_{L^2}$

  e.g. $\Omega = [0,1]^2$, $a(u,v) = \int_\Omega \nabla u \cdot \nabla v + uv\, dx$, $\ell(v) = \int_\Omega f v\, dx$, $f \in L^2(\Omega)$, $L = (-\Delta + id)$
- $u_h \in X_h \subset X$ solves $a(u_h, v_h) = \ell(v_h)\ \forall v_h \in X_h$ (finite elements on quasiuniform triangulation $\mathcal{T}$)

<u>Thm:</u> $\|u - u_h\|_{L^2} \le C h \|R(u_h)\|_{H^{-1}}$

Pf: Let w solve the dual problem $a(v,w) = (u-u_h, v)_{L^2}\ \forall v \in X$ (see Aubin-Nitsche)

$$\Rightarrow \|u-u_h\|^2_{L^2} = (u-u_h, u-u_h)_{L^2} = a(u-u_h, w) = a(u-u_h, w-w_h) \quad \forall w_h \in X_h$$

$$= \langle Lu - Lu_h, w - w_h\rangle_{H^{-1},H^1_0} = R(u_h)(w_h - w) \le \|R(u_h)\|_{H^{-1}} \|w_h - w\|_{H^1}$$

$w_h = I_h w$

$$\le C\|R(u_h)\|_{H^{-1}}\, h\, |w|_{H^2} \le C\|R(u_h)\|_{H^{-1}}\, h\, \|u-u_h\|_{L^2} \qquad \square$$

Using the following techniques, this can be further refined so that on each triangle T the local h_T is used instead of h.

Adaptivity: local error estimator

Let $a(u,v)=(f,v)_{L^2}$ $\forall v\in H_0^1(\Omega)$ with $a(u,v)=\langle Lu,v\rangle_{H^{-1},H_0^1}$ coercive & bounded, $Lu=-\operatorname{div}(A\nabla u)+b\cdot\nabla u+cu$, $f\in L^2(\Omega)$.

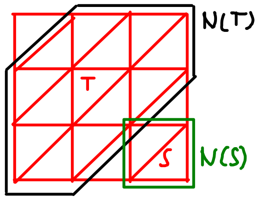
1) $\|R(u_h)\|_{H^{-1}}$ difficult to compute 2) we want error estimator for each element

$\Rightarrow$ we will replace $\|R(u_h)\|_{H^{-1}}^2$ by $\eta_h(u_h)^2=\sum_{T\in\mathcal{T}}\eta_T(u_h)^2$ with

$$\eta_T(u_h)^2 := h_T^2\,\|Lu_h-f\|_{L^2(T)}^2+h_T\,\|[A\nabla u_h\cdot n]\|_{L^2(\partial T)}^2$$

$[\cdot]$ = jump across edge

and show <u>reliability</u> and (almost) <u>efficiency</u>. <u>Ingredients</u>:



(a) $\frac{1}{\rho}$-quasi-uniform triangulation $\mathcal{T}$,

$N(T)=\bigcup\{T_\ell\in\mathcal{T}\mid \overline{T_\ell}\cap\overline{T}\neq\emptyset\}$ for $T\in\mathcal{T}$, $N(S)=T_\ell\cup T_j$ for $S=\overline{T_\ell}\cap\overline{T_j}$, $h_S=\max_{T\in N(S)}h_T$

(b) Galerkin-orthogonality $\langle R(u_h),\varphi\rangle_{H^{-1},H_0^1}=a(u_h-u,\varphi)=a(u_h-u,\varphi-\varphi_h)$

(c) $\exists$ (quasi-) interpolation $\mathcal{I}_h: H_0^1(\Omega)\to X_h\subset H_0^1(\Omega)$ such that $\forall K\in\mathcal{T}$ i.e. $\leq C(\rho)\cdot\ldots$

$\cdot\ \|\mathcal{I}_h v-v\|_{H^1(K)}\lesssim\|\nabla v\|_{L^2(N(K))}$ $\cdot\ \|\mathcal{I}_h v-v\|_{L^2(K)}\lesssim h_K\|\nabla v\|_{L^2(N(K))}$ (see later)

(d) trace estimate $\|v\|_{L^2(S)}\lesssim h_T^{-\frac{1}{2}}\|v\|_{L^2(T)}+h_T^{\frac{1}{2}}\|\nabla v\|_{L^2(T)}$ $\forall T\in\mathcal{T}$, $S\in\partial T$, $v\in H^1(T)$ homework

(e) Poincaré-inequality $\|v\|_{L^2(T)}\lesssim h_T\|v\|_{H^1(T)}$ $\forall T\in\mathcal{T}$, $v\in H_0^1(T)$ homework

Adaptivity: reliability local error estimator

<u>Thm</u>: Let u_h be the FE solution. There exists $C = C(\Omega, \rho) > 0$ with $\|R(u_h)\|^2_{H^{-1}} \leq C \sum_{K \in \mathcal{T}} \eta_K(u_h)^2$.

Pf: $\langle R(u_h), \varphi \rangle_{H^{-1}, H_0^1} \overset{(b)}{=} a(u_h - u, \varphi - \varphi_h) = a(u_h, \varphi - \varphi_h) - (f, \varphi - \varphi_h)_{L^2(\Omega)}$

$$= \sum_{K \in \mathcal{T}} \int_K \nabla(\varphi - \varphi_h) \cdot A \nabla u_h + (\varphi - \varphi_h)(b \cdot \nabla u_h + c u_h - f)\, dx$$

$$= \sum_{K \in \mathcal{T}} (L u_h - f, \varphi - \varphi_h)_{L^2(K)} + \int_{\partial K} (\varphi - \varphi_h)(A \nabla u_h) \cdot n \, dx$$

$$= \sum_{K \in \mathcal{T}} (L u_h - f, \varphi - \varphi_h)_{L^2(K)} + \frac{1}{2} \int_{\partial K} (\varphi - \varphi_h) [(A \nabla u_h) \cdot n] \, dx$$

