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Hodge operators and groups of isometries of diagonalizable symmetric bilinear forms in characteristic two

Linus Kramer [a], Markus J. Stroppel [b,*]

[a] *Mathematisches Institut, Fachbereich Mathematik und Informatik, Universität Münster, Einsteinstraße 62, 48149 Münster, Germany*
[b] *LExMath, Fakultät 8, Universität Stuttgart, 70550 Stuttgart, Germany*

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We study groups of isometries of non-alternating symmetric bilinear forms on vector spaces of characteristic two, and actions of these groups on exterior powers of the space, viewed as modules over algebras generated by Hodge operators.

* Corresponding author.
E-mail addresses: linus.kramer@uni-muenster.de (L. Kramer), stroppel@mathematik.uni-stuttgart.de (M.J. Stroppel).

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Introduction

In [10], we have introduced Hodge operators using diagonalizable σ-hermitian forms on vector spaces over a field $\boldsymbol{F}$ of arbitrary characteristic; here σ is an automorphism of $\boldsymbol{F}$ with $\sigma^2 = \mathrm{id}$, we allow $\sigma = \mathrm{id}$ (in fact, we concentrate on that case in the present paper). If the dimension of that space is an even number $n = 2\ell$ then these operators help to understand exceptional homomorphisms between groups of semi-similitudes; these homomorphisms can be interpreted as representations of the group of semi-similitudes of the given form on the ℓ-th exterior power of the space, where the latter is turned into a module over an algebra $\boldsymbol{K}_\ell$ generated by the Hodge operator (see [10, Sect. 2]). That algebra turns out to be a composition algebra if $\sigma \neq \mathrm{id}$ or if $\operatorname{char} \boldsymbol{F} \neq 2$, but it will be inseparable if $\sigma = \mathrm{id}$ and $\operatorname{char} \boldsymbol{F} = 2$: in that case, we obtain $\boldsymbol{K}_\ell \cong \boldsymbol{F}[X]/(X^2 - \delta)$, for some $\delta \in \boldsymbol{F}$.

In the case where $\sigma = \mathrm{id}$ and the characteristic is two, the forms in question are not the ones that lead to classical groups: we then use a bilinear form that is symmetric but not alternating (see 1.2 below); the definitions of classical groups (i.e., symplectic, unitary, or orthogonal groups) in characteristic two employ non-degenerate forms that are either alternating, or σ-hermitian with $\sigma \neq \mathrm{id}$, or quadratic forms.

Therefore, the inseparable case is treated in a cursory way only in [10]. However, it leads to phenomena that appear to be interesting, if only as a marked contrast to the results in [10]: for instance, the orthogonal group with respect to the bilinear form may be a simple group (see 3.2.a below) or abelian (see 3.2.b), or it may act on a submodule of the exterior power with a rather large nilpotent kernel, and inducing a simple group on that submodule (see 3.1). We treat this inseparable case in a more detailed manner in the present notes, with a focus on $\ell = 2$ (and $n = 4$) because interesting phenomena are already apparent in this dimension (and the Klein quadric provides some extra geometric intuition).

1. Symmetric bilinear and diagonal quadratic forms in characteristic two

We recall basic facts about forms over fields of characteristic two, and fix some notation.

1.1 Notation. Let $\boldsymbol{F}$ be any field of characteristic 2, let V be a vector space of dimension n over $\boldsymbol{F}$, and let $h\colon V \times V \to \boldsymbol{F}$ be a non-degenerate symmetric bilinear form. Moreover, assume that there exists an orthogonal basis $v_1, \ldots, v_n$ with respect to h (see 1.2 below).

As $\dim V$ is assumed to be finite, our assumption that h be not degenerate is equivalent to the fact that (by a slight abuse of notation) we may view h as a linear isomorphism onto the dual space $V^\vee$,

$$h\colon V \to V^\vee : v \mapsto h(v, -)$$

see e.g. [9, Ch. I, §2]. Consider now the exterior algebra $\bigwedge V$, cf. [8, VI 9]. We note that $\bigwedge$ is a functor on vector spaces and (semi)linear maps, cp. [10, 1.6]. Moreover, there is a natural isomorphism $(\bigwedge V)^\vee \cong \bigwedge(V^\vee)$, so we may write unambiguously $\bigwedge V^\vee$. Explicitly, we have $\langle f_1 \wedge \cdots \wedge f_\ell, w_1 \wedge \cdots \wedge w_\ell\rangle = \det(\langle f_i, w_k\rangle)$ for $f_i \in V^\vee$ and $w_k \in V$, see [11, I.5.6] or[1] [2, § 8, Thme. 1, p. 102]. Applying the functor $\bigwedge$ to $h\colon V \to V^\vee$, we obtain $\bigwedge h\colon \bigwedge V \to \bigwedge V^\vee$; we interpret this as a bilinear form $\bigwedge h$ on the exterior algebra $\bigwedge V$. Using the explicit formula above, we find

$$\textstyle\bigwedge h(v_1 \wedge \cdots \wedge v_\ell, w_1 \wedge \cdots \wedge w_\ell) = \bigwedge^\ell h(v_1 \wedge \cdots \wedge v_\ell, w_1 \wedge \cdots \wedge w_\ell) = \det(h(v_i, w_j)).$$

In particular, the form $\bigwedge^\ell h$ is symmetric because transposition does not change the determinant.

The quadratic form $q \coloneqq q_h\colon V \to \boldsymbol{F}\colon v \mapsto h(v,v)$ is a semilinear map, its companion is the Frobenius endomorphism $\varphi\colon \boldsymbol{F} \to \boldsymbol{F}\colon x \mapsto x^2$, considered as an isomorphism from $\boldsymbol{F}$ onto the field $\boldsymbol{F}^\square \coloneqq \{x^2 \mid x \in \boldsymbol{F}\}$ of squares. The kernel of q consists of all vectors that are isotropic with respect to h. In particular, it contains $V^\perp$. For each vector space complement W for that kernel in V, the restriction of q to W is an injective map, and $\dim_{\boldsymbol{F}} W = \dim_{\boldsymbol{F}^\square} q(V)$.

Let $\Gamma\mathrm{L}(V)$ be the group of semilinear bijections of V onto itself. For $\gamma \in \Gamma\mathrm{L}(V)$, let $\alpha_\gamma \in \mathrm{Aut}(\boldsymbol{F})$ denote the companion of γ. In the spirit of Dieudonné's notation, we consider the group

$$\Gamma\mathrm{O}(V,h) \coloneqq \left\{\gamma \in \Gamma\mathrm{L}(V) \;\middle|\; \exists r_\gamma \in \boldsymbol{F}^\times\, \forall v,w \in V\colon h(\gamma(v),\gamma(w)) = r_\gamma\, \alpha_\gamma(h(v,w))\right\}$$

of *semi-similitudes* of the form h, the subgroup $\mathrm{GO}(V,h) \coloneqq \Gamma\mathrm{O}(V,h) \cap \mathrm{GL}(V)$ of (linear) similitudes, and the group $\mathrm{O}(V,h) \coloneqq \left\{\gamma \in \mathrm{GL}(V) \;\middle|\; \forall v,w \in V\colon h(\gamma(v),\gamma(w)) = h(v,w)\right\}$ of isometries. Note that Dieudonné would refer to the latter group as a unitary group, he reserved the term "orthogonal" for groups of isometries of quadratic forms.

For a quadratic form $q\colon V \to \boldsymbol{F}$ we have

$$\Gamma\mathrm{O}(V,q) \coloneqq \left\{\gamma \in \Gamma\mathrm{L}(V) \;\middle|\; \exists r_\gamma \in \boldsymbol{F}^\times\, \forall v \in V\colon q(\gamma(v)) = r_\gamma\, \alpha_\gamma(q(v))\right\},$$

$\mathrm{GO}(V,q) \coloneqq \Gamma\mathrm{O}(V,q) \cap \mathrm{GL}(V)$, and $\mathrm{O}(V,q) \coloneqq \left\{\gamma \in \mathrm{GL}(V) \;\middle|\; \forall v \in V\colon q(\gamma(v)) = q(v)\right\}$.

If the form in question is not zero then the *multiplier* r_γ of a linear similitude γ is determined by γ, and $\gamma \mapsto r_\gamma$ is a group homomorphism.

The following result dates back to [1, Th. 6, p. 392], see also [5, I § 10, p. 20 with I § 8, p. 15] or [15, Theorem 6.3.1]. For the reader's convenience, we include a proof.

1.2 Lemma. *Let $h\colon V \times V \to \boldsymbol{F}$ be a symmetric bilinear form on a vector space V of finite dimension n over a field $\boldsymbol{F}$ with* $\operatorname{char} \boldsymbol{F} = 2$. *Let $k \coloneqq \dim_{\boldsymbol{F}^\square} q_h(V)$.*

[1] The treatment in [2] is quite different from that in later editions [4].

a. *If h is non-zero then there exists an orthogonal basis for V with respect to h if, and only if, the form h is not alternating, i.e., if there exists $v \in V$ with $q_h(v) \neq 0$.*

b. *If h is not alternating then there exists an orthogonal basis $v_1, \dots, v_n$ of V such that $q_h(v_1), \dots, q_h(v_k)$ is an $\boldsymbol{F}^{\square}$-basis for $q_h(V)$.*

Proof. We abbreviate $q \coloneqq q_h$ as in 1.1. If there exists an orthogonal basis then the Gram matrix with respect to that basis is diagonal, and will be zero if the form is alternating. Conversely, assume that there exists $v \in V$ with $q(v) \neq 0$. If $V^{\perp} \neq \{0\}$, we choose any basis for $V^{\perp}$, and any vector space complement W to $V^{\perp}$ with $v \in W$. The restriction of h to W is not degenerate, and not alternating. It suffices to show that there is an orthogonal basis for W. Assume that $w_1, \dots, w_k$ are pairwise orthogonal vectors in W with $q(w_i) \neq 0$. Then these vectors are linearly independent, the restriction of h to their span W_k is not degenerate, and $W_k^{\perp} \cap W$ is a complement to W_k in W.

We proceed by induction on $\dim(W_k^{\perp} \cap W)$: If the restriction of h to $W_k^{\perp} \cap W$ is either zero or not alternating, we apply the induction hypothesis. It remains to study the case where there exist $x, y \in W_k^{\perp} \cap W$ with $h(x, y) = 1$ and $q(x) = 0 = q(y)$. Put $w_{k+1} \coloneqq w_k + q(w_k)y$, $w_{k+2} \coloneqq w_k + x + q(w_k)y$, and $\tilde{w}_k \coloneqq w_k + x$. Straightforward computation yields $q(w_{k+1}) = q(w_{k+2}) = q(\tilde{w}_k) = q(w_k) \neq 0$, and that the vectors $\tilde{w}_k, w_{k+1}, w_{k+2} \in \{w_1, \dots, w_{k-1}\}^{\perp}$ are pairwise orthogonal.

So $w_1, \dots, w_{k-1}, \tilde{w}_k, w_{k+1}, w_{k+2}$ is an orthogonal basis for $W_k + \boldsymbol{F}x + \boldsymbol{F}y$. Applying the induction hypothesis to $(W_k + \boldsymbol{F}x + \boldsymbol{F}y)^{\perp} \cap W$ finishes the proof of assertion a.

For assertion b, we choose any orthogonal basis and re-order it in such a way that the values of q at the first k basis vectors are linearly independent over $\boldsymbol{F}^{\square}$. $\square$

1.3 Lemma. *Let $h \colon V \times V \to \boldsymbol{F}$ be a symmetric bilinear form over a field $\boldsymbol{F}$ with* $\operatorname{char} \boldsymbol{F} = 2$*, and let $q \coloneqq q_h \colon V \to \boldsymbol{F} \colon x \mapsto h(x, x)$ be the corresponding quadratic form. If q is anisotropic then the groups $\mathrm{O}(V, q)$ and $\mathrm{O}(V, h)$ are both trivial, and each one of the sets $\boldsymbol{L} \coloneqq \mathrm{GO}(V, q) \cup \{0\}$ and $\boldsymbol{L}_h \coloneqq \mathrm{GO}(V, h) \cup \{0\}$ forms a subfield of the endomorphism ring $\mathrm{End}_{\boldsymbol{F}}(V)$.*

Putting $r_0 \coloneqq 0$, we extend the multiplier map to a homomorphism $r \colon \boldsymbol{L} \to \boldsymbol{F}$ of fields. This yields isomorphisms from $\boldsymbol{L}$ and from $\boldsymbol{L}_h$, respectively, onto subfields of $\boldsymbol{F}$.

