Transformation Groups
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Fibrations and Coset Spaces for Locally Compact Groups

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Abstract
Let G be a topological group and let $K, L \subseteq G$ be closed subgroups, with $K \subseteq L$. We prove that if L is a locally compact pro-Lie group, then the map $q : G/K \to G/L$ is a Serre fibration, and a Hurewicz fibration if G is paracompact. As an application of this, we obtain two older results by Skljarenko, Madison and Mostert.

Keywords Locally compact groups · Homogeneous spaces · Fibrations

1 Introduction

Suppose that G is a topological group and that $K, L \subseteq G$ are closed subgroups with $K \subseteq L$. If we endow the coset spaces G/K and G/L with the quotient topologies, we have a natural continuous and open G-equivariant map

$$q : G/K \to G/L$$

and the question is how q behaves from the viewpoint of homotopy theory: is q a locally trivial bundle? Is it a fibration? If the natural map $G \to G/L$ admits local sections, then q is indeed a locally trivial bundle, see Lemma 3.1 below. In particular, q is a locally trivial bundle and a Hurewicz fibration if G is a Lie group. We cannot expect such a result for general locally compact groups, as the following example shows.

Example Put $H = \mathrm{SU}(2)$ and $G = H^{\mathbb{N}}$. Then G is a compact connected and locally connected group. The center of G is the compact totally disconnected group $L = \{\pm 1\}^{\mathbb{N}}$. The natural map $q : G \to G/L$ does not admit local sections, since otherwise G would be locally homeomorphic to the space $G/L \times L$, which is not locally connected. In particular, q is not a locally trivial bundle.

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In the case that G is a compact group, Madison and Mostert proved in 1969 [10] that $q : G/K \to G/L$ is always a Hurewicz fibration (Madison attributes the proof to Mostert). This applies in particular to the example above. A modern proof for their result is given in Theorem 2.8 in [9]. Already in 1963, Skljarenko [14] had proved that q is a Hurewicz fibration if G is a locally compact group. Apparently his results were not widely noticed at that time. In his 1970 PhD thesis [18], Wigner claimed that for every topological group G, the map $q : G \to G/L$ is a fibration, provided that $L \subseteq G$ is a locally compact subgroup. However, his proof appears to have gaps. We comment on this at the end of our article. Our main result is as follows.

Theorem *Let G be a topological group and let $K, L \subseteq G$ be closed subgroups, with $K \subseteq L$. If L is a locally compact pro-Lie group, then the map*

$$q : G/K \to G/L$$

is a Serre fibration. If G is paracompact, then q is a Hurewicz fibration.

As a consequence of this result, we recover the results by Skljarenko, Madison and Mostert. Our proof follows ideas similar to the proofs in [10] and [9], and it relies on Palais' Slice Theorem [13] and an important result by Antonyan [2]. It is rather different from Skljarenko's proof [14], which uses well-ordered chains of Lie quotients.

2 Generalities on Bundles and Fibrations

Throughout, all spaces and topological groups are assumed to be Hausdorff unless stated otherwise. Other topological assumptions will be stated explicitly.

We recall some terminology about bundles. A *bundle* over B is a continuous surjective map $p : E \to B$, where E and B are Hausdorff spaces. We call E the *total space*, B the base space, and for $b \in B$ we denote by $E_b = p^{-1}(b)$ the *fiber* over b. A *morphism* between bundles $p : E \to B$ and $p' : E' \to B$ is a continuous map $f : E \to E'$ with $p' \circ f = p$:

$$\begin{array}{ccc} E & \xrightarrow{f} & E' \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p'} \\ B & = & B. \end{array}$$

The morphisms of bundles over B form a category in an obvious way. A bundle is called *trivial* if it is isomorphic to a bundle of the form $\mathrm{pr}_1 : B \times F \to B$.

For $A \subseteq B$ we put $E_A = p^{-1}(A)$. The restriction $p|_{E_A} : E_A \to A$ is then a bundle over A. The bundle $E \to B$ is called *locally trivial* if every $b \in B$ has a neighborhood $V \subseteq B$ such that the restriction $E_V \to V$ is trivial.