Hölder

$$\leq \sum_{K \in \mathcal{T}} \|L u_h - f\|_{L^2(K)} \|\varphi - \varphi_h\|_{L^2(K)} + \frac{1}{2} \|[A \nabla u_h \cdot n]\|_{L^2(\partial K)} \|\varphi - \varphi_h\|_{L^2(\partial K)}$$

$$\overset{(d)}{\lesssim} \sum_{K \in \mathcal{T}} \|L u_h - f\|_{L^2(K)} \|\varphi - \varphi_h\|_{L^2(K)} + \|[A \nabla u_h \cdot n]\|_{L^2(\partial K)} \left(h_K^{-\frac{1}{2}} \|\varphi - \varphi_h\|_{L^2(K)} + h_K^{\frac{1}{2}} |\varphi - \varphi_h|_{H^1(K)} \right)$$

pick $\varphi_h = \mathcal{I}_h \varphi$

$$\overset{(c)}{\lesssim} \sum_{K \in \mathcal{T}} \left(\|L u_h - f\|_{L^2(K)} h_K + \|[A \nabla u_h \cdot n]\|_{L^2(\partial K)} h_K^{\frac{1}{2}} \right) \|\nabla \varphi\|_{L^2(N(K))}$$

Cauchy-Schwarz

$$\lesssim \sqrt{\sum_{K \in \mathcal{T}} h_K^2 \|L u_h - f\|^2_{L^2(K)} + h_K \|[A \nabla u_h \cdot n]\|^2_{L^2(\partial K)}} \sqrt{\sum_{K \in \mathcal{T}} \|\nabla \varphi\|^2_{L^2(N(K))}}$$

$$\sum_{K \in \mathcal{T}} \|\nabla \varphi\|^2_{L^2(N(K))} \lesssim \sum_{K \in \mathcal{T}} \|\nabla \varphi\|^2_{L^2(K)} = \|\nabla \varphi\|^2_{L^2(\Omega)} \leq \|\varphi\|^2_{H^1(\Omega)}$$

$$\|R(u_h)\|_{H^{-1}} = \sup_{\varphi \in H_0^1(\Omega)} \frac{\langle R(u_h), \varphi \rangle_{H^{-1}, H_0^1}}{\|\varphi\|_{H^1}} \lesssim \sqrt{\sum_{K \in \mathcal{T}} \eta_K^2(u_h)} \qquad \square$$

Finite Elements: elliptic PDEs (2D Poisson equation)

Adaptivity: efficiency local error estimator

Set $\cdot\ r \in L^2(\Omega)$, $r|_T = (Lu_h - f)|_T \ \forall T \in \mathcal{T}$, $\cdot\ \bar{r} \in L^2(\Omega)$, $\bar{r}|_T = \frac{1}{|T|}\int_T r\,dx$

$\cdot\ j \in L^2(S)$ $\forall$ edges S, $j|_S = [A\nabla u_h \cdot n]$, $\cdot\ \bar{j} \in \mathcal{P}_1(S)$ $\forall S$ L^2-projection of j

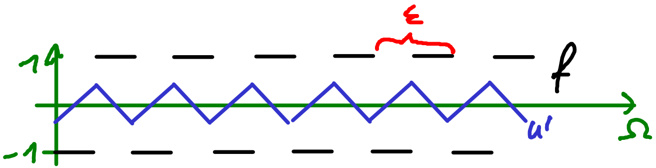
$$\Rightarrow \eta_T(u_h)^2 = h_T^2 \|r\|^2_{L^2(T)} + h_T \sum_{S \in \partial T} \|j\|^2_{L^2(S)}$$

Thm: $\eta_T(u_h)^2 \lesssim \|u_h - u\|^2_{H^1(N(T))} + h_T^2 \|r - \bar{r}\|^2_{L^2(N(T))} + h_T \|j - \bar{j}\|^2_{L^2(\partial T)}$. local efficiency!

$$\Rightarrow \eta_h(u_h)^2 = \sum_{T \in \mathcal{T}} \eta_T(u_h)^2 \lesssim \|u_h - u\|^2_{H^1(\Omega)} + \sum_{T \in \mathcal{T}} \left(h_T^2 \|r - \bar{r}\|^2_{L^2(N(T))} + h_T \|j - \bar{j}\|^2_{L^2(\partial T)} \right)$$

$\Rightarrow$ $\eta_h(u_h)$ is efficient up to oscillation terms

The oscillation terms cannot be ignored:



Ex: $a(u,v) = \int_\Omega \nabla u \cdot \nabla v\,dx$, $a(u,v) = (f,v)_{L^2}$ $\forall v \in H_0^1(\Omega)$,

$a(u_h, v_h) = (f, v_h)_{L^2}$ $\forall v_h \in X_h =$ linear FE-space.

Let $\varepsilon << h_T$ $\Rightarrow (f, v_h)_{L^2(\Omega)} = 0 \Rightarrow u_h = 0$, but $u'(x) = \begin{cases} x - \varepsilon i, & x \in \varepsilon i + [-\varepsilon/4, \varepsilon/4], i \in \mathbb{N} \\ -x - \varepsilon(i + \frac{1}{2}), & \text{else} \end{cases}$

$$\eta_h(u_h)^2 = h^2 \|f\|^2_{L^2(\Omega)} = h^2 |\Omega|$$

$$\|u_h - u\|^2_{H^1} \approx \|\nabla(u_h - u)\|^2_{L^2} = \|\nabla u\|^2_{L^2} = \frac{\varepsilon^2}{3 \cdot 4^2} |\Omega| \not> \eta_h(u_h)^2 \quad !$$

Adaptivity: efficiency local error estimator (proof)

Pf: $h_S^{\frac{1}{2}}\|j\|_{L^2(S)} \lesssim \|u_h-u\|_{H^1(N(S))} + h_S\|r-\bar r\|_{L^2(N(S))} + h_S^{\frac{1}{2}}\|j-\bar j\|_{L^2(S)}$, because ...

