Winter Meeting on

Bruhat-Tits buildings

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Lectures on

Metric Aspects of Euclidean Buildings

Lecture I – CAT(0) Geometry

Let (X,d) be a metric space. A <u>geodesic</u> is a map $\gamma : J \to X$, where $J \subseteq \mathbb{R}$ is a closed interval, such that $d(\gamma(s),\gamma(t)) = |s-t|$ holds for all $s,t \in J$. We call X <u>geodesic</u> iff any two points $p,q \in X$ can be joined by a geodesic. We call γ (or $\gamma(J)$) a segment / ray / line if $J=[a,b]$ / $J=[0,\infty)$ / $J=\mathbb{R}$. X has <u>extensible geodesics</u> if every segment is contained in a line.

<u>Ex</u>
- Banach spaces are geodesic with extensible geodesics
- every connected metric graph (1-dim. simpl. complex, edges have length 1) is geodesic
- every cplt Riemannian manifold is geodesic

Put $\mathbb{E}^n = (\mathbb{R}^n, d)$ $\quad d(p,q)^2 = \sum_{i=1}^{n} (p_i - q_i)^2$

$\mathbb{S}^n$ unit n-sphere with <u>spherical metric</u> $\cos d(p,q) = \sum_{i=0}^{n} p_i q_i$

<u>The CAT(0) Condition</u> (Gromov: <u>C</u>artan–<u>A</u>lexandrov–<u>T</u>oponogov)

Let X be a metric space. Given $a,b,c \in X$, choose $\bar{a},\bar{b},\bar{c} \in \mathbb{E}^2$ such that

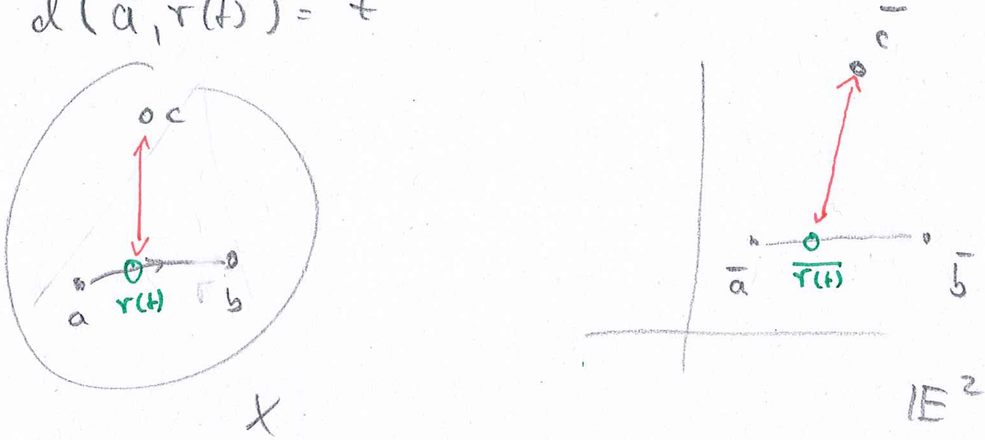
$$\begin{aligned} d(a,b) &= d(\bar{a},\bar{b}) \\ d(b,c) &= d(\bar{b},\bar{c}) \\ d(c,a) &= d(\bar{c},\bar{a}) \end{aligned}$$

(we call $\bar{a},\bar{b},\bar{c}$ a <u>comparison triangle</u>).

X satisfies the <u>CAT(0) condition</u> if for all $a,b,c \in X$ there is a geodesic γ from a to b (depending on a,b but <u>not</u> on c!)

such that for all $t \in [0, d(a,b)]$,

$$d(c, \gamma(t)) \leq d(\bar{c}, \overline{\gamma(t)})$$ where $\overline{\gamma(t)}$ is the unique point on the line from $\bar{a}$ to $\bar{b}$ with $d(\bar{a}, \overline{\gamma(t)}) = t$