The set $q(V)$ is a vector space over the field $r(\boldsymbol{L})$. If $1 \in q(V)$ then $r(\boldsymbol{L}) \subseteq q(V)$.

Proof. Clearly we have $\mathrm{O}(V, h) \leq \mathrm{O}(V, q)$ and $\mathrm{GO}(V, h) \leq \mathrm{GO}(V, q)$. Interpreting q as a semi-linear map from V to the vector space $\boldsymbol{F}$ over $\boldsymbol{F}^{\square}$ we see that q is injective by our assumption that q is anisotropic. Therefore, the group $\mathrm{O}(V, q)$ is trivial, and so is the subgroup $\mathrm{O}(V, h)$. It follows that the multiplier map $r \colon \mathrm{GO}(V, q) \to \boldsymbol{F}^{\times}$ is injective. In particular, the group $\mathrm{GO}(V, q)$ is isomorphic to a subgroup of $\boldsymbol{F}^{\times}$, and thus commutative.

The set $\boldsymbol{L} \coloneqq \mathrm{GO}(V, q) \cup \{0\} \subseteq \mathrm{End}_{\boldsymbol{F}}(V)$ is closed under multiplication, and each member of $\boldsymbol{L} \setminus \{0\}$ has a multiplicative inverse. We extend the multiplier map by putting $r_0 \coloneqq 0$; this extension is still multiplicative.

For $\lambda, \mu \in \boldsymbol{L}$, we note $q\big((\lambda+\mu)(v)\big) = q\big(\lambda(v)+\mu(v)\big) = q\big(\lambda(v)\big)+q\big(\mu(v)\big) = r_\lambda q(v) + r_\mu q(v) = (r_\lambda + r_\mu)q(v)$. So $\boldsymbol{L}$ is closed under addition, and the multiplier map is additive.

Now assume $\lambda, \mu \in \mathrm{GO}(V,h) \le \mathrm{GO}(V,q)$; we need to show $\lambda+\mu \in \boldsymbol{L}_h := \mathrm{GO}(V,h) \cup \{0\}$. The multiplier $(r_\lambda)^2$ of $\lambda^2 \in \mathrm{GO}(V,h)$ is also the multiplier of $r_\lambda \,\mathrm{id} \in \mathrm{GO}(V,h)$, so injectivity of the multiplier map yields $\lambda^2 = r_\lambda \,\mathrm{id}$. Using $\operatorname{char} \boldsymbol{F} = 2$ and commutativity of $\mathrm{GO}(V,h)$, we obtain

$$\begin{aligned} h\big(\lambda(v),\mu(w)\big) + h\big(\mu(v),\lambda(w)\big) &= r_\lambda^{-1} h\big(\lambda^2(v), \lambda\mu(w)\big) + r_\mu^{-1} h\big(\mu^2(v), \mu\lambda(w)\big) \\ &= h\big(v, \lambda\mu(w)\big) + h\big(v, \mu\lambda(w)\big) = 0. \end{aligned}$$

This yields $h\big((\lambda+\mu)(v),(\lambda+\mu)(w)\big) = h\big(\lambda(v),\lambda(w)\big) + h\big(\lambda(v),\mu(w)\big) + h\big(\mu(v),\lambda(w)\big) + h\big(\mu(v),\mu(w)\big) = r_\lambda\, h(v,w) + r_\mu\, h(v,w) = (r_\lambda + r_\mu)\, h(v,w)$, as required.

Finally, assume that there exists $v \in V$ with $q(v) = 1$. For $\lambda \in \mathrm{GO}(V,q)$, we obtain $q(\lambda(v)) = r_\lambda q(v) = r_\lambda$, and $q(V) = \{r_\lambda \mid \lambda \in \boldsymbol{L}\}$ follows. □

Clearly, we have $\boldsymbol{F}\mathbf{1} \le \boldsymbol{L}_h \le \boldsymbol{L}$, where $\mathbf{1}$ denotes the identity (matrix). It may well happen that $\boldsymbol{F}\mathbf{1} \ne \boldsymbol{L}_h \ne \boldsymbol{L}$, see 3.4d below.

1.4 Remark. Let $\boldsymbol{K} = \boldsymbol{F} + \boldsymbol{F}z$ be the local algebra of degree 2 over a field $\boldsymbol{F}$ with $\operatorname{char} \boldsymbol{F} = 2$, with $z^2 = 0$. For a symmetric matrix $S \in \boldsymbol{F}^{3\times3}$, let $g\colon \boldsymbol{K}^3 \times \boldsymbol{K}^3 \to \boldsymbol{K}$ and $\bar{g}\colon \boldsymbol{F}^3 \times \boldsymbol{F}^3 \to \boldsymbol{F}$, respectively, be the bilinear forms with Gram matrix S. Every matrix $B \in \boldsymbol{K}^{3\times3}$ can be written as $B = A + zX$ with $A, X \in \boldsymbol{F}^{3\times3}$. Computing $B^\intercal SB = A^\intercal SA + z(A^\intercal SX + X^\intercal SA)$ we see that B belongs to $\mathrm{O}(\boldsymbol{K}^3, g)$ precisely if $A \in \mathrm{O}(\boldsymbol{F}^3, \bar{g})$ and $A^\intercal SX$ is symmetric.

E.g., for $S = \begin{pmatrix} 1&0&0\\0&1&0\\0&0&c \end{pmatrix}$ with $c \in \boldsymbol{F} \smallsetminus \boldsymbol{F}^\square$ we obtain $\mathrm{O}(\boldsymbol{F}^3, \bar{g}) = \left\{ \begin{pmatrix} \mathbf{1}+aJ & 0 \\ 0 & 1 \end{pmatrix} \,\middle|\, a \in \boldsymbol{F} \right\}$, where $J := \begin{pmatrix} 1&1\\1&1 \end{pmatrix}$, and then $\mathrm{O}(\boldsymbol{K}^3, g) = \left\{ \begin{pmatrix} (\mathbf{1}+aJ)(\mathbf{1}+zS) & zc(\mathbf{1}+aJ)v \\ zv^\intercal & 1+zd \end{pmatrix} \,\middle|\, a,d \in \boldsymbol{F}, v \in \boldsymbol{F}^2, S = S^\intercal \in \boldsymbol{F}^{2\times2} \right\}$.

In Section 3, we will repeatedly need the following.

1.5 Lemma. *Let $\boldsymbol{K}$ be a commutative local ring with $1+1=0$ in $\boldsymbol{K}$, and let* $\mathrm{SL}_2(\boldsymbol{K}) := \left\{ \begin{pmatrix} a&b\\c&d \end{pmatrix} \,\middle|\, a,b,c,s \in \boldsymbol{K}, ad-bc = 1 \right\}$. *We write* $\boldsymbol{i} := \begin{pmatrix} 0&1\\1&0 \end{pmatrix}$, $L_x := \begin{pmatrix} 1&0\\x&1 \end{pmatrix}$, $U_x := \begin{pmatrix} 1&x\\0&1 \end{pmatrix}$,

$$\hat{L}_x = \begin{pmatrix} 1+x & 0 & x \\ 0 & 1 & 0 \\ x & 0 & 1+x \end{pmatrix}, \quad \hat{U}_x = \begin{pmatrix} 1+x & x & 0 \\ x & 1+x & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \text{and} \quad T := \begin{pmatrix} 1&1&1\\1&1&0\\1&0&1 \end{pmatrix}.$$

The bilinear forms g and g' on $\boldsymbol{K}^3$ are given by $g(x,y) = x_1y_1 + x_2y_3 + x_3y_2$ and $g'(x,y) = \sum_{k=1}^{3} x_ky_k$, respectively.

a. *The set $\{L_x \mid x \in \boldsymbol{K}\} \cup \{U_x \mid x \in \boldsymbol{K}\}$ generates $\mathrm{SL}_2(\boldsymbol{K})$.*

b. *The set* $\{\hat{L}_x \mid x \in \boldsymbol{K}\} \cup \{\hat{U}_x \mid x \in \boldsymbol{K}\}$ *generates a group* $\hat{\Sigma}$ *isomorphic to* $\mathrm{SL}_2(\boldsymbol{K})$. *Indeed,* $T^{-1}\hat{L}_x T = \left(\begin{smallmatrix} 1 & 0 \\ 0 & L_x \end{smallmatrix}\right)$, $T^{-1}\hat{U}_x T = \left(\begin{smallmatrix} 1 & 0 \\ 0 & U_x \end{smallmatrix}\right)$, *and* $T^{-1}\,\hat{\Sigma}\,T = \Sigma := \left\{ \left(\begin{smallmatrix} 1 & 0 \\ 0 & A \end{smallmatrix}\right) \,\middle|\, A \in \mathrm{SL}_2(\boldsymbol{K}) \right\}$.

c. *Now assume that the maximal ideal* M *of* $\boldsymbol{K}$ *has the property* $MM = \{0\}$. *Then* $\Xi := \left\{ \left(\begin{smallmatrix} 1 & u^\intercal \boldsymbol{i} \\ u & \mathbf{1} \end{smallmatrix}\right) \,\middle|\, u \in M^2 \right\}$ *and* $\Theta := \left\{ \left(\begin{smallmatrix} a & 0 \\ 0 & \mathbf{1} \end{smallmatrix}\right) \,\middle|\, a \in 1 + M \right\}$ *are subgroups, and* $\Xi\Theta$ *is an elementary abelian* 2*-group. The groups* Ξ *and* Θ *are both normalized by* Σ, *the semi-direct product*

$$\Xi\Theta\Sigma = \left\{ \begin{pmatrix} a & au^\intercal \boldsymbol{i}\, C \\ u & C \end{pmatrix} \,\middle|\, a \in 1 + M,\, u \in M^2,\, C \in \mathrm{SL}_2(\boldsymbol{K}) \right\}$$

equals $\mathrm{O}(\boldsymbol{K}^3, g)$, *and* $T(\Xi\Theta\Sigma)T^{-1} = \mathrm{O}(V, g')$.

Here M^2 *denotes the set of columns with two entries from* M.

Proof. Assertion a is well known ([7, Th. 4.3.9], see [16, p. 22] for the case where $\boldsymbol{K}$ is a field). Since matrix algebra over a local ring is less popular, we provide a direct argument (actually, in a form that works for every local ring): Let $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\boldsymbol{K})$. Then a and c cannot both be non-invertible. If a is invertible, then

$$\begin{pmatrix} 1 & 0 \\ a^{-1}(c-1) & 1 \end{pmatrix} \begin{pmatrix} 1 & a-1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & a^{-1}(1+b-a) \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

If c is invertible, then

$$\begin{pmatrix} 1 & c^{-1}(a-1) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} \begin{pmatrix} 1 & c^{-1}(d-1) \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Assertion b follows by direct computations.

Now assume $MM = \{0\}$. Then M is the kernel of the Frobenius endomorphism, and $\boldsymbol{K}^\square = \{x^2 \mid x \in \boldsymbol{K}\} \cong \boldsymbol{K}/M$ is a field. We interpret the quadratic form $q := q_g$ as a semilinear map from $\boldsymbol{K}^3$ to the vector space $\boldsymbol{K}$ over $\boldsymbol{K}^\square$. The kernel of q is $\{(x_1, x_2, x_3)^\intercal \in \boldsymbol{K}^3 \mid x_1^2 = 0\} = \{(x_1, x_2, x_3)^\intercal \in \boldsymbol{K}^3 \mid x_1 \in M\}$, and contains $\{(0, x_2, x_3)^\intercal \mid x_2, x_3 \in \boldsymbol{K}\}$. That kernel is invariant under each isometry in $\mathrm{O}(V, g)$.