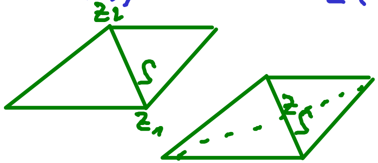
- extend $\bar j$ linearly from S to $N(S)$ such that $\|\bar j\|_{L^2(N(S))} + h_S\|\nabla \bar j\|_{L^2(N(S))} \lesssim h_S^{\frac{1}{2}}\|\bar j\|_{L^2(S)}$ homework
- Let $\varphi \in W_0^{1,\infty}(N(S))$ for edge S with

  $\varphi \geq 0$, $\int_S p^2\varphi\, dx \sim \int_S p^2\, dx \;\forall p \in \mathcal{P}_1$, $\|\varphi\|_{L^\infty(N(S))} + h_S\|\nabla\varphi\|_{L^\infty(N(S))} \lesssim 1$

  Ex: (edge-bubble) · $\varphi = \psi_{z_1}\psi_{z_2}$ (Verfürth)

  ψ_z = hat function in z

  · $\varphi = \psi_z$ (Dörfler)



$$\|\bar j\|^2_{L^2(S)} \lesssim \int_S \bar j\, \underbrace{(\bar j\varphi)}_{\psi_S}\, dx = \int_S j\,\psi_S\, dx + \int_S (\bar j - j)\,\psi_S\, dx$$

- $\left|\int_S j\,\psi_S\, dx\right| \lesssim \|r\|_{L^2(N(S))}\|\psi_S\|_{L^2(N(S))} + \|u-u_h\|_{H^1(N(S))}\|\psi_S\|_{H^1(N(S))}$, since

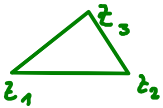
  $\int_S j\,\psi_S\, dx = a(u_h-u,\psi_S) - \int_{N(S)} r\,\psi_S\, dx$
- $\|\psi_S\|_{L^2(N(S))} \overset{\text{Hölder}}{\leq} \|\bar j\|_{L^2(N(S))}\|\varphi\|_{L^\infty(N(S))} \lesssim h_S^{\frac{1}{2}}\|\bar j\|_{L^2(S)}$ and

  $\|\psi_S\|_{H^1(N(S))} \leq \|\psi_S\|_{L^2(N(S))} + \|\nabla\psi_S\|_{L^2(N(S))} \lesssim h_S^{1/2}\|\bar j\|_{L^2(S)} + \|\bar j\,\nabla\varphi\|_{L^2(N(S))} + \|\varphi\,\nabla\bar j\|_{L^2(N(S))}$

  $\leq h_S^{1/2}\|\bar j\|_{L^2(S)} + \|\bar j\|_{L^2(N(S))}\|\nabla\varphi\|_{L^\infty(N(S))} + \|\nabla\bar j\|_{L^2(N(S))}\|\varphi\|_{L^\infty(N(S))} \lesssim h_S^{-\frac{1}{2}}\|\bar j\|_{L^2(S)}$
- $\|j\|_{L^2(S)} \leq \|j-\bar j\|_{L^2(S)} + \|\bar j\|_{L^2(S)} \lesssim \|j-\bar j\|_{L^2(S)} + \dfrac{\int_S j\,\psi_S\, ds}{\|\bar j\|_{L^2(S)}} + \|\bar j - j\|_{L^2(S)}\dfrac{\|\psi_S\|_{L^2(S)}}{\|\bar j\|_{L^2(S)}}$ use (d)+(a), the above & following estimate for $\|r\|_{L^2(N(S))}$

Adaptivity: efficiency local error estimator (proof cont.)

- $h_T \|r\|_{L^2(T)} \lesssim \|u_h - u\|_{H^1(T)} + h_T \|r - \bar{r}\|_{L^2(T)}$, because ...
  - $h_T \|r\|_{L^2(T)} \le h_T \|\bar{r}\|_{L^2(T)} + h_T \|r - \bar{r}\|_{L^2(T)}$
  - Let $\psi \in W_0^{1,\infty}(T)$ with $\int_T \psi \, dx = 1$, $\|\nabla \psi\|_{L^\infty(T)} \lesssim h_T^{-1}$

<u>Ex</u>: (element bubble) · $\psi = c\,\psi_{z_1}\psi_{z_2}\psi_{z_3}$  (Verfürth)

· $\psi = c\,\psi_z$  (Dörfler)

$$\|\bar{r}\|^2_{L^2(T)} = \int_T \bar{r}\,(\bar{r}\psi)\,dx \le \|\bar{r}\|_{H^{-1}(T)} \|\bar{r}\psi\|_{H^1(T)} \lesssim \|\bar{r}\|_{H^{-1}(T)} \|\nabla(\bar{r}\psi)\|_{L^2(T)}$$

Hölder → $\le \|\bar{r}\|_{H^{-1}(T)} \|\bar{r}\|_{L^2(T)} \|\nabla\psi\|_{L^\infty(T)} \lesssim h_T^{-1} \|\bar{r}\|_{H^{-1}(T)} \|\bar{r}\|_{L^2(T)}$

$$\Longrightarrow \quad \|\bar{r}\|_{L^2(T)} \lesssim h_T^{-1} \|\bar{r}\|_{H^{-1}(T)} \le h_T^{-1} \|r\|_{H^{-1}(T)} + h_T^{-1} \|r - \bar{r}\|_{H^{-1}(T)}$$

- $\|r\|_{H^{-1}(T)} = \sup\limits_{w \in H_0^1(T)} \dfrac{\langle r, w \rangle_{H^{-1}, H_0^1}}{\|w\|_{H^1(T)}} = \sup\limits_{w \in H_0^1(T)} \dfrac{a(u_h - u, w)}{\|w\|_{H^1(T)}} \lesssim \|u_h - u\|_{H^1(T)}$

- $\|r - \bar{r}\|_{H^{-1}(T)} = \sup\limits_{w \in H_0^1(T)} \dfrac{\langle r - \bar{r}, w \rangle_{H^{-1}, H_0^1}}{\|w\|_{H^1(T)}} \le \sup\limits_{w \in H_0^1(T)} \dfrac{\|r - \bar{r}\|_{L^2(T)} \|w\|_{L^2(T)}}{\|w\|_{H^1(T)}} \overset{(e)}{\lesssim} h_T \|r - \bar{r}\|_{L^2(T)}$

- $\eta_T(u_h)^2 = h_T^2 \|r\|^2_{L^2(T)} + h_T \sum_{S \in \partial T} \|j\|^2_{L^2(S)}$

$$\lesssim \|u_h - u\|^2_{H^1(T)} + h_T^2 \|r - \bar{r}\|^2_{L^2(T)} + \sum_{S \in \partial T} \Big(\|u_h - u\|^2_{H^1(N(S))} + h_S^2 \|r - \bar{r}\|^2_{L^2(N(S))} + h_S \|j - \bar{j}\|^2_{L^2(S)} \Big) \quad \square$$

Adaptivity: Clément interpolation

The Bramble-Hilbert lemma for nodal interpolation only holds if H^m (or more generally $W^{p,m}$) $\subset C^\ell$ (where typically $\ell=0$). For all other cases, Clément interpolation was invented!

