Two-transitive Lie groups

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Abstract. Using a characterization of parabolics in reductive Lie groups due to Furstenberg, elementary properties of buildings, and some algebraic topology, we give a new proof of Tits' classification of 2-transitive Lie groups.

Among many other results, Tits classified in [41] all 2-transitive Lie groups. His proof is based on Dynkin's classification of maximal complex subalgebras in complex simple Lie algebras; it is long and depends on consideration of various cases. Since the resulting list of groups is also long (at least in the affine case), it is clear that there cannot be a very short proof of the full classification. On the other hand, Lie theory has developed since the time [41] was written. In particular, Tits himself changed the picture through his theory of buildings (as he pointed out, his paper [41] was one of the motivations for him to invent buildings). The language, the methods, and the terminology have changed since then, and it is natural to look for a new (and shorter) proof of Tits' classification. Note also that the proof presented in [41], IV F 1.2, p. 222, does not cover certain real forms of exceptional groups—a footnote on p. 223 asserts that Tits found a proof for these cases, too, after the manuscript went into print; see also loc.cit., p. 240. The details were never published.

Almost at the same time as Tits, Borel [4] determined all 1-connected spaces X which admit a 2- or 3-transitive Lie group action; however, Borel did not classify the corresponding groups. His proof relies on spectral sequences, Freudenthal's theory of ends, and on the results of Borel and De Siebenthal about homogeneous spaces of positive Euler characteristic.

In this paper, we give a complete proof for Tits' classification. The main ingredients are a characterization of parabolics in Lie groups due to Furstenberg, elementary properties of buildings, some algebraic topology (certainly more elementary than the machinery employed in Borel's work [4]), and representation theory of semisimple (compact) Lie groups.

As we remarked before, the only published proof for the classification is [41]. The classification in the affine case is also stated (but not proved) in Völklein [47], and some

remarks on the strategy of Tits' original proof can be found in Salzmann et al. [36], 96.15 and 96.16.

Related results for other classes of groups are Knop's classification [26] of 2-transitive actions of algebraic groups over algebraically closed fields in arbitrary characteristic (which is achieved by quite different methods), and the classification of all finite 2-transitive groups; see Dixon-Mortimer [11], 7.7 for a description of these groups. In the course of our classification, we recover Knop's result for the special case of complex algebraic groups.

The main results of the classification are as follows.

Theorem A.   *Let G be a locally compact, σ-compact topological transformation group acting effectively and 2-transitively on a space X which is not totally disconnected. Then G is a Lie group and $X \cong G/G_x$ is a connected manifold. The connected component G° is simple if and only if X is compact.*

Note that it is not enough to assume that the group G is locally compact; the full homeomorphism group of any topological manifold, endowed with the discrete topology, satisfies all the other conditions of the theorem.

Theorem B.   *If (G, X) is as in Theorem A, and if X is compact, then X is either the point set of a projective space, or the set of all absolute points of a polarity (of index 1) in a projective space. In the first case, G is (a finite extension of) the little projective group, and in the second case, G is (a finite extension of) the centralizer of the polarity in the little projective group of the projective space.*

The projective spaces in question real, complex, quaternionic, or octonionic, and the possibilities for the groups G are explicitly determined, see Theorem 3.3.

Theorem C.   *If (G, X) is as in Theorem A, and if X is noncompact, then $X \cong \mathbb{R}^m$ is a real vector space, and G is a semidirect product $G = G_x \ltimes \mathbb{R}^m$, where $G_x \leqq \mathrm{GL}_m \mathbb{R}$ is a linear group acting transitively on the nonzero vectors.*

We determine explicitly the connected linear groups which act transitively on the nonzero vectors in Theorem 6.17.

The dichotomy that either X is compact and G° is simple, or that X is noncompact and G of affine type is proved by ideas similar to Borel's, but with a modest amount of algebraic topology. The classification of transitive linear groups depends very much on representation theory.

In case where X is compact, the key ingredient is a characterization of parabolics due to Furstenberg. This characterization was used by Burns-Spatzier [9] in order to classify compact connected buildings with strongly transitive automorphism groups. Using Furstenberg's result, the classification is reduced to a problem about the W-valued distance in spherical buildings. Here, we need some elementary properties of buildings (but the proof does not depend on the classification of spherical Moufang buildings). We rely of course on the classification of real simple Lie groups and their structure theory.

Outline of the classification. The first section collects some basic material on 2-transitive permutations groups. Then we show that a 2-transitive locally compact group (which satisfies some additional hypotheses) is automatically a Lie group. Up to this point, our proof is more or less the same as Tits' original proof; these results can also be found in Salzmann et al. 96.15. After this point, we follow a different line than Tits. First we consider the case where the space X which G acts on is compact. This case is much easier than the noncompact case, since one can use a convenient criterion due to Furstenberg which characterizes parabolics in Lie groups. Using this result, it is not difficult to show that G is essentially a simple (noncompact) Lie group, and that X is a vertex set of the building Δ belonging to G. If the real rank of the group G is at least 2, the building has to be a projective space, and this leads to a full classification in the compact case.

The noncompact case is more involved. Here we use some algebraic topology to prove that X is contractible if it is noncompact. Once this is proved, it is not difficult to see that X is a real vector space, and that G acts through affine-linear transformations. The task is then to classify linear Lie groups acting transitively on nonzero vectors in a real vector space. Such a group acts transitively on the half-rays in the vector space, and we can use the classification of compact Lie groups acting transitively on spheres. In the last section we classify those 2-transitive groups in our list which are Moufang sets. This re-proves and generalizes the classification by Kalscheuer, Tits, and Grundhöfer of all sharply 2-transitive Lie groups.

1. Preliminaries

In this section we collect a few general and simple facts about 2-transitive groups. An *action* of a group G on a nonempty set X is a homomorphism $G \to \mathrm{Sym}(X)$ of G into the symmetric group of X. For $x \in X$, we denote the stabilizer of x by $G_x = \{g \in G \mid g(x) = x\}$. For a subset $U \subseteq G$ we put $U \cdot x = \{g(x) \mid g \in U\}$. If G acts transitively on X (i.e. if $X = G \cdot x$ for some $x \in X$) and if $H \leq G$ is a subgroup, then H acts transitively on X if and only if $G = G_x H$. The *kernel* of an action is the subgroup $G_{[X]} = \bigcap \{G_x \mid x \in X\}$. The action is *effective* if $G_{[X]} = 1$ (often this is called a *faithful* action). If the action is effective, G can be identified with its image in $\mathrm{Sym}(X)$. A transitive action is *regular* if $G_x = 1$ holds for all $x \in X$.

The action of G on X is called *k-transitive*, for $|X| \geq k \geq 2$, if G acts transitively on the set of all k-tuples $(x_1, \ldots, x_k) \in X^k$ with pairwise distinct entries. Clearly, a $(k+1)$-transitive group acts also k-transitively; in particular, it acts transitively on X. If $|X| \geq 3$, then G acts 2-transitively on X if and only if the stabilizer G_x of $x \in X$ acts transitively on $X \setminus \{x\}$ for every $x \in X$. A transitive action (of G on X) is *primitive* if the stabilizer G_x is a maximal subgroup of G, i.e. if $G_x < H \subseteq G$ implies $H = G$.

1.1. Lemma. *If G acts primitively on X, and if $N \trianglelefteq G$ is a normal subgroup, then either $N \leq G_x$ (and thus $N \leq G_{[X]}$), or N acts transitively on X.*

Proof.   See e.g. Dixon-Mortimer [11], Theorem 1.6A(v).   □

The following criterion for primitivity is very convenient. Let $R \subseteq X \times X$ be an equivalence relation, i.e. $xRyRz$ implies xRz, $\mathrm{id}_X \subseteq R$, and $\iota(R) = R$, where $\iota(x, y) = (y, x)$.

Let us call $\mathrm{id}_X = \{(x, x) \mid x \in X\}$ and $X \times X$ the *trivial equivalence relations*. An equivalence relation is *G-invariant* if $g(R) = R$ holds for all $g \in G$ (with respect to the diagonal action of G on $X \times X$).

1.2. Lemma. *A transitive action of G on X is primitive if and only if $X \times X$ contains no nontrivial G-invariant equivalence relations.*

Proof. See e.g. Jacobson [24], Theorem 1.12, or Dixon-Mortimer [11], Corollary 1.5A. □

If G is 2-transitive, then G acts transitively on $Y = X \times X \backslash \mathrm{id}_X$, hence there are no nontrivial *G*-invariant equivalence relations. Thus, a 2-transitive action is primitive.

1.3. Lemma. *Suppose that G acts 2-transitively on X. Then the stabilizer G_x is a maximal subgroup of G.* □

1.4. Lemma. *Suppose that G acts primitively and effectively on X. If $1 \neq A \leq G$ is an abelian normal subgroup, then G is a semidirect product $G = G_x \ltimes A$. The group A acts regularly on X, and if we identify A with X via the evaluation map $ev_x : a \mapsto a(x)$, then each $g \in G_x$ acts by conjugation,*

$$ga(x) = gag^{-1}g(x) = gag^{-1}(x).$$

In particular, G_x acts transitively on $A \backslash \{1\}$ if G acts 2-transitively on X.

Proof. By Lemma 1.3, *A* acts transitively on *X*. A transitive and effective abelian group acts regularly. Thus $A_x = G_x \cap A = 1$, and $G = G_x A$ is a semidirect product. □

In the next lemma, we identify *G* with its image in $\mathrm{Sym}(X)$. The result is used in Section 3.

1.5. Lemma. *Suppose that G acts primitively and effectively on X. If G is not cyclic of prime order, $G \not\cong \mathbb{Z}/p$, then the centralizer of G in the symmetric group is trivial, $\mathrm{Cen}_{\mathrm{Sym}(X)}(G) = 1$. In particular, G is centerless, and the normalizer $\mathrm{Nor}_{\mathrm{Sym}(X)}(G)$ is contained in the automorphism group of G.*

Proof. Let $C = \mathrm{Cen}_{\mathrm{Sym}(X)}(G)$, and assume that $C \neq 1$. Then *C* is a nontrivial normal subgroup in the primitive group $CG \leq \mathrm{Sym}(X)$, hence *C* acts transitively on *X* by Lemma 1.3. Let $g \in G_x$. Then $c(x) = cg(x) = gc(x)$ holds for all $c \in C$, so *g* fixes the orbit $C \cdot x = X$ elementwise. It follows that *G* acts regularly on *X*. Since the action is primitive, *G* has no proper subgroups, thus $G \cong \mathbb{Z}/p$, for some prime *p*. □

2. Two-transitive Lie groups and locally compact groups

We fix some topological terminology. A homeomorphism is denoted by '$\cong$', and a homotopy equivalence by '$\simeq$'. *Unless stated otherwise, all spaces are assumed to be Hausdorff.* An *n-manifold* is a second countable metrizable space which is locally homeomorphic to $\mathbb{R}^n$. In a topological group we assume always that the singletons are closed, so the groups themselves are regular topological spaces. A *Lie group* is a second countable topo-

logical group which is at the same time a smooth manifold, such that the group operations are smooth maps. A *transformation group* is a pair (G, X) consisting of a topological group G, acting as a group of homeomorphisms on a topological space $X \neq \emptyset$, such that the map $G \times X \to X$ is continuous. If G is a Lie group, we call (G, X) a *Lie transformation group*. The connected component of the identity in a topological group G is denoted G°.

A space is called *σ-compact* if it is a countable union of compact subsets. With our conventions, Lie groups are σ-compact. If (G, X) is a transitive transformation group, and if $x \in X$, then the map $G \to G/G_x$ is open, and thus the natural map $G/G_x \to X$ is continuous. We need the following sharpening of this fact, cp. Salzmann et al. [36], 96.8.

2.1. Proposition. *Let (G, X) be a transitive transformation group, and let $x \in X$. If G and X are locally compact, and if G is σ-compact, then the natural map $G/G_x \to X$ is a homeomorphism; in particular, the evaluation map $ev_x : G \to X$, $g \mapsto g(x)$ is open.* $\square$

The Approximation Theorem states that every locally compact group can be approximated by Lie groups in the following sense, cp. Salzmann et al. [36], 93.8. Note that every open subgroup of G contains the connected component G°, and that G° is a closed normal subgroup of G. However, G° need not be open if G is not a Lie group.

2.2. Approximation Theorem. *Let G be a locally compact group.*

(1) *There exists an open (and σ-compact) subgroup $V \leqq G$ such that V/G° is compact.*

(2) *Given a subgroup V as in (1), and given any neighborhood U of $1 \in G$, there exists a compact normal subgroup $N \trianglelefteq V$ of V with $N \subseteqq U$, such that V/N is a Lie group.* $\square$

$$
\begin{array}{ccccccc}
& & & 1 & & & \\
& & & \downarrow & & & \\
& & & N & & \text{compact} & \\
& & & \downarrow & & & \\
1 & \longrightarrow & G^\circ & \longrightarrow & V & \longrightarrow V/G^\circ \longrightarrow & 1 \\
& & & \downarrow & & & \\
& & & V/N & \text{Lie} & & \\
& & & \downarrow & & & \\
& & & 1 & & &
\end{array}
$$

Theorem 2.4 below is essentially due to Tits [41]; the present proof is a simplified version of the proof given in Salzmann et al. [36], 96.15. A space is called *totally disconnected* if the component of every point is trivial. The following useful result is due to Eilenberg [13], 3.1.

2.3. Lemma. *Let X be a connected space, and let*

$$Y = X \times X \backslash \mathrm{id}_X = \{(x, y) \mid x, y \in X, x \neq y\}.$$

Let $\iota(x, y) = (y, x)$. If Y is not connected, then Y has precisely two (necessarily open) components A, B, and $\iota(A) = B$. □

2.4. Theorem. *Let (G, X) be an effective topological transformation group, where G is σ-compact and locally compact, and X is locally compact and not totally disconnected. Suppose that G acts 2-transitively on X. Then G is a Lie group and $X \cong G/G_x$ is a smooth and connected manifold. Every open subgroup $V \leqq G$ acts primitively on X; in particular, G° acts primitively on X. The group G° is noncompact and has at most three orbits in $X \times X$.*

Proof. First we prove that X is connected. Define an equivalence relation R on X by setting xRy if x, y are contained in a connected subset. Since X is not totally disconnected, $R \neq \mathrm{id}_X$. This relation is G-invariant, since G acts by homeomorphisms on X. Thus $R = X \times X$ by Lemma 1.2, and hence X is connected.