We note $C^\intercal \boldsymbol{i} C = (\det C)\, \boldsymbol{i}$ for each matrix $C \in \boldsymbol{K}^{2\times 2}$ and $v^\intercal \boldsymbol{i}\, v = 0$ for each $v \in \boldsymbol{K}^2$. For any $a \in \boldsymbol{K}$, any $u, w \in \boldsymbol{K}^2$, and $C \in \boldsymbol{K}^{2\times 2}$ we write $w^\intercal = (w_1, w_2)$ and consider the matrix

$$A_{a,u,w,C} := \begin{pmatrix} a & w^\intercal \\ u & C \end{pmatrix}.$$

If the kernel of q is invariant under that matrix, we have $0 = q\big(A_{a,u,w,C}(0,1,0)^\intercal\big) = w_1^2$ and also $0 = q\big(A_{a,u,w,C}(0,0,1)^\intercal\big) = w_2^2$, so $w \in M^2$. Then $ww^\intercal = 0$.

Now $A_{a,u,w,C} \in \mathrm{O}(V,g)$ yields the conditions $a^2 = 1$, $wa = C^\intercal \boldsymbol{i}\, u$, and $C^\intercal \boldsymbol{i}\, C = \boldsymbol{i}$. This leads to $a^{-1} = a \in 1 + M$, $\det C = 1$, and $u \in M^2$.

Conversely, we have $A_{a,u,w,C} \in \mathrm{O}(V,g)$ for each choice of $a \in 1 + M$, $C \in \mathrm{SL}_2(\boldsymbol{K})$, $u \in M^2$ and $w := C^\intercal \boldsymbol{i}\, ua$. So

$$\mathrm{O}(V,g) = \left\{ \begin{pmatrix} a & au^\intercal \boldsymbol{i}\, C \\ u & C \end{pmatrix} \middle|\ a \in 1 + M,\, C \in \mathrm{SL}_2(\boldsymbol{K}),\, u \in M^2 \right\}.$$

It is easy to see that both $\Theta := \{A_{a,0,0,\mathbf{1}} \mid a \in 1 + M\}$ and Ξ are elementary abelian 2-groups. A straightforward calculation yields that they centralize each other, and that they are normalized by Σ. Then $\Theta\Xi$ is an elementary abelian subgroup, and normalized by Σ, as well. Thus $\mathrm{O}(V,g) = \Xi\Theta\Sigma$ follows.

We note $T^\intercal = T$ and $T^2 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & \boldsymbol{i} \end{smallmatrix}\right)$, so T^2 is the Gram matrix for g. So $T\,\mathrm{O}(V,g)\,T^{-1}$ equals the group $\mathrm{O}(V,g')$ of isometries with respect to the form with Gram matrix $(T^{-1})^\intercal T^2 T^{-1} = \mathbf{1}$. □

2. The Hodge operator in characteristic two

Fundamental definitions and results about Hodge operators have been worked out in some detail in [10]. We repeat the fundamental facts (with simplifications due to the concentration on a special case) and refer to [10] for details and proofs.

2.1 The Pfaffian form. Recall that $\dim \bigwedge^n V = 1$ if $\dim V = n$. We fix an isomorphism $b\colon \bigwedge^n V \to \boldsymbol{F}$. For each positive integer $\ell \leq n$ the map b induces an isomorphism $\mathrm{Pf}\colon \bigwedge^{n-\ell} V \to \bigwedge^\ell V^\vee$ given by

$$\mathrm{Pf}(v_1 \wedge \cdots \wedge v_{n-\ell})(w_1 \wedge \cdots \wedge w_\ell) = b(v_1 \wedge \cdots \wedge v_{n-\ell} \wedge w_1 \wedge \cdots \wedge w_\ell)\,.$$

This is the *Pfaffian form*, see [8, VI 10 Problems 23–28, VIII 12 Problem 42]. As $\mathrm{char}\,\boldsymbol{F} = 2$, the resulting bilinear map Pf on $\bigwedge V$ is symmetric,

$$\mathrm{Pf}(v_1 \wedge \cdots \wedge v_{n-\ell}, w_1 \wedge \cdots \wedge w_\ell) = \mathrm{Pf}(w_1 \wedge \cdots \wedge w_\ell, v_1 \wedge \cdots \wedge v_{n-\ell})\,.$$

If $n = 2\ell$ is even then $\mathrm{Pf}(v_1 \wedge \cdots \wedge v_\ell, v_1 \wedge \cdots \wedge v_\ell) = 0$ holds for each $v_1 \wedge \cdots \wedge v_\ell$, so Pf is an alternating form on $\bigwedge^\ell V$.

2.2 Remarks. For $n = 4$ and $\ell = 2$ we are dealing with the space $\bigwedge^2 \boldsymbol{F}^4$ that carries the Klein quadric. The *quadratic form* Pq defining the Klein quadric is also referred to as a Pfaffian form (cf. [6] and [14] where this form is denoted by q), and Pf is the polar form of that quadratic form. Under the present assumption $\mathrm{char}\,\boldsymbol{F} = 2$, the polar form Pf carries less information than the quadratic form Pq; in fact, for any diagonal quadratic form k (i.e., such that the polar form of k is zero) the sum $\mathrm{Pq} + k$ also has Pf as polar form. Note that Pf is alternating because it is the polar form of a quadratic form in characteristic two.

If one interprets the elements of $\bigwedge^2 \boldsymbol{F}^4$ as alternating matrices then there exists a scalar $s \in \boldsymbol{F}^\times$ such that $\mathrm{Pq}(X)^2 = s \det X$ holds for each $X \in \bigwedge^2 \boldsymbol{F}^4$, cf. [3, §5 no. 2, Prop. 2, p. 84]; the scalar s reflects the choice of basis underlying that interpretation.

We identify $x_{12} v_1 \wedge v_2 + x_{13} v_1 \wedge v_3 + x_{14} v_1 \wedge v_4 + x_{23} v_2 \wedge v_3 + x_{24} v_2 \wedge v_4 + x_{34} v_3 \wedge v_4$ with

$$X = \begin{pmatrix} 0 & x_{12} & x_{13} & x_{14} \\ -x_{12} & 0 & x_{23} & x_{24} \\ -x_{13} & -x_{23} & 0 & x_{34} \\ -x_{14} & -x_{24} & -x_{34} & 0 \end{pmatrix}$$

and compute $\det X = (x_{12}x_{34} - x_{13}x_{24} + x_{14}x_{23})^2$; up to a scalar (viz., the value $b(v_1 \wedge v_2 \wedge v_3 \wedge v_4)$ above), the value of Pq at X is obtained as $x_{12}x_{34} - x_{13}x_{24} + x_{14}x_{23}$, and the polar form of Pq maps (X, Y) to the corresponding scalar multiple of $x_{12}y_{34} + x_{13}y_{24} + x_{14}y_{23} + x_{23}y_{14} + x_{24}y_{13} + x_{34}y_{12}$.

See [16, 12.14] for an interpretation of Pq in terms of the exterior algebra.

2.3 The Hodge operator. We now consider the composite

$$\boldsymbol{J} \coloneqq \mathrm{Pf}^{-1} \circ \bigwedge h \colon \ \bigwedge^\ell V \xrightarrow[\cong]{\bigwedge h} \bigwedge^\ell V^\vee \xrightarrow[\cong]{\mathrm{Pf}^{-1}} \bigwedge^{n-\ell} V \,.$$

This semilinear isomorphism is the *Hodge operator*. It depends, of course, on h and on b but not on the choice of basis.

2.4 Explicit computation. Suppose that $v_1, \ldots, v_n$ is an orthogonal basis of V. For $\bigwedge^\ell V$ we use the basis vectors $v_{i_1} \wedge \cdots \wedge v_{i_\ell}$ with ascending $i_1 < \cdots < i_\ell \le n$. Then $\bigwedge^\ell h(v_1 \wedge \cdots \wedge v_\ell, -)$ is a linear form on $\bigwedge^\ell V$ which annihilates each one of those basis vectors, except for $v_1 \wedge \cdots \wedge v_\ell$; in fact

$$\bigwedge^\ell h(v_1 \wedge \cdots \wedge v_\ell, v_1 \wedge \cdots \wedge v_\ell) = h(v_1, v_1) \cdots h(v_\ell, v_\ell) \,.$$

In other words: $\bigwedge^\ell h$ is again diagonalizable. It then also follows that $\bigwedge^\ell h$ is not degenerate. The linear form $\mathrm{Pf}(v_{\ell+1} \wedge \cdots \wedge v_n)$ annihilates the same collection of basis ℓ-vectors, and

$$\mathrm{Pf}(v_{\ell+1} \wedge \cdots \wedge v_n, v_1 \wedge \cdots \wedge v_\ell) = b(v_1 \wedge \cdots \wedge v_n) \,.$$

Therefore

$$\boldsymbol{J}(v_1 \wedge \cdots \wedge v_\ell) = v_{\ell+1} \wedge \cdots \wedge v_n \, \frac{h(v_1, v_1) \cdots h(v_\ell, v_\ell)}{b(v_1 \wedge \cdots \wedge v_n)} \,.$$

Note that this last formula is correct only if $v_1, \ldots, v_n$ is an orthogonal basis, and cannot be used if $v_1 \wedge \cdots \wedge v_\ell$ corresponds to a subspace U of V such that the restriction of h to U is degenerate.

2.5 The square of the Hodge operator. *Let H be the Gram matrix of h with respect to the orthogonal basis $v_1, \dots, v_n$. The square of $\boldsymbol{J}$ is a linear automorphism of $\bigwedge^\ell V$, and we find that*

$$\boldsymbol{J}^2 = \delta_\ell \, \mathbf{1} \quad \text{where} \quad \delta_\ell := \tfrac{\det(H)}{b(v_1 \wedge \cdots \wedge v_n)^2} \,.$$

Recall that $\det(H)$ depends on the choice of basis; the invariant would be the square class $\operatorname{disc}(h) \in \boldsymbol{F}^\times / \boldsymbol{F}^\boxtimes$ of $\det(h(v_i, v_j))$. (Here $\boldsymbol{F}^\boxtimes$ denotes the multiplicative group of the field $\boldsymbol{F}^\square := \{s^2 \mid s \in \boldsymbol{F}\}$ of all squares in $\boldsymbol{F}$.) However, the whole expression depends only on h and b. Replacing the isomorphism $b \colon \bigwedge^n V \to \boldsymbol{F}$ changes $\boldsymbol{J}$ by a factor and $\boldsymbol{J}^2$ by the square of that factor. In particular, the isomorphism type of the algebra $\boldsymbol{K}_\ell$ introduced in 2.7 below does not depend on the choice of b.

2.6 Lemma. *For all $x, y \in \bigwedge^\ell V$ we have*

a. $\operatorname{Pf}(\boldsymbol{J}(x), y) = \bigwedge^\ell h(x, y)$,

b. $\operatorname{Pf}(\boldsymbol{J}(x), \boldsymbol{J}(y)) = \delta_\ell \operatorname{Pf}(y, x)$,

c. $\bigwedge^\ell h(\boldsymbol{J}(x), y) = \delta_\ell \operatorname{Pf}(x, y)$,

d. $\bigwedge^\ell h(\boldsymbol{J}(x), \boldsymbol{J}(y)) = \delta_\ell \bigwedge^\ell h(x, y)$.

From now on, assume $n = 2\ell$. Then $\boldsymbol{J}$ is an $\boldsymbol{F}$-linear endomorphism of $\bigwedge^\ell V$. We are going to use $\boldsymbol{J}$ to give $\bigwedge^\ell V$ the structure of a right module over an associative algebra of dimension 2 over $\boldsymbol{F}$.

2.7 The algebra $\boldsymbol{K}_\ell$. Take $\delta_\ell = \frac{\det(H)}{b(v_1 \wedge \cdots \wedge v_n)^2}$ as in 2.5 and put $\boldsymbol{K}_\ell := \boldsymbol{F}[\boldsymbol{j}_\ell]/(\boldsymbol{j}_\ell^2 - \delta_\ell)$.