Thm: For $\Omega \subset \mathbb{R}^d$, $m \geq 0$, $p \in [1,\infty]$, $X_h = \{ v \in C^0(\Omega) \mid v|_T \in \mathcal{P}_k(T)\ \forall T \in \mathcal{T},\ v|_{\partial\Omega} = 0 \}$

there exists a linear operator $I_h : L^1(\Omega) \to X_h$ such that

depends on quasiuniformity

$$\| D^t (v - I_h v) \|_{L^q(T)} \lesssim h_T^{\,s - \frac{d}{p} - t + \frac{d}{q}} \| D^s v \|_{L^p(N(T))}$$

$\forall v \in W^{m,p}(\Omega)$, $T \in \mathcal{T}$, $0 \leq t \leq s \leq \min(m, k+1)$ $\quad s - \frac{d}{p} \geq t - \frac{d}{q}$.

In particular, $\| v - I_h v \|_{L^2(T)} \lesssim h_T \| \nabla v \|_{L^2(N(T))}$ & $\| \nabla (v - I_h v) \|_{L^2(T)} \lesssim \| \nabla v \|_{L^2(N(T))}$

Here for 1st order Lagrange finite elements for $H^1_0(\Omega)$:

Def: (Clément interpolation) M_i shall be the support of the piecewise affine hat function in node z_i

- $P_{M_i} : L^2(M_i) \to X_h(M_i) = \{ v_h|_{M_i} \mid v_h \in X_h \}$ is defined as the L^2-orthogonal projection, i.e.

$$\| v - P_{M_i} v \|_{L^2(M_i)} = \inf_{v_h \in X_h(M_i)} \| v - v_h \|_{L^2(M_i)}$$

why?

($P_{M_i} v$ is the solution $u_h \in X_h(M_i)$ of $(u_h, v_h)_{L^2(M_i)} = (v, v_h)_{L^2(M_i)}\ \forall v_h \in X_h(M_i)$, which is well-defined by LM)

- $I_h v = \sum_{z_i \in \mathcal{V} \cap \mathring{\Omega}} (P_{M_i} v)(z_i)\, \varphi_i$ for φ_i the hat function in z_i

Thm: The Clément interpolation satisfies $\| v - I_h v \|_{L^2(T)} \lesssim h_T \| \nabla v \|_{L^2(N(T))}$ & $\| \nabla (v - I_h v) \|_{L^2(T)} \lesssim \| \nabla v \|_{L^2(N(T))}$

Adaptivity: Clément interpolation (proof)

Pf.: For an inner element T_ℓ we have

partition of unity; triangle & Hölder ineq.; sum over triangle nodes

$$\|v - I_h v\|_{L^2(T_\ell)} = \Big\|\sum_i (v - P_{M_i} v(z_i))\psi_i\Big\|_{L^2(T_\ell)} \le \sum_i \|v - P_{M_i} v(z_i)\|_{L^2(T_\ell)}$$

$$\le \sum_i \|v - P_{M_i} v\|_{L^2(T_\ell)} + \|P_{M_i} v - P_{M_i} v(z_i)\|_{L^2(T_\ell)} \le \sum_i \|v - P_{M_i} v\|_{L^2(M_i)} + \|P_{M_i} v - P_{M_i} v(z_i)\|_{L^2(T_\ell)}$$

- with $\bar v_M = \frac{1}{|M|}\int_M v\,dx$ one has

pull back to reference domain, Poincaré, transformation back to M

$$\|v - P_M v\|^2_{L^2(M)} \le \|v - \bar v_M\|^2_{L^2(M)} \lesssim h_M^2 \|\nabla(v - \bar v_M)\|^2_{L^2(M)} = h_M^2 \|\nabla v\|^2_{L^2(M)}$$

- $$\|P_{M_i} v - P_{M_i} v(z_i)\|_{L^2(T_\ell)} \le \sum_k \|P_{M_i} v(z_k) - P_{M_i} v(z_i)\|_{L^2(T_\ell)} = \sum_k |T_\ell|^{1/2} |P_{M_i} v(z_k) - P_{M_i} v(z_i)|$$

$$\le \sum_k |T_\ell|^{1/2} h_{T_\ell} \|\nabla P_{M_i} v\|_{L^\infty(T_\ell)} = \sum_k h_{T_\ell} \|\nabla P_{M_i} v\|_{L^2(T_\ell)} \lesssim h_{T_\ell} \|\nabla P_{M_i} v\|_{L^2(T_\ell)}$$

"inverse inequality" (homework)

- $$\|\nabla P_M v\|_{L^2(T_\ell)} = \|\nabla(P_M v - \bar v_M)\|_{L^2(T_\ell)} = \|\nabla P_M(v - \bar v_M)\|_{L^2(T_\ell)} \le \|\nabla P_M(v - \bar v_M)\|_{L^2(M)} \lesssim \frac{1}{h_M}\|P_M(v - \bar v_M)\|_{L^2(M)}$$

$\|P_M\| \le 1$

$$\le \|v - \bar v_M\|_{L^2(M)} / h_M \lesssim \|\nabla v\|_{L^2(M)}$$

- $$\|\nabla(v - I_h v)\|_{L^2(T_\ell)} = \Big\|\sum_i (\nabla v)\psi_i + (v - P_{M_i} v(z_i))\nabla\psi_i\Big\|_{L^2(T_\ell)} \lesssim \sum_i \|\nabla v\|_{L^2(T_\ell)} + \|v - P_{M_i} v(z_i)\|_{L^2(T_\ell)} / h_{T_\ell}$$

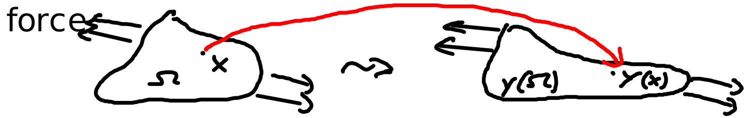
$$\lesssim \sum_i \|\nabla v\|_{L^2(M_i)} \lesssim \|\nabla v\|_{L^2(N(T_\ell))} \qquad \square$$

Perspective: nonlinear elliptic PDE - elasticity

A classical example of a nonlinear elliptic PDE comes from the description of the deformation of elastic (material) bodies.

Def: - In the undeformed, stress-free state the elastic body occupies a sufficiently smooth, open, bounded, connected domain $\Omega \subset IR^3$. This is called the reference configuration.