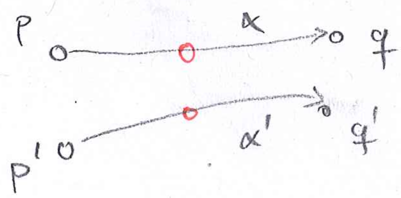


Examples

- $\mathbb{E}^n$ is CAT(0) for all $n \geq 0$
- Banach spaces are CAT(0) iff they are Hilbert spaces
- a Riem. Mnf. is CAT(0) iff it is 1-connected and has nonpositive sectional curvature
- a metric graph is CAT(0) iff it is a tree

Facts

- The metric in a CAT(0) space is convex:



$d(p,q) = \lambda \qquad d(p',q') = \lambda'$

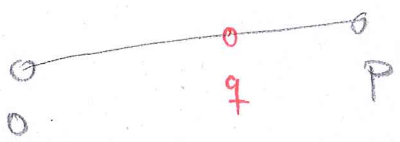
$$\Rightarrow d(\alpha(\lambda t), \alpha'(\lambda' t)) \leq (1-t)\, d(p,p') + t \cdot d(q,q')$$

$\forall\, t \in [0,1]$

In particular, geodesics are unique

- Every CAT(0) space is contractible: pick $o \in X$, put $h_t(p) = q$ $\quad d(o,q) = t \cdot d(o,p)$

q on geodesic from p to o



- (X,d_X), (Y,d_Y) metric, $d=\sqrt{d_X^2+d_Y^2}$ on $X\times Y$. Then $X\times Y$ CAT(0) $\Leftrightarrow$ X, Y CAT(0)

- If X, Y are CAT(0) and if $\varphi: X \to Y$ is a locally isometric map, then φ is an isometric embedding. (Suffices to prove this for $X=[a,b]$)

- X CAT(0), $K \subseteq X$ convex $\Rightarrow$ K CAT(0)

- X CAT(0) $\rightsquigarrow$ the metric completion of X is CAT(0).

Let X be a metric space, $A \subseteq X$ bounded. Put $\mathrm{rad}(A) = \inf\{r \mid A \subseteq \bar{B}_r(p)$ for some $p\in X\}$. We call $p \in X$ a <u>center for A</u> if $A \subseteq B_r(p)$ with $r = \mathrm{rad}(A)$.

<u>Thm</u> (Bruhat-Tits) In a complete CAT(0) space every bounded set has a unique center.

[o in the middle between p and q, $p,q\in X$, $a \in A$

$$d(o,a)^2 \le \frac{1}{2}\left(d(p,a)^2 + d(q,a)^2\right) - \frac{1}{4} d(p,q)^2$$

$\rightsquigarrow$ uniqueness]

<u>Cor</u> If a group Γ acts isometrically on a complete CAT(0) space X and if some Γ-orbit is bounded (e.g. if Γ is finite) then Γ has a fixed point.

<u>The boundary at infinity</u> Let X be CAT(0),
let $r:[0,\infty)\to X$ be a ray. Put

$$\beta_r(p) = \lim_{t\to\infty}\left(d(r(t),p) - \underbrace{d(r(t),r(0))}_{=t}\right)$$

Then β_r is a continuous function on X, the <u>Busemann function</u> determined by r.

$$\beta_r - \beta_{r'} = \text{const} \iff \sup\{d(r(t),r'(t)) \mid t\geq 0\} < \infty$$
$$\iff r, r' \text{ have } \underline{\text{finite distance}}.$$

Put $\partial_\infty X = \{\text{rays in } X\}/_{\text{finite dist}} \cong \{\text{Busem fcts}\}/_{\mathbb{R}}$

<u>Gromov embedding</u>

$$\begin{tikzcd} X \arrow[r] \arrow[dr, "\mathrm{Gro}"'] & C(X) \arrow[d] \\ & C(X)/_{\mathbb{R}} \end{tikzcd} \qquad \begin{tikzcd} p \arrow[r, mapsto] \arrow[dr, mapsto, "\mathrm{Gro}"'] & d(p,-) \arrow[d, mapsto] \\ & d(p,-)+\mathbb{R} \end{tikzcd}$$