Now we show that $G^\circ \neq 1$. If $G^\circ = 1$, then G is totally disconnected, and thus zero-dimensional, see Hewitt-Ross [21], Thm. 3.5. Therefore, $X \cong G/G_x$ is also zero-dimensional, see loc.cit., Thm. 7.11, and X contains arbitrarily small open and closed subsets. This contradicts the fact that X is connected.

Let $Y = \{(x, y) \mid x, y \in X, x \neq y\}$. This is a locally compact space, and G acts transitively on Y; the evaluation map $ev_{(x,y)} : G \to Y$ is open by Proposition 2.1. Let $V \leqq G$ be an open subgroup. Since V contains the normal subgroup G°, it acts transitively on X by Lemma 1.1. For every $(x, y) \in Y$, the orbit $ev_{(x,y)}(V) = V \cdot (x, y)$ is open. By Lemma 2.3, Y has at most two components A, B, so V has at most two orbits A, B in Y. Moreover, $\iota(A) = B$, so $X \times X$ contains no nontrivial V-invariant equivalence relation, i.e. the action of V on X is primitive by Lemma 1.2.

Now choose a cocompact open subgroup $V \leqq G$ as in the Approximation Theorem 2.2 (1), and choose a small neighborhood $U \subseteq V$ of the identity such that $U \cdot x \neq X$. Then U cannot contain a proper normal subgroup $N \trianglelefteq V$, since otherwise $N \cdot x = X$. Therefore, $V^\circ = G^\circ$ is an open (Lie) subgroup of G by Theorem 2.2, and G/G° is discrete and (by σ-compactness of G) countable.

If the open subgroup G° is compact, then its orbits in $X \times X$ are compact. Since there are at most three different orbits A, B, id_X, this would imply that $A \cup B = X \times X \setminus \mathrm{id}_X$ is compact, a contradiction to the fact that $X \setminus \{x\}$ is noncompact (because X is connected). □

For the remainder of this section, we assume that G is a Lie transformation group acting on a connected manifold $X \cong G/G_x$, and that this action is effective and 2-transitive.

We need two results about the connectivity of complements of discrete subsets.

2.5. Lemma. *Let M be a connected manifold of positive dimension. If M is compact or if $\dim(M) \geqq 2$, then $M \setminus \{x\}$ is path connected for all $x \in M$.*

Proof. If $\dim(M) \geqq 2$, then $H_1(M, M \setminus \{x\}) = 0 = \tilde{H}_0(M)$, and the exact sequence

$$\to H_1(M, M \setminus \{x\}) \to \tilde{H}_0(M \setminus \{x\}) \to \tilde{H}_0(M) \to 0$$

shows that $\tilde{H}_0(M \backslash \{x\}) = 0$, so $M \backslash \{x\}$ is also path connected. If $\dim(M) = 1$ and if M is compact, then $M \cong \mathbb{S}^1$ is a circle, and $M \backslash \{x\} \cong \mathbb{R}$ is path connected. $\quad\square$

2.6. Lemma. *If X is compact or if $\dim(X) > 1$, then the connected component G° acts also 2-transitively on X.*

Proof. The connected component $G^\circ \leqq G$ is a normal subgroup, hence G° acts transitively on X by Lemma 1.3. The subgroup $G^\circ \leqq G$ is open, hence $(G^\circ)_x$ is open in G_x. Let $y \in X \backslash \{x\}$. The evaluation map $ev_y : g \mapsto g(y)$, $G_x \to X \backslash \{x\}$ is open by Proposition 2.1; since $X \backslash \{x\}$ is connected, $ev_y\big((G^\circ)_x\big) = X \backslash \{x\}$, i.e. $(G^\circ)_x$ acts transitively on $X \backslash \{x\}$. $\quad\square$

2.7. Lemma. *If G has an abelian normal subgroup $A \neq 1$, then $A \cong \mathbb{R}^n$ is a vector group.*

Proof. The group A acts regularly on X by Lemma 1.4, hence A is connected (by Proposition 2.1). A connected abelian n-dimensional Lie group is of the form $\mathbb{R}^k / \mathbb{Z}^k \times \mathbb{R}^{n-k}$, for some $k \in \{0, \ldots, n\}$, see Onishchik-Vinberg [33], Ch. 1, §2, Prop. 3, p. 29. If $k \geqq 1$, then $\mathbb{R}^k / \mathbb{Z}^k$ contains elements of order l for all $l \in \mathbb{N}$. This is not possible by Lemma 1.4 (all elements of A different from the identity have to be conjugate under G_x), so $k = 0$.

In particular, the existence of a nontrivial proper abelian normal subgroup implies that X is noncompact. The converse will be proved in Section 4: if X is noncompact, then G has a nontrivial proper abelian normal subgroup.

2.8. Proposition. *If G has no nontrivial proper normal abelian subgroup, then the connected component G° is simple.*

Proof. Assume that G has no nontrivial proper abelian normal subgroup. Let $\sqrt{G}$ denote the radical of G, i.e. the largest normal connected solvable subgroup, see Onishchik-Vinberg [33], Ch. 1, §4, 6°. Let $1 < Z < \cdots < \sqrt{G}$ denote the derived series of $\sqrt{G}$. If $\sqrt{G} \neq 1$, then Z is a nontrivial normal abelian subgroup in G, since it is characteristic in $\sqrt{G}$. Thus $\sqrt{G} = 1$. Note also that $\sqrt{G} = \sqrt{G^\circ}$. Therefore G° is semisimple and centerless and thus a direct product of simple Lie groups, see Salzmann et al. [36], 94.23.

If $\dim(X) \geqq 2$ or if X is compact, then G° acts 2-transitively on X by Lemma 2.6. If $\dim(X) = 1$ and if X is noncompact, then $X \cong \mathbb{R}$. By Brouwer's result, no centerless semisimple Lie group acts transitively on $\mathbb{R}$, see Salzmann et al. [36], 96.30, so this case cannot occur.

Assume now that $\dim(G) \geqq 2$, and that $G^\circ = G_1 \times G_2$, with $G_1 \neq 1$ simple. Suppose that $G_2 \neq 1$. Then G_1 and G_2 are normal in G° and thus transitive on X. Suppose that $g_1 \in G_1$ fixes x, and that $g_2 \in G_2$. Then $g_2(x) = g_2 g_1(x) = g_1\big(g_2(x)\big)$, so g_1 fixes $g_2(x)$ for all $g_2 \in G_2$. Since $G_2 \cdot x = X$, this implies that $g_1 = 1$, i.e. G_1 acts regularly on X. We identify X with G_1; then the action of G_1 on X is the standard left regular action $(g, x) \mapsto gx$ of G_1 on itself. The centralizer of this action $\mathrm{Cen}_{\mathrm{Sym}(X)}(G_1)$ is isomorphic to G_1 with the action $(g, x) \mapsto xg^{-1}$. It follows that $G_2 \cong G_1$, with the action $(g_1, g_2)(x) = g_1 x g_2^{-1}$. The stabilizer of $1 \in X$ is the diagonal subgroup $\{(g, g) \mid g \in G_1\}$; its orbits in X are the conjugacy classes of G_1. But a centerless simple Lie group contains a torus $\mathbb{R}/\mathbb{Z}$; in particular, it contains

elements of arbitrary finite order, and thus infinitely many conjugacy classes. This is a contradiction to the 2-transitivity of G°. Therefore $G_2 = 1$ and $G = G_1$ is simple. $\square$

If we assume in addition that X is compact, then there is a shortcut in the last step of the proof above: if G° is a product of at least two simple factors, then each simple factor acts regularly, and thus G° is compact, a contradiction to Theorem 2.4.

3. The case when X is compact

In this section we classify all pairs (G, X), where G is a Lie group acting effectively and 2-transitively on a compact connected manifold X. By 2.6, 2.7, and 2.8, G° is a simple centerless Lie group which acts 2-transitively on X. Recall the definition of a parabolic subgroup in an arbitrary semisimple Lie group H (of noncompact type). Suppose that

$$H = KAU$$

is an Iwasawa decomposition for H (the unipotent subgroup U is denoted 'N' in most books on Lie groups; we use U, since N has a different meaning with BN-pairs), see Helgason [20], Ch. IX, §1, Thm. 1.3. Let $K_0 = \mathrm{Cen}_K(A)$ denote the K-centralizer of A (this is the *reductive anisotropic kernel* of H, see Tits [42], p. 39). Then

$$B = K_0 A U$$

is a subgroup, a *minimal parabolic subgroup* of H, see Helgason [20], Ch. IX, §1, Warner [48], p. 56. A *parabolic subgroup* is any subgroup of H which is conjugate to an overgroup of B. There are precisely $2^{\dim(A)}$ conjugacy classes of parabolics in H (including the classes of H and B), see Warner [48], Thm. 1.2.1.1. If we put $N = \mathrm{Nor}_H(K_0 A)$, then (B, N) is a BN-pair of rank $\dim(A)$ for the group H, see Warner [48], p. 68.

3.1. Theorem. *Let $H \leqq \mathrm{PSL}_{m+1}\mathbb{R}$ be a closed semisimple subgroup, and consider the standard projective action of $\mathrm{PSL}_{m+1}\mathbb{R}$ on the real projective space $\mathbb{R}\mathrm{P}^m$. Let $X \subseteq \mathbb{R}\mathrm{P}^m$ be a closed H-invariant subset. Assume that the following two conditions are satisfied.*

(a) *If $\emptyset \neq Y \subseteq X$ is closed and H-invariant, then $Y = X$.*

(b) *If $\emptyset \neq Z \subseteq X \times X$ is closed and H-invariant, then $Z \cap \mathrm{id}_X \neq \emptyset$.*

Then $X = H/H_x$ is a homogeneous space of H, and H_x is a parabolic subgroup of H.

Proof.   Condition (a) says that the action is *minimal* in the sense of Furstenberg [14], p. 278–279, and condition (b) says that it is *proximal*; by the very definition, it is what Furstenberg calls *projective*. By Proposition 4.3 in loc.cit., p. 280, there exists an H-equivariant map from the *maximal Furstenberg boundary* $B(H)$ of H onto X. Since we assumed that H is semisimple, the discussion in loc.cit., p. 280 shows that there is an H-equivariant homeomorphism $B(H) \cong H/B$ between the Furstenberg boundary and the coset space H/B, where $B \leqq H$ is a minimal parabolic. The space $H/B \cong K/K_0$ is compact (it is the chamber set of the canonical building $\Delta(H)$ associated to H), and thus H_x contains a conjugate of B, i.e. H_x is a parabolic. $\square$

3.2. Proposition. *If X is compact, and if G is a simple connected Lie group acting effectively and 2-transitively on X, then the stabilizer $P = G_x$ is a maximal parabolic subgroup of the simple group G.*

Proof. Let $\mathfrak{p}$ denote the Lie algebra of P, with $k = \dim_{\mathbb{R}}(\mathfrak{p})$. The G-normalizer of $\mathfrak{p}$ contains P; by maximality of P, it coincides with P. Consider the adjoint action of G on the Grassmann manifold $\mathrm{Gr}_k(\mathfrak{g})$ of all k-dimensional real subspaces of the Lie algebra $\mathfrak{g}$ of G. The stabilizer of $\mathfrak{p} \in \mathrm{Gr}_k(\mathfrak{g})$ is P, as we just saw. The Grassmannian embeds G-equivariantly into the real projective space $\mathbf{P}(\bigwedge^k \mathfrak{g})$ of the k-th exterior power of the real vector space $\mathfrak{g}$. The G-orbit of $\bigwedge^k \mathfrak{p} \in \mathbf{P}(\bigwedge^k \mathfrak{g})$ is therefore G-equivariantly homeomorphic to $X = G/P$. Since G acts 2-transitively, the conditions (a) and (b) in Furstenberg's Theorem 3.1 are clearly satisfied, hence P is a parabolic subgroup, and, by 2-transitivity on X, a maximal one. $\square$

Recall that the *real rank* of a semisimple Lie group H is the real vector space dimension of $\mathfrak{a}$, where $\mathfrak{h} = \mathfrak{k} + \mathfrak{a} + \mathfrak{u}$ is an Iwasawa decomposition of the Lie algebra $\mathfrak{h}$ of H.

3.3. Theorem. *Let (G, X) be an effective 2-transitive transformation group. Assume that G is locally compact and σ-compact, and that X is compact and not totally disconnected. Then G is a Lie group and the connected component G° is simple and noncompact. There exists a subgroup $\hat{G}$ of the automorphism group of G° such that either $G = G^{\circ}$, or $G = \hat{G}$. Both $\hat{G}$ and G° act 2-transitively on X. Topologically, the space X is either a sphere or a projective space over $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, or $\mathbb{O}$. The possibilities for the pair (G°, X) are as follows.*

(a) *If the real rank of G° is 1, then X is a sphere, and there are the following possibilities.*

G°	X	$\hat{G}$	$|\hat{G}/G^{\circ}|$	
$\mathrm{PEO}_{n,1}\mathbb{R}$	$\mathbb{S}^{n-1}$	$\mathrm{PO}_{n,1}\mathbb{R}$	2	$(n \geqq 2)$
$\mathrm{PSU}_{n,1}\mathbb{C}$	$\mathbb{S}^{2n-1}$	$\mathrm{P\Gamma U}_{n,1}\mathbb{C}$	2	$(n \geqq 2)$
$\mathrm{PU}_{n,1}\mathbb{H}$	$\mathbb{S}^{4n-1}$	$\mathrm{PU}_{n,1}\mathbb{H}$	1	$(n \geqq 2)$
$\mathrm{F}_{4(-20)}$	$\mathbb{S}^{15}$	$\mathrm{F}_{4(-20)}$	1	

The simple subgroup $\mathrm{PEO}_{n,1}\mathbb{R} = (\mathrm{PO}_{n,1}\mathbb{R})^{\circ}$ is the subgroup generated by all Eichler (*or Siegel*) transformations, see e.g. Hahn-O'Meara [19], p. 214. Alternatively, the group $\mathrm{PEO}_{n,1}\mathbb{R}$ can be described as the commutator group of $\mathrm{PO}_{n,1}\mathbb{R}$, or as the connected component $(\mathrm{PO}_{n,1}\mathbb{R})^{\circ}$.