2.8 Definition. For $v \in \bigwedge^\ell V$ we put $v\,\boldsymbol{j}_\ell := \boldsymbol{J}(v)$. In this way, the space $\bigwedge^\ell V$ becomes an $\mathrm{O}(V,h)$-$\boldsymbol{K}_\ell$-bimodule, i.e., it becomes a right module over $\boldsymbol{K}_\ell$ and $\mathrm{O}(V,h)$ acts $\boldsymbol{K}_\ell$-linearly from the left. Choosing an orthogonal basis $v_1, \dots, v_n$ for V with a fixed ordering, we obtain a basis B for $\bigwedge^\ell V$ consisting of all $v_{j_1} \wedge \cdots \wedge v_{j_\ell}$ where $(j_1, \dots, j_\ell)$ is an increasing sequence of length ℓ in $\{1, \dots, n\}$. The sequences with $j_1 = 1$ form a subset B_1 of B, and $\boldsymbol{J}$ maps each element of B_1 to one of $B \setminus B_1$. Moreover, the set B_1 forms a basis for the $\boldsymbol{K}_\ell$-module $\bigwedge^\ell V$, showing that the latter is a free module.

2.9 The bilinear form on the module. We define $g \colon \bigwedge^\ell V \times \bigwedge^\ell V \to \boldsymbol{K}_\ell$ by

$$g(u,v) := \bigwedge^\ell h(u,v) + \bigwedge^\ell h(u, v\boldsymbol{j}_\ell)\, \boldsymbol{j}_\ell^{-1} = \bigwedge^\ell h(u,v) + \boldsymbol{j}_\ell \,(-1)^\ell \operatorname{Pf}(u,v)\,;$$

see 2.6 for the description on the right hand side. This expression is $\boldsymbol{K}_\ell$-bilinear.

Note that $\boldsymbol{K}_\ell$ is not a field, in general: we need the more general concept of bilinear forms over rings.

2.10 Proposition. *The form g is diagonalizable.*

For a general proof (and a general formula) see [10, 2.7]. Actually, if $\boldsymbol{K}_\ell$ is a field (of characteristic two, and $\sigma = \mathrm{id}$ by our assumptions) it suffices to note that g is not alternating, see 1.2. For the applications in Section 3 below, we give a special statement explicitly:

2.11 Example. Let v_1, v_2, v_3, v_4 be an orthogonal basis for V, with respect to h. Then $w_2 := v_1 \wedge v_2$, $w_3 := v_1 \wedge v_3$, $w_4 := v_1 \wedge v_4$ form an orthogonal basis for the free $\boldsymbol{K}_2$-module $\bigwedge^2 V$, with respect to g. Explicitly, we have $g(w_2, w_2) = h(v_1, v_1)h(v_2, v_2)$, $g(w_3, w_3) = h(v_1, v_1)h(v_2, v_2)$, and $g(w_4, w_4) = h(v_1, v_1)h(v_4, v_4)$.

From the definition of g it is clear that $\mathrm{O}(V, h)$ preserves g and that the group $\Gamma\mathrm{O}(V, h)$ of semi-similitudes (see 1.1) acts by semi-similitudes of g, see [10, 1.8]. For $\ell = \frac{1}{2}\dim(V)$ we have thus constructed a homomorphism $\eta_\ell \colon \Gamma\mathrm{O}(V, h) \to \Gamma\mathrm{O}(\bigwedge^\ell V, g)$.

2.12 Lemma. *The kernel of η_ℓ is trivial.*

Proof. That kernel consists of all scalar multiples $s\mathbf{1}$ of the identity $\mathbf{1}$, where $s^2 = 1$. Since $\operatorname{char} \boldsymbol{F} = 2$, this yields $s = 1$. □

We will call $\boldsymbol{K}_\ell$ *split* whenever it contains divisors of zero. This extends the established terminology for composition algebras. Recall that $\boldsymbol{K}_\ell$ is split precisely if δ_ℓ is a square: $\delta_\ell = s^2$ for some $s \in \boldsymbol{F}^\times$ (and this happens precisely if h has discriminant 1).

In that case, we may assume $s = 1$ without loss of generality. In fact, if we replace our isomorphism $b \colon \bigwedge^n V \to \boldsymbol{F}$ by sb then the Hodge operator $\boldsymbol{J}$ is replaced by $\boldsymbol{J}\,s^{-1} \colon X \mapsto \boldsymbol{J}(X)s^{-1}$, and we have $(\boldsymbol{J}\,s^{-1})^2 = \mathbf{1}$ while the algebra $\boldsymbol{K}_\ell$ remains the same. If $s = 1$ then $z := 1 + \boldsymbol{j}_\ell \in \boldsymbol{K}_\ell$ satisfies $z^2 = 2z = 0$, and is nilpotent. Thus $\boldsymbol{K}_\ell \cong \boldsymbol{F}[X]/(X^2)$ is a local ring if $\boldsymbol{K}_\ell$ is split. Recall that a local ring is a ring in which the set of non-invertible elements is closed under addition (we allow the case where that set consists of 0 alone).

2.13 Lemma. *Let $W := \bigwedge^\ell V$, and assume that $\boldsymbol{K}_\ell$ is split.*

- **a.** *The maximal ideal in $\boldsymbol{K}_\ell$ is generated by a nilpotent element z. The submodule Wz and the quotient module W/Wz are isomorphic via $\rho_z \colon w + Wz \mapsto wz$.*
- **b.** *The restriction of the form g to the subspace Wz is trivial.*
- **c.** *The $\boldsymbol{K}_\ell$-submodule Wz is invariant under $\eta(\Gamma\mathrm{O}(V, h))$. Thus we obtain a homomorphism $\eta_\ell^o \colon \Gamma\mathrm{O}(V, h) \to \Gamma\mathrm{L}(Wz)$.*
- **d.** *The group induced by $\ker \eta_\ell^o$ on W is an elementary abelian 2-group, acting trivially on W/Wz.*

Proof. The first three assertions are taken from [10, 3.6]. For the last assertion, we note that elements of $\ker \eta_\ell^o$ act trivially on W/Wz because ρ_z is a module homomorphism. So $\ker \eta_\ell^o$ is isomorphic to a subgroup of $\mathrm{Hom}_{\boldsymbol{F}}(W/Wz, Wz)$. □

The precise structure of $\mathrm{O}(V,h)$ and $\ker \eta^o_\ell$ depends on the defect of the form h.

3. The four-dimensional cases

We focus on the case where $\ell = 2$ and $n = 2\ell = 4$, and write $\boldsymbol{K} := \boldsymbol{K}_2$. Either $\boldsymbol{K}$ splits and is isomorphic to the local algebra $\boldsymbol{F}[X]/(X^2)$, or we have an inseparable quadratic field extension $\boldsymbol{K}|\boldsymbol{F}$. Recall from 1.1 that $q\colon V \to \boldsymbol{F}\colon x \mapsto h(x,x)$ is a φ-semilinear map, where $\boldsymbol{F}$ is considered as a vector space over the subfield $\boldsymbol{F}^\square$ of squares, and $\varphi\colon \boldsymbol{F} \to \boldsymbol{F}^\square\colon s \mapsto s^2$ is the Frobenius endomorphism. Note that $\mathrm{O}(V,h)$ is contained in $\mathrm{O}(V,q)$. We distinguish cases according to $\dim_{\boldsymbol{F}^\square} q(V) \in \{1,2,3,4\}$; recall that $\dim_{\boldsymbol{F}} \ker q = 4 - \dim_{\boldsymbol{F}^\square} q(V) \in \{3,2,1,0\}$ is called the defect of q. At several places we will use the fact that the orthogonal group of an anisotropic diagonal quadratic form is trivial if the ground field has characteristic 2; cf. [5, § 16, p. 35]. Recall from 2.13 that the restriction $g|_{Wz\times Wz}$ is trivial if z is nilpotent (of course, this is of interest only if $\boldsymbol{K}$ splits).

We use the standard basis e_1, e_2, e_3, e_4 for $V = \boldsymbol{F}^4$, and write $W := \bigwedge^2 V$.

We will also use the basis $b_1 := (1,1,1,1)^\intercal = e_1+e_2+e_3+e_4$, $b_2 := (1,1,0,0)^\intercal = e_1+e_2$, $b_3 := (1,0,1,0)^\intercal = e_1+e_3$, $b_4 := (0,0,0,1)^\intercal = e_4$.

3.1 Proposition. *If q has defect 3, then $\boldsymbol{K}$ splits and $\mathrm{O}(V,h) \cong \big(\mathrm{SL}_2(\boldsymbol{F}) \ltimes \boldsymbol{F}^2\big) \times \boldsymbol{F}$. The normal subgroup $\Xi \cong \boldsymbol{F}^2 \times \boldsymbol{F}$ is the kernel of the action on Wz.*

Note that $\boldsymbol{K} = \boldsymbol{F} + \boldsymbol{F}z \cong \boldsymbol{F}[X]/(X^2)$ is not a field; the subfield $\boldsymbol{F} < \boldsymbol{K}$ is the image under the endomorphism $\psi\colon \boldsymbol{K} \to \boldsymbol{K}\colon r + sz \mapsto r$. Using matrix descriptions with respect to the $\boldsymbol{K}$-basis $b_1 \wedge b_4$, $b_2 \wedge b_4$, $b_3 \wedge b_4$, we apply ψ to each matrix entry and obtain an endomorphism of the group $\mathrm{GL}_3(\boldsymbol{K})$. Let $\Psi \cong \mathrm{GL}_3(\boldsymbol{F})$ be the image under that endomorphism; in the chosen coordinates, this subgroup $\Psi < \mathrm{GL}_3(\boldsymbol{K})$ consists of all invertible 3×3 matrices with entries from $\boldsymbol{F}$.

We obtain $\mathrm{SL}_2(\boldsymbol{F}) \cong \eta^o(\mathrm{O}(V,h)) = \mathrm{O}(W,g) \cap \Psi < \mathrm{O}(W,g) \cong \mathrm{SL}_2(\boldsymbol{K}) \cong \mathrm{SL}_2(\boldsymbol{F}) \times \mathrm{SL}_2(\boldsymbol{F})$. Every multiplier is a square, the group of similitudes is $\mathrm{GO}(V,h) = \boldsymbol{F}^\times\, \mathrm{O}(V,h)$.

Proof. If q has defect 3 then $\dim_{\boldsymbol{F}^\square} q(V) = 1$ and we may (upon replacing h by a scalar multiple of h, cp. 1.2b) assume that $h(x,y) = x^\intercal y$ is the description in coordinates x, y with respect to the standard basis. This form h has Witt index 2. In coordinates v, w with respect to the basis b_1, b_2, b_3, b_4 above, it is given by the map $\tilde{h}$, with

$$\tilde{h}(v,w) := v^\intercal \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} w = v_1w_4 + v_2w_3 + v_3w_2 + v_4w_1 + v_4w_4\,.$$

The orthogonal group $\mathrm{O}(V,\tilde{h})$ leaves the quadratic form $\tilde{h}(v,v) = v_4^2$ invariant. Thus it fixes the linear form with matrix $(0,0,0,1)$, and using suitable block matrices we obtain

$$\mathrm{O}(V,\tilde{h}) \le \left\{ \begin{pmatrix} A & x \\ 0 & a \end{pmatrix} \,\middle|\, a \in \boldsymbol{F}^\times, A \in \mathrm{GL}_3(\boldsymbol{F}), x \in \boldsymbol{F}^3 \right\}.$$

Now one computes easily that

$$\mathrm{O}(V,\tilde{h}) = \left\{ \begin{pmatrix} 1 & t^\intercal \boldsymbol{i} & c \\ 0 & B & Bt \\ 0 & 0 & 1 \end{pmatrix} \,\middle|\, B \in \mathrm{SL}_2(\boldsymbol{F}), t \in \boldsymbol{F}^2, c \in \boldsymbol{F} \right\}$$

where $\boldsymbol{i} := \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$.