A *section* over $A \subseteq B$ is a continuous map $s : A \to E_A$, with $p \circ s = \mathrm{id}_A$. A section over a neighborhood U of a point $b \in B$ is called a *local section*.

Let $p : E \to B$ be a bundle and let Z be a topological space. The bundle has the *homotopy lifting property (HLP)* with respect to Z if the following holds: given any two continuous maps $f : Z \times [0, 1] \to B$ and $\tilde{f}_0 : Z \times \{0\} \to E$ such that

$p(\tilde{f}_0(z,0)) = f(z,0)$, there exists a continuous extension $\tilde{f} : Z \times [0,1] \to E$ of $\tilde{f}_0$ with $p \circ \tilde{f} = f$. This situation is illustrated by the commutative diagram

$$\begin{array}{ccc} Z \times \{0\} & \xrightarrow{\tilde{f}_0} & E \\ \downarrow & \overset{\tilde{f}}{\nearrow} & \downarrow p \\ Z \times [0,1] & \xrightarrow{f} & B. \end{array}$$

A bundle which has the HLP for every space Z is called a *Hurewicz fibration*. It is called a *Serre fibration* if it has the HLP for every cube $[0,1]^m$. In this case, it has the HLP with respect to every CW complex, cp. [3] Theorem 6.4, Chapter VII. Every locally trivial bundle is a Serre fibration; such a bundle is a Hurewicz fibration if B is paracompact, see [4] Theorem 4.2, Chapter XX.

3 Quotients by Locally Compact Groups

Suppose that G is a topological group, and that $K, L \subseteq G$ are closed subgroups with $K \subseteq L$. If we endow G/K and G/L with the quotient topologies with respect to the canonical maps $G \to G/K$ and $G \to G/L$, then the canonical map $q : G/K \to G/L$ is a continuous open G-equivariant map whose fibers are homeomorphic to L/K.

We could not find a reference for this fact, so we supply a proof. We consider the commutative diagram

$$\begin{array}{ccccc} L & \hookrightarrow & G & = & G \\ p_1 \downarrow & & p_2 \downarrow & & p_3 \downarrow \\ L/K & \overset{i}{\hookrightarrow} & G/K & \xrightarrow{q} & G/L. \end{array}$$

The vertical maps p_1, p_2, p_3 are quotient maps, and L carries the subspace topology. This shows already that i and q are continuous. The maps p_1, p_2, p_3 are also open, cp. Lemma 6.2 in [15], and the quotients are Hausdorff, cp. Proposition 6.6 in *loc.cit.* If $U \subseteq G/K$ is open, then $q(U) = p_3(p_2^{-1}(U))$ is also open, hence q is an open map. We claim next that $i \circ p_1$ is an open map onto its image. Let $W \subseteq G$ be open and put $V = L \cap W$. Then $VK/K = (WK/K) \cap (L/K)$, i.e. $i(p_1(V)) = p_2(W) \cap L/K$. Hence $i \circ p_1$ is an open map onto its image, and thus i maps L/K homeomorphically onto its image in G/K. Finally, all fibers of q are homeomorphic under the G-action on G/K.

A *Lie group* is a group which is at the same time a differentiable manifold, such that the multiplication and inversion maps are smooth. We do not assume that Lie groups are second countable; indeed, the identity component G° of a Lie group G is automatically open and second countable, and G is metrizable and paracompact, cp. Proposition 9.1.15 and Theorem 10.3.25 in [7]. We recall also that closed subgroups of Lie groups are again Lie groups. We refer to the excellent book [7] for generalities about Lie groups.