Let $\hat{X}$ = closure of $\mathrm{Gro}(X)$ w.r.t. topology of uniform convergence on bounded sets

<u>Thm</u> X complete CAT(0) $\Rightarrow \hat{X} = \mathrm{Gro}(X) \,\dot{\cup}\, \{\text{Busem fcts}\}/_{\mathbb{R}}$

$\rightsquigarrow$ topology on $\bar{X} = X \cup \partial_\infty X \cong \hat{X}$

Arzela-Ascoli: X proper $\Rightarrow \bar{X}$ compact.

This is the <u>cone topology</u> on $X \cup \partial_\infty X$

$\Rightarrow$ <u>Bordification</u> / <u>"Compactification"</u> of a CAT(0) space

Note: $\mathrm{Iso}(X)$ acts on $\bar{X}$

Application: $(A_i)_{i\in\mathbb{N}} \subseteq X$ nested sequence of closed sets, X proper
$\Rightarrow \bigcap A_i \neq \emptyset$ or the A_i have a common point at infinity.

Angles Let X be $CAT(0)$, $o \in X$, $p, q \in X - \{o\}$.
Let α, β be geodesics from o to p and q, resp.

Define the *angle* $\measuredangle_o(p,q) = \measuredangle_o(\alpha, \beta)$ by

$$\sin\left(\tfrac{1}{2}\measuredangle_o(p,q)\right) = \lim_{t\to 0} \frac{d(\alpha(t), \beta(t))}{2t}$$

Fact: $\measuredangle_o$ is a continuous pseudometric on $X - \{o\}$

Let $\Sigma_o X$ = metric completion of $(X - \{o\}, \measuredangle_o)$

This is the *space of directions* at o, a complete metric space.

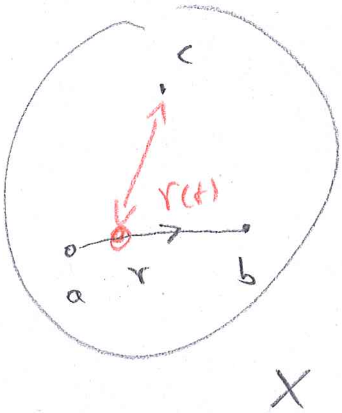
The CAT(1) condition Let X be a metric space

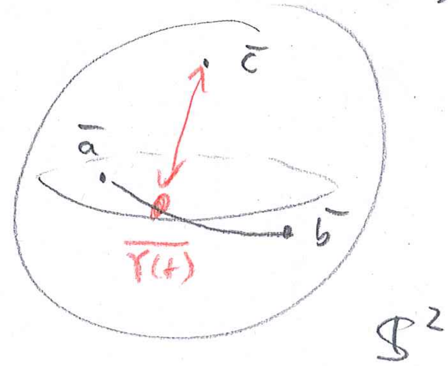
Given $a, b, c \in X$ with $d(a,b) + d(b,c) + d(c,a) < \pi$
we find $\bar{a}, \bar{b}, \bar{c} \in \mathbb{S}^2$ at same distances
$d(a,b) = d(\bar{a}, \bar{b})$, etc.

X satisfies the *CAT(1) condition* if for all such $a,b,c \in X$,
there is a geodesic γ from a to b, such that for all
not depending on c
$0 \le t \le d(a,b)$,

$$d(\gamma(t), c) \le d(\overline{\gamma(t)}, \bar{c})$$

spherical distance!





Examples • $\mathbb{S}^n$ is for all $n \geq 0$ CAT(1)

• Every CAT(0) space is CAT(1) (!)

• Put $d_\pi = \min\{d, \pi\}$. Then (X,d) is CAT(1) $\Leftrightarrow$ (X, d_π) is CAT(1)

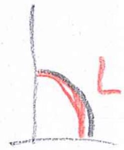
Two constructions

(1) Let (X,d) be a metric space. The cone over X is $CX = X \times [0,\infty)/_\sim$ with metric

$$d(xt, x't')^2 = t^2 + t'^2 - 2tt' \cos d_\pi(x,x')$$

Identify $x0 = x'0$ for all $x, x' \in X$.