Note that the elements with trivial B form an elementary abelian subgroup Ξ. In fact, we have an isomorphism

$$\xi\colon \boldsymbol{F}^2 \times \boldsymbol{F} \to \Xi\colon (t_1,t_2,t_3) \mapsto \begin{pmatrix} 1 & t_2 & t_1 & t_3 + t_1t_2 \\ 0 & 1 & 0 & t_1 \\ 0 & 0 & 1 & t_2 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The map ξ is $\boldsymbol{F}$-linear if we let the scalars act via the rule $(t_1,t_2,t_3)\cdot s := (t_1s, t_2s, t_3s^2)$. In particular, the dimension of $\boldsymbol{F}^2 \times \boldsymbol{F} \cong \Xi$ becomes $2 + \dim_{\boldsymbol{F}^\square} \boldsymbol{F}$ which will be greater than 3 whenever the field $\boldsymbol{F}$ is not perfect.[2] The group

$$\Sigma := \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \mathrm{GL}_4(\boldsymbol{F}) \,\middle|\, B \in \mathrm{SL}_2(\boldsymbol{F}) \right\} \cong \mathrm{SL}_2(\boldsymbol{F})$$

of block matrices normalizes Ξ and acts in the expected way: it fixes $\xi(\{0\}^2 \times \boldsymbol{F})$ pointwise and induces the usual action on the set

$$\xi\left(\left\{(t_1,t_2,t_1t_2) \,\middle|\, t_1,t_2 \in \boldsymbol{F}\right\}\right) = \left\{ \begin{pmatrix} 1 & t_2 & t_1 & 0 \\ 0 & 1 & 0 & t_1 \\ 0 & 0 & 1 & t_2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \,\middle|\, t_1,t_2 \in \boldsymbol{F} \right\}.$$

However, that set is not a subgroup; there is no Σ-invariant subgroup complement to $\xi(\{0\}^2 \times \boldsymbol{F})$.

We fix the isomorphism $b\colon \bigwedge^4 V \to \boldsymbol{F}$ in such a way that $\boldsymbol{J}(e_{\pi(1)} \wedge e_{\pi(2)}) = e_{\pi(3)} \wedge e_{\pi(4)}$ for each permutation π of $\{1,2,3,4\}$; recall that the standard basis e_1,e_2,e_3,e_4 is an ortho*normal* basis with respect to the form h in the present case. In particular, we now find $\delta = 1$, the algebra $\boldsymbol{K}$ splits, and $z := 1 + \boldsymbol{j}$ is nilpotent.

Using the basis $b_1,\dots,b_4$ from above, we obtain the $\boldsymbol{K}$-basis $b_1 \wedge b_4$, $b_2 \wedge b_4$, $b_3 \wedge b_4$ for W. With respect to that basis, the Gram matrix for g is $\left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{smallmatrix}\right)$. The elements of $\mathrm{O}(W,g)$ are thus described (with respect to the same basis) by the block matrices $\left(\begin{smallmatrix} a & u^\intercal \boldsymbol{i} C \\ u & C \end{smallmatrix}\right)$, with $a \in 1 + Fz$, $u \in (Fz)^2$, and $C \in \mathrm{SL}_2(\boldsymbol{K})$, see 1.5c. We observe that

[2] This phenomenon also plays its role in the study of duality of symplectic quadrangles, cf. [12] and [13].

$$\begin{array}{lllllll} Y_1 & := & (b_1 \wedge b_4)z & = & (e_1 \wedge e_4 + e_2 \wedge e_4 + e_3 \wedge e_4)z & = & b_1 \wedge b_4 + b_2 \wedge b_3\,, \\ Y_2 & := & (b_2 \wedge b_4)z & = & (e_1 \wedge e_4 + e_2 \wedge e_4)z & = & b_1 \wedge b_2\,, \\ Y_3 & := & (b_3 \wedge b_4)z & = & (e_1 \wedge e_4 + e_3 \wedge e_4)z & = & b_1 \wedge b_3 \end{array}$$

form a basis for Wz.

Evaluating $\xi(t_1, t_2, t_3) \in \Xi$ at Y_1, Y_2, and Y_3, we see that Ξ acts trivially on Wz, and then also acts trivially on W/Wz, see 2.13d. For $B \in \mathrm{SL}_2(\boldsymbol{F})$ we find that $\left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right) \in \Sigma$ maps $Y_1a_1 + Y_2a_2 + Y_3a_3$ to $Y_1a_1 + Y_2a_2' + Y_3a_3'$ with $(a_2', a_3')^{\intercal} = B(a_2, a_3)^{\intercal}$. In other words, the image of that element of Σ under η^o is described by the block matrix $\left(\begin{smallmatrix} 1 & 0 \\ 0 & B \end{smallmatrix}\right)$.

This action of $\mathrm{SL}_2(\boldsymbol{F})$ is an action by isometries of the $\boldsymbol{F}$-bilinear form k on Wz defined by $k(Y_1a_1 + Y_2a_2 + Y_3a_3, Y_1x_1 + Y_2x_2 + Y_3x_3) := a_1x_1 + a_2x_3 + a_3x_2$; see 1.5c (applied with $M = \{0\}$). Note that $\Sigma \cong \mathrm{SL}_2(\boldsymbol{F})$ induces the full group $\mathrm{O}(Wz, k)$. However, the form k is not g^o because $g^o \equiv 0$, see 2.13.

The range $q(V)$ of the quadratic form q is just $\boldsymbol{F}^{\square}$. So every similitude of q has an element of $\boldsymbol{F}^{\boxtimes}$ as multiplier, and belongs to $\boldsymbol{F}^{\times}\,\mathrm{O}(V, q)$. From $\mathrm{GO}(V, h) \le \mathrm{GO}(V, q)$ it then follows that $\mathrm{GO}(V, h) = \boldsymbol{F}^{\times}\,\mathrm{O}(V, h)$. □

3.2 Proposition. *Assume that q has defect 2. In coordinates with respect to the $\boldsymbol{K}$-basis $e_1 \wedge e_2$, $e_1 \wedge e_3$, $e_1 \wedge e_4$, let $\Psi \cong \mathrm{GL}_3(\boldsymbol{F})$ be the subgroup of $\mathrm{GL}_3(\boldsymbol{K})$ consisting of matrices with entries from $\boldsymbol{F}$.*

- **a.** *If $\boldsymbol{K}$ is not split, then $\mathrm{O}(V, h) \cong \mathrm{SL}_2(\boldsymbol{F}) \cong \eta(\mathrm{O}(V, h)) = \mathrm{O}(W, g) \cap \Psi < \mathrm{O}(W, g) \cong \mathrm{SL}_2(\boldsymbol{K})$. The field $\boldsymbol{L}_h$ is an inseparable quadratic extension field of $\boldsymbol{F}\mathbf{1}$.*
- **b.** *If $\boldsymbol{K}$ is split, then $\mathrm{O}(V, h) \cong \boldsymbol{F}^3$ is abelian, and its action on Wz is neither trivial nor faithful. We have $\mathrm{O}(W, g) \cong \boldsymbol{F} \times \big((\boldsymbol{F}^3 \times \boldsymbol{F}^2) \rtimes \boldsymbol{F}\big)$.*
  Again, the field $\boldsymbol{L}_h$ is an inseparable quadratic extension field of $\boldsymbol{F}\mathbf{1}$.

In both cases, we have $\boldsymbol{L} = \boldsymbol{L}_h$, and that field is a quadratic extension of $\boldsymbol{F}\mathbf{1}$.

Proof. If $\dim_{\boldsymbol{F}^{\square}} q(V) = 2$, we may assume $h(x, y) = x_1c_1y_1 + x_2c_2y_2 + x_3c_3y_3 + x_4c_4y_4$, where c_3 and c_4 lie in $c_1\boldsymbol{F}^{\square} + c_2\boldsymbol{F}^{\square}$, and c_1, c_2 are linearly independent over $\boldsymbol{F}^{\square}$.

If $c_3 \in \boldsymbol{F}^{\square}c_1$ we may assume $c_3 = c_1$. If $c_3 \notin \boldsymbol{F}^{\square}c_1$ then there exist $s, t \in \boldsymbol{F}$ with $c_2 = s^2c_1 + t^2c_3$. Then $T := \left(\begin{smallmatrix} tc_3 & s \\ sc_1 & t \end{smallmatrix}\right)$ is invertible, and $T^{\intercal}\left(\begin{smallmatrix} c_1 & 0 \\ 0 & c_3 \end{smallmatrix}\right)T = \left(\begin{smallmatrix} c_1^* & 0 \\ 0 & c_2 \end{smallmatrix}\right)$ holds with $c_1^* := (tc_3)^2c_1 + (sc_1)^2c_3$. Thus $f_1 := tc_3e_1 + sc_1e_3$, $f_2 := e_2$, $f_3 := se_1 + te_3$, $f_4 := e_4$ is an orthogonal basis, and the Gram matrix for h with respect to that basis has diagonal entries c_1^*, c_2, c_2, c_4. Repeating the argument, we obtain that either there exists a diagonal Gram matrix with three identical diagonal entries (if $c_4 \notin \boldsymbol{F}^{\square}c_1^*$), or there exists a diagonal Gram matrix with two pairs of identical diagonal entries.

Up to similitude, we may thus assume that the Gram matrix (with respect to the standard basis) is one of

$$H_1 := \begin{pmatrix} m & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad H_2 := \begin{pmatrix} m & 0 & 0 & 0 \\ 0 & m & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

respectively, with $m \in \boldsymbol{F} \smallsetminus \boldsymbol{F}^\square$.

a. If the form h is described by H_1 then its discriminant m is not a square, and $\boldsymbol{K}$ is not split; in fact, we have $\boldsymbol{K} = \boldsymbol{F}(\boldsymbol{j}) \cong \boldsymbol{F}[X]/(X^2 - m)$, with $\boldsymbol{j}^2 = m \notin \boldsymbol{F}^\square$. With respect to the $\boldsymbol{K}$-basis $e_1 \wedge e_2$, $e_1 \wedge e_3$, $e_1 \wedge e_4$, the Gram matrix for g is $\left(\begin{smallmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{smallmatrix}\right)$. From 1.5c (applied with $M = \{0\}$) we know that $\mathrm{O}(W, g) \cong \mathrm{SL}_2(\boldsymbol{K})$. As m is a square in $\boldsymbol{K}$, the set $q_g(W)$ coincides with $\boldsymbol{K}^\square$, and every multiplier of q_g is a square. So $\mathrm{GO}(W, q_g) = \boldsymbol{K}^\times \mathrm{O}(W, q_g)$, and $\mathrm{GO}(W, g) = \boldsymbol{K}^\times \mathrm{O}(W, g)$.

In order to understand the group $\mathrm{O}(V, h)$, we first study the quadratic form given by $q(x) = h(x, x) = mx_1^2 + x_2^2 + x_3^2 + x_4^2 = mx_1^2 + (x_2 + x_3 + x_4)^2$. As m is not a square in $\boldsymbol{F}$, the kernel of q is the hyperplane $\left\{ (0, x_2, x_3, x_4)^\intercal \;\middle|\; x_2 + x_3 + x_4 = 0 \right\}$. This hyperplane is invariant under $\mathrm{O}(V, h)$; it will be convenient to use the basis $b_1 := e_1$, $b_2 := e_2 + e_3 + e_4$, $b_3 := e_2 + e_3$, $b_4 := e_2 + e_4$. With respect to that basis, the Gram matrix of h is the block matrix $\left(\begin{smallmatrix} N & 0 \\ 0 & \boldsymbol{i} \end{smallmatrix}\right)$, where $N := \left(\begin{smallmatrix} m & 0 \\ 0 & 1 \end{smallmatrix}\right)$ is a Gram matrix for the norm form of $\boldsymbol{K}|\boldsymbol{F}$, and $\boldsymbol{i} := \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. In coordinates with respect to that basis, the isometry group of h consists of the block matrices of the form $\left(\begin{smallmatrix} E & 0 \\ 0 & A \end{smallmatrix}\right)$ with the 2×2 identity matrix E, and $A \in \mathrm{SL}_2(\boldsymbol{F})$. In particular, we find $\mathrm{O}(V, h) \cong \mathrm{SL}_2(\boldsymbol{F})$.