Lemma 3.1 *Let G be a topological group with closed subgroups $K \subseteq L \subseteq G$. Suppose that the canonical map $G \to G/L$ admits a local section. Then $q : G/K \to G/L$ is a locally trivial bundle. This holds in particular if G is a Lie group.*

Proof The group G acts transitively on G/L and equivariantly on $q : G/K \to G/L$. The existence of a local section near some point $x \in G/L$ therefore implies the existence of a local section near every point $y \in G/L$. Let $s : U \to G$ be a local section, where $U \subseteq G/L$ is a nonempty open subset. Then the map $h : U \times L/K \to W = q^{-1}(U)$ that maps $(u, \ell K)$ to $s(u)\ell K$ is a homeomorphism, with inverse $w \longmapsto (q(w), s(q(w))^{-1}w)$, which makes the diagram

$$\begin{array}{ccc} U \times L/K & \xrightarrow{h} & W \\ {\scriptstyle \mathrm{pr}_1}\downarrow & & \downarrow{\scriptstyle q|_W} \\ U & = & U \end{array}$$

commute. For the last claim, we note that $G \to G/L$ admits local sections if G is a Lie group, cp. Corollary 10.1.11 in [7] or Theorem 3.58 in [17]. □

We recall that a continuous surjective map is called *proper* (or *perfect*) if it is closed and has compact fibers.

Lemma 3.2 *Let G be a topological group with locally compact subgroups $K \subseteq H \subseteq G$. If H/K is compact, then the canonical map $q : G/K \to G/H$ is proper.*

Proof Every $h \in H$ has a compact neighborhood D_h in H. Since H/K is compact, there are finitely many elements $h_1, \ldots, h_k \in H$ such that $p(D_{h_1}) \cup \cdots \cup p(D_{h_k}) = H/K$, where $p : H \to H/K$ is the canonical map. We put $C = D_{h_1} \cup \cdots \cup D_{h_k}$ and we note that $H = KC$. Suppose that $A \subseteq G/K$ is closed, with closed preimage B in G. The preimage of $q(A) \subseteq G/H$ in G is then the closed set $BH = BC$, cp. Lemma 3.19 in [15], hence $q(A)$ is closed. The fibers of q are homeomorphic to H/K and therefore compact. □

Lemma 3.3 *Let G be a topological group, let $(I, \leq)$ be a nonempty linearly ordered set and let $(H_i)_{i\in I}$ be a family of locally compact subgroups of G, such that $H_i \subseteq H_j$ holds for all $i \leq j$. Put $H = \bigcap_{i\in I} H_i$. The natural maps $G/H_i \to G/H_j$ make up a projective system in the category of topological spaces and continuous maps. If the quotients H_i/H are compact, then the natural map*

$$G/H \to \varprojlim G/H_i.$$

is a homeomorphism.

Proof One model for the projective limit $\varprojlim G/H_i$ consists of all elements $(g_i H_i)_{i\in I}$ of $\prod_{i\in I} G/H_i$ with the property that $g_i H_j = g_j H_j$ holds for i, j with $i \leq j$, cp. Appendix Two in [4]. Given such an element $(g_i H_i)_{i\in I}$, the collection of compact sets $\{g_i H_i/H \subseteq G/H \mid i \in I\}$ has the finite intersection property and contains therefore a common element $gH \in G/H$. Then $g_i H_i = gH_i$ holds for all i. We consider the natural continuous map $f : G/H \to \varprojlim G/H_i$ that maps gH to $(gH_i)_{i\in I}$. This map f has a two-sided inverse t that maps $(gH_i)_{i\in I}$ to $gH = \bigcap_{i\in I} gH_i$. In particular, f is a continuous bijection. We claim that f is a closed map, and hence a homeomorphism.

By Lemma 3.2, the maps $p_i : G/H \to G/H_i$ are proper. Therefore the product map $p : \prod_{i\in I} G/H \to \prod_{i\in I} G/H_i$ is also a proper map by Theorem 3.7.9 in [5], and hence closed. The diagonal map $d : G/H \to \prod_{i\in I} G/H$ is also closed, cp. Corollary 2.3.21 in *loc.cit*. Hence $f = p \circ d$ is closed. □

Let G be a topological group, let X be a space and suppose that the group G acts (from the right) on the set X. We say that this action is a *right transformation group* if the action map $X \times G \to X$ is continuous. For each element $x \in X$, we denote its stabilizer by $G_x = \{g \in G | xg = x\}$. Left transformation groups are defined analogously. We recall some terminology and results due to Palais [13].