- The new position of a point $x \in \Omega$ after the deformation is denoted $y(x)$. $y: \Omega \to IR^3$ is called the deformation, $F = Dy: \Omega \to IR^{3'3}$ is called the deformation gradient.



- The coordinate $x \in \Omega$ is called Lagrangean coordinate, i.e. every considered quantity (material density, elastic forces, etc.) at a position $y(x)$ in the deformed material is represented as a function of the original position x of the material point.

- The representation of considered quantities as a function of the new position $y(x)$ is called representation in Eulerian coordinates.

Perspective: nonlinear elliptic PDE - energy density

Def: · A material is called <u>hyperelastic</u> if the total work performed to deform the material is stored in form of mechanical potential energy within the material and if the energy density at location x only depends on x and $F(x)$.

· The <u>stored energy</u> thus has the form $\int_\Omega W(x, F(x))\,dx$ with the stored energy function $W: \Omega \times \mathbb{R}^{3\times 3} \to \mathbb{R}$.

· W is called
- <u>homogeneous</u> if $W(x,F) = W(F)$ $\forall\, x \in \Omega,\ F \in \mathbb{R}^{3\times 3}$
- <u>orientation-preserving</u> if $W(x,F) = \infty$ for $\det F \le 0$
- <u>isotropic</u> if $W(x,F) = W(x, FR)$ $\forall R \in SO(3)$ — first rotate, then stretch
  (i.e. material is equally stiff in all directions)
- <u>rigid body motion invariant</u> if $W(x,I) = 0$ & $W(x, RF) = W(x,F)$ $\forall R \in SO(3)$ — first stretch, then rotate
  (i.e. rigid rotation / translation costs no energy)
- <u>of p-growth</u> if $\exists\, c > 0 : W(x,F) \ge c\,|F|^p - c$

In the following, W shall have all these properties.

Ex: $W(F) = a|F|^p + b \det F^{-q}$ if $\det F > 0$, ∞ else

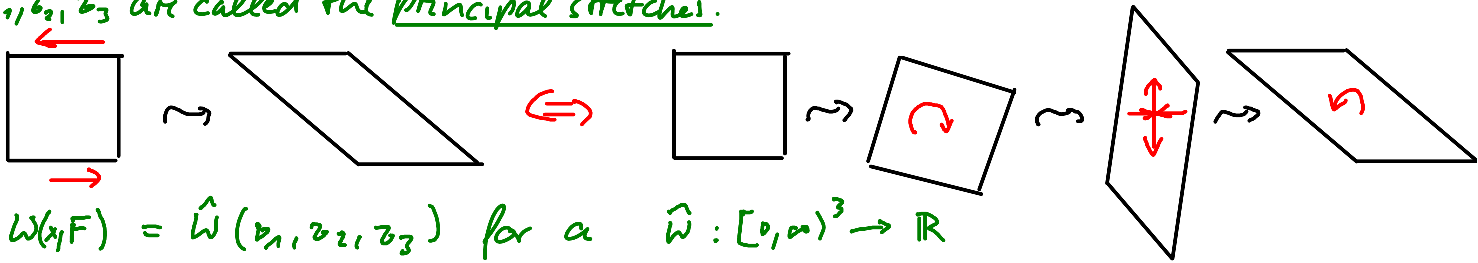
Frobenius norm $|F|^2 = tr(F^T F)$

Perspective: nonlinear elliptic PDE - principal stretches

Rem: By first order Taylor expansion, any deformation y can be written as an affine transformation $y(z) \approx y(x) + F(x)(z-x)$ in a neighbourhood of a point x. $|Fv|$ describes the new length of a vector $v \in \mathbb{R}^3$ after the deformation, $|\operatorname{cof} F n|$ ($\operatorname{cof} A = \det A \, A^{-T}$) the area change of the plane with normal n, $\det F$ the volume change.

Def: Let $F(x) = U \Sigma V$ be the singular value decomposition, then $y(z) \simeq y(x) - F(x)\,x + U\Sigma V z$ is a rotation V, subsequent stretch/compression along the coordinate directions by the singular values $\sigma_1, \sigma_2, \sigma_3$, a subsequent rotation U, and a final translation. $\sigma_1, \sigma_2, \sigma_3$ are called the principal stretches.

Ex:



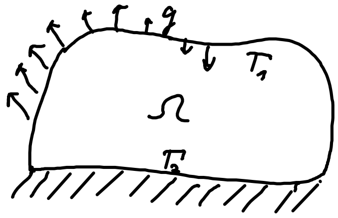
Thm: $W(x,F) = \hat{W}(\sigma_1, \sigma_2, \sigma_3)$ for a $\hat{W} : [0,\infty)^3 \to \mathbb{R}$

Pf: $W(x,F) = W(x, U^T F V^T) = W(x, \Sigma) = W(\Sigma)$ $\square$

I.e. the stored energy only depends on the principal stretches!

Perspective: nonlinear elliptic PDE - variational problem

Let the reference configuration have Lipschitz boundary, the body shall be fixed at $\Gamma_2 \subset \partial\Omega$ with positive measure, at $\Gamma_1 = \partial\Omega \setminus \Gamma_2$ a force density (stress = force per area) $g : \Gamma_1 \to \mathbb{R}^3$ is applied.



if this is violated, set $E[y] = \infty$

The free energy of a deformation y with $y|_{\Gamma_2} = id|_{\Gamma_2}$ is $E[y] = \int_\Omega W(Dy(x))\,dx - \int_{\Gamma_1} y(x)\cdot g(x)\,dx$.

The resulting deformation is a minimizer of the free energy.

minimization problem: $\min\{ E[y] \mid y|_{\Gamma_2} = id|_{\Gamma_2} \}$ ← most natural form of elliptic problems

weak formulation: $y \in W^{1,p}(\Omega)$ satisfies $y|_{\Gamma_2} = id|_{\Gamma_2}$ &

$A:B = tr(A^T B)$ — $\int_\Omega DW(Dy) : D\phi \, dx - \int_{\Gamma_1} \phi \cdot g \, dx = 0 \quad \forall \phi \in W^{1,p}(\Omega)$ with $\phi|_{\Gamma_2} = 0$

strong formulation: y satisfies

$$\underbrace{\mathrm{div}}_{\text{rowwise divergence}}\Big(\underbrace{DW}_{\text{1. Piola-Kirchhoff-stress tensor}}(Dy)\Big) = 0 \quad \text{in } \Omega$$

conservation of linear momentum

$$y(x) = x \quad \text{on } \Gamma_2$$

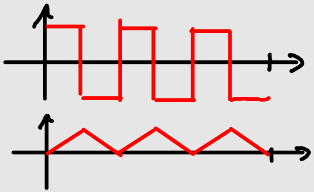
$$DW(Dy)\cdot n = g \quad \text{on } \Gamma_1$$

Perspective: nonlinear elliptic PDE - weak convergence

Def: Let X, X^* be normed vector space & dual space, $(x_i)_{i\in\mathbb{N}} \subset X$, $(y_i)_{i\in\mathbb{N}} \subset X^*$.