As in 1.5a, we generate the group $\mathrm{SL}_2(\boldsymbol{F})$ by the matrices L_x and U_x, with $x \in \boldsymbol{F}$. Transforming $\left(\begin{smallmatrix} E & 0 \\ 0 & L_x \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} E & 0 \\ 0 & U_x \end{smallmatrix}\right)$ back into the description with respect to standard coordinates, we obtain that $\mathrm{O}(V, h)$ is generated by the matrices $\tilde{A}_x := \tilde{T}^{-1} \left(\begin{smallmatrix} E & 0 \\ 0 & A_x \end{smallmatrix}\right) \tilde{T}$ and $\tilde{B}_x := \tilde{T}^{-1} \left(\begin{smallmatrix} E & 0 \\ 0 & B_x \end{smallmatrix}\right) \tilde{T}$, where $\tilde{T} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & T \end{smallmatrix}\right)$, $\tilde{A}_x = \left(\begin{smallmatrix} 1 & 0 \\ 0 & \hat{L}_x \end{smallmatrix}\right)$, and $\tilde{B}_x = \left(\begin{smallmatrix} 1 & 0 \\ 0 & \hat{U}_x \end{smallmatrix}\right)$, with $x \in \boldsymbol{F}$.

With respect to the $\boldsymbol{K}$-basis $e_1 \wedge e_2$, $e_1 \wedge e_3$, $e_1 \wedge e_4$ for W, we then find that the action of these elements on W is described by the matrices $\hat{L}_x$ and $\hat{U}_x$, respectively, with $x \in \boldsymbol{F}$. The same matrices, but with $x \in \boldsymbol{K}$ instead of $x \in \boldsymbol{F}$, generate $\mathrm{O}(W, g) \cong \mathrm{SL}_2(\boldsymbol{K})$; see 1.5c. Therefore, the image of $\mathrm{O}(V, h)$ under η equals $\mathrm{O}(W, g) \cap \mathrm{GL}_3(\boldsymbol{F})$.

For each similitude $\varphi \in \mathrm{GO}(V, h)$, the multiplier r_φ lies in the range $q(V) = \boldsymbol{F}^\square + \boldsymbol{F}^\square m$ because that range contains 1. We note that $q(V)$ forms the quadratic extension field $\boldsymbol{F}^\square(m) \cong \boldsymbol{F}(\sqrt{m}) \cong \boldsymbol{K}$ of $\boldsymbol{F}^\square$. Every element $a^2 + b^2 m \in \boldsymbol{F}^\square(m) \smallsetminus \{0\}$ is the multiplier of some similitude of the form h; in coordinates with respect to the basis b_1, b_2, b_3, b_4, the block matrix $\left(\begin{smallmatrix} A & 0 \\ 0 & A \end{smallmatrix}\right)$ with $A := \left(\begin{smallmatrix} a & b \\ bm & a \end{smallmatrix}\right) \in \mathrm{GL}_2(\boldsymbol{F})$ describes a similitude with multiplier $a^2 + b^2 m$. So $\boldsymbol{L} = \boldsymbol{L}_h = \boldsymbol{F}\mathbf{1} + \boldsymbol{F} \left(\begin{smallmatrix} 0 & 1 \\ m & 0 \end{smallmatrix}\right)$.

b. Now consider the case where h is described by H_2. Then the discriminant is a square, and $\boldsymbol{K}$ is split. We normalize such that $\boldsymbol{j}^2 = 1$, and $\boldsymbol{K} = \boldsymbol{F} + \boldsymbol{F}z \cong \boldsymbol{F}[X]/(X^2)$ holds for $z := 1 + \boldsymbol{j}$. The kernel of the quadratic form given by $q(x) = h(x, x)$ is spanned by $d_1 := e_1 + e_2$ and $d_2 := e_3 + e_4$. We use $d_3 := e_1$ and $d_4 := e_4$ to extend this to a basis for V. With respect to that basis, the Gram matrix for h is the block matrix $\left(\begin{smallmatrix} 0 & N \\ N & N \end{smallmatrix}\right)$. Again, $N = \left(\begin{smallmatrix} m & 0 \\ 0 & 1 \end{smallmatrix}\right)$ but this is no longer a Gram matrix for the norm of $\boldsymbol{K}|\boldsymbol{F}$. However, from $2 = \dim_{\boldsymbol{F}^\square} q(V) = \dim(\boldsymbol{F}^\square + \boldsymbol{F}^\square m)$ we infer that $1, m$ are linearly independent over $\boldsymbol{F}^\square$, and N is the Gram matrix of an anisotropic quadratic form on $\boldsymbol{F}^2$. Thus the isometry group of that form is trivial.

In the chosen coordinates, a block matrix $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ describing an isometry of h will leave the kernel of q invariant, so $C = 0$. Invariance of q then gives $\left(\begin{smallmatrix} 0 & 0 \\ 0 & N \end{smallmatrix}\right) = \left(\begin{smallmatrix} A^\intercal & 0 \\ B^\intercal & D^\intercal \end{smallmatrix}\right) \left(\begin{smallmatrix} 0 & 0 \\ 0 & N \end{smallmatrix}\right) \left(\begin{smallmatrix} A & B \\ 0 & D \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & 0 \\ 0 & D^\intercal N D \end{smallmatrix}\right)$, and D is an element of the trivial isometry group of the quadratic form described by N. So $D = \mathbf{1}$. Now invariance of h implies $\left(\begin{smallmatrix} 0 & N \\ N & N \end{smallmatrix}\right) = \left(\begin{smallmatrix} A^\intercal & 0 \\ B^\intercal & \mathbf{1} \end{smallmatrix}\right) \left(\begin{smallmatrix} 0 & N \\ N & N \end{smallmatrix}\right) \left(\begin{smallmatrix} A & B \\ 0 & \mathbf{1} \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & A^\intercal N \\ NA & NB + B^\intercal N + N \end{smallmatrix}\right)$, and $A = \mathbf{1}$ follows.

In the chosen coordinates, the isometry group of h therefore consists of the block matrices of the form $\left(\begin{smallmatrix} \mathbf{1} & B \\ 0 & \mathbf{1} \end{smallmatrix}\right)$ with $B \in \boldsymbol{F}^{2\times 2}$ such that NB is a symmetric matrix. Transforming this description into coordinates with respect to the standard basis e_1, e_2, e_3, e_4, we obtain $\mathrm{O}(V, h) = \left\{ \left(\begin{smallmatrix} \mathbf{1}+aJ & bJ \\ mbJ & \mathbf{1}+cJ \end{smallmatrix}\right) \,\middle|\, a, b, c \in F \right\}$, where $J = \left(\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right)$. This shows that $\mathrm{O}(V, h) \cong \boldsymbol{F}^3$ is an elementary abelian 2-group.

Now $w_1 := e_1 \wedge e_4$, $w_2 := e_1 \wedge e_3$, $w_3 := e_1 \wedge e_2$ is a $\boldsymbol{K}$-basis for W, with $\boldsymbol{j} w_1 = e_2 \wedge e_3$, $\boldsymbol{j} w_2 = e_2 \wedge e_4$, $\boldsymbol{j} w_3 = m e_3 \wedge e_4$. With respect to the K-basis w_1, w_2, w_3, the action of $\left(\begin{smallmatrix} \mathbf{1}+aJ & bJ \\ mbJ & \mathbf{1}+cJ \end{smallmatrix}\right) \in \mathrm{O}(V, h)$ is described by the matrix

$$\begin{pmatrix} 1+a+c+z(ac+mb^2) & a+c+z(a+ac+mb^2) & zmb \\ a+c+z(a+ac+mb^2) & 1+a+c+z(ac+mb^2) & zmb \\ zb & zb & 1 \end{pmatrix},$$

and the action on Wz, with respect to the $\boldsymbol{F}$-basis

$$\begin{aligned} w_1 z &:= (e_1 \wedge e_4) z = e_1 \wedge e_4 + e_2 \wedge e_3, \\ w_2 z &:= (e_1 \wedge e_3) z = e_1 \wedge e_3 + e_2 \wedge e_4, \\ w_3 z &:= (e_1 \wedge e_2) z = e_1 \wedge e_2 + e_3 \wedge e_4, \end{aligned}$$

is described by the matrix

$$\begin{pmatrix} 1+a+c & a+c & 0 \\ a+c & 1+a+c & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

This shows that the action of $\mathrm{O}(V, h)$ on Wz is not faithful, the kernel of the restriction η^o is

$$\ker \eta^o = \left\{ \begin{pmatrix} \mathbf{1} + aJ & bJ \\ bJ & \mathbf{1} + aJ \end{pmatrix} \,\middle|\, a \in \boldsymbol{F} \right\}.$$

In coordinates with respect to the basis $w_1, w_3, w_1 + w_3$, the form g is described by the matrix $\left(\begin{smallmatrix} m & 0 & m \\ 0 & 1 & 0 \\ m & 0 & 0 \end{smallmatrix}\right)$. The matrix describing an isometry λ of g with respect to that basis has all entries from F if, and only if, that matrix is in $\left\{\left(\begin{smallmatrix}1&0&0\\0&1&0\\a&0&1\end{smallmatrix}\right)\,\middle|\, a\in \boldsymbol{F}\right\}$. With respect to the basis w_1, w_2, w_3, such an isometry λ is thus described by a matrix of the form $\left(\begin{smallmatrix}\mathbf{1}+aJ&0\\0&1\end{smallmatrix}\right)$. Using 1.4 we now infer that the elements of $\mathrm{O}(W,g)$ are exactly those K-linear maps that are described (with respect to w_1, w_2, w_3) by matrices of the form

$$\psi(a,u,S,t) := \begin{pmatrix}(\mathbf{1}+aJ)(\mathbf{1}+zS) & zm(\mathbf{1}+aJ)u\\ zu^{\intercal} & 1+zt\end{pmatrix},$$

where $a, t\in \boldsymbol{F}$, $u\in \boldsymbol{F}^2$, and S is a symmetric matrix with entries from $\boldsymbol{F}$. So

$$\mathrm{O}(W,g) = \left\{\psi(a,u,S,t)\,\middle|\, a,t\in F,\ u\in F^2,\ S=S^{\intercal}\in F^{2\times 2}\right\};$$

the multiplication is given by

$$\psi(a,u,S,s)\,\psi(b,v,T,t) = \psi\left(a+b, u+v+bJu, S+T+\mathrm{tr}(S)b\boldsymbol{i}, s+t\right).$$

The subgroups $\Theta := \{\psi(a,0,0,0) \mid a\in \boldsymbol{F}\}$ and $\mathrm{Z} := \{\psi(0,0,0,s)\mid s\in\boldsymbol{F}\}$ are both in a natural way isomorphic to the additive group of $\boldsymbol{F}$. Note that Z is contained in the center

$$\left\{\psi(0,u,S,s)\,\middle|\, s\in\boldsymbol{F}, u\in JF^2, S=S^{\intercal}\in\boldsymbol{F}^{3\times 3}, \mathrm{tr}(S)=0\right\}$$

of $\mathrm{O}(W,g)$. The subgroups

$$\Xi_2 := \{\psi(0,u,0,0)\mid u\in F^2\}\cong \boldsymbol{F}^2 \quad\text{and}\quad \Xi_3 := \{\psi(0,0,S,0)\mid S=S^{\intercal}\in\boldsymbol{F}^{2\times 2}\}\cong\boldsymbol{F}^3$$

are both normal in $\mathrm{O}(W,g)$, and $\mathrm{O}(W,g)$ is the direct product of Z with the semidirect product of $\Xi_2\times\Xi_3$ with the subgroup Θ; conjugation by $\psi(a,0,0,0)$ maps $\psi(0,u,S,0)\in \Xi_2\,\Xi_3$ to $\psi\left(0,\, u+aJu,\, S+\mathrm{tr}(S)(a\boldsymbol{i}+a^2J), 0\right)$.

Note that η maps $\left(\begin{smallmatrix}\mathbf{1}+aJ & bJ\\ mbJ & \mathbf{1}+cJ\end{smallmatrix}\right)$ to $\psi\left(a+c, \left(\begin{smallmatrix}b\\b\end{smallmatrix}\right), \left(\begin{smallmatrix}a^2+b^2m & a+a^2+b^2m\\ a+a^2+b^2m & a^2+b^2m\end{smallmatrix}\right), 0\right)$.