Then X is CAT(1) $\Leftrightarrow$ CX is CAT(0)



(2) Let X, Y be CAT(1) spaces, $L = \{(c,s) \mid c,s \geq 0,\ c^2+s^2 = 1\}$

$X * Y = X \times L \times Y/_\sim$ is the spherical join, with

write $(x,y,c,s) = (xc, ys)$ $x0 = x'0$ $y0 = y0'$ $\forall x,x' \in X$ $y,y' \in Y$

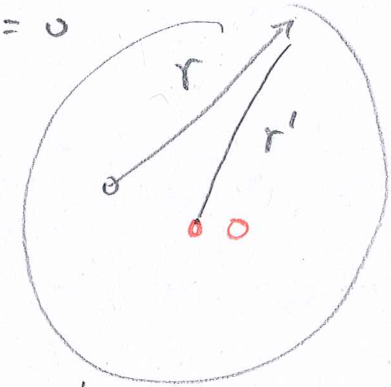
$$\cos d((xc,ys),(x'c',y's')) = c \cdot c' \cdot \cos d_\pi(x,x') + ss' \cdot \sin d_\pi(y,y')$$

Then $X * Y$ is again CAT(1)

Thm If X is CAT(0), then $\Sigma_0 X$ is CAT(1)

The Tits metric on $\partial_\infty X$ Given a ray r in a complete CAT(0) space, let $[r] = \{\text{rays at finite distance from } r\}$

$\rightsquigarrow \partial_\infty X = \{[r] \mid r \text{ ray}\}$. Given $o \in X$, there is a unique $r' \in [r]$ with $r'(0) = o$



Put $\sphericalangle_o([r],[\delta]) = \sphericalangle_o(r', \delta')$

$r' \in [r]$, $\delta' \in [\delta]$ $r'(0) = \delta'(0) = o$

and $\sphericalangle([r],[\delta]) = \sup\{\sphericalangle_o([r],[\delta]) \mid o \in X\}$

Then $\sphericalangle$ is a metric on X, the Tits metric

Thm $(\partial_\infty X, \sphericalangle)$ is CAT(1). The identity map $(\partial_\infty X, \sphericalangle) \to (\partial_\infty X, \text{cone-topology})$ is continuous.

But: T regular locally finite tree $\rightsquigarrow$ $\partial_\infty T$ is Cantor set in cone topology, discrete in Tits topology.

Thm X CAT(0) with extensible geodesics,

$\partial_\infty X \cong A * B$ in Tits metric $\Leftrightarrow$

$X = Y \times Z$ Y, Z CAT(0)

Lecture II - Spherical and affine buildings

Let V be a set, Δ a collection of finite subsets of V such that (i) $\bigcup \Delta = V$ (ii) $a \subseteq b \in \Delta \Rightarrow a \in \Delta$

Then Δ is called a simplicial complex with vertex set V.

If Δ_1, Δ_2 are simplicial complexes, on vertex sets V_1, V_2 ($V_1 \cap V_2 = \emptyset$) then $\Delta_1 * \Delta_2 = \{ a \cup b \mid a \in \Delta_1, b \in \Delta_2 \}$ is the (simplicial) join, again a simplicial complex.

The link of $a \in \Delta$ is $\mathrm{lk}(a) = \{ b \in \Delta \mid a \cap b = \emptyset, a \cup b \in \Delta \}$

Example • Γ - a group $\Gamma_1, \dots, \Gamma_m$ (distinct) subgroups

$V = \Gamma/\Gamma_1 \cup \dots \cup \Gamma/\Gamma_m$ $\quad a \subseteq V$ finite set of cosets is a simplex iff $\bigcap a \neq \emptyset$

This is the nerve complex $\Delta = N(\Gamma, \{\Gamma_i\})$. Note that Γ acts on Δ

The geometric realization Let Δ be a simplicial complex with vertex set V. For $a \in \Delta$ put

$$|a| = \{ \sum_{v \in a} v s_v \mid s_v \in [0,1], \sum_{v \in a} s_v = 1 \} \subseteq \mathbb{R}^a \subseteq \bigoplus_V \mathbb{R}$$

and $|\Delta| = \bigcup \{ |a| \mid a \in \Delta \} \subseteq \bigoplus_V \mathbb{R}$. This is the geometric realization of Δ.