Definition 3.4 Let $X \times G \to X$ be a (right) transformation group. For subsets $A, B \subseteq X$ we put $G_{A,B} = \{g \in G \mid (Ag) \cap B \neq \varnothing\}$. If A and B are open, then $G_{A,B}$ is an open subset of G. We say that the action is a *Cartan action* if X is completely regular and if every point $x \in X$ has an open neighborhood U such that $G_{U,U}$ has compact closure in G. This condition implies that G is locally compact.

Lemma 3.5 *Let G be a topological group with a locally compact subgroup $L \subseteq G$, and let $N \trianglelefteq L$ be a closed normal subgroup of L. Then the natural right action of L/N on G/N is a Cartan action.*

Proof First of all, G/N is completely regular, cp. II.8.14 in [6]. Let $W \subseteq G$ be an open symmetric identity neighborhood in G such that $WW \cap L$ has compact closure in L. For $g \in G$ we put $U = gWN/N \subseteq G/N$. Then U is an open neighborhood of gN. Let $\ell \in L$. If $U\ell N \cap U \neq \varnothing$, then $gw_1\ell n = gw_2$ holds for certain $w_1, w_2 \in W$ and $n \in N$, whence $\ell n \in WW$ and $\ell N \in WWN/N$. The right-hand side has compact closure in L/N. □

Definition 3.6 Let $X \times H \to X$ be a transformation group and let $K \subseteq H$ be a closed subgroup. A subset $S \subseteq X$ is called a *K-kernel* if there exists an H-equivariant map $f : SH \to K\backslash H$ such that $f^{-1}(K) = S$. If in addition SH is open in X, then S is called a *K-slice*. For $x \in X$, a *slice at* x is an H_x-slice that contains x, where H_x denotes the stabilizer of x.

Palais proved that for Cartan actions $X \times H \to X$ of Lie groups there exist H_x-slices for every $x \in X$, cp. Theorem 2.3.3 in [13]. Abels and Lütkepohl later considered the existence of slices in a somewhat different setting [1]. We recall that a group action $X \times H \to X$ is called *free* if $H_x = \{e\}$ holds for all $x \in X$. As an immediate consequence of Palais' theorem applied to free actions, we get the following result.

Theorem 3.7 (Palais) *Let $X \times H \to X$ be a Cartan action of a Lie group H. If the action is free, then the orbit map $q : X \to X/H$ admits local sections. Moreover, q is a locally trivial bundle.*

Proof The map q is an H-principal bundle in the terminology of [13]. By *op.cit.* Theorem 2.3.3, there exists a slice S at every point $x \in X$. This means in our situation that there is a subset $S \subseteq X$ such that $SH \subseteq X$ is open, and a continuous equivariant map $f : SH \to H$ with $S = f^{-1}(e)$. This implies that $f(xh) = h$ for all $x \in S$,

$h \in H$. Hence the map $S \times H \to SH$ that maps (x, h) to xh is a homeomorphism, with inverse $xh \mapsto (xhf(xh)^{-1}, f(xh))$.

The set $q(SH) = SH/H$ is open in X/H and the restriction $S \times H \cong SH \to SH/H$ is a trivial bundle. Finally, the map $s : SH/H \to S$ that maps xH to x is continuous and hence a local section for q. □

Corollary 3.8 *Let G be a topological group with closed subgroups $P \subseteq Q \subseteq G$. Assume that N is a compact normal subgroup of Q, and that Q/N is a Lie group. Then the bundle $p : G/PN \to G/Q$ is locally trivial. In particular, p is a Serre fibration, and a Hurewicz fibration if G is paracompact.*