(b) *If the real rank k of G° is at least 2, then X is a projective space of rank k, and there are only the following possibilities.*

G°	X	$\hat{G}$	$|\hat{G}/G^{\circ}|$	
$\mathrm{PSL}_{k+1}\mathbb{R}$	$\mathbb{R}\mathrm{P}^k$	$\mathrm{PGL}_{k+1}\mathbb{R}$	2	$(k \text{ odd})$
$\mathrm{PSL}_{k+1}\mathbb{R}$	$\mathbb{R}\mathrm{P}^k$	$\mathrm{PSL}_{k+1}\mathbb{R}$	1	$(k \text{ even})$
$\mathrm{PSL}_{k+1}\mathbb{C}$	$\mathbb{C}\mathrm{P}^k$	$\mathrm{P\Gamma L}_{k+1}\mathbb{C}$	2	
$\mathrm{PSL}_{k+1}\mathbb{H}$	$\mathbb{H}\mathrm{P}^k$	$\mathrm{PSL}_{k+1}\mathbb{H}$	1	
$\mathrm{E}_{6(-26)}$	$\mathbb{O}\mathrm{P}^2$	$\mathrm{E}_{6(-26)}$	1	$(k = 2)$

Note that there are the following isomorphisms of simple Lie groups:

$$\mathrm{PSL}_2\mathbb{R} \cong \mathrm{PSU}_{1,1}\mathbb{R} \cong \mathrm{PEO}_{2,1}\mathbb{R},$$

$$\mathrm{PSL}_2\mathbb{C} \cong \mathrm{PEO}_{3,1}\mathbb{R},$$

$$\mathrm{PU}_{1,1}\mathbb{H} \cong \mathrm{PEO}_{4,1}\mathbb{R},$$

$$\mathrm{PSL}_2\mathbb{H} \cong \mathrm{PEO}_{5,1}\mathbb{R}.$$

Proof of Theorem 3.3.   We know already by 2.6, 2.7, and 2.8 that $H = G^\circ$ is a simple centerless Lie group which acts 2-transitively on X, and by Proposition 3.2 the stabilizer $P = H_x$ is a maximal parabolic subgroup. If the real rank of H is 1, then Table V in Ch. X, p. 518 in Helgason [20] shows that (H, X) is as in (a). Note that a group with a BN-pair of rank 1 is the same as a 2-transitive group; thus, each simple Lie group of real rank 1 is a 2-transitive group. Of course, this follows also by direct inspection of the list of groups.

Now suppose that the real rank k of H is at least 2. Then we consider the irreducible spherical building $\Delta = \Delta(H)$ of H. Theorem 4.3 in the next section shows that Δ is the building associated to a projective space (of finite rank, as all buildings related to Lie groups are of finite rank), and that the parabolic H_x is the stabilizer of a point in this projective space.

Then H is one of the groups $\mathrm{PSL}_{k+1}\mathbb{F}$, for $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$, and Δ is a projective geometry of rank k over $\mathbb{F}$, or $k = 2$, and Δ is the Cayley plane and $H = \mathrm{E}_{6(-26)}$. This follows either from the topological Fundamental Theorem of Projective Geometry (as proved by Kolmogorov [27], see also Kühne-Löwen [30] and the survey by Grundhöfer-Löwen [17]), or from Cartan's classification of simple Lie groups and Tits' classification [42] of their Tits diagrams (unjustly sometimes called "Satake diagrams") and their relative diagrams, see Helgason [20], Table VI, Ch. X, p. 532. In any case, we obtain the list of groups G° and spaces X in (b).

By Lemma 1.5, G is a subgroup of the automorphism group of $H = G^\circ$. The outer automorphism groups of all simple Lie groups have been determined by Takeuchi [39], see also Wolf [49], 8.8, p. 263. In case (b), we have the additional condition that G preserves the conjugacy class of P, and thus G has to act by type-preserving automorphisms on the projective space. Thus, we obtain the list of groups $\hat{G}$ given in (a) and (b).   □

3.4. Corollary. *If G is 3-transitive, then $X \cong \mathbb{S}^{n-1}$ and $G \cong \mathrm{PO}_{n,1}\mathbb{R}$, for $n \geqq 2$, or $G \cong \mathrm{PEO}_{n,1}\mathbb{R}$ and $n \geqq 3$. None of these groups is 4-transitive.*

If the action is sharply 3-transitive (i.e. regular on triples of pairwise distinct points), then $G \cong \mathrm{PGL}_2\mathbb{R}$ or $G \cong \mathrm{PSL}_2\mathbb{C}$.

Proof. If the building $\Delta = \Delta(G^\circ)$ is a projective geometry of rank $k \geqq 2$, then there exists triples of collinear points and triples of noncollinear points, so G cannot be 3-transitive. Therefore, a 3-transitive group is one of the groups given in Theorem 3.3 (a). The unitary groups $\mathrm{P\Gamma U}_{n,1}\mathbb{C}$ and $\mathrm{PU}_{n,1}\mathbb{H}$ cannot act 3-transitively if $n \geqq 2$: choose three linearly independent vectors u, v, w with

$$(u|v) = (v|w) = (w|u) \quad \text{and} \quad (u|u) = (v|v) = (w|w) = 0$$

(we denote the corresponding hermitian form by $(x|y) = -\bar{x}_0 y_0 + \bar{x}_1 y_1 + \cdots + \bar{x}_n y_n$). Then $x = u + v\mathtt{i}$ is also isotropic, but no semilinear map fixing the subspace spanned by $\{u, v\}$ can move the subspace spanned by x to the subspace spanned by w. There is a more geometric way to see this, which is also valid for the group $F_{4(-20)}$. These groups can be viewed as subgroups of $P\Gamma L_{n+1}\mathbb{C}$, $PGL_{n+1}\mathbb{H}$, acting on the set $\mathscr{Q}$ of absolute points of a hyperbolic polarity in the projective geometry (for the group $F_{4(-20)}$, the corresponding projective space is the Cayley plane $PG_2(\mathbb{O})$, see [36], Ch. 1). If we fix two distinct points $x, y \in \mathscr{Q}$, then we fix the line L joining these two points. Now $L \cap \mathcal{O}$ is a proper subset of $\mathscr{Q}$, and $L \cap \mathscr{Q} \neq \{x, y\}$ if $n \geqq 2$; in fact, $L \cap \mathscr{Q} \cong \mathbb{S}^{\dim_{\mathbb{R}} \mathbb{F} - 1}$ for $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$. This excludes the unitary groups and $F_{4(-20)}$.

The group $PEO_{2,1}\mathbb{R} \cong PSL_2\mathbb{R}$ is not 3-transitive; the remaining orthogonal groups are 3-transitive, as can be checked. Finally, none of these groups is 4-transitive: Let $p, q, r, s \in \mathscr{Q}$ be four distinct points such that the lines pq and rs intersect in an interior point of $\mathscr{Q}$. Then r, s cannot be moved to two points r', s' (fixing p, q) such that $r's'$ intersects pq in an exterior point. If the action is sharply 3-transitive, then $\dim(G) = 3\dim(X)$, and thus only the groups $PGL_2\mathbb{R}$ and $PSL_2\mathbb{C}$ remain. $\quad\square$

3.5. Corollary. *If X is a complex homogeneous space (i.e. if X is a complex manifold and if G preserves the complex structure), then $X = \mathbb{C}P^n$ and $G \cong PSL_{n+1}\mathbb{C}$.*

Proof. Suppose that X is a complex manifold, and that the G-action preserves the complex structure. Then $\dim(X)$ is even. In case (a) of Theorem 3.3, $X \cong \mathbb{S}^{2k}$ is a sphere, and $G^\circ = PEO_{2k+1,1}\mathbb{R}$. The compact subgroup $K = SO(2k+1)$ acts transitively on X, and the isotropy representation of K_x on the tangent space $T_x X \cong \mathbb{R}^{2k}$ is the standard action of $SO(2k)$. If $k \geqq 2$, then this action is not complex linear. Thus $k = 1$ and $G^\circ = PEO_{3,1}\mathbb{R} \cong PSL_2\mathbb{C}$.

In case (b) of 3.3, X is an even-dimensional real projective space or a complex or quaternionic projective space. Again we consider the isotropy representation of K_x for a suitable compact subgroup $K \leqq G^\circ$. For $\mathbb{R}P^{2k}$, we have $K = SO(2k+1)$, and $K_x = O(2k)$ acts in the standard way on $T_x X \cong \mathbb{R}^{2k}$. This action preserves no complex structure on the tangent space, not even for $k = 1$. For $X = \mathbb{H}P^n$ we take $K = Sp(n+1)$; then $K_x = Sp(n) \cdot Sp(1)$ acts in the standard way (from the left and right) on $T_x X \cong \mathbb{H}^n$; again, there is no invariant complex structure for all $k \geqq 1$. Thus the only possibility is that $G^\circ = PSL_{n+1}\mathbb{C}$ in the standard action on $\mathbb{C}P^n$. Complex conjugation is not $\mathbb{C}$-linear, hence $G = G^\circ$. $\quad\square$

4. Digression: two-transitive actions on buildings

In this section we consider the following situation. Δ is a building (thick, and of arbitrary, possibly infinite rank), and G is a group of (type-preserving) automorphisms of Δ. We assume that the action of G on one type of residues of Δ is 2-transitive. This is a purely combinatorial problem, and we make no topological assumptions. After I had finished a first version of this section, Bernhard Mühlherr pointed out to me that the book by Brouwer-Cohen-Neumaier [8] contains related results.

Recall that a *Coxeter diagram* is an undirected labeled graph whose nodes are labeled by some set I. For each pair $i, j \in I$, there is a number $m_{ij} = m_{ji} \in \mathbb{N} \cup \{\infty\}$, with $m_{ii} = 1$, and $m_{ij} \geqq 2$ if $i \neq j$. If the nodes i, j are not adjacent (joined by an edge), then $m_{ij} = 2$. If the nodes i, j are adjacent, and if $m_{ij} \geqq 4$, then the edge joining them is labeled with the number m_{ij}. The corresponding *Coxeter group* is the group generated by a set $S = \{s_i \mid i \in I\}$ of generators labeled by I, with relations $(s_i s_j)^{m_{ij}} = 1$ (for $m_{ij} \neq \infty$). The pair (W, S) is called a *Coxeter system*. The Coxeter system is *irreducible* if its Coxeter graph is connected. Reducible Coxeter systems are products (in a natural sense). For $J \subseteq I$, the subgroup generated by $S_J = \{s_J \mid j \in J\}$ is denoted W_J; one can prove that the pair (W, S_J) is again a Coxeter system. We refer to the books by Bourbaki [5] and Humphreys [22], and to Ch. 2 in Ronan [35]. Recall the Coxeter diagram $\mathbf{A}_k$ which is defined as
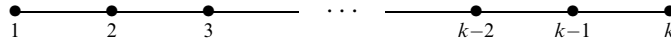
We denote the limit of these Coxeter diagrams (as k goes to infinity) by $\mathbf{A}_\omega$; the corresponding Coxeter diagram is thus
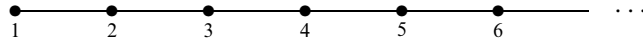
The next result can also be extracted from Cooperstein [10], cp. Brouwer-Cohen-Neumaier [8], Thm. 10.2.3, p. 300.

4.1. Proposition. *Let $(W, \{s_i \mid i \in I\})$ be an irreducible Coxeter system, where I has (possibly infinite) cardinality κ. Let J be a subset of I and assume that W acts 2-transitively on W/W_J. Then $\kappa \leqq \aleph_0$, and either the Coxeter system is of type $\mathbf{A}_k$, with $J = \{1\}$ or $J = \{k\}$, or of type $\mathbf{A}_\omega$, and $J = I \backslash \{1\}$ (the nodes of the diagram are labeled as above).*

Proof. The proof is very simple. We divide it into four basic steps.

(1) *J has corank 1, i.e. $I \backslash J$ is a singleton.*

Since W_J has to be a maximal subgroup, $I \backslash J$ is a singleton, which we denote by $\{1\}$. Thus we have $W = W_J \cup W_J s_1 W_J$. Now we use the standard description of shortest double coset representatives: every element w which has the property that all reduced expressions of this element start and end with s_1 represents a unique double coset $W_J w W_J$.

(2) *In the Coxeter diagram, the node 1 has at most one neighbor.*

Assume that node 1 has two different neighboring nodes, say $2, 3$. Then $s_1 s_2 s_3 s_1$ is a reduced word representing a third double coset, $W_J s_1 s_2 s_3 s_1 W_J$ (note that the order of $s_2 s_3$ is not important for this argument).

(3) *No node in the Coxeter digram has more than two neighbors.*

Assume otherwise; let $i \neq 1$ be a node with three different neighbors. Let 1-2-3-$\cdots$-$(i-1)$-i be a shortest path in the Coxeter diagram, and let j, k be distinct neighbors of node i different from $i-1$. Then $s_1 s_2 \cdots s_{i-1} s_i s_j s_k s_i s_{i-1} \cdots s_1$ is a reduced word which represents

another shortest double coset representative. This implies already that I is finite or countable, and that the Coxeter diagram of W is a string diagram (with at least one end).

(4) *The Coxeter diagram has no multiple bonds, $m_{ij} \leqq 3$.*

Assume that 1-2- $\cdots$ -i-j is a shortest path in the diagram, and that i, j are joined by an edge labeled $m_{ij} \geqq 4$. Then $s_i s_j$ has order at least 4. The reduced word $s_1 s_2 \cdots s_{i-1} s_i s_j s_i s_{i-1} \cdots s_1$ represents another shortest double coset representative.