With the weak topology ($A \subseteq |\Delta|$ closed $\Leftrightarrow$ $A \cap |a|$ closed in $|a|$ for all $a \in \Delta$), $|\Delta|$ is a CW complex

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Coxeter groups and Coxeter complexes

Let I be a finite set (m_{ij}) a matrix with

$m_{ii} = 1$ $m_{ij} = m_{ji} \in \mathbb{N}_{\geq 2} \cup \{\infty\}$ for $i \neq j$. Then

$W = \langle I \mid (ij)^{m_{ij}} = 1 \quad i,j \in I \rangle$ is a Coxeter grp.
(W,I) Coxeter system

Fact: $J \subseteq I \Rightarrow \langle J \rangle \leq W$ is again a Coxeter group w.r.t. J and $(\langle J \rangle, J)$ is a Coxeter system.

For $i \in I$ put $W^i = \langle I - \{i\} \rangle \leq W$. Then

$\Sigma = \Sigma(W,I) = N(W, \{W^i \mid i \in I\})$ is the Coxeter complex of (W,I)

Fact $\Sigma = \Sigma_1 * \Sigma_2$ simplicial join $\Leftrightarrow$

$I = J \dot\cup K$, $[J,K] = 1 \Leftrightarrow W = \langle J \rangle \times \langle K \rangle$

Then (W,I) is called reducible.

A definition of buildings

Let (W,I) be a Coxeter system, $\Sigma = \Sigma(W,I)$ its Coxeter complex. Let Δ be a simplicial complex and $\mathcal{C}$ a collection of simplicial injections $\varphi: \Sigma \to \Delta$. We call $(\Delta, \mathcal{C})$ a building of type (W,I) with atlas $\mathcal{C}$ if

(i) $\forall a,b \in \Delta \; \exists \varphi \in \mathcal{C} \quad a,b \in \varphi(\Sigma)$
(ii) $\forall w \in W \; \varphi \in \mathcal{C} \quad \varphi \circ w \in \mathcal{C}$
(iii) $\forall \varphi, \psi \in \mathcal{C} \; \exists w \in W$ s.t. $\psi \circ w = \varphi$ on $A = \varphi^{-1}(\psi(\Sigma))$

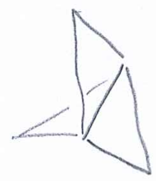
$$\begin{tikzcd} & \Delta & \\ \Sigma \arrow[ur, "\varphi"] & & \Sigma \arrow[ul, "\psi"'] \\ & A \arrow[ul] \arrow[ur, "w"'] & \end{tikzcd}$$

<u>Example</u> $\Delta = \Sigma$, $\mathcal{C} = W$ (acting on Σ)

<u>Terminology</u> $\mathcal{C}(\Sigma)$ – <u>apartment</u> $\mathcal{C}$– <u>chart</u>

$C \in \Delta$ maximal simplex = <u>chamber</u>
$a \in \Delta$ codim 1 simplex = <u>panel</u>

Δ <u>thick</u> $\Leftrightarrow$ every panel is in at least 3 chambers
(always in at least 2 chambers)



Δ of <u>irreducible type</u> $\Leftrightarrow$ (W,I) not reducible
$\Leftrightarrow$ $\Delta \neq \Delta_1 * \Delta_2$

Δ <u>spherical</u> $\Leftrightarrow$ W <u>finite</u> (Δ need <u>not</u> be finite!)