Proof The Lie group action $G/N \times Q/N \to G/N$ is free, hence the bundle $G/N \to (G/N)/(Q/N) \cong G/Q$ has local sections by Theorem 3.7. Let $s : U \to G/N$ be a section over an open set $U \subseteq G/Q$. We note that the map $G/N \times Q/PN \to G/PN$ that maps (gN, qPN) to $gqPN$ is continuous, because arrow (1) in the diagram

$$\begin{array}{ccc} G \times Q & \xrightarrow{\text{multiply}} & G \\ {\scriptstyle (1)}\downarrow & & \downarrow \\ G/N \times Q/PN & \longrightarrow & G/PN. \end{array}$$

is a product of open maps and hence open. Then the map $h : U \times Q/PN \to G/PN$ that maps (gQ, qPN) to $s(gQ)qPN$ is a trivialization of the bundle $G/PN \to G/Q$ over U.

Every locally trivial bundle is a Serre fibration by Theorem 4.2 in Chapter XX of [4]. If G is paracompact, then G/Q is also paracompact by Corollary 1.5 in [2] and hence $G/PN \to G/Q$ is a Hurewicz fibration by Theorem 4.2 in Chapter XX of [4]. □

Definition 3.9 We call a topological group L a *locally compact pro-Lie group* if every identity neighborhood $U \subseteq L$ contains a compact normal subgroup N such that L/N is a Lie group. In this case, every closed subgroup $H \subseteq L$ is also a locally compact pro-Lie group. Indeed, $H/H \cap N \cong HN/N \subseteq L/N$ is then a closed subgroup of the Lie group L/N, and therefore a Lie group. We refer to the excellent book [8] on pro-Lie groups, in particular to Definition 3.25 and Remark 1.31.

The solution of Hilbert's 5th Problem due to Iwasawa, Gleason, Yamabe, Montgomery and Zippin says that every locally compact group L which is almost connected (i.e. L/L° is compact) is a locally compact pro-Lie group. Moreover, every locally compact group has an open almost connected subgroup which is a locally compact pro-Lie group. References are p. 175 in [11] or Theorem 6.0.11 in [16], and Theorem 4.4 in [9].

Now we get to our main result.

Theorem 3.10 *Let G be a topological group and let $K, L \subseteq G$ be closed subgroups, with $K \subseteq L$. Assume that L is a locally compact pro-Lie group. Then the bundle*

$$q : G/K \to G/L$$

is a Serre fibration. If G is paracompact, then q is a Hurewicz fibration.

Proof Let Z be a topological space and let $f : Z \times [0, 1] \to G/L$ and $\tilde{f}_0 : Z \times \{0\} \to G/K$ be continuous maps such that $q \circ \tilde{f}_0(z, 0) = f(z, 0)$. We need to study the lifting problem

$$\begin{array}{ccc} Z \times \{0\} & \xrightarrow{\tilde{f}_0} & G/K \\ \big\downarrow{\scriptstyle i} & \overset{\tilde{f}}{\nearrow} & \big\downarrow{\scriptstyle p} \\ Z \times [0, 1] & \xrightarrow{f} & G/L, \end{array}$$

where either $Z = [0, 1]^m$ is a cube, or G is paracompact and Z is any Hausdorff space. For this, we consider the collection $\mathcal{H}$ consisting of all pairs (H, h), where $H \subseteq G$ is a closed subgroup with $K \subseteq H \subseteq L$ such that H/K is compact, and $h : Z \times [0, 1] \to G/H$ is a continuous map that solves an intermediate lifting problem, i.e. h makes the following diagram commute

$$\begin{array}{ccc} Z \times \{0\} & \xrightarrow{\tilde{f}_0} & G/K \\ \Big\downarrow{\scriptstyle i} & & \downarrow \\ & & G/H \\ & \overset{h}{\nearrow} & \downarrow \\ Z \times [0, 1] & \xrightarrow{f} & G/L, \end{array}$$

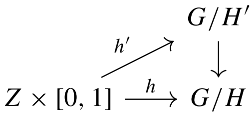
where the vertical arrows on the right are the natural maps. We turn $\mathcal{H}$ into a poset by declaring $(H, h) \le (H', h')$ if $H \supseteq H'$ and if the diagram

$$\begin{array}{ccc} & & G/H' \\ & \overset{h'}{\nearrow} & \downarrow \\ Z \times [0, 1] & \xrightarrow{h} & G/H \end{array}$$

commutes. Our aim is now to show that there is an element $(K, \tilde{f}) \in \mathcal{H}$.