By (1)–(4), the diagram has only simple bonds and does not branch, and 1 is an end node. $\square$

Now we consider 2-transitive actions on buildings. Since we allow infinite rank, we view buildings as *chamber systems*, see Tits [45] or Ronan [35]. A chamber system is a set $\mathscr{C}$, endowed with a collection $\{\sim_i \,|\, i \in I\}$ of equivalence relations. For a subset $J \subseteq I$, the equivalence relation generated by $\{\sim_j \,|\, j \in J\}$ is denoted $\sim_J$. Given a Coxeter system $(W, \{s_i \,|\, i \in I\})$, a *building*

$$\Delta = (\mathscr{C}, \{\sim_i \,|\, i \in I\}, \delta, W, \{s_i \,|\, i \in I\})$$

is a chamber system $\mathscr{C}$, endowed with a metric $\delta : \mathscr{C} \times \mathscr{C} \to W$ taking its values in a Coxeter group (subject to certain axioms which can be found e.g. in Ronan's book [35], Ch. 3). An automorphism of a building (in this sense) is a permutation of $\mathscr{C}$ which preserves the equivalence classes (and thus the metric δ). A *J-residue* in a building is a J-equivalence class.

For $i \in I$, let V_i denote the set of all residues of type $I \setminus \{i\}$ in the building. Define an *incidence relation* $*$ as follows: two residues are incident if their intersection is nonempty. The datum $\Gamma = (\{V_i\}_{i \in I}, *)$ determines a *diagram geometry*. Suppose now that the building is of type $\mathbf{A}_\kappa$, for $2 \leqq \kappa \leqq \omega$. Put $\mathscr{P} = V_1$ and $\mathscr{L} = V_2$ (the nodes are labeled as before).

4.2. Theorem. *If $\kappa \geqq 2$, then the geometry $(\mathscr{P}, \mathscr{L}, *)$ obtained from an $\mathbf{A}_\kappa$-building is a projective space.*

Proof. The proof indicated by Tits in [45], p. 540 applies (also for the case $\kappa = \omega$). $\square$

4.3. Theorem. *Let Δ be a thick irreducible building over some type set I, let G be a group of type preserving automorphisms of Δ acting 2-transitively on the collection of all J-residues, for some J. Then Δ is of type $\mathbf{A}_k$ or $\mathbf{A}_\omega$; the corresponding diagram geometry can be identified with a projective space (possibly of infinite rank), and G acts 2-transitively on the points of this space.*

Proof. Let X, Y, Z be distinct J-residues. Since the stabilizer of X is a maximal subgroup, J has corank 1, i.e. $I \setminus J$ is a singleton. There is a unique element $w \in \delta(X \times Y) \subseteq W$ such that $\ell(w)$ minimizes the lengths $\ell(\delta(x, y))$, where x, y run through all chambers in X, Y, see Scharlau-Dress [12]. Since the action is 2-transitive on the J-residues, the same w gives the minimal distance for elements in Z. Thus $W = W_J \cup W_J w W_J$; in particular, W acts 2-transitively on W/W_J. By Proposition 4.1, Δ is of type $\mathbf{A}_k$ or $\mathbf{A}_\omega$. $\square$

5. The case when X is noncompact

In this section we assume that G is a 2-transitive effective Lie transformation group acting on X, and that X is noncompact. We show that in this case $X \cong \mathbb{R}^n$ is a real vector space, and that G splits as a semidirect product $G = G_x \ltimes \mathbb{R}^n$. In view of Lemma 2.7 and Proposition 2.8, it suffices to prove that G° cannot be simple if X is noncompact. We need some more facts about transformation groups. For a group H acting on a set X, we denote the fixed point set by

$$X^H = \{x \in X \mid H \cdot x = \{x\}\}.$$

Recall the Malcev-Iwasawa Theorem, cp. Salzmann et al., 93.10.

5.1. Malcev-Iwasawa Theorem. *Let G be a connected locally compact group.*

(1) *There exists a maximal compact subgroup $K \leqq G$. If $L \leqq G$ is a compact subgroup, then $gLg^{-1} \leqq K$ for a suitable $g \in G$.*

(2) *There exist closed subgroups $A_1, \ldots, A_m \leqq G$ isomorphic to $(\mathbb{R}, +)$, such that G is directly spanned as $G = A_1 A_2 \cdots A_m K$ (i.e. every $g \in G$ can in a unique way be written as a product $g = a_1 \cdots a_m k$, with $a_i \in A_i$ and $k \in K$). In particular, G is homeomorphic to $K \times \mathbb{R}^m$, and $G/K \cong \mathbb{R}^m$ is contractible.*

We also need the following result.

5.2. Lemma. *Let K be a compact connected Lie group. Then K is abelian if and only if $\pi_3(K) = 0$.*

Proof. It is a well-known result that $\pi_3(H) \cong \mathbb{Z}$ holds for every almost simple compact Lie group H, see Onishchik [32], §17, Theorem 2, p. 257. A compact connected Lie group is topologically a product of compact almost simple Lie groups and a torus; it is abelian if and only if no nontrivial simple factors exist. □

5.3. Corollary. *Let G be an almost simple Lie group. If G is not locally isomorphic to $SL_2\mathbb{R}$, then $\pi_3(G) \neq 0$.*

Proof. Since we consider only the third homotopy group, we may assume that G is centerless. Let $K \leqq G$ be a maximal compact subgroup. A direct inspection of Table V in Helgason [20], Ch. X, p. 518 or Onishchik-Vinberg [33], Table 9, p. 312 shows that K is not abelian, provided that $G \neq PSL_2\mathbb{R}$. Since we have a homotopy equivalence $K \simeq G$ by the Malcev-Iwasawa Theorem 5.1, the claim follows. □

For a proof of Whitehead's Theorem below see Spanier [38], Ch. 7, Sec. 6, Corollary 24.

5.4. Theorem. *Let $f : X \to Y$ be a continuous map. If X and Y have the homotopy type of CW-complexes (e.g. if X and Y are manifolds, see Bredon [6], V, 1.6) and if the induced map $\pi(f)_\bullet : \pi_\bullet(X) \to \pi_\bullet(Y)$ is an isomorphism, then f is a homotopy equivalence.*

Combining these two results, we obtain the next lemma.

5.5. Lemma. *Let $Z = H/H_z$ be a homogeneous space of a Lie group H. Assume that H and H_z are connected. Let $K \leqq H$ be a maximal compact subgroup, such that K_z is a maximal compact subgroup of H_z. Then there is a homotopy equivalence $K/K_z \simeq H/H_z$.*

Proof. The spaces H/K and H_z/K_z are contractible by the Malcev-Iwasawa Theorem 5.1. A fibration of CW-complexes with contractible fibres (with a contractible base, resp.) induces a homotopy equivalence between the base and the total space (the fibre and the total space, resp.) by Whitehead's Theorem 5.4. Consider the fibrations

$$
\begin{array}{ccccc}
 & & K/K_z & & \\
 & & \downarrow f & \searrow^{gf} & \\
H_z/K_z & \longrightarrow & H/K_z & \xrightarrow{g} & H/H_z \\
 & & \downarrow & & \\
 & & H/K & &
\end{array}
$$

The maps f and g are homotopy equivalences, and so is their composite gf. □

The following theorem can be proved under much weaker assumptions; however, the present version, which is taken from Bredon's book [6] suffices for our purposes.

5.6. Theorem. *Let X be a connected noncompact n-manifold, and let K be a compact Lie group acting smoothly or locally smoothly on X. Assume that K has an $(n-1)$-dimensional orbit. Then there are only the following two possibilities.*

(a) *$X/K \cong \mathbb{R}$ and every K-orbit is principal. In this case, X decomposes as $X \cong K/K_x \times \mathbb{R}$, with trivial K-action on $\mathbb{R}$.*

(b) *$X/K \cong [0, \infty)$, and there exists precisely one nonprincipal orbit $K \cdot z = Z \cong K/K_z$. Let $K_x \leqq K_z$ be the stabilizer of a point $x \in X \backslash Z$. Then $K_z/K_x \cong \mathbb{S}^m$ is a sphere, and X is equivariantly homeomorphic to the $(m+1)$-vector bundle associated to the orthogonal sphere bundle $K_z/K_x \to K/K_x \to K/K_z$.*

Proof. For the concept of a locally smooth action see Bredon's book [6], Ch. IV. A smooth action of a compact Lie group is locally smooth by loc.cit., VI, 2.4; by IV, 3.1, principal orbits exist. Our claim is thus a slight reformulation of loc.cit., IV, 8.1. □

5.7. Corollary. *If $n \geqq 2$ and if K has a fixed point, then X is equivariantly homeomorphic to euclidean space $\mathbb{R}^n$, and K acts transitively on $\mathbb{S}^{n-1}$.*

Proof. In this case, $Z = \{z\}$. A vector bundle over Z is the same as a real vector space $\mathbb{R}^n$. □

The proof of the next lemma was kindly pointed out by R. Löwen.

5.8. Lemma. *Let M be a connected manifold of dimension $n = \dim(M) \geqq 3$. If $D \subseteq M$ is a closed discrete subset, then the inclusion $M \backslash D \hookrightarrow M$ induces an isomorphism of fundamental groups.*

Proof. Assume first that $D = \{x_0\}$ is a singleton. Let $\phi : (\mathbb{R}^n, 0) \to (M, x_0)$ be a coordinate chart and put $B_\varepsilon = \{\phi(x) \,|\, |x| \leq \varepsilon\}$. Then $M = B_2 \cup (M \backslash B_1)$, and $M \backslash B_1 \simeq M \backslash \{x_0\}$. Moreover, $B_2 \cap (M \backslash B_1) = B_2 \backslash B_1 \simeq \mathbb{S}^{n-1}$ is 1-connected, and we can apply the Seifert-Van Kampen Theorem to the diagram
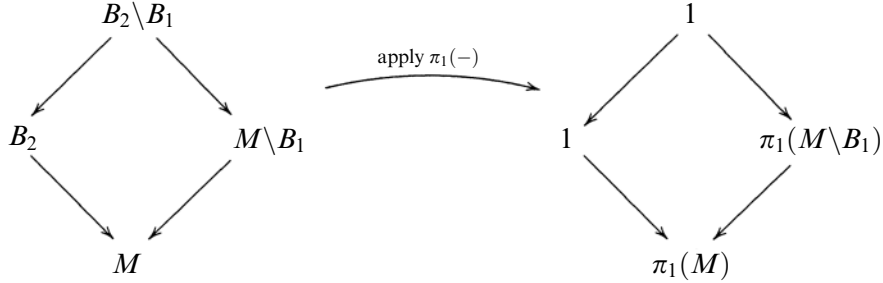
to conclude that $\pi_1(M \backslash \{x_0\}) \to \pi_1(M)$ is an isomorphism (see for example Bredon [7], Ch. III, Cor. 9.5). An easy induction shows that $\pi_1(M \backslash D) \to \pi_1(M)$ is an isomorphism, provided that D is finite.

Assume now that $D \subseteq M$ is an arbitrary closed discrete set not containing the base point. We show first that $\pi_1(M \backslash D) \to \pi_1(M)$ is surjective. So let $\beta : [0,1] \to M$ be a path representing an element $[\beta]_{\pi_1(M)}$ of $\pi_1(M)$, and let U be a relatively compact open neighborhood of the image of β. Then $U \cap D$ is finite, and by our previous discussion, $\pi_1(U \backslash D) \to \pi_1(U)$ is an isomorphism. Let $\alpha : [0,1] \to U \backslash D$ be a path representing an element $[\alpha]_{\pi_1(U \backslash D)}$ in $\pi_1(U \backslash D)$ whose image $[\alpha]_{\pi_1(U)}$ in $\pi_1(U)$ represents the same element $[\beta]_{\pi_1(U)}$ as β. Then α and β represent the same element in $\pi_1(M)$, i.e. $[\alpha]_{\pi_1(M)} = [\beta]_{\pi_1(M)}$.

Next we prove that the map is injective. Let γ be a path representing an element of $\pi_1(M \backslash D)$, and assume that its image $[\gamma]_{\pi_1(M)}$ is homotopic to the constant path. Fix such a homotopy $F : [0,1] \times [0,1] \to M$, and let V be a relatively compact open neighborhood of the image of F. Then γ represents the identity element in $\pi_1(V)$. By our previous discussion, $[\gamma]_{\pi_1(V \backslash D)}$ is also the identity element, and thus $[\gamma]_{\pi_1(M \backslash D)}$ is the identity element. $\square$

5.9. Definition. Let G be a connected Lie transformation group acting transitively on a manifold X. Assume that the following holds: for every $x \in X$, the fixed point set $X^{G_x} = \{y \in X \,|\, G_x \cdot y = \{y\}\}$ of G_x is discrete (and necessarily closed), and G_x acts transitively on its complement $X \backslash X^{G_x}$. Then we call G *almost 2-transitive* on X.

5.10. Proposition. *Let G be a connected Lie group acting almost 2-transitively on a noncompact 1-connected n-manifold X, with $n \geqq 3$. Then G acts 2-transitively on X, and $X \cong \mathbb{R}^n$ is contractible.*

Proof. By assumption, $\pi_0(G) = 1 = \pi_1(X)$. From the exact sequence

$$\pi_1(X) \to \pi_0(G_x) \to \pi_0(G)$$

we see that G_x is connected, and $X \backslash X^{G_x}$ is 1-connected by Lemma 5.8. Let $y \in X \backslash X^{G_x}$, so $G_x/G_{x,y} \cong X \backslash X^{G_x}$. The exact sequence

$$\pi_1(X \backslash X^{G_x}) \to \pi_0(G_{x,y}) \to \pi_0(G_x)$$

shows that $G_{x,y}$ is connected. We choose a maximal compact subgroup $K \leqq G$ such that $K_x \leqq G_x$ and $K_{x,y} \leqq G_{x,y}$ are maximal compact subgroups. Note that each of these groups is connected.