As in case a, the set $q(V)$ coincides with the subfield $\boldsymbol{F}^{\square}(m)$ of $\boldsymbol{F}$, and $\boldsymbol{F}^{\square}(m)\smallsetminus\{0\}$ is the set of all multipliers of similitudes of h; in coordinates with respect to the basis d_1, d_2, d_3, d_4, the block matrix $\left(\begin{smallmatrix}A&0\\0&A\end{smallmatrix}\right)$ with $A := \left(\begin{smallmatrix}a&b\\bm&a\end{smallmatrix}\right)\in\mathrm{GL}_2(\boldsymbol{F})$ describes a similitude with multiplier a^2+b^2m. So $\boldsymbol{L}_h = \boldsymbol{L} = \boldsymbol{F}\mathbf{1}+\boldsymbol{F}\left(\begin{smallmatrix}0&1\\m&0\end{smallmatrix}\right)$ is an extension of degree 2 over $\boldsymbol{F}\mathbf{1}$. □

3.3 Proposition. *If q has defect 1, then $\mathrm{O}(V,h)\cong\boldsymbol{F}$ is abelian, and $\boldsymbol{K}$ is not split. We have $\mathrm{O}(V,h)\cong(\boldsymbol{F},+)$, and $\eta(\mathrm{O}(V,h)) < \mathrm{O}(W,g)\cong(\boldsymbol{K},+)$. There is a basis v_1, v_2, v_3, v_4 for V such that h has the Gram matrix*

$$H := \begin{pmatrix}0&1&0&0\\1&1&0&0\\0&0&c_3&0\\0&0&0&c_4\end{pmatrix}.$$

With respect to the basis $v_1 \wedge v_3$, $v_2 \wedge v_3$, $v_1 \wedge v_2$ for W, the elements of $\eta(\mathrm{O}(V,h))$ are those elements of $\mathrm{O}(W,g)$ that are described by matrices with entries from $\boldsymbol{F}$.

For each one of the forms h, q_h, and g, every multiplier is a square, the respective groups of similitudes are $\mathrm{GO}(V,h) = \boldsymbol{F}^\times \mathrm{O}(V,h)$ and $\mathrm{GO}(W,g) = \boldsymbol{K}^\times \mathrm{O}(W,g)$. However, the multipliers of the form q_g are the non-zero elements of the quadratic extension $\boldsymbol{K}^\square(c_3)$ over $\boldsymbol{K}^\square$.

Proof. If $\dim_{\boldsymbol{F}^\square} q(V) = 3$ then there is $u_1 \in V \setminus \{0\}$ with $h(u_1,u_1) = 0$. As h is not degenerate, there exists $u_2 \in V$ with $h(u_1,u_2) = 1$. We write $s := h(u_2,u_2)$. If $s = 0$ then $\dim\ker q > 1$, contradicting our assumption that the range of q has dimension 3. We multiply the form by s^{-1} and then replace u_1 by su_1; we may thus assume $s = 1$.

The restriction of h to $\boldsymbol{F}u_1 + \boldsymbol{F}u_2$ is not degenerate, so $\{u_1,u_2\}^\perp$ forms a vector space complement for that subspace in V. If the restriction of h to that complement were isotropic then $\dim\ker q$ would be greater than 1.

So the restriction of h to $\{u_1,u_2\}^\perp$ is anisotropic, and diagonalizable by 1.2. Choosing an orthonormal basis u_3, u_4 for $\{u_1,u_2\}^\perp$, we obtain that the Gram matrix for h with respect to the basis u_1,u_2,u_3,u_4 is H; the discriminant of h is represented by $\delta := c_3c_4$. Now $\delta \notin \boldsymbol{F}^\square$; otherwise, the vector $u_3c_4 + u_4\sqrt{c_3c_4}$ would be isotropic, and $\dim\ker q$ would be greater than 1. So $\boldsymbol{K} \cong \boldsymbol{F}(\sqrt{\delta})$ is not split. We note that $c_3 \notin \boldsymbol{K}^\square$; in fact $c_3 = \left(r + t\sqrt{\delta}\right)^2$ with $r,t \in \boldsymbol{F}$ would imply $0 = 1 + c_3(r/c_3)^2 + c_4t^2$, contradicting the fact that $\dim\ker q = 1$.

The group $\mathrm{O}(V,h)$ leaves invariant the quadratic form q. Therefore the subspace $\ker q$ is invariant under $\mathrm{O}(V,h)$, and so is the orthogonal space $\ker q^\perp$, which is spanned by $\{u_1,u_3,u_4\}$. Using these facts facilitates to see that the elements of $\mathrm{O}(V,h)$ are described (with respect to the basis u_1,u_2,u_3,u_4) by matrices of the form $\left(\begin{smallmatrix} U_x & 0 \\ 0 & \mathbf{1} \end{smallmatrix}\right)$ with $x \in \boldsymbol{F}$, $E = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, and U_x as in 1.5a. So $\mathrm{O}(V,h) \cong (\boldsymbol{F},+)$.

The vectors $v_1 := u_1 + u_2$, $v_2 := u_2$, $v_3 := u_3$, $v_4 := u_4$ form an orthogonal basis. In coordinates with respect to that basis, the group $\mathrm{O}(V,h)$ consists of the block matrices $\tilde{U}_x := \left(\begin{smallmatrix} \hat{U}_x & 0 \\ 0 & 1 \end{smallmatrix}\right)$, with $x \in \boldsymbol{F}$. With respect to the $\boldsymbol{K}$-basis $w_1 := v_1 \wedge v_3$, $w_2 := v_2 \wedge v_3$, $w_3 := v_1 \wedge v_2$, the Gram matrix for the bilinear form g is $\left(\begin{smallmatrix} c_3 & 0 & 0 \\ 0 & c_3 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right)$. The form g is isotropic (as prophesied by [10, 2.9]); in fact, the vector $w_1 + w_2$ is isotropic with respect to g.

In order to determine $\mathrm{GO}(W,g)$, we consider the quadratic form q_g obtained by evaluating g on the diagonal. That quadratic form has one-dimensional kernel because $c_3 \notin \boldsymbol{K}^\square$. So $\boldsymbol{K}(w_1 + w_2)$ is that kernel, and both $\boldsymbol{K}(w_1 + w_2)$ and $(w_1 + w_2)^\perp = \boldsymbol{K}(w_1 + w_2) + \boldsymbol{K}w_3$ are invariant under the group $\mathrm{GO}(W,g)$. It turns out that the elements of $\mathrm{GO}(W,g)$ are described (with respect to the basis w_1,w_2,w_3) by matrices of the form $a\hat{U}_x$, with $a \in \boldsymbol{K}^\times$ and $\hat{U}_x$ as in 1.5, with $x \in \boldsymbol{K}$. This yields $\mathrm{O}(W,g) \cong (\boldsymbol{K},+)$, and the group $\mathrm{GO}(W,g)$ is the direct product of that group by the multiplicative group $\boldsymbol{K}^\times$. In coordinates with respect to the basis $w_1 + w_2, w_2, w_3$, the members of $\mathrm{O}(W,q_g)$ are described by matrices of the form $\left(\begin{smallmatrix} 1 & b \\ 0 & \mathbf{1} \end{smallmatrix}\right)$ with $b \in K^2$, and $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & c_3 \\ 0 & 1 & 0 \end{pmatrix}$ describes a similitude of q_g with multiplier c_3, so $\boldsymbol{K}^\square(c_3)$ is the field of multipliers for q_g.

For each similitude of h, the multiplier lies in $q(V) = \boldsymbol{F}^\square + \boldsymbol{F}^\square c_3 + \boldsymbol{F}^\square c_4$. If $m \in q(V) \smallsetminus \boldsymbol{F}^\square$ were a multiplier of some similitude then $q(V)$ would be a vector space over the field $\boldsymbol{F}^\square(m)$. Since this is impossible, we obtain $\mathrm{GO}(V,h) = \boldsymbol{F}^\times \mathrm{O}(V,h)$, and $\mathrm{GO}(V,q) = \boldsymbol{F}^\times \mathrm{O}(V,q)$. □

3.4 Proposition. *If q has defect 0 then both $\mathrm{O}(V,q)$ and $\mathrm{O}(V,h)$ are trivial, and the following assertions hold.*

- **a.** *If $q(V)$ is a subfield of $\boldsymbol{F}$ then the subfield $\boldsymbol{L} := \mathrm{GO}(V,q) \cup \{0\}$ of $\mathrm{End}_{\boldsymbol{F}}(V)$ is a totally inseparable extension of degree 4 and exponent 1 over $\boldsymbol{F}\mathbf{1}$, and $r(\boldsymbol{L}) = q(V)$. In all other cases, we have $\boldsymbol{L} = \boldsymbol{F}\mathbf{1}$.*
- **b.** *If there exists $\delta \in \mathrm{GO}(V,h) \smallsetminus \boldsymbol{F}\mathbf{1}$ then the discriminant of h is represented by an element of the field $\boldsymbol{F}^\square(r_\delta)$.*
- **c.** *If the discriminant of h is trivial then the field $\boldsymbol{L}_h := \mathrm{GO}(V,h) \cup \{0\}$ coincides with $\boldsymbol{L}$, and $r(\boldsymbol{L}_h) = r(\boldsymbol{L}) = q(V)$.*
- **d.** *If the discriminant of h is not trivial but $q(V)$ is a subfield of $\boldsymbol{F}$ then the field $\boldsymbol{L}_h$ is an inseparable extension of degree 2 over $\boldsymbol{F}\mathbf{1}$, and $\boldsymbol{F}^\square < r(\boldsymbol{L}_h) < r(\boldsymbol{L}) = q(V)$.*
- **e.** *If $1 \in q(V)$ but $q(V)$ is not a subfield of $\boldsymbol{F}$ then $\boldsymbol{L}_h = \boldsymbol{L} = \boldsymbol{F}\mathbf{1}$, and the discriminant of h is not trivial.*

Proof. Let v_1, v_2, v_3, v_4 be an orthogonal basis for V with respect to h. Passing to a scalar multiple of the form, we may assume $q(v_1) = 1$, then both h and q are described by the diagonal matrix with diagonal entries 1, $a := q(v_2)$, $b := q(v_3)$, and $c := q(v_4)$. The discriminant of h is the coset in $\boldsymbol{F}^\times / \boldsymbol{F}^\boxtimes$ that is represented by $\det H = abc$, and $1, a, b, c$ are linearly independent over $\boldsymbol{F}^\square$ because q is anisotropic.

Recall from 1.3 that $\boldsymbol{L}$ and $\boldsymbol{L}_h$ are subfields of the endomorphism algebra $\mathrm{End}_{\boldsymbol{F}}(V)$, and that the multiplier map $r\colon \boldsymbol{L} \to \boldsymbol{F}$ is a field homomorphism. For $\lambda \in \mathrm{GO}(V,q)$, the set $\boldsymbol{E}_\lambda := \boldsymbol{F}\mathbf{1} + \boldsymbol{F}\lambda$ is a subfield of $\boldsymbol{L}$, and $q(V)$ is a vector space over the field $r(\boldsymbol{E}_\lambda) = \{r_\gamma \mid \gamma \in \boldsymbol{F}\mathbf{1} + \boldsymbol{F}\lambda\} = \boldsymbol{F}^\square(r_\lambda)$.

If $\boldsymbol{L} \neq \boldsymbol{F}\mathbf{1}$, we choose $\lambda \in \mathrm{GO}(V,q)$ such that the multiplier r_λ is not a square. Then $q(V)$ has dimension 2 over $r(\boldsymbol{E}_\lambda)$. The restriction of h to $\boldsymbol{E}_\lambda v_1 = \boldsymbol{F}v_1 + \boldsymbol{F}\lambda(v_1)$ is not degenerate because (the restriction of) q is anisotropic.