- x_i <u>converges weakly</u> to $x \in X$ ($x_i \longrightarrow x$) if $\langle x_i, y\rangle_{X,X^*} \xrightarrow[i\to\infty]{} \langle x,y\rangle \; \forall y \in X^*$
- y_i <u>converges weakly-*</u> to $y \in X^*$ ($y_i \xrightarrow{*} y$) if $\langle x, y_i\rangle_{X,X^*} \xrightarrow[i\to\infty]{} \langle x,y\rangle \; \forall x \in X$
- X is called <u>reflexive</u> if $(X^*)^*$ is isometrically isomorphic to X.
- X is called <u>separable</u> if it contains a countable dense subset.

Ex:
- $L^p(\Omega)$ is reflexive & separable for $p \in (1,\infty)$ $\left((L^p)^{**} = (L^q)^* = L^p \text{ for } \frac{1}{p}+\frac{1}{q}=1\right)$
- $W^{m,p}(\Omega)$ is reflexive & separable for $p \in (1,\infty)$
- $x_i = \operatorname{sgn} \circ \sin(i\,\cdot) \longrightarrow 0$ in $L^p((0,1))$
- $\tilde{x}_i = \int x_i \, dt \longrightarrow 0$ in $W^{1,p}(0,1)$



Thm:
- $x_i \longrightarrow x \quad \Longrightarrow \quad x_i \longrightarrow x$
- $x_i \longrightarrow x \quad \Longrightarrow \quad \{x_i\}$ is bounded
- $f_i \longrightarrow f$ in $W^{1,p}(\Omega)$, $p > \dim(\Omega) \quad \Longrightarrow \quad f_i \longrightarrow f$ in $C^0(\overline{\Omega})$ for subsequence

"Sobolev-embedding"

Perspective: nonlinear elliptic PDE - replacement for Lax-Milgram

__Thm:__ (Banach-Alaoglu) Let $A \subset X^*$ be a bounded subset of the dual space to a separable vector space X, then A contains a weakly-$*$ converging sequence.

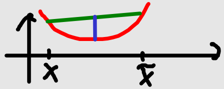
__Cor:__ For X reflexive & X^* separable, any bounded $A \subset X$ contains a weakly converging sequence.

__Thm:__ (Ball) Let $f_n \longrightarrow f$ in $W^{1,p}(\Omega;\mathbb{R}^3)$ with $p > 3$. Then

$\det Df_n \longrightarrow \det Df$ in $L^{p/3}(\Omega)$ and $\operatorname{cof} Df_n \longrightarrow \operatorname{cof} Df$ in $L^{p/2}(\Omega;\mathbb{R}^{3\times 3})$.

= collection of all 2×2 subdeterminants

__Thm:__ (Mazur) Let $x_n \longrightarrow x$, then there exists a sequence $\tilde{x}_n = \sum\limits_{i=n} a_{in} x_i$ of convex combinations (i.e. $a_{in} \in [0,1]$, $\sum\limits_{i=1}^{n} a_{in} = 1$) with $\tilde{x}_n \longrightarrow x$.

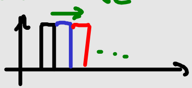
__Thm:__ (Jensen) Let $f: X \to \mathbb{R}$ convex (i.e. $f\left(\frac{x+\tilde{x}}{2}\right) \le \frac{f(x)+f(\tilde{x})}{2}$ $\forall x, \tilde{x} \in X$ 

) and $\sum\limits_{i=1}^{n} a_i x_i$ a convex combination, then $f\left(\sum\limits_{i=1}^{n} a_i x_i\right) \le \sum\limits_{i=1}^{n} a_i f(x_i)$.

__Thm:__ Let $f_n \longrightarrow f$ in $L^1(\Omega)$, then there exists a subsequence $f_{n_i} \longrightarrow f$ pointwise almost everywhere.

__Rem:__ Subsequence is important! $f_n^k(x) = \begin{cases} 1, & x \in \left[\frac{k}{n}, \frac{k+1}{n}\right] \\ 0 & \text{else} \end{cases}$; $f_1 = f_1^0, f_2 = f_2^0, f_3 = f_2^1, f_4 = f_3^0, \dots \xrightarrow{L^1} 0$,

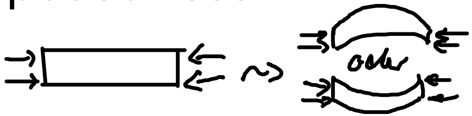
but not pointwise a.e.



__Thm:__ (Fatou) Let $f_n \ge 0$, then $\int_\Omega \liminf\limits_{n\to\infty} f_n \, dx \le \liminf\limits_{n\to\infty} \int_\Omega f_n \, dx$

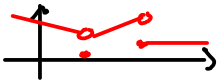
Perspective: nonlinear elliptic PDE - well-posedness

Rem: In general, a solution is not unique, e.g.



Def.: · W is called polyconvex if it can be written as a convex function of the subdeterminants of F, $W(F) = \hat{W}(F, \operatorname{cof} F, \det F)$ with $\hat{W}$ convex.

· $f : X \to \mathbb{R}$ is called lower semi-continuous, if $f(x) \le \liminf_{n\to\infty} f(x_n) \ \forall x_n \to x$



lower semi-continuous

Assumption: W shall be nonnegative, lower semi-continuous, polyconvex, of p-growth with $p > 3$.

Thm: $y \mapsto \int_\Omega W(Dy)\,dx$ is lower semi-continuous with respect to weak convergence in $W^{1,p}(\Omega)$.