Let $\mathbb{F}_2$ denote the field with two elements. Then $H_k(X; \mathbb{F}_2) = 0 = H_k(X \backslash X^{G_x}; \mathbb{F}_2)$ for all $k \geqq n$, because X and $X \backslash X^{G_x}$ are noncompact manifolds, see Bredon [7], Ch. VI, Cor. 7.12, e.g. Moreover, $\dim\big(H_n(X, X \backslash X^{G_x}; \mathbb{F}_2)\big) = |X^{G_x}|$ (by excision), and the exact sequence

$$\to H_n(X; \mathbb{F}_2) \to H_n(X, X \backslash X^{G_x}; \mathbb{F}_2) \to H_{n-1}(X \backslash X^{G_x}; \mathbb{F}_2) \to$$

shows that $\dim\big(H_{n-1}(X \backslash X^{G_x}; \mathbb{F}_2)\big) \geqq |X^{G_x}|$. But $X \backslash X^{G_x} \simeq K_x/K_{x,y}$ is homotopy equivalent to a compact connected manifold of dimension strictly less than n by Lemma 5.5. Thus $\dim(K_x/K_{x,y}) = n - 1$, and $H_{n-1}(K_x/K_{x,y}; \mathbb{F}_2) \cong \mathbb{F}_2$. In particular, $|X^{G_x}| = 1$, and thus G_x acts transitively on $X \backslash \{x\}$, i.e. G is 2-transitive. The orbit $K_x \cdot y \cong K_x/K_{x,y}$ has codimension 1 in X. By Corollary 5.7, $X \cong \mathbb{R}^n$. $\square$

5.11. Proposition. *Let G be a 2-transitive Lie group acting on a noncompact connected manifold X of dimension $\dim(X) = n \geqq 3$. Then X is 1-connected, and thus by Proposition 5.10, $X \cong \mathbb{R}^n$ is contractible.*

Proof. Replacing G by the universal cover of its connected component, we may assume that G is 1-connected. Thus we have an isomorphism $\pi_1(X) \cong \pi_0(G_x)$. Let $H = (G_x)^\circ$. The exact sequence

$$\to \pi_1(G) \to \pi_1(G/H) \to \pi_0(H) \to$$

shows that $Z = G/H$ is 1-connected, and $Z = G/H \xrightarrow{p} G/G_x = X$ is the universal covering of X. Let $F = p^{-1}(x)$. We claim that H acts transitively on $Z \backslash F$. Let $z \in Z \backslash F$. Then $p(H \cdot z) = H \cdot p(z) = X \backslash \{x\}$, because H is open in G_x and because $X \backslash \{x\}$ is connected. Thus the H-orbits in $Z \backslash F$ are n-dimensional and hence open. But $Z \backslash F$ is connected, so $H \cdot z = Z \backslash F$. Thus G acts almost 2-transitively on G/H. By Proposition 5.10, the action is in fact 2-transitive and thus $H = G_x$. $\square$

Finally, we consider the low-dimensional cases.

5.12. Lemma. *Let G be a 2-transitive Lie group acting on a noncompact connected manifold X of dimension $\dim(X) = n \leqq 2$. Then $X \cong \mathbb{R}^n$ is contractible.*

Proof. The only noncompact connected 1-manifold is $\mathbb{R}$.

If $\dim(X) = 2$, then $H = G^\circ$ acts 2-transitively by Lemma 2.6. Assume that X is a noncompact surface. If H is not simple, then $X \cong \mathbb{R}^2$ by Lemma 2.7. Assume that H is simple. Since H acts 2-transitively, $\dim(H) \geqq 4$, and in particular $H \neq \mathrm{PSL}_2\mathbb{R}$. Let K be a maximal compact subgroup of H. Since $H \neq \mathrm{PSL}_2\mathbb{R}$, the group K is not abelian, and thus

$\pi_3(H) = \pi_3(K) \neq 0$ by Corollary 5.3. Since X is a noncompact surface, the only homotopy group of X which is possibly nontrivial is the fundamental group. The exact sequence

$$\to \pi_4(X) \to \pi_3(H_x) \to \pi_3(H) \to \pi_3(X) \to$$

shows that $\pi_3(H_x) \neq 0$, hence H_x contains a connected nonabelian compact subgroup $L \leqq (H_x)^\circ$ by Lemma 5.2. Choose $y \in X \backslash X^L$. Then $L \cdot y \cong \mathbb{S}^1$ is a circle, and $X \cong \mathbb{R}^2$ by Corollary 5.7. (We will see below that this case is impossible.) $\square$

5.13. Corollary. *If G is an effective 2-transitive Lie group acting on X and if X is noncompact, then G° is not simple.*

Proof.   By Proposition 5.11 and Lemma 5.12, the space X is contractible. Assume that $H = G^\circ$ is simple. Since X is contractible, the stabilizer H_x is connected, and $H_x \hookrightarrow H$ is a homotopy equivalence by Whitehead's Theorem 5.4. Thus H_x contains a maximal compact subgroup K of G. But the maximal compact subgroups in a noncompact simple Lie group are maximal subgroups, see Helgason [20], Ch. VI, Ex. A3(iv), p. 276 and p. 567, hence $H_x = K$, so $X = H/K$ is a Riemannian symmetric space, and H preserves the metric; in particular, H has infinitely many orbits on $X \times X$, contradicting Theorem 2.4. $\square$

Combining this result with our previous analysis, we obtain the following final result for the noncompact case.

5.14. Theorem. *Let G be a locally compact and σ-compact group acting effectively and 2-transitively on a locally compact, noncompact, not totally disconnected space X. Then G is a Lie group and has a normal vector subgroup $\mathbb{R}^n \trianglelefteq G$ acting regularly on $X \cong \mathbb{R}^n$. Moreover, $G = G_x \ltimes \mathbb{R}^n$ is a semidirect product of $\mathbb{R}^n$ and a point stabilizer G_x.*

If $n = 1$, then $G = \mathrm{AGL}_1\mathbb{R} = \mathrm{GL}_1\mathbb{R} \ltimes \mathbb{R}$.

If $n \geqq 2$, then the connected stabilizer $(G_x)^\circ$ is a reductive linear Lie group acting transitively on the nonzero vectors of $\mathbb{R}^n$. All possibilities for $(G_x)^\circ$ (and thus for $G^\circ = (G_x)^\circ \ltimes \mathbb{R}^n$) are determined in Theorem 6.17. None of these actions is 3-transitive. If G acts sharply 2-transitive (i.e. if the two-point stabilizers are trivial), then $n = 1, 2, 4$, and $G_x = (G_x)^\circ$ is one of the groups given in Corollary 7.4.

If X is a complex manifold and if G preserves the complex structure, then G° is one of the groups given in Theorem 6.17 (d). $\square$

In particular, we obtain Knop's result [26] for characteristic 0:

5.15. Corollary. *If G is a complex connected Lie group which acts complex analytically and 2-transitively on a complex manifold X, then either*

(a) $X \cong \mathbb{C}\mathrm{P}^n$ *and* $G = \mathrm{PSL}_{n+1}\mathbb{C}$, *or*

(b) $X \cong \mathbb{C}^n$ *and* $G_x \cong \mathrm{SL}_n\mathbb{C}$ *or* $G_x \cong \mathrm{GL}_n\mathbb{C}$, *or*

(c) $X \cong \mathbb{C}^{2n}$ *and* $G_x \cong \mathrm{Sp}_{2n}\mathbb{C}$ *or* $G_x \cong \mathrm{Sp}_{2n}\mathbb{C} \cdot \mathbb{C}^*$.

In case (b) *or* (c), $G \cong G_x \ltimes \mathbb{C}^n$.

Note however that in the case $X \cong \mathbb{C}^n$, there exist several real, noncomplex 2-transitive groups by Theorem 6.17.

6. Transitive groups acting on $\mathbb{R}^m \backslash 0$

In this section we classify all closed linear Lie groups acting transitively on the non-zero vectors of a finite dimensional real vector space. The main ingredients in our proof are the classification of compact connected linear Lie groups acting transitively on spheres, and the representation theory of compact Lie groups.

6.1. Theorem. *Let $m \geq 2$, and let $K \leq \mathrm{SO}(m)$ be a closed connected subgroup. If K acts transitively on the sphere $\mathbb{S}^{m-1} = \left\{ x \in \mathbb{R}^m \,\middle|\, |x| = 1 \right\}$, then K is (up to automorphisms of $\mathrm{SO}(m)$) one of the following groups.*

K	$\mathbb{R}^m$	m
$\mathrm{SO}(n)$	$\mathbb{R}^n$	n
$\mathrm{SU}(n)$	$\mathbb{C}^n$	$2n$
$\mathrm{U}(n)$	$\mathbb{C}^n$	$2n$
$\mathrm{Sp}(n)$	$\mathbb{H}^n$	$4n$
$\mathrm{Sp}(n) \cdot \mathrm{U}(1)$	$\mathbb{H}^n$	$4n$
$\mathrm{Sp}(n) \cdot \mathrm{Sp}(1)$	$\mathbb{H}^n$	$4n$
G_2	$\mathrm{Pu}(\mathbb{O})$	7
$\mathrm{Spin}(7)$	$\mathbb{O}$	8
$\mathrm{Spin}(9)$	$\mathbb{O} \oplus \mathbb{O}$	16

Besides the standard inclusions between the classical groups

$$(e.g. \ \mathrm{Sp}(k) \leq \mathrm{SU}(2k) \leq \mathrm{U}(2k) \leq \mathrm{SO}(4k) \ etc.),$$

the inclusions in the special dimensions 7, 8, *and* 16 *are as follows* (*note that* $\mathrm{SU}(4) \cong \mathrm{Spin}(6)$ *and* $\mathrm{Sp}(2) \cong \mathrm{Spin}(5)$).
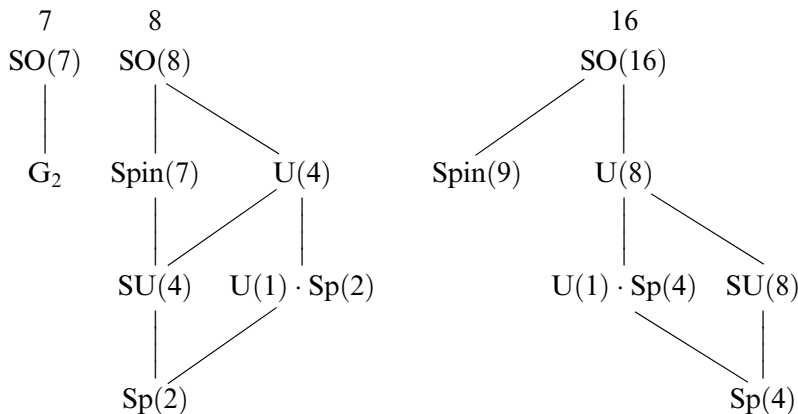
All exceptional phenomena in dimensions 7, 8, 16 *are related to the Cayley plane* $\mathrm{PG}_2\mathbb{O}$; *see Salzmann et al.* [36], *Ch. I.*

Proof.   We could use the classification of compact Lie groups acting transitively on spheres, due to Montgomery-Samelson [31] and Borel [2], [3]; see also Poncet [34], Besse [1], 7.13, Onishchik [32], §18, Theorem 3 (i), Salzmann et al. [36], 96.20–23, and Kramer [28], Ch. 6. However, we actually need only the classification of transitive subgroups of $\mathrm{SO}(n)$, which is much easier and follows directly from Onishchik's classification of factorizations of $\mathrm{SO}(n)$, see [32], p. 227 and [15], II, §4.5, p. 144. $\square$

6.2. Corollary.   *Suppose that* $m \geqq 3$. *Then the commutator group* $K' = [K, K]$ *acts also transitively on* $\mathbb{S}^{m-1}$.

Proof.   This follows either by direct inspection of the list, or by a simple homotopy-theoretic argument, see Onishchik [32], §5, Prop. 9, or Grundhöfer-Knarr-Kramer [18], Lem. 1.3. $\square$

6.3. Proposition.   *Let* V *be a finite dimensional real vector space of dimension* $m \geqq 3$, *and let* $H \leqq \mathrm{GL}(V)$ *be a closed connected group. Suppose that* H *acts transitively on the nonzero vectors of* V. *Then* H *is reductive, and consequently* $H' = [H, H]$ *is a semisimple subgroup of* $\mathrm{SL}(V)$. *If* $K \leqq H$ *is a maximal compact subgroup and if* $|-|$ *is a* K-*invariant norm on* V, *then* K *and* $[K, K]$ *act transitively on the sphere* $\mathbb{S}^{m-1} = \{x \in V \mid |x| = 1\}$. *Consequently,* K *is one of the groups given in Theorem* 6.1.

Proof.   Clearly, H acts irreducibly on $\mathbb{R}^m$, and thus H is reductive, see Salzmann et al. [36], 95.2, 95.6. Let $K \leqq H$ be a maximal compact subgroup, let $|-|$ be a K-invariant norm, and let v be a vector with $|v| = 1$. There is a homotopy equivalence $K/K_v \simeq H/H_v \cong \mathbb{R}^m \backslash \{0\}$ by Lemma 5.5. Thus $K/K_v \simeq \mathbb{S}^{m-1}$, and in particular $\dim(K/K_v) = m - 1$. By domain invariance, $K \cdot x = \mathbb{S}^{m-1}$, and by Corollary 6.2, the commutator group $[K, K]$ is also transitive on $\mathbb{S}^{m-1}$. $\square$

6.4. Corollary.   *There exists a simple subgroup* $H_1 \leqq H$ *such that* $K \cap H_1$ *acts transitively on* $\mathbb{S}^{m-1}$.

Proof.   By Proposition 6.3, $[K, K] \leqq [H, H]$ acts transitively on $\mathbb{S}^m$. Theorem 6.1 shows in each case that K has a simple factor which acts transitively on $\mathbb{S}^{m-1}$. We can choose a simple factor $H_1 \leqq [H, H]$ containing this simple factor of K. $\square$

Now we determine all possibilities for the semisimple group $[H, H]$. The following facts will be useful.