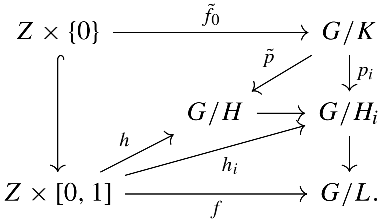
Claim 1 *The set $\mathcal{H}$ is nonempty.* Let $N \trianglelefteq L$ be a compact subgroup such that L/N is a Lie group, and put $H = KN$. We note that $H/K \cong N/N \cap K$ is compact, and we apply Corollary 3.8 with $(K, L, N) = (P, Q, N)$. Thus $G/H \to G/L$ is a Serre fibration, and a Hurewicz fibration if G is paracompact. Hence f has a lift h such that $(H, h) \in \mathcal{H}$, provided that $Z = [0, 1]^m$, or for a general Z if G is paracompact.

Claim 2 *If $(H, h) \in \mathcal{H}$ with $H \neq K$, then there exists $(H', h') \in \mathcal{H}$ with $(H, h) < (H', h')$.* Since $H \subseteq L$ is closed, H is a locally compact pro-Lie group, cp. Definition 3.9. Since H/K is Hausdorff, there is an open identity neighborhood $W \subseteq H$ such that $WK \neq H$. Let $N \subseteq W$ be a compact normal subgroup of H such that H/N is a Lie group. We put $H' = KN$. Applying Corollary 3.8 to $(K, H, N) = (P, Q, N)$, we see that $G/H' \to G/H$ is a Serre fibration, and a Hurewicz fibration if G is paracompact. Hence h has a lift $h' : Z \times [0, 1] \to G/H'$ such that $(H', h') \in \mathcal{H}$, provided that $Z = [0, 1]^m$, or for a general Z if G is paracompact. The diagram



commutes, whence $(H, h) < (H', h')$.

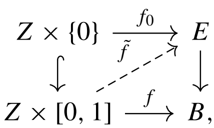
Claim 3 *The poset* $(\mathcal{H}, \leq)$ *is inductive, i.e. every linearly ordered subset has an upper bound.* Let $(I, \leq)$ be a linearly ordered set, with $(H_i, h_i) \in \mathcal{H}$, such that $(H_i, h_i) \leq (H_j, h_j)$ holds for $i \leq j$. Put $H = \bigcap_{i \in I} H_i$. From the fact that H_i/K is compact and from the maps $H/K \to H_i/K \to H_i/H$ we see that H_i/H and H/K are also compact. By Lemma 3.3, G/H is the projective limit of the projective system formed by the maps $G/H_j \to G/H_i$. From the universal property of the projective limit we obtain for each $i \in I$ a commutative diagram



Here $p_i : G/K \to G/H_i$ is the natural map, $p = \varprojlim p_j$ and $h = \varprojlim h_j$. The natural map $\tilde{p} : G/K \to G/H$ also makes the upper-right triangle commute. Therefore $p = \tilde{p}$, whence $(H, h) \in \mathcal{H}$.

Now we finish the proof. Since $\mathcal{H}$ is a nonempty inductive poset by Claim 1 and Claim 3, it has maximal elements by Zorn's Lemma. Let (H, h) be a maximal element in $\mathcal{H}$. By Claim 2, $H = K$ and hence $\tilde{f} = h$ solves the lifting problem.