Pf: Let $y_n \to y$ in $W^{1,p}(\Omega)$; wlog let $\liminf_{n\to\infty} \int_\Omega W(Dy_n)\,dx = \lim_{n\to\infty} \int_\Omega W(Dy_n)\,dx$. else consider subsequence

$$\overset{\text{Ball}}{\Longrightarrow} \omega_n := (Dy_n, \operatorname{cof} Dy_n, \det Dy_n) \xrightarrow{L^p \times L^{p/2} \times L^{p/3}} (Dy, \operatorname{cof} Dy, \det Dy) =: \omega$$

$$\overset{\text{Mazur}}{\Longrightarrow} \tilde{\omega}_n = \sum_{i=n}^{N_n} a_{in}\,\omega_i \xrightarrow{L^p \times L^{p/2} \times L^{p/3}} \omega \quad \Longrightarrow \quad \tilde{\omega}_{n_i} \to \omega \text{ pointwise a.e.:}$$

$$\int_\Omega W(Dy)\,dx = \int_\Omega \hat{W}(\omega)\,dx \le \int_\Omega \liminf_{i\to\infty} \hat{W}(\tilde{\omega}_{n_i})\,dx \overset{\text{Fatou}}{\le} \liminf_{i\to\infty} \int_\Omega \hat{W}(\tilde{\omega}_{n_i})\,dx$$

$$\overset{\text{Jensen}}{\le} \liminf_{i\to\infty} \int_\Omega \sum_{j=n_i}^{N_{n_i}} a_{jn_i}\,\hat{W}(\omega_j)\,dx = \liminf_{i\to\infty} \sum_{j=n_i}^{N_{n_i}} a_{jn_i} \int_\Omega W(Dy_j)\,dx = \lim_{n\to\infty} \int_\Omega W(Dy_n)\,dx \quad \square$$

Perspective: nonlinear elliptic PDE - well-posedness II

Thm: Let $g \in L^q(\Gamma_1)$, $\frac{1}{q}+\frac{1}{p}=1$. E has a minimizer y with $y|_{\Gamma_2} = id|_{\Gamma_2}$.

Pf: "direct method of the calculus of variations"

0) $E \not\equiv \infty$ and E is bounded below

$$E[id] < \infty \;\&\; E[y] \geq C\|Dy\|^p_{L^p} - C - \underbrace{\int_{\Gamma_1} g\cdot y\,dx}_{\leq \|g\|_{L^q(\Gamma_1)}\|y\|_{L^p(\Gamma_1)} \leq \hat{C}\|g\|_{L^q(\Gamma_1)}\|y\|_{W^{1,p}(\Omega)}} \overset{\text{Poincaré}}{\geq} \tilde{C}\|y\|^p_{W^{1,p}} - C - \hat{C}\|g\|_{L^q}\|y\|_{W^{1,p}} \quad (*)$$

$$\geq -C - \left(1-\frac{1}{p^{p/(p-1)}}\right)\sqrt[p-1]{\frac{(\hat{C}\|g\|_{L^q})^p}{\tilde{C}}}$$

1) Consider "minimizing sequence" $y_1, y_2, \ldots$ with $E[y_n] \longrightarrow \inf_y E[y]$

2) Show that a subsequence converges to some y (in a sense to be chosen adequately):

$$E[y_n] < C \overset{(*)}{\Longrightarrow} \|y_n\|_{W^{1,p}} \text{ is bounded} \Longrightarrow y_{n_i} \rightharpoonup y \text{ in } W^{1,p}(\Omega;\mathbb{R}^3)$$

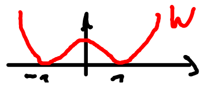
3) Show lower semi-continuity wrt. the above convergence, i.e. $E[y] \leq \liminf_{i\to\infty} E[y_{n_i}] \leq \inf E$:

previous thm. + $\int_{\Gamma_1} g\cdot y_n\,dx \longrightarrow \int_{\Gamma_1} g\cdot y\,dx$ (trace theorem) $\square$

Ex: How can direct method fail?

Rem: (Poly-)convexity is essential! E.g. $y_n(x) = \frac{|nx - \text{round}(nx)|}{n} \xrightarrow{W^{1,p}} 0 \equiv y$,

but $\int_0^1 W(y_n')\,dx = 0 \not\to 1 = \int_0^1 W(y')\,dx$ for $W(a) = (a^2-1)^2$!



Perspective: nonlinear elliptic PDE - discretization

As usual for finite elements we choose a finite element space X_h and define the discrete approximation y_h to $y \in \operatorname{argmin} E$ as $y_h \in \underset{X_h}{\operatorname{argmin}} E$.

In the following we use first order Lagrange elements.

<u>Thm</u>: E has a minimizer in X_h.

Pf: Direct method in X_h; it only remains to show that $y_h \in X_h$, $y_h \xrightarrow{W^{1,p}} y \Rightarrow y \in X_h$.

This follows from $y_h \longrightarrow y$ in C^0. $\square$

Does y_h converge to y as $h \to 0$?

Define
$$E_h[y] = \begin{cases} E[y], & \text{if } y \in X_h \\ \infty & \text{else} \end{cases}$$

$$\Rightarrow \quad y_h \in \operatorname{argmin} E_h[y]$$

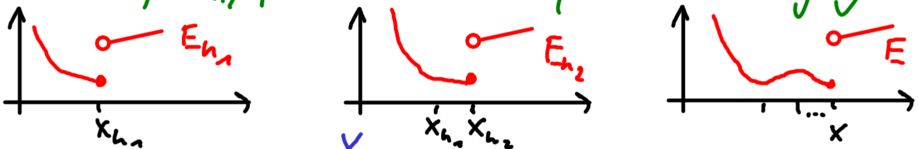
Perspective: nonlinear elliptic PDE - Gamma convergence

Def: · Fix a notion of convergence on space X. A functional $E_h : X \to \mathbb{R}$ Γ-converges to $E : X \to \mathbb{R}$ ($\Gamma(X)-\lim_{h\to 0} E_h = E$), if $\forall h_i \to 0$

- $\forall\, x_i \xrightarrow[i\to\infty]{X} x$: $\liminf_{i\to\infty} E_{h_i}[x_i] \geq E[x]$ "liminf-inequality"
- $\forall\, x \in X\ \exists\, x_i \xrightarrow{X} x$: $\limsup_{i\to\infty} E_{h_i}[x_i] \leq E[x]$ "limsup-inequality"

· The functionals E_h are called equicoercive if $\exists\, C \subset X$ sequentially compact with $\operatorname{argmin} E_h \subset C\ \forall h$.