6.5. Theorem.   *Let* $H \leqq \mathrm{SL}_m\mathbb{R}$ *be a closed semisimple subgroup. After conjugation with some element in* $\mathrm{SL}_m\mathbb{R}$, *the Cartan decomposition of* $\mathfrak{sl}_m\mathbb{R} = \mathfrak{so}(m) \oplus \mathfrak{p}$ *is also a Cartan decomposition of* $\mathfrak{h}$,

$$\mathfrak{h} = \big(\mathfrak{h} \cap \mathfrak{so}(m)\big) \oplus (\mathfrak{h} \cap \mathfrak{p}).$$

Proof.   See Onishchik-Vinberg [33], Ch. 5, Theorem 4, p. 261. $\square$

6.6. Lemma. *Let* $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ *be a Cartan decomposition of a simple noncompact Lie algebra. Then*

$$[\mathfrak{p}, \mathfrak{p}] = \mathfrak{k}.$$

In particular, if $\mathfrak{l} \leqq \mathfrak{g}$ *is a subalgebra containing* $\mathfrak{p}$*, then* $\mathfrak{l} = \mathfrak{g}$*.*

Proof. Let $\mathfrak{h} = [\mathfrak{p}, \mathfrak{p}] \leqq \mathfrak{k}$. Using the Jacobi identity, it is clear that $[\mathfrak{k}, \mathfrak{h}] \leqq \mathfrak{h}$. Since $[\mathfrak{p}, \mathfrak{h}] \leqq \mathfrak{p}$, we conclude that $\mathfrak{h} + \mathfrak{p}$ is an ideal in $\mathfrak{g}$, hence $\mathfrak{h} + \mathfrak{p} = \mathfrak{g} = \mathfrak{k} + \mathfrak{p}$. This is a direct sum decomposition with $\mathfrak{h} \leqq \mathfrak{k}$, therefore $\mathfrak{h} = \mathfrak{k}$. $\square$

We use the following facts from representation theory. We rely on Tits [43] and [44], on the Reference Chapter in Onishchik-Vinberg [33], and on Chapter 9 in Salzmann et al. [36].

6.7. Remarks on representation theory. Associated to a real semisimple Lie group G is its weight lattice; every weight π is a linear combination with integral coefficients of the so-called fundamental weights π_i. The dominant weights (i.e. the weights where all coefficients are nonnegative) correspond bijectively to the equivalence classes of (finite dimensional) complex irreducible representations of G. We denote the complex irreducible module associated to the dominant weight π by $R(\pi)$. The Galois group of $\mathbb{C}/\mathbb{R}$ acts on the weight lattice via $\pi \mapsto \bar{\pi}$; if $\pi = \bar{\pi}$ then $R(\pi)$ admits a real or a quaternionic structure (i.e. a complex semilinear endomorphism ϕ with $\phi^2 = \mathrm{id}$ resp. $\phi^2 = -\mathrm{id}$). There is a corresponding map β from the invariant weights into the Brauer group $\mathrm{Br}(\mathbb{R}) = \{\mathbb{R}, \mathbb{H}\}$. There are three types of real irreducible representations of G:

(1) If $\pi \neq \bar{\pi}$, then π is of complex type and $_\mathbb{R}R(\pi) = R(\pi)$ is a real irreducible G-module.

(2) If $\pi = \bar{\pi}$ and $\beta_\pi = \mathbb{R}$, then π is of real type. There exists a unique real irreducible G-module $_\mathbb{R}R(\pi)$, and $R(\pi) \cong {}_\mathbb{R}R(\pi) \otimes \mathbb{C}$.

(3) If $\pi = \bar{\pi}$ and $\beta_\pi = \mathbb{H}$, then π is of quaternionic type. In this case, $_\mathbb{R}R(\pi) = R(\pi)$ is a real irreducible module, and $R(\pi)$ admits a quaternionic structure.

The Reference Chapter in Onishchik-Vinberg [33] contains tables which we will frequently use to explain how exterior powers $\bigwedge^k V$, symmetric powers $S^k V$, and tensor products $V^{\otimes k}$ of complex irreducible modules V can be decomposed.

We fix some notation. We denote the ring of $n \times n$-matrices with entries in F by $F(n)$. The notation for Lie groups and algebras is as in Onishchik-Vinberg [33]. In particular, we denote the quaternion unitary group $\mathrm{U}_n\mathbb{H}$ by $\mathrm{Sp}(n)$. Let

$$S(n) = \{X \in \mathbb{R}(n) \mid X^T = X\} \quad \text{and} \quad S_0(n) = \{X \in S(n) \mid \mathrm{tr}(X) = 0\}.$$

The Cartan decomposition of $\mathfrak{sl}_m\mathbb{R}$ is $\mathfrak{sl}_m\mathbb{R} = \mathfrak{so}(m) \oplus S_0(m)$.

6.8. Lemma. *Let* $H \leqq \mathrm{SO}(n)$ *be a subgroup. Then there is a standard action of* H *on* $S_0(n)$ *which is given by* $X \mapsto hXh^T$*. The following actions of subgroups of* $\mathrm{SO}(n)$ *on* $S_0(n)$ *are* $\mathbb{R}$*-irreducible for* $n \geqq 2$*:*

H	n
$SO(n)$	n
G_2	7
$Spin(7)$	8

Proof. For the orthogonal groups, this follows from the fact that every quadratic form can be diagonalized by conjugation with an element of $SO(n)$. If $0 \neq U \leq S_0(n)$ is an $SO(n)$-invariant subspace, then U intersects the space A consisting of diagonal traceless matrices nontrivially. Moreover, the $SO(n)$-stabilizer of A induces the symmetric group $Sym(n)$ on the diagonal matrices. Thus U contains A, whence $U = S_0(n)$. (The action of $SO(n)$ is in fact a *polar action*; it is the isotropy representation of the Riemannian symmetric space $SL_n\mathbb{R}/SO(n)$. The principal orbits are isoparametric submanifolds; the orbit types form the building $\Delta(SL_n\mathbb{R})$, i.e. the real projective geometry of rank $n-1$.)

For the groups G_2 and $Spin(7)$ we use representation theory. The complex module corresponding to the action of G_2 on the set $Pu(\mathbb{O}) \cong \mathbb{R}^7$ of pure octonions is $R(\pi_1)$, and

$$S(7) \otimes \mathbb{C} \cong S^2 R(\pi_1) = R(2\pi_1) \oplus \mathbb{C}.$$

The Galois group of $\mathbb{C}/\mathbb{R}$ acts trivially on the weights, and all representations of G_2 are of real type, see Tits [43], p. 42. Thus $S_0(7) \cong {}_\mathbb{R}R(2\pi_1)$ is a real irreducible G_2-module.

The reasoning for $Spin(7)$ is similar. The complex module for the 8-dimensional representation is $R(\pi_3)$, and $S^2 R(\pi_3) = R(2\pi_3) \oplus \mathbb{C}$, see Onishchik-Vinberg [33], p. 301. This module is of real type, see Tits [43], p. 31, hence ${}_\mathbb{R}R(2\pi_3) \otimes \mathbb{C} \cong R(2\pi_3)$. Thus $S_0(8) \cong {}_\mathbb{R}R(2\pi_3)$ is a real irreducible $Spin(7)$-module. $\square$

Now we determine all closed noncompact semisimple subgroups of $SL_m\mathbb{R}$ which contain one of the compact groups in Theorem 6.1.

6.9. Proposition. *Let $H \leq SL_m\mathbb{R}$ be a closed noncompact semisimple subgroup, for $m \geq 3$. If $SO(m) < H$, then $H = SL_m\mathbb{R}$. If $m = 7$ and if $G_2 < H$, then $H = SL_7\mathbb{R}$. If $m = 8$ and if $Spin(7) < H$, then $H = SL_8\mathbb{R}$.*

Proof. Let $\mathfrak{h}$ denote the Lie algebra of H. We may assume that $\mathfrak{h} \leq \mathfrak{sl}_m\mathbb{R} = \mathfrak{so}(n) \oplus S_0(n)$ is embedded as in Theorem 6.5. Let $K = SO(m) \cap H$. Then $\mathfrak{h} \cap S_0(m)$ is a nonzero K-module. Since we assume that $K = SO(m)$ (resp. that $G_2 \leq K$ or that $Spin(7) \leq K$), it follows from Lemma 6.8 that $\mathfrak{h} \cap S_0(m) = S_0(m)$, and thus $\mathfrak{h} = \mathfrak{sl}_m\mathbb{R}$ by Lemma 6.6. $\square$

The case of $Spin(9)$ acting on $\mathbb{O} \oplus \mathbb{O}$ is different. Consider the affine Cayley plane $AG_2\mathbb{O}$, and let $H \leq GL_{16}\mathbb{R}$ denote the stabilizer of the origin in its collineation group. Then $H \cong (Spin_{9,1}\mathbb{R})^\circ$, and $H \cap F_4 \cong Spin(9)$, see Salzmann et al. [36], Sec. 15, in particular 15.6, and p. 628.

6.10. Proposition. *Let $H \leq SL_{16}\mathbb{R}$ be a closed noncompact semisimple subgroup with $Spin(9) < H$. Then H is one of the groups $(Spin_{9,1}\mathbb{R})^\circ$ or $SL_{16}\mathbb{R}$.*

Proof. The idea is to decompose the Spin(9)-module $S_0(16)$ into irreducible submodules. We denote the Lie algebra of Spin(9) $\leqq$ SO(16) by $\mathfrak{spin}(9) \leqq \mathfrak{so}(16)$. Let $V = \mathbb{O} \oplus \mathbb{O}$ denote the 'natural' Spin(9)-module. Then $S(16) \cong S^2 V$.

The Galois group of $\mathbb{C}/\mathbb{R}$ acts trivially on all weights of Spin(9), and all fundamental representations of Spin(9) are of real type, see Tits [43], p. 31. We label the fundamental weights of Spin(9) in the standard way (as in Onishchik-Vinberg [33], p. 293 or in Tits [43], p. 30). The complex module $R(\pi_4)$ associated to the weight π_4 is the Spin-module, $R(\pi_4) \cong V \otimes \mathbb{C}$. By Onishchik-Vinberg [33], p. 301, we have a decomposition

$$S^2 R(\pi_4) \cong R(2\pi_4) \oplus R(\pi_1) \oplus \mathbb{C}.$$

Therefore we obtain a decomposition into real irreducible modules

$$S^2 V \cong {}_{\mathbb{R}}R(2\pi_4) \oplus {}_{\mathbb{R}}R(\pi_1) \oplus \mathbb{R} \quad \text{and} \quad S_0(16) \cong {}_{\mathbb{R}}R(2\pi_4) \oplus {}_{\mathbb{R}}R(\pi_1).$$

The dimensions of the modules ${}_{\mathbb{R}}R(\pi_1) \cong \mathbb{R}^9$, and ${}_{\mathbb{R}}R(2\pi_4)$ are 9 and 126, respectively, see Onishchik-Vinberg [33], p. 301.

Now suppose that $H \leqq \mathrm{SL}_{16}\mathbb{R}$ is embedded as in Theorem 6.5. Since Spin(9) $\leqq H$, the Lie algebra $\mathfrak{h}$ of H is a Spin(9)-module. Consider the nonzero Spin(9)-module $M = \mathfrak{h} \cap S_0(16)$. If $M = S_0(16)$, then $\mathfrak{h} = \mathfrak{sl}_{16}\mathbb{R}$ by Lemma 6.6. Since we know that $\mathfrak{so}_{9,1}\mathbb{R} \leqq \mathfrak{sl}_{16}\mathbb{R}$, the case $M = {}_{\mathbb{R}}R(\pi_1)$ is possible, and $[M, M] = \mathfrak{spin}(9)$ in this case, so $\mathfrak{h} = \mathfrak{so}_{9,1}\mathbb{R}$ is uniquely determined. Finally, suppose that $M = {}_{\mathbb{R}}R(2\pi_4)$. Then $\dim(\mathfrak{spin}(9) + M) = 162$; no such semisimple Lie algebra (with $\mathfrak{spin}(9)$ as maximal compact subalgebra) exists. $\square$

Now we consider semisimple groups which contain SU(n). Let

$$H(n) = \{X \in \mathbb{C}(n) \mid \overline{X}^T = X\}$$

denote the set of hermitian matrices, and let $H_0(n) = \{X \in H(n) \mid \mathrm{tr}(X) = 0\}$. Note that $\mathfrak{su}(n) = \mathtt{i}H_0(n)$ and $\mathfrak{u}(n) = \mathtt{i}H(n)$.

Let $\Theta = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$ and $\mathtt{I} = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$. We identify $\mathbb{C}$ with the subalgebra of $\mathbb{R}(2)$ spanned by $1, \mathtt{I}$; this subalgebra has a vector space complement $\mathbb{C}\Theta$ which is spanned by Θ and $\mathtt{I}\Theta$. Every matrix $X \in \mathbb{R}(n) \otimes \mathbb{R}(2) \cong \mathbb{R}(2n)$ decomposes uniquely as

$$X = X_1 \otimes 1 + X_2 \otimes \mathtt{I} + (X_3 \otimes 1 + X_4 \otimes \mathtt{I})\Theta.$$

We call $X_1 \otimes 1 + X_2 \otimes \mathtt{I}$ the *complex part* of X and $X_3 \otimes \Theta + X_4 \otimes \mathtt{I}\Theta$ the *anti-complex part* of X. There is a natural injection $\mathbb{C}(n) \hookrightarrow \mathbb{R}(2n)$ which is given by $A + B\mathtt{i} \mapsto A \otimes 1 + B \otimes \mathtt{I}$, for $A, B \in \mathbb{R}(n)$, and $A + B\mathtt{i} \in H(n)$ if and only if $A \in S(n)$ and $B \in \mathfrak{so}(n)$. More generally, we have

$$(X_1 \otimes 1 + X_2 \otimes \mathtt{I} + X_3 \otimes \Theta + X_4 \otimes \mathtt{I}\Theta)^T = X_1^T \otimes 1 - X_2^T \otimes \mathtt{I} + X_3^T \otimes \Theta + X_4^T \otimes \mathtt{I}\Theta.$$

Using this identity, it is not hard to show that $S_0(2n) = H_0(n) \oplus S(n) \otimes \mathbb{C}\Theta$. The corresponding decomposition into SU(n)-modules is

$$S_0(2n) \cong \mathfrak{su}(n) \oplus S^2 V,$$

where $V \cong \mathbb{C}^n$ is the natural module (the symmetric power of V being taken over $\mathbb{C}$!). Clearly, $\mathfrak{su}(n)$ is an irreducible real $\mathrm{SU}(n)$-module for all $n \geqq 2$. By Onishchik-Vinberg [33], p. 300, the complex module $S^2 R(\pi_1) \cong R(2\pi_1)$ is irreducible for all $n \geqq 2$; and $2\pi_1$ is of complex type, $_{\mathbb{R}}R(2\pi_1) = R(2\pi_1)$, provided that $n \geqq 3$. Thus $S^2 V$ is a real irreducible $\mathrm{SU}(n)$-module for $n \geqq 3$.