<u>Fact</u> $(\Delta, \mathcal{C})$ bldg $\Rightarrow$ ex. unique <u>maximal</u> <u>atlas</u> $\mathcal{C}_{max} \supseteq \mathcal{C}$
s.t. $(\Delta, \mathcal{C}_{max})$ is a bldg.

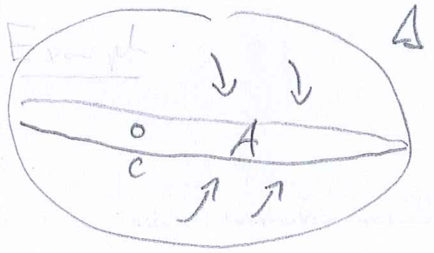
<u>Thm</u> (Tits) Δ thick irreducible spherical type, $\dim \Delta \geq 2$
$\Rightarrow$ ex G, G_i $i \in I$ sd $\Delta \cong N(G, \{G_i\})$
G simple algebraic / classical group (+ a few more ...)
G_i maximal parabolics

<u>Ex</u> $G = SL_{n+1}(K)$ G_i stabilizer of $e_0 K \oplus \dots \oplus e_i K$
$e_0, \dots, e_n$ standard basis , Δ <u>Flag complex</u> of K^{n+1}

$\rightsquigarrow$ Fundamental Theorem of Projective Geometry

<u>The retraction</u> Let Δ be a building, $A \subseteq \Delta$ an apartment, $c \in A$ a chamber. Given $b \in \Delta$, there exists an apartment A' with $c, b \in A'$ and a unique simplicial isomorphism $\varphi : A' \longrightarrow A$ fixing c pointwise. Put $\rho(b) = \varphi(b)$. There is a resulting well-defined retraction $\rho : \Delta \longrightarrow A$ fixing A pointwise. Put

$$\rho_{A,c} = \rho$$



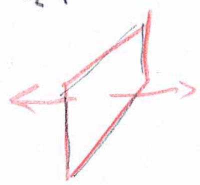
<u>Remark</u> Δ a building, $a \in \Delta \Rightarrow lk(a)$ is again a building.

<u>Fact</u> (W,I) of spherical type $\Leftrightarrow |\Sigma| \cong \mathbb{S}^m$

$$m = \# I - 1$$

Euclidean reflection groups

Let $\mathcal{H}$ be a collection of affine hyperplanes in $\mathbb{E}^m$ with (i) $\mathcal{H}$ is locally finite ($\forall p \in \mathbb{E}^m\ \exists \varepsilon > 0$ s.t. $\{H \in \mathcal{H} \mid B_\varepsilon(p) \cap H \neq \emptyset\}$ is finite) and (ii) r_H reflection along $H \in \mathcal{H}$ $\Rightarrow r_H(\mathcal{H}) = \mathcal{H}$



H

Put $W = \langle r_H \mid H \in \mathcal{H} \rangle$. We call W a euclidean reflection group.

Thm $\mathbb{E}^m$ has a unique W-invariant decomposition

$$\mathbb{E}^m = X_{-k} \times \dots \times X_0 \times \dots \times X_\ell$$

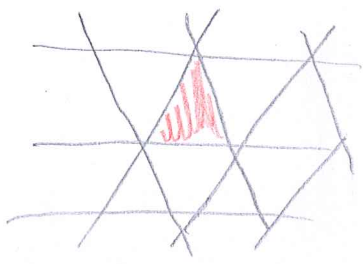
$X_i \cong \mathbb{E}^{m_i}$ such that:

$i < 0 \Rightarrow W$ acts as a finite irreducible reflection group W_i on X_i. W_i is a spherical Coxeter group of irred. type (and any such W_i may occur)

$i = 0 \Rightarrow W$ does not act at all

$i > 0 \Rightarrow W$ acts as $W_i \ltimes \Gamma$, W_i finite irred. reflection group as above, $\Gamma \cong \mathbb{Z}^{m_i}$ a W_i-invariant lattice. Then $W_i \ltimes \Gamma = \tilde{W}_i$ is called an affine Coxeter group (this is indeed a Coxeter group!)