Pick any $w \in (\boldsymbol{E}_\lambda v_1)^\perp \smallsetminus \{0\}$. Then $d := q(w)$ is not in $r(\boldsymbol{E}_\lambda) = q(\boldsymbol{E}_\lambda(v_1))$ because q is injective. Now $q(V) = q(\boldsymbol{E}_\lambda) + q(\boldsymbol{E}_\lambda)d = \boldsymbol{F}^\square + \boldsymbol{F}^\square r_\lambda + \boldsymbol{F}^\square d + \boldsymbol{F}^\square r_\lambda d$ coincides with the field $\boldsymbol{F}^\square(r_\lambda, d)$ generated by r_λ and d over $\boldsymbol{F}^\square$. This is a totally inseparable extension of degree 4 and exponent 1.

Conversely, assume that $q(V)$ is a subfield of $\boldsymbol{F}$. Then there are $s_1, s_a, s_b, s_c \in \boldsymbol{F}$ with $ab = s_1^2 + s_a^2 a + s_b^2 b + s_c^2 c$, and $s_c \neq 0$ because $1, a, b, ab$ are linearly independent over $\boldsymbol{F}^\square$. Replacing the fourth basis vector v_4 by $s_1 v_1 + s_a v_2 + s_b v_3 + s_c v_4$ now yields a basis for V, and with respect to that basis, the form q is described by the diagonal matrix with diagonal entries $1, a, b, ab$. In those coordinates, the matrices

$$A := \begin{pmatrix} 0 & a & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & a \\ 0 & 0 & 1 & 0 \end{pmatrix} \text{ and } B := \begin{pmatrix} 0 & 0 & b & 0 \\ 0 & 0 & 0 & b \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

represent similitudes of q, and taking multipliers maps $\boldsymbol{L} = \boldsymbol{F}\mathbf{1} + \boldsymbol{F}A + \boldsymbol{F}B + \boldsymbol{F}AB$ onto $\boldsymbol{F}^\square(a, b) = q(V)$. This completes the proof of assertion a.

Now assume $\lambda \in \mathrm{GO}(V, h)$, and that λ is not a scalar multiple of the identity. Then $q(V)$ is a subfield of $\boldsymbol{F}$, and $r_\lambda \in q(V) \smallsetminus \boldsymbol{F}^\square$. We put $w_1 := v_1$, $w_2 := \lambda(v_1)$, and pick $w_3 \in \{w_1, w_2\}^\perp \smallsetminus \{0\}$. We observe that the restriction of the form h to $\boldsymbol{F}w_1 + \boldsymbol{F}w_2$ is not degenerate. As λ is a similitude of h, the subspace $\boldsymbol{F}w_3 + \boldsymbol{F}\lambda(w_3)$ is the orthogonal complement for $\boldsymbol{F}w_1 + \boldsymbol{F}w_2$.

With respect to the basis $w_1, w_2, w_3, w_4 := \lambda(v_3)$, the form h is described by the Gram matrix

$$H' := \begin{pmatrix} 1 & s & 0 & 0 \\ s & r_\lambda & 0 & 0 \\ 0 & 0 & q(w_3) & t \\ 0 & 0 & t & r_\lambda q(w_3) \end{pmatrix},$$

where $s := h(w_1, w_2)$, and $t = h(w_3, w_4)$. The discriminant of h is represented by $\det H' = (r_\lambda + s^2)(q(w)^2 r_\lambda + t^2) = q(w)^2 r_\lambda^2 + (st)^2 + (s+t)^2 r_\lambda \in \boldsymbol{F}^\square(r_\lambda)$, as claimed in assertion b.

We now use the orthogonal basis v_1, v_2, v_3, v_4, again. If the discriminant of h is trivial then abc is a square in $\boldsymbol{F}$, and with respect to the basis $v_1, v_2, v_3, \sqrt{abc}\, c^{-1} v_4$, the form h has the diagonal Gram matrix with diagonal entries $1, a, b, ab$. The endomorphisms A and B from above then give similitudes of h, with multipliers a and b, respectively. This yields $\boldsymbol{L}_h = \boldsymbol{L}$ and $r(\boldsymbol{L}_h) = r(\boldsymbol{L}) = q(V)$, so assertion c is proved.

Now assume that $q(V)$ is a subfield of $\boldsymbol{F}$, so $c \in \boldsymbol{F}^\square(a, b)$. The discriminant of h is represented by abc, and by assertion b that element lies in each subfield $\boldsymbol{F}^\square(r_\delta)$, where $\delta \in \mathrm{GO}(h) \smallsetminus \boldsymbol{F}\mathbf{1}$. If the discriminant is not trivial then abc is not a square, and we have $\{r_\delta \mid \delta \in \mathrm{GO}(h)\} \subseteq \boldsymbol{F}^\square(abc)$. Again, there exist $s_1, s_a, s_b, u \in \boldsymbol{F}$ with $c = s_1^2 + s_a^2 a + s_b^2 b + u^2 ab$, and $u \neq 0$ because $1, a, b, c$ are linearly independent over $\boldsymbol{F}^\square$. Replacing the last basis vector v_4 by $u^{-1} v_4$, we achieve $u = 1$, and

$$ab = s_1^2 + s_a^2 a + s_b^2 b + c\,. \tag{$*$}$$

We claim that there exists $\gamma \in \mathrm{GO}(V, h)$ with multiplier $r_\gamma = abc$. In order to find the images of the basis vectors v_1, v_2, v_3, v_4 under γ, we use the relation $(*)$ to compute

$$\begin{aligned} q(\gamma(v_1)) &= abc &&= (ab + s_1^2)^2 + (s_b b + s_1 s_a)^2 a + (s_a a + s_1 s_b)^2 b + s_1^2 c\,, \\ q(\gamma(v_2)) &= a^2 bc &&= a^2\big((s_b b + s_1 s_a)^2 + (b + s_a^2)^2 a + (s_1 + s_a s_b)^2 b + s_a^2 c\big)\,, \\ q(\gamma(v_3)) &= ab^2 c &&= b^2\big((s_a a + s_1 s_b)^2 + (s_1 + s_a s_b)^2 a + (a + s_b^2)^2 b + s_b^2 c\big)\,, \\ q(\gamma(v_4)) &= abc^2 &&= c^2\big(s_1^2 + s_a^2 a + s_b^2 b + c\big)\,. \end{aligned}$$

As q is injective, this yields

$$\begin{aligned}
\gamma(v_1) &= (ab+s_1^2)v_1+(s_bb+s_1s_a)v_2+(s_aa+s_1s_b)v_3+s_1v_4\,,\\
\gamma(v_2) &= a\big((s_bb+s_1s_a)v_1+(b+s_a^2)v_2+(s_1+s_as_b)v_3+s_av_4\big)\,,\\
\gamma(v_3) &= b\big((s_aa+s_1s_b)v_1+(s_1+s_as_b)v_2+(a+s_b^2)v_3+s_bv_4\big)\,,\\
\gamma(v_4) &= c\big(s_1v_1+s_av_2+s_bv_3+v_4\big)\,.
\end{aligned}$$

Straightforward computations (using the relation $(*)$, again) yield that $\gamma(v_1),\gamma(v_2)$, $\gamma(v_3),\gamma(v_4)$ is an orthogonal basis with respect to h, and thus γ is indeed a similitude with multiplier abc. We obtain $\boldsymbol{L}_h=\boldsymbol{F}\mathbf{1}+\boldsymbol{F}\gamma$ and the proof of assertion d is complete.

If $q(V)$ is not a subfield of $\boldsymbol{F}$ then assertion a yields $\boldsymbol{L}=\boldsymbol{F}\mathbf{1}$, and $\boldsymbol{L}_h=\boldsymbol{L}$ follows from $\boldsymbol{F}\mathbf{1}\le\boldsymbol{L}_h\le\boldsymbol{L}$. The discriminant is non-trivial by assertion c, and the last assertion e is established. □

We collect the results obtained for the different cases:

3.5 Theorem. *Assume* $\operatorname{char}\boldsymbol{F}=2$ *and* $\ell=2$. *Then one of the following holds.*

- **a.** *If q has defect 3, then $\boldsymbol{K}$ splits and $\mathrm{O}(V,h)\cong\big(\mathrm{SL}_2(\boldsymbol{F})\ltimes\boldsymbol{F}^2\big)\times\boldsymbol{F}$. The normal subgroup $\Xi\cong\boldsymbol{F}^2\times\boldsymbol{F}$ is the kernel of the action on Wz. See 3.1.*
  We obtain $\mathrm{SL}_2(\boldsymbol{F})\cong\eta^o(\mathrm{O}(V,h))<\mathrm{O}(W,g)\cong\mathrm{SL}_2(\boldsymbol{K})\cong\mathrm{SL}_2(\boldsymbol{F})\times\mathrm{SL}_2(\boldsymbol{F})$.
  Every multiplier is a square, so $\boldsymbol{L}_h=\boldsymbol{L}=\boldsymbol{F}\mathbf{1}$, and $\mathrm{GO}(V,h)=\boldsymbol{F}^\times\,\mathrm{O}(V,h)$.
- **b.** *If q has defect 2 and $\boldsymbol{K}$ is not split, then $\mathrm{O}(V,h)\cong\mathrm{SL}_2(\boldsymbol{F})$. See 3.2.a.*
  We have $\mathrm{SL}_2(\boldsymbol{F})\cong\eta(\mathrm{O}(V,h))<\mathrm{O}(W,g)\cong\mathrm{SL}_2(\boldsymbol{K})$.
  The field $\boldsymbol{L}_h=\boldsymbol{L}$ is an inseparable quadratic extension field of $\boldsymbol{F}\mathbf{1}$.
- **c.** *If q has defect 2 and $\boldsymbol{K}$ is split, then $\mathrm{O}(V,h)\cong\boldsymbol{F}^3$ is abelian, and its action on Wz is neither trivial nor faithful. See 3.2.b.*
  We have $\mathrm{O}(W,g)\cong\boldsymbol{F}\times\big((\boldsymbol{F}^3\times\boldsymbol{F}^2)\rtimes\boldsymbol{F}\big)$.
  Again, the field $\boldsymbol{L}_h=\boldsymbol{L}$ is an inseparable quadratic extension field of $\boldsymbol{F}\mathbf{1}$.
- **d.** *If q has defect 1, then $\mathrm{O}(V,h)\cong\boldsymbol{F}$ is abelian, and $\boldsymbol{K}$ is not split. See 3.3.*
  We have $\mathrm{O}(V,h)\cong(\boldsymbol{F},+)$, and $\eta(\mathrm{O}(V,h))<\mathrm{O}(W,g)\cong(\boldsymbol{K},+)$.
  The field $\boldsymbol{L}_h$ equals $\boldsymbol{F}\mathbf{1}$, and the group of similitudes is $\mathrm{GO}(V,h)=\boldsymbol{F}^\times\,\mathrm{O}(V,h)$.
- **e.** *If q has defect 0, then q is anisotropic and both $\mathrm{O}(V,q)$ and $\mathrm{O}(V,h)$ are trivial. See 3.4.*
  - *If $q(V)$ is a subfield of $\boldsymbol{F}$ then $\boldsymbol{L}$ is a totally inseparable extension of degree 4 and exponent 1 over $\boldsymbol{F}\mathbf{1}$, and $r(\boldsymbol{L})=q(V)$.*
  - *If $q(V)$ is not a subfield then the discriminant is not trivial, and $\boldsymbol{L}_h=\boldsymbol{L}=\boldsymbol{F}\mathbf{1}$.*
  - *If the discriminant is trivial then $q(V)=r(\boldsymbol{L})$, and $\boldsymbol{L}_h=\boldsymbol{L}$ has degree 4 over $\boldsymbol{F}\mathbf{1}$.*
  - *If the discriminant is not trivial but $q(V)$ is a field then $\boldsymbol{L}_h$ is an inseparable extension of degree 2 over $\boldsymbol{F}\mathbf{1}$, and $\boldsymbol{F}\mathbf{1}\lneqq\boldsymbol{L}_h\lneqq\boldsymbol{L}$.* □

Declaration of competing interest

There is no competing interest.

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Data availability

No data was used for the research described in the article.

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