As a consequence, we recover a result due to Skljarenko, cp. Theorem 15 in [14]. We first note the following fact, which is used implicitly in *op.cit.* Suppose that $p : E \to B$ is a bundle, and that the base space is a coproduct $B = \coprod_{s \in S} B_s$ (a disjoint union of closed and open subspaces B_s). Assume that for each $s \in S$, the space E_{B_s} is also a coproduct, $E_{B_s} = \coprod_{t \in T} E_{s,t}$. If each map $E_{s,t} \to B_s$ is a Hurewicz fibration, then $E \to B$ is a Hurewicz fibration. Indeed, if we are given a lifting problem



we may decompose Z into the disjoint closed and open subsets $Z_{s,t} \times \{0\} = f_0^{-1}(E_{s,t})$.

Then $f(Z_{s,t} \times [0,1]) \subseteq B_s$ (because $[0,1]$ is connected and B_s is closed and open), and it suffices to solve the lifting problems

$$\begin{array}{ccc} Z_{s,t} \times \{0\} & \xrightarrow{f_0} & E_{s,t} \\ \big\downarrow & \overset{\tilde f_{s,t}}{\nearrow} & \big\downarrow \\ Z_{s,t} \times [0,1] & \xrightarrow{f} & B_s \end{array}$$

for all s, t to obtain a global solution $\tilde{f}$. The following is Theorem 15 in [14]. In the case that G is compact, the result is also due to Madison and Mostert [10], cp. Theorem 2.8 in [9]. □

Theorem 3.11 (Skljarenko) *Let G be a locally compact group and let $K, L \subseteq G$ be closed subgroups such that $K \subseteq L$. Then the natural map $G/K \to G/L$ is a Hurewicz fibration.*

Proof We essentially follow Skljarenko's proof [14]. Let $H \subseteq G$ be an open subgroup which is a locally compact pro-Lie-group, cp. Definition 3.9. We consider the double cosets $HgL \in H\backslash G/L$, which partition G into disjoint open subsets. Since $G \to G/L$ is an open map, the image HgL/L of HgL in G/L is open. For each $\ell \in L$ and $g \in G$ we have a commutative diagram

$$\begin{array}{ccc} H/H \cap g\ell K(g\ell)^{-1} & \xrightarrow{(3)} & H/H \cap gLg^{-1} \\ \big\downarrow{\scriptstyle (1)} & & \big\downarrow{\scriptstyle (2)} \\ Hg\ell K/K & \xrightarrow{(4)} & HgL/L, \end{array}$$

where the two vertical arrows are H-equivariant homeomorphisms. The horizontal arrow (3) is a Hurewicz fibration by Theorem 3.10, because H is a locally compact pro-Lie group. Hence (4) is also a Hurewicz fibration. Since $G/K = \bigcup\{Hg\ell K/K \mid g \in G, \ell \in L\}$ and $G/L = \bigcup\{HgL/L \mid g \in G\}$, the claim follows by the remarks preceeding this theorem. □

Remark 3.12 As we mentioned in the introduction, Wigner claims in Prop. 2 in [18] that $G \to G/L$ is a fibration if L is a locally compact subgroup of the topological group G. However, there are several issues with the proof.

The author argues first that the result is true if L is compact (and tacitly uses a deep result due to Gleason on the existence of local sections for compact Lie transformation groups). So far the proof follows ideas similar to Madison and Mostert [10] and appears to be correct, albeit very brief.

For general locally compact groups, Wigner argues that L has an open subgroup L' which has a compact normal subgroup $L'' \trianglelefteq L'$ such that L'/L'' is a Lie group. Then he uses a result due to Serre in order to show that $G/L'' \to G/L'$ is a locally trivial bundle. However, Serre's proof is only valid for compact Lie group actions. This part could be fixed by invoking Palais' results [13] instead. Once this is done, it follows that $G \to G/L'$ is a fibration, since it is the composite of two fibrations $G \to G/L'' \to G/L'$.

The last claim is that the map $G/L' \to G/L$ is a covering map since the fiber L/L' is discrete. However, no argument is given for this claim, so we put it as an open question.

Problem 3.13 Let G be a topological group and let $K, L \subseteq G$ be locally compact subgroups, with $K \subseteq L$ open in L. Is $G/K \to G/L$ a fibration?

If the answer to the problem is affirmative, we may drop the assumption that L is a pro-Lie group in Theorem 3.10.

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