Fact For an affine Coxeter group $\tilde{W} = W \ltimes \Gamma$, $\Gamma \cong \mathbb{Z}^m$ the Coxeter complex "is" $\mathbb{E}^m$, $|\Sigma| \cong \mathbb{E}^m$



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An *affine building* is a building whose Coxeter group is affine.

Lemma Let X be a set, $d: X \times X \to \mathbb{R}$ a map.
Let $\mathcal{A}$ be a collection of subsets of X with

(i) $A \in \mathcal{A} \Rightarrow (A, d)$ is a metric space

(ii) $\forall p, q \in A$ ex $A \in \mathcal{A}$ $p, q \in A$

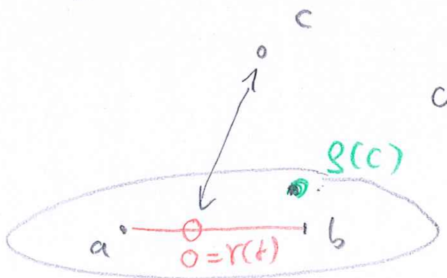
(iii) $\forall A \in \mathcal{A}, o \in A \ \exists \ \rho : X \to A$ retraction $(\rho^2 = \rho, \rho(X) = A)$

s.t. $\forall p \in A \ q \in X$
$$d(o, \rho(q)) = d(o, q)$$
$$d(p, \rho(q)) \leq d(p, q)$$

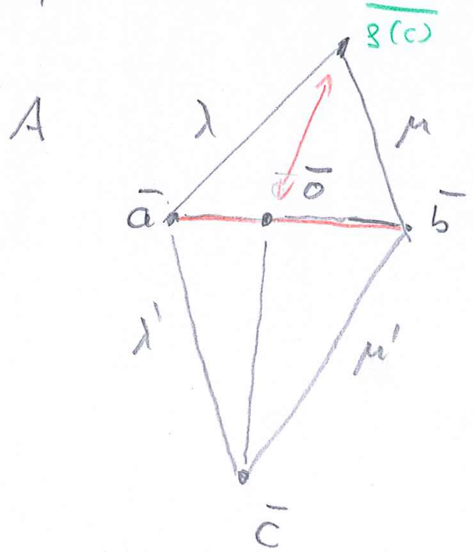
Then X is a metric space. If each $A \in \mathcal{A}$ is CAT(0) (CAT(1), resp.) then X is CAT(0) (CAT(1))

Pf Triangle inequality:
$$d(p,q) + d(q,o) \geq d(p, \rho(q)) + d(\rho(q), o) \geq d(p, o)$$

CAT(0) Condition:



$$d(o,c) = d(o, \rho(c)) \leq d(\bar{o}, \overline{\rho(c)}) \leq d(\bar{o}, \bar{c})$$



$\lambda' \geq \lambda$

$\mu' \geq \mu$

$\Rightarrow d(\bar{c}, \bar{o}) \geq d(\overline{\rho(c)}, \bar{o})$

(Similarly for CAT(1))

Thm The polymetric realization of an affine (spherical) building Δ admits a W-compatible CAT(0) (CAT(1)) metric.

Pf For an apt $A \subseteq \Delta$ we put the standard euclidean / spherical metric on $|A| \cong \mathbb{E}^m$ / $|A| \cong S^m$. This metric is W-invariant, hence we obtain a well defined map $d: |\Delta| \times |\Delta| \to \mathbb{R}$ which restricts to a metric on each apartment.