6.11. Proposition. *Let $H \leqq \mathrm{SL}_{2n}\mathbb{R}$ be a closed semisimple noncompact subgroup with $\mathrm{SU}(n) < H$, for $n \geqq 3$. Then H is one of the groups $\mathrm{SL}_n\mathbb{C}$, $\mathrm{Sp}_{2n}\mathbb{R}$, or $\mathrm{SL}_{2n}\mathbb{R}$.*

Proof.   We assume that $\mathfrak{h} \leqq \mathfrak{sl}_{2n}\mathbb{R}$ is embedded as in Theorem 6.5. Thus $\mathfrak{h} \cap S_0(2n)$ is a nonzero $\mathrm{SU}(n)$-module. We showed above that $S_0(2n) = H_0(n) \oplus S(n) \otimes \mathbb{C}\Theta$ is a decomposition into real irreducible $\mathrm{SU}(n)$-modules. The real dimensions of these two modules are $n^2 - 1$ and $n^2 + n$, respectively, so they are not isomorphic. Moreover, the following subalgebras exist:

$$\mathfrak{sl}_n\mathbb{C} = \mathfrak{su}(n) \oplus H_0(n) \quad \text{and} \quad \mathfrak{sp}_{2n}\mathbb{R} = \mathfrak{u}(n) \oplus S(n) \otimes \mathbb{C}\Theta.$$

Since

$$[H_0(n), H_0(n)] = \mathfrak{su}(n), \quad [S(n) \otimes \mathbb{C}\Theta, S(n) \otimes \mathbb{C}\Theta] = \mathfrak{u}(n) \quad \text{and}$$

$$[S_0(2n), S_0(2n)] = \mathfrak{so}(2n)$$

by Lemma 6.6, the algebras $\mathfrak{sl}_n\mathbb{C}$, $\mathfrak{sp}_{2n}\mathbb{R}$, and $\mathfrak{sl}_{2n}\mathbb{R}$ are the only possibilities for $\mathfrak{h}$. $\square$

Now we consider the case where $\mathfrak{sp}(n) \leqq \mathfrak{h}$, for $n \geqq 2$. Let $V = \mathbb{C}^{2n} = \mathbb{H}^n$ denote the natural $\mathrm{Sp}(n)$-module. The complex $\mathrm{SU}(2n)$-module corresponding to $\mathfrak{su}(2n)$ is

$$\mathfrak{su}(n) \otimes \mathbb{C} \cong \mathfrak{sl}_{2n}\mathbb{C} = \{X \in \mathbb{C}(2n) \mid \mathrm{tr}(X) = 0\}.$$

Thus we have to consider the complex $\mathrm{Sp}(n)$-module $V \otimes V$. By Onishchik-Vinberg [33], p. 302, $V \otimes V = R(\pi_1) \otimes R(\pi_1) \cong R(\pi_2) \oplus R(2\pi_1) \oplus \mathbb{C}$. The Galois group of $\mathbb{C}/\mathbb{R}$ acts trivially on the fundamental weights, and π_k is of real type if and only if k is even, see Tits [43], p. 34. Thus π_2 and $2\pi_1$ are of real type. Moreover, $R(2\pi_1) = \mathfrak{sp}_{2n}\mathbb{C}$, and we obtain a decomposition into real irreducible $\mathrm{Sp}(n)$-modules

$$\mathfrak{su}(2n) \cong \mathfrak{sp}(n) \oplus {}_{\mathbb{R}}R(\pi_2).$$

The dimension of $_{\mathbb{R}}R(\pi_2)$ is $(2n+1)(n-1)$, and $\dim(\mathfrak{sp}(n)) = (2n+1)n$; in particular, the modules are not isomorphic. The complex $\mathfrak{sp}(n)$-module $S(2n) \otimes \mathbb{C} \cong S^2 R(\pi_1)$ decomposes as

$$S^2 R(\pi_1) \cong R(2\pi_1),$$

see Onishchik-Vinberg [33], p. 302. Thus the decomposition into real irreducible modules is

$$S^2 V \cong \mathfrak{sp}(n) \oplus \mathfrak{sp}(n)$$

and

$$S_0(4n) \cong \mathfrak{sp}(n) \oplus \mathfrak{sp}(n) \oplus \mathfrak{sp}(n) \oplus {}_\mathbb{R}R(\pi_2).$$

6.12. Proposition. *Let $H \leqq \mathrm{SL}_{4n}\mathbb{R}$ be a closed noncompact semisimple subgroup with $\mathrm{Sp}(n) < H$, for $n \geqq 2$. Then H is one of the groups $\mathrm{SL}_n\mathbb{H}$, $\mathrm{SL}_n\mathbb{H} \cdot \mathrm{Sp}(1)$, $\mathrm{SL}_{2n}\mathbb{C}$, $\mathrm{Sp}_{2n}\mathbb{C}$, $\mathrm{SL}_{4n}\mathbb{R}$, or $\mathrm{Sp}_{4n}\mathbb{R}$.*

Proof. We assume that $\mathfrak{h} \leqq \mathfrak{sl}_{4n}\mathbb{R}$ is embedded as in Theorem 6.5. Thus $\mathfrak{h} \cap S_0(4n)$ is a nonzero $\mathrm{Sp}(n)$-module. Note that the subalgebras $\mathfrak{sl}_n\mathbb{H}$, $\mathfrak{sp}_{2n}\mathbb{C}$, $\mathfrak{sl}_{2n}\mathbb{C}$ and $\mathfrak{sp}_{4n}\mathbb{R}$ exist in $\mathfrak{sl}_{4n}\mathbb{R}$. We have

$$\mathfrak{sl}_n\mathbb{H} \cap S_0(4n) \cong {}_\mathbb{R}R(\pi_2),$$

$$\mathfrak{sp}_{2n}\mathbb{C} \cap S_0(4n) \cong \mathfrak{sp}(n),$$

$$\mathfrak{sl}_{2n}\mathbb{C} \cap S_0(4n) \cong \mathfrak{sp}(n) \oplus {}_\mathbb{R}R(\pi_2),$$

$$\mathfrak{sp}_{4n}\mathbb{R} \cap S_0(4n) \cong \mathfrak{sp}(n) \oplus \mathfrak{sp}(n),$$

$$\mathfrak{sl}_{4n}\mathbb{R} \cap S_0(4N) \cong S_0(4n).$$

Let $a \in \mathrm{Sp}(1) = \mathrm{Cen}_{\mathrm{SO}(4n)}\mathrm{Sp}(n)$ be a pure element, $a + \bar{a} = 0$. Then a defines a real symplectic structure ω_a on $\mathbb{R}^{4n} = \mathbb{H}^n$ by $\omega_a(u, v) = \mathrm{Re}(\bar{u}^T a v)$. Thus we see that all copies of $\mathfrak{sp}(n) \leqq S_0(4n)$ are conjugate under the group $\mathrm{Sp}(1)$. In view of Lemma 6.6, we immediately obtain the following results.

If $\mathfrak{h} \cap S_0(4n) = {}_\mathbb{R}R(2\pi_1)$, then $\mathfrak{sl}_n\mathbb{H} \leqq \mathfrak{h}$ and thus $\mathfrak{h} = \mathfrak{sl}_n\mathbb{H}$ or $\mathfrak{h} = \mathfrak{sl}_n\mathbb{H} \oplus \mathfrak{sp}(1)$.

If $\mathfrak{h} \cap S_0(4n) \cong \mathfrak{sp}(n)$, then $\mathfrak{h}$ is conjugate to the algebra $\mathfrak{sp}_{2n}\mathbb{C}$.

If $\mathfrak{h} \cap S_0(4n) \cong \mathfrak{sp}(n) \oplus {}_\mathbb{R}R(2\pi_1)$, then $\mathfrak{h}$ is conjugate to $\mathfrak{sl}_{2n}\mathbb{C}$.

If $\mathfrak{h} \cap S_0(4n) \cong \mathfrak{sp}(n) \oplus \mathfrak{sp}(n)$, then $\mathfrak{h}$ is conjugate to $\mathfrak{h} = \mathfrak{sp}_{4n}\mathbb{R}$.

If $\mathfrak{h} \cap S_0(4n) = S_0(4n)$, then $\mathfrak{h} = \mathfrak{sl}_{4n}\mathbb{R}$.

Thus, if $\mathfrak{h} \cap S_0(4n)$ contains $\mathfrak{sp}(n) \oplus {}_\mathbb{R}R(\pi_2)$ or $\mathfrak{sp}(n) \oplus \mathfrak{sp}(n)$, then $\mathfrak{su}(2n) \leqq \mathfrak{h}$, and we showed in Proposition 6.11 that in this case either $\mathfrak{h} = \mathfrak{sp}_{4n}\mathbb{R}$ or $\mathfrak{h} = \mathfrak{sl}_{2n}\mathbb{C}$.

There are no other cases to be considered. $\square$

Finally, we discuss the low dimensional cases.

6.13. Proposition. *Let $H \leqq \mathrm{SL}_4\mathbb{R}$ be a closed semisimple noncompact Lie group containing $\mathrm{SU}(2)$. Then H is one of the groups $\mathrm{SL}_2\mathbb{C}$, $\mathrm{Sp}_4\mathbb{R}$, or $\mathrm{SL}_4\mathbb{R}$.*

Proof. As we noted before Proposition 6.11, we have a decomposition $S_0(4) \cong \mathfrak{su}(2) \oplus S^2V$, where $V = \mathbb{C}^2 = R(\pi_1)$ is the natural module (the symmetric power is taken over $\mathbb{C}$). Moreover, $S^2R(\pi_1) \cong R(2\pi_1)$, and $\beta_{2\pi_1} = \mathbb{R}$. Thus ${}_\mathbb{R}R(2\pi_1) = \mathfrak{su}(2)$,

and $S_0(4) \cong \mathfrak{su}(2) \oplus \mathfrak{su}(2) \oplus \mathfrak{su}(2)$. Similarly as in Proposition 6.12, the centralizer $\mathrm{Sp}(1) = \mathrm{Cen}_{\mathrm{SO}(4)}\mathrm{SU}(2)$ acts transitively on the copies of $\mathfrak{su}(2)$ in $S_0(4)$. Thus we have only the cases

$$\mathfrak{h} \cap S_0(4) \cong \mathfrak{su}(2) \quad \text{and} \quad \mathfrak{h} \cong \mathfrak{sl}_2\mathbb{C},$$

$$\mathfrak{h} \cap S_0(4) \cong \mathfrak{su}(2) \oplus \mathfrak{su}(2) \quad \text{and} \quad \mathfrak{h} \cong \mathfrak{sp}_4\mathbb{R},$$

$$\mathfrak{h} \cap S_0(4) \cong S_0(4) \quad \text{and} \quad \mathfrak{h} = \mathfrak{sl}_4\mathbb{R}. \quad \square$$

We summarize these results as follows.

6.14. Theorem. *Let $G \leqq \mathrm{GL}_m\mathbb{R}$ be a closed subgroup, for $m \geqq 3$. Suppose that G acts transitively on the projective space $\mathbb{R}\mathrm{P}^{m-1}$. Let $L = [G, G]$ denote the commutator group of G. Then $G \leqq \mathrm{Nor}_{\mathrm{GL}_m\mathbb{R}}(L)$, and there is a split short exact sequence*

$$1 \to L \to \mathrm{Nor}_{\mathrm{GL}_m\mathbb{R}}(L) \overset{\leftarrow}{\to} N \to 1.$$

The following list gives all possibilities for L and the factor group $N = \mathrm{Nor}_{\mathrm{GL}_m\mathbb{R}}(L)/L$. We put $\mathbb{R}_> = \{r \in \mathbb{R} \mid r > 0\}$.

L	m	N
$\mathrm{SO}(2n)$	$2n$	$\mathbb{R}_>$
$\mathrm{SO}(2n+1)$	$2n+1$	$\mathbb{R}^*$
$\mathrm{SU}(n)$	$2n \ (n \geqq 3)$	$\mathbb{C}^*$
$\mathrm{Sp}(n)$	$4n$	$\mathrm{SO}(3) \cdot \mathbb{R}_>$
$\mathrm{Sp}(n) \cdot \mathrm{Sp}(1)$	$4n$	$\mathbb{R}_>$
G_2	7	$\mathbb{R}^*$
$\mathrm{Spin}(7)$	8	$\mathbb{R}_>$
$\mathrm{Spin}(9)$	16	$\mathbb{R}_>$
$\mathrm{SL}_{2n}\mathbb{R}$	$2n$	$\mathbb{R}_>$
$\mathrm{SL}_{2n+1}\mathbb{R}$	$2n+1$	$\mathbb{R}^*$
$\mathrm{Sp}_{2n}\mathbb{R}$	$2n$	$\mathbb{R}_>$
$\mathrm{SL}_n\mathbb{C}$	$2n$	$\mathbb{C}^* \rtimes \mathbb{Z}/2$
$\mathrm{Sp}_{2n}\mathbb{C}$	$4n$	$\mathbb{C}^* \rtimes \mathbb{Z}/2$
$\mathrm{SL}_n\mathbb{H}$	$4n$	$\mathrm{SO}(3) \cdot \mathbb{R}_>$
$\mathrm{SL}_n\mathbb{H} \cdot \mathrm{Sp}(1)$	$4n$	$\mathbb{R}_>$
$\mathrm{Spin}_{9,1}\mathbb{R}$	16	$\mathbb{R}_>$

Proof. The possibilities for the group L were determined in Propositions 6.9–6.12. In each case, it is not difficult to determine the normalizer and to construct a splitting of the exact sequence. $\square$

The proof of the next lemma is straight-forward; it can be used to derive a list of all connected transitive groups.