Thm: If $\Gamma(X)-\lim_{h\to 0} E_h = E$ and the E_h are equicoercive, then any sequence x_{h_i} of minimizers of E_{h_i} possesses a subsequence converging to a minimizer of E.



but no convergence rate! This would depend on the shape of the energy in the minimum (e.g. locally quadratic or quartic).

Pf: · equicoercivity $\Rightarrow x_{h_i} \xrightarrow{X} x \in X$ for a subsequence

· for all $\hat{x} \in X$ there exists $x_i \xrightarrow{X} \hat{x}$ with

$$E[\hat{x}] \geq \limsup_{i\to\infty} E_{h_i}[x_i] \geq \limsup_{i\to\infty} E_{h_i}[x_{h_i}] \geq \liminf_{i\to\infty} E_{h_i}[x_{h_i}] \geq E[x] \qquad \square$$

Perspective: nonlinear elliptic PDE - convergence

<u>Thm</u>: Let
- Ω Lipschitz, Γ_1, $\Gamma_2 = \partial\Omega \setminus \Gamma_1$ Lipschitz,
- $W : \mathbb{R}^{3\times 3} \to \mathbb{R}$ polyconvex, continuous, of p-growth with $p > 3$,
- $W(F) \le C|F|^p + C$ for a $C > 0$,
- $g \in L^q(\Gamma_1)$, $\frac{1}{q} + \frac{1}{p} = 1$,
- X_h $\frac{1}{\beta}$-quasiuniform first order Lagrange finite element space with grid width h,
- $E : W^{1,p}(\Omega) \to \mathbb{R} \cup \{\infty\}$, $E[y] = \begin{cases} \int_\Omega W(\nabla y)\,dx - \int_{\Gamma_1} g \cdot y\,dx, & \text{if } y|_{\Gamma_2} = id \\ \infty & \text{else} \end{cases}$

  $E_h : W^{1,p}(\Omega) \to \mathbb{R} \cup \{\infty\}$, $E_h[y] = \begin{cases} E[y], & \text{if } y \in X_h \\ \infty & \text{else} \end{cases}$.

usually we want $W(F) = \infty$ if $\det F \leq 0$, but this makes argument much more complicated

Then $\Gamma(\text{weak convergence in } W^{1,p}(\Omega))-\lim_{h\to 0} E_h = E$.

<u>Cor</u>: Every sequence $y_{h_i} \in \operatorname{argmin} E_{h_i}$, $h_i \xrightarrow{i\to\infty} 0$, contains a subsequence $y_{h_{i_n}} \xrightarrow[h\to\infty]{W^{1,p}} y \in \operatorname{argmin} E$.

Pf: $E[id] \ge E_{h_i}[y_{h_i}] = E[y_{h_i}] \overset{(*)}{\Longrightarrow} \|y_{h_i}\|_{W^{1,p}}$ is uniformly bounded

$\Rightarrow E_{h_i}$ is equicoercive wrt. weak convergence in $W^{1,p}(\Omega)$. $\square$

Perspective: nonlinear elliptic PDE - convergence proof

Pf : Let $h_i \to 0$.

<u>liminf-ineq.</u> : Let y_i be a sequence with $y_i \xrightarrow{W^{1,p}} y$, then the lower semi-continuity of E implies:

$$E[y] \leq \liminf_{i\to\infty} E[y_i] \leq \liminf_{i\to\infty} E_{h_i}[y_i]$$

<u>limsup-ineq.</u> : Let $y \in W^{1,p}(\Omega)$ be given.

- $W^{2,p}(\Omega)$ is dense in $W^{1,p}(\Omega) \Longrightarrow \forall n \in \mathbb{N}\ \exists\, y^n \in W^{2,p}(\Omega)$ with $\|y^n - y\|_{W^{1,p}} \leq \frac{1}{n}$.
- Set $y_i^n = I_{h_i} y^n \Longrightarrow \|y_i^n - y\|_{W^{1,p}} \leq \|y_i^n - y^n\|_{W^{1,p}} + \|y^n - y\|_{W^{1,p}} \leq C h_i \|y^n\|_{W^{2,p}} + \frac{1}{n}$
- Let i_n such that $C h_{i_n} \|y^n\|_{W^{2,p}} \leq \frac{1}{n}$, and define the sequence $y_i \in X_{h_i}$, $i \in \mathbb{N}$, as $(y_1^1, y_2^1, \dots, y_{i_2-1}^1, y_{i_2}^2, y_{i_2+1}^2, \dots, y_{i_3-1}^2, y_{i_3}^3, y_{i_3+1}^3, \dots)$
- $y_i \xrightarrow{W^{1,p}} y$ (thus also $y_i \longrightarrow y$), since $\|y_i - y\|_{W^{1,p}} \leq \frac{2}{n}$ for all y_i after $y_{i_n}^n$
- choose subsequence y_{i_k} such that $\lim_{k\to\infty} E_{h_{i_k}}[y_{i_k}] = \limsup_{i\to\infty} E_{h_i}[y_i]$ and such that $Dy_{i_k} \longrightarrow Dy$ pointwise a.e.
- still to show : $\lim_{k\to\infty} E_{h_{i_k}}[y_{i_k}] \leq E[y]$

Perspective: nonlinear elliptic PDE - convergence proof (cont.)

We have

- $W(Dy_{i_k}) \longrightarrow W(Dy)$ pointwise a.e.
- $C|Dy_{i_k}|^p + C \xrightarrow{L^1} C|Dy|^p + C$, since $\quad y_{h_{i_k}} \to y$ in $W^{1,p}$

  $$\int_\Omega \Big| C|Dy_{i_k}|^p + C - C|Dy|^p - C \Big| dx = C\int_\Omega \Big| |Dy_{i_k}|^p - |Dy|^p \Big| dx \xrightarrow[k\to\infty]{} 0$$

- $\int_\Omega W(Dy_{i_k})\,dx \longrightarrow \int_\Omega W(Dy)\,dx$ by Lebesgue's dominated convergence
- $\int_{\Gamma_N} g\cdot y_{i_k}\,dx \longrightarrow \int_{\Gamma_N} g\cdot y\,dx$, since $y_{i_k}|_{\Gamma_N} \xrightarrow{L^p} y|_{\Gamma_N}$

$\Longrightarrow \quad E_{h_{i_k}}[y_{i_k}] \longrightarrow E[y] \qquad \square$