Given $A \subseteq \Delta$ apt, $o \in |A|$ choose a chamber $c \in A$ with $o \in |c|$, consider $\rho: \Delta \to A \leadsto \rho: |\Delta| \to |A|$

Let $p \in |A|$, $q \in |\Delta|$, choose $A' \subseteq \Delta$ with $c \in A'$ and $q \in |A'|$. Then $d(o,q) = d(o, \rho(q))$ since $\rho: |A'| \to |A|$ is an isometry. Let γ be a geodesic in some apt. $|A''|$ from p to q. Since ρ is simplicial, $\rho \circ \gamma$ is a broken geodesic from p to $\rho(q)$

$\Rightarrow d(p, \rho(q)) \leq d(p,q)$ $\square$

Thm Let Δ be an affine building. Then $\partial_\infty |\Delta|$ (in the Tits metric) is a spherical building Δ_∞, the spherical building at infinity. The apts of Δ_∞ are the Tits boundaries of the apts of $\mathcal{C}_{max}$.

Thm Let Δ be an affine building, let $\mathbb{E}^l \xrightarrow{\varphi} |\Delta|$ be a local isometric map. Then there is an apt A (w.r.t $\mathcal{C}_{max}$) such that $\mathbb{E}^l \to |A|$ embeds isometrically.

Lecture III - Characterizations

<u>Fact</u>. Let Δ be an affine building, let v be a vertex. Then $|lk(v)| \cong \Sigma_v |\Delta|$

In general, if $a \in \Delta$ and if $o \in |a|$ is an interior point, then

$$\Sigma_o |\Delta| \cong \mathbb{S}^l * |lk(a)| \quad \text{spherical join}$$

$$l = \dim(a) - 1$$

In particular, $\Sigma_o |\Delta|$ is <u>for all</u> $o \in |\Delta|$ the CAT(1) realization of a spherical building — in fact, a building of type W, where $\overline{W} = W \rtimes \Gamma$ is the affine Coxeter group.

<u>Fact</u> Δ_∞ thick $\Leftrightarrow$ exists $o \in |\Delta|$ with $\Sigma_o |\Delta|$ thick <u>of type W</u>.

<u>Thm</u> (Bruhat-Tits) Suppose that Δ is affine, thick, of dimension ≥ 3. Then Δ is of algebraic origin (comes alg. group / classical group over valued field or skew field)

Now to characterizations.

Thm Let Δ be a spherical building, let $A \subseteq sd(\Delta)$ be a convex subcomplex ($|A| \subseteq |\Delta|$ convex)

$sd(a)$

a

Vertices in sd = nonempty simplices

simplices in sd = asc. chains

$|sd(\Delta)| = |\Delta|$

If for every $o \in |A|$ there is $p \in |A|$ st $d(o,p) = \pi$, then there exist a thick spherical building Δ' s.t.

$$|A| = \mathbb{S}^{\ell} * |\Delta'| \qquad \text{unique}$$

Note: this says something even for $A = sd(\Delta)$!

Thm Let Δ be an affine build, $A \subseteq sd(\Delta)$, $|A| \subseteq |\Delta|$ convex. If $|A|$ has extensible geodesics, then unique

$$|A| = \mathbb{E}^n \times \underbrace{|T_1| \times \dots \times |T_k|}_{\text{thick trees}} \times \underbrace{|\Delta_1| \times \dots \times |\Delta_\ell|}_{\text{affine, thick of dim} \geq 2}$$

Thm (Kleiner / Balser-Lytchak)

X locally compact CAT(0) with extensible geodesics, $\dim(X) = m$, $\forall p,q \in X \; \exists F \subseteq X \quad p,q \in F$, $F \cong \mathbb{E}^m \Rightarrow X = \mathbb{E}^n \times |T_1| \times \dots \times \quad |\Delta_\ell|$

(as above

Thm (Charney-Lytchak) X CAT(0) piecewise euclid cell cplx with extensible geodesics. If every ray is in a discrt set of lines, then

$$X \cong \mathbb{E}^n \times |T_1| \cdots \times |\Delta_e| \quad \text{as before}$$

Thm (Leeb) Let X be a locally compact CAT(0) space with extensible geodesics. If $\partial_\infty X$ is the CAT(1) realisation of a thick spherical bld. of dim ≥ 1 of irreducible type, then X is either a Riemannian symmetric space or the geometric realization of an affine building.

Metric aspects of Euclidean buildings – literature and further reading.

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