6.15. Lemma. *Let $S \leqq \mathbb{H}^*$ be a closed noncompact connected 1-dimensional sub-group. Up to conjugation, S is of the form*

$$S = S_a = \{e^{t(1+\mathrm{i}a)} \mid t \in \mathbb{R}\} = \{xe^{\mathrm{i}a\ln(x)} \mid x \in \mathbb{R}_>\}$$

for some real number $a \in \mathbb{R}$. The group is central in $\mathbb{H}^*$ if and only if $a = 0$, and $S_0 = \mathbb{R}_>$.

Proof. Let $\mathfrak{s} \leqq \mathbb{R} \oplus \mathfrak{su}(2)$ denote the Lie algebra of S. After conjugation with a suitable element $g \in \mathbb{H}^*$, we may assume that $\mathfrak{s} \leqq \mathbb{R} \oplus \mathfrak{so}(2) \cong \mathbb{R} \oplus \mathrm{i}\mathbb{R}$. Let $(x, \mathrm{i}y) \in \mathfrak{s}$ be a generator. Then $\exp(tx, \mathrm{i}ty) = e^{tx}e^{\mathrm{i}ty}$. Since we assumed that S is not compact, $x \neq 0$ and we may put $(x, y) = (1, a)$. $\quad\square$

In the 2-dimensional case, there are the following possibilities.

6.16. Lemma. *Let $H \leqq \mathrm{SL}_2\mathbb{R}$ be a connected group acting transitively on the nonzero vectors. Then $H = \mathrm{SL}_2\mathbb{R}$, or H is conjugate to $\mathbb{C}^* \leqq \mathrm{SL}_2\mathbb{R}$.*

Proof. We have $3 = \dim(\mathrm{SL}_2\mathbb{R}) \geqq \dim(H) \geqq 2$, and H is reductive. Thus H is abelian if $H \neq \mathrm{SL}_2\mathbb{R}$. In the abelian case, H acts regularly and is thus homeomorphic to $\mathbb{C}^*$; in particular, it contains a torus $\mathrm{SO}(2)$. The connected centralizer of $\mathrm{SO}(2)$ is $\mathbb{C}^* = \mathrm{Cen}_{\mathrm{SL}_2\mathbb{R}}\mathrm{SO}(2)$. $\quad\square$

Combining the results of this section, we obtain the following final result.

6.17. Theorem. *Let $H \leqq \mathrm{GL}_m\mathbb{R}$ be closed connected subgroup which acts transitively on $\mathbb{R}^m\setminus\{0\}$. Up to conjugation, H is one of the groups listed in (a), (b), (c) below.*

(a) $[H, H]$ *is compact and $m \geqq 3$.*

$[H, H]$	m	H	
$\mathrm{SO}(n)$	n	$\mathrm{SO}(n) \cdot \mathbb{R}_>$	
$\mathrm{SU}(n)$	$2n$	$\mathrm{SU}(n) \cdot S_a$, $\mathrm{SU}(n) \cdot \mathbb{C}^*$	$(+)$
$\mathrm{Sp}(n)$	$4n$	$\mathrm{Sp}(n) \cdot S_a$, $\mathrm{Sp}(n) \cdot \mathbb{C}^*$	$(+)$
$\mathrm{Sp}(n) \cdot \mathrm{Sp}(1)$	$4n$	$\mathrm{Sp}(n) \cdot \mathbb{H}^*$	
G_2	7	$G_2 \cdot \mathbb{R}_>$	
$\mathrm{Spin}(7)$	8	$\mathrm{Spin}(7) \cdot \mathbb{R}_>$	
$\mathrm{Spin}(9)$	16	$\mathrm{Spin}(9) \cdot \mathbb{R}_>$	

Here, a can be any real number.

(b) $[H, H]$ *is noncompact and $m \geqq 3$.*

$[H, H]$	m	H	
$\mathrm{SL}_n\mathbb{R}$	n	$\mathrm{SL}_n\mathbb{R}$, $\mathrm{SL}_n\mathbb{R} \cdot \mathbb{R}_>$	
$\mathrm{SL}_n\mathbb{C}$	$2n$	$\mathrm{SL}_n\mathbb{C}$, $\mathrm{SL}_n\mathbb{C} \cdot \mathrm{U}(1)$, $\mathrm{SL}_n\mathbb{C} \cdot S_a$, $\mathrm{GL}_n\mathbb{C}$	$(+)$
$\mathrm{SL}_n\mathbb{H}$	$4n$	$\mathrm{SL}_n\mathbb{H}$, $\mathrm{SL}_n\mathbb{H} \cdot \mathrm{U}(1)$, $\mathrm{SL}_n\mathbb{H} \cdot S_a$, $\mathrm{SL}_n\mathbb{H} \cdot \mathbb{C}^*$	$(+)$
$\mathrm{SL}_n\mathbb{H} \cdot \mathrm{Sp}(1)$	$4n$	$\mathrm{SL}_n\mathbb{H} \cdot \mathrm{Sp}(1)$, $\mathrm{SL}_n\mathbb{H} \cdot \mathbb{H}^*$	
$\mathrm{Sp}_{2n}\mathbb{R}$	$2n$	$\mathrm{Sp}_{2n}\mathbb{R}$, $\mathrm{Sp}_{2n}\mathbb{R} \cdot \mathbb{R}_>$	
$\mathrm{Sp}_{2n}\mathbb{C}$	$4n$	$\mathrm{Sp}_{2n}\mathbb{C}$, $\mathrm{Sp}_{2n}\mathbb{C} \cdot \mathrm{U}(1)$, $\mathrm{Sp}_{2n}\mathbb{C} \cdot S_a$, $\mathrm{Sp}_{2n}\mathbb{C} \cdot \mathbb{C}^*$	$(+)$
$\mathrm{Spin}_{9,1}\mathbb{R}$	16	$\mathrm{Spin}_{9,1}\mathbb{R}$, $\mathrm{Spin}_{9,1}\mathbb{R} \cdot \mathbb{R}_>$	

Again, a can be any real number.

(c) *For $m = 1, 2$ there are only the following possibilities.*

m	H	
1	$\mathbb{R}^*$	
2	$\mathbb{C}^*$	$(+)$
	$\mathrm{SL}_2\mathbb{R},\ \mathrm{SL}_2\mathbb{R} \cdot \mathbb{R}_>$	

(d) *If H preserves a complex structure on $\mathbb{R}^m$, then H is one of the groups in* (a), (b), (c) *marked with* $(+)$.

Proof. For $m \geq 3$, the possibilities for the semisimple group $L = [H, H]$ are determined in Theorem 6.14. A direct inspection (combined with Lemma 6.15) yields the list. For $m = 2$ we use Lemma 6.16, and the case $m = 1$ is trivial. Finally, (d) follows by direct inspection of the actions. $\square$

Using the results in this section, it is not difficult to obtain a list of linear groups acting transitively on the point set $\mathbb{FP}^k$ of a projective space over $\mathbb{F} = \mathbb{C}, \mathbb{H}$, see Völklein [47], Satz 2. Also, the possibilities for closed, but not necessarily connected groups can be determined using 6.14. We leave this to the reader.

7. Locally compact Moufang sets

A *Moufang set* is a tripel (G, U, X), where G is a 2-transitive permutation group acting on X, and $U \trianglelefteq G_x$ is a normal subgroup of a stabilizer G_x acting regularly on $X \backslash \{x\}$. Moufang sets were introduced by Tits in [46]; they are also known as *split doubly transitive groups*. Note that the special case $G_x = U$ is the same as a *sharply 2-transitive group*. See also Kramer [29], Section 1.8.

In this section, we determine all Moufang sets, where G is an effective 2-transitive Lie group and U is closed in G_x. We call such a Moufang set a *locally compact and connected Moufang set*. According to the classification of 2-transitive Lie groups, we distinguish three cases: the case where G° is simple and of rank 1, the case where G° is simple and of higher rank, and the affine case where $X = \mathbb{R}^n$.

Recall from Section 3 the *Iwasawa decomposition* of a simple Lie group

$$H = KAU$$

and the corresponding minimal parabolic

$$B = K_0 AU,$$

where $K_0 = \mathrm{Cen}_K(A)$ is the reductive anisotropic kernel.

7.1. Proposition. *Let G be a 2-transitive Lie group, with $G^\circ = H$ simple and of rank 1, as in Theorem 3.3 (a). As above, let U denote the unipotent radical of a minimal parabolic $B \subseteq H$. Then $(G, U, H/B)$ is a locally compact connected Moufang set.*

Proof. This is clear from the Iwasawa decomposition. Let $X = G/G_x = H/B$, and let $y \in X \setminus \{x\}$. Then $H_{x,y} = B_y = K_0 A$. The group U is normal in $B = H_x$, and intersects the two-point stabilizer $K_0 A$ trivially. Thus it acts regularly on $X \setminus \{x\}$. $\square$

If G° is a simple 2-transitive Lie group of rank at least 2, then there is no way of making $X = G/G_x$ into a Moufang set. The following proof was pointed out by Hendrik Van Maldeghem, replacing a more topological (and more complicated) argument of mine. Let X be the point set of a desarguesian projective space of rank at least 2, or the point space of a projective Moufang plane. Let G be a group of automorphisms of the projective space, containing all elations. Then (G, X) cannot be made into a Moufang set. To see this, let $p \in X$ and assume that $U \trianglelefteq G_p$ is a normal subgroup acting regularly on $X \setminus \{p\}$. Let $u \in U$, and let τ be an elation with center p. Then $u\tau u^{-1}$ is also an elation with center p, and so is the commutator $[u, \tau]$. If we choose u, τ in such a way that u does not fix the axis of τ (which is possible, since the rank of the projective space is at least 2), then $[u, \tau] \in U$ is a nontrivial elation with center p. Since U is normal, U contains all elations with center p. These elations form an abelian normal subgroup of G_p which is, however, not regular on X.

7.2. Proposition. *None of the groups in Theorem* 3.3 (b) *can be made into a Moufang set.* $\square$

Finally, we consider the question of uniqueness in the case where $H = G^\circ$ is simple and of rank 1. The question is thus if G_x admits a regular normal subgroup V different from U. Since $X \setminus \{x\}$ is connected, we have $V \trianglelefteq B = H_x = K_0 A U$. So $V \cap K_0 A = 1$ (by regularity of V) and $K_0 A V = B$ (by transitivity). Let $\mathfrak{u}$ denote the Lie algebra of U, and $\mathfrak{v}$ the Lie algebra of V. We have to prove that $\mathfrak{v} = \mathfrak{u}$. We decompose the Lie algebra $\mathfrak{b}$ of B into irreducible $K_0 A$-modules. As a $K_0 A$-module, $\mathfrak{k}_0 \oplus \mathfrak{a}$ decomposes as $[\mathfrak{k}_0, \mathfrak{k}_0] \oplus \mathrm{Cen}(\mathfrak{k}_0) \oplus \mathfrak{a}$. Direct inspection shows that no $K_0 A$-submodule of $\mathfrak{u}$ is isomorphic to a $K_0 A$-submodule of $\mathfrak{k}_0 \oplus \mathfrak{a}$. This proves uniqueness of $\mathfrak{u}$.

7.3. Lemma. *Each of the groups in Theorem* 3.3 (a) *is in a unique way a locally compact and connected Moufang set, i.e. the data* (G, X) *determine the closed regular normal subgroup* $U \trianglelefteq G_x$ *uniquely.* $\square$

It remains to consider the case where X is noncompact. Then G is a semidirect product $G = G_x \ltimes \mathbb{R}^m$, and $U \trianglelefteq G_x$ is a normal subgroup acting regularly on the nonzero vectors in $\mathbb{R}^m$. It follows that the chosen maximal compact subgroup $K \subseteq U$ acts regularly on the sphere $\mathbb{S}^{m-1}$. Direct inspection of the list in Theorem 6.1 shows that this happens only for $m = 1, 2, 4$, and $[U, U]$ is one of the groups 1, $\mathrm{SO}(2)$, $\mathrm{Sp}(3)$. We obtain the following result, which is originally due to Kalscheuer [25] and was re-proved by Tits [40] and Grundhöfer [16].

7.4. Theorem. *Let* $U \subseteq \mathrm{GL}_m \mathbb{R}$ *be a closed subgroup which acts regularly (i.e. sharply transitively) on* $\mathbb{R}^m \setminus \{0\}$. *Then* $m = 1, 2, 4$, *and* U *is one of the groups* $\mathbb{R}^*$, $\mathbb{C}^*$, $\mathbb{H}^* = \mathrm{Sp}(1) \cdot S_0$, *or* $\mathrm{Sp}(1) \cdot S_a$, *for* $a \neq 0$.

Proof. The result follows by direct inspection of the tables in Theorem 6.17. $\square$

Since $U \trianglelefteq H$ is a normal subgroup, it remains to determine the normalizer $\mathrm{Nor}_{\mathrm{GL}_m\mathbb{R}}(U)$. The following table shows the corresponding quotients $\mathrm{Nor}_{\mathrm{GL}_m\mathbb{R}}(U)/U$.

U	m	$\mathrm{Nor}_{\mathrm{GL}_m\mathbb{R}}(U)/U$
$\mathbb{R}^*$	1	1
$\mathbb{C}^*$	2	$\mathbb{Z}/2$
$\mathbb{H}^*$	4	$\mathrm{SO}(3) \cdot \mathbb{R}_>$
$\mathrm{Sp}(1) \cdot S_a$	4	$\mathbb{R}_> \ (a \neq 0)$

Using this, it is not difficult to determine the locally compact and connected Moufang sets for noncompact X. Also, it is easy to see that the pair (G, X) determines U for $m = 1, 2$. This is *not* true for $m = 4$: if $H = \mathrm{Sp}(1) \cdot \mathbb{C}^*$ is the group consisting of all maps $x \mapsto hxc$, for $h \in \mathrm{Sp}(1)$ and $c \in \mathbb{C}^*$, then $\mathrm{Sp}(1) \cdot S_a = U \trianglelefteq H$ is a regular normal subgroup for any choice of $a \in \mathbb{R}$.

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