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# A note on commutators in compact semisimple Lie algebras

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*Dedicated to Jacques Tits*

Given any two elements $A$, $B$ in a compact semisimple Lie algebra, we show that there exist elements $X$, $Y$, $Z$ such that

$$A = [X, Y] \quad \text{and} \quad B = [X, Z].$$

The proof uses Cartan subalgebras and their root systems. We also review some related problems about Cartan subalgebras in compact semisimple Lie algebras.

Gotô's commutator theorem [1949; Hofmann and Morris 2020, Corollary 6.56] states that in a compact connected semisimple Lie group $G$, every element is a commutator. There is an infinitesimal version of Gotô's theorem which says that every element in a compact semisimple Lie algebra $\mathfrak{g}$ is a commutator, see [Hofmann and Morris 2007, Theorem A3.2]. The proof given in loc. cit., which uses Kostant's convexity theorem, is attributed to K.-H. Neeb. Other proofs were given later by D'Andrea and Maffei [2016] and Malkoun and Nahlus [2016; 2017]. We prove the following somewhat stronger result by elementary means.

**Theorem 1.** *Let $\mathfrak{g}$ be a semisimple compact Lie algebra and let $A, B \in \mathfrak{g}$. Then there is a regular element $X \in \mathfrak{g}$ with*

$$A, B \in [X, \mathfrak{g}] = \mathrm{ad}(X)(\mathfrak{g}).$$

Our Key Lemma 6, which is the main step of the proof, uses a variant of Jacobi's method, see [Kleinsteuber et al. 2004; Malkoun and Nahlus 2016, Appendix B][1] and [Wildberger 1993]. In the course of the proof we show in Corollary 7 that every linear subspace $W \subseteq \mathfrak{g}$ of codimension at most 2 contains a Cartan subalgebra.

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[1] Note that [Malkoun and Nahlus 2016] and [Malkoun and Nahlus 2017] differ considerably.

We refer to the books [Adams 1969; Bröcker and tom Dieck 1985; Helgason 1978; Hilgert and Neeb 2012; Hofmann and Morris 2020; Tits 1983] for general facts about semisimple compact Lie algebras.

**Definition 2.** A finite dimensional real semisimple Lie algebra $\mathfrak{g}$ is called *compact* if its Killing form $\langle -, - \rangle$ is negative definite. In this case its *adjoint group*

$$G = \langle \exp(\mathrm{ad}(X)) \mid X \in \mathfrak{g} \rangle$$

is compact and

$$|X| = \sqrt{-\langle X, X \rangle}$$

is a $G$-invariant euclidean norm on $\mathfrak{g}$. In what follows, orthogonality in $\mathfrak{g}$ will always refer to the Killing form. The *centralizer* of $A \in \mathfrak{g}$ is the Lie subalgebra

$$\mathrm{Cen}_{\mathfrak{g}}(A) = \{X \in \mathfrak{g} \mid [X, A] = 0\}.$$

**Lemma 3.** *Let* $\mathfrak{g}$ *be a compact semisimple Lie algebra and let* $A \in \mathfrak{g}$. *Then* $\mathfrak{g}$ *decomposes* (*as a* $\mathrm{Cen}_{\mathfrak{g}}(A)$*-module*) *orthogonally as*

$$\mathfrak{g} = \mathrm{Cen}_{\mathfrak{g}}(A) \oplus [A, \mathfrak{g}].$$

*Proof.* Let $X, Y \in \mathfrak{g}$. If $X$ centralizes $A$, then

$$\langle X, [A, Y] \rangle = \langle [X, A], Y \rangle = 0,$$

whence $X \in [A, \mathfrak{g}]^{\perp}$. Conversely, if $X \in [A, \mathfrak{g}]^{\perp}$, then

$$0 = \langle X, [A, Y] \rangle = \langle [X, A], Y \rangle$$

holds for all $Y$ and thus $[X, A] = 0$. This shows that $\mathrm{Cen}_{\mathfrak{g}}(A) = [A, \mathfrak{g}]^{\perp}$. Since the Killing form is negative definite, $\mathfrak{g} = \mathrm{Cen}_{\mathfrak{g}}(A) \oplus [A, \mathfrak{g}]$. The Jacobi identity shows that $[X, [A, \mathfrak{g}]] \subseteq [A, \mathfrak{g}]$ for $X \in \mathrm{Cen}_{\mathfrak{g}}(A)$, hence this is a decomposition of $\mathfrak{g}$ into $\mathrm{Cen}_{\mathfrak{g}}(A)$-modules. □

We recall some facts about the structure of compact semisimple Lie algebras, which can be found in [Adams 1969; Bröcker and tom Dieck 1985; Helgason 1978; Hilgert and Neeb 2012; Hofmann and Morris 2020].

**Facts 4.** Let $\mathfrak{g}$ be a compact semisimple Lie algebra. We call a maximal abelian subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ a *Cartan subalgebra*. All Cartan subalgebras in $\mathfrak{g}$ are conjugate under the action of $G$, see [Helgason 1978, Theorem V.6.4] or [Hofmann and Morris 2020, Theorem 6.27]. The dimension of $\mathfrak{h}$ is called the *rank* of $\mathfrak{g}$. Let $\mathfrak{h} \subseteq \mathfrak{g}$ be a Cartan subalgebra. Then

$$T = \{\exp(\mathrm{ad}(H)) \mid H \in \mathfrak{h}\}$$

is a *maximal torus* in $G$. As a $T$-module, the Lie algebra $\mathfrak{g}$ decomposes as an orthogonal direct sum of irreducible $T$-modules,

$$\mathfrak{g} = \mathfrak{h} \oplus \sum_{\alpha \in \Phi^+} L_\alpha,$$

see [Hofmann and Morris 2020, Chapter 6]. The *positive real roots* $\alpha \in \Phi^+$ are certain nonzero linear forms $\alpha : \mathfrak{h} \to \mathbb{R}$. Each $T$-module $L_\alpha$ is 2-dimensional and carries a complex structure $\boldsymbol{i}$ such that $L_\alpha \cong \mathbb{C}$ and

$$\exp(\mathrm{ad}(H))(X) = \exp(2\pi \boldsymbol{i} \alpha(H))X$$

holds for all $H \in \mathfrak{h}$, $\alpha \in \Phi^+$ and $X \in L_\alpha$. Hence $H \in \mathfrak{h}$ acts on $L_\alpha$ as

$$\mathrm{ad}(H)(X) = [H, X] = 2\pi \boldsymbol{i} \alpha(H) X.$$

The positive real roots separate the points in $\mathfrak{h}$, i.e., $\bigcap\{\ker(\alpha) \mid \alpha \in \Phi^+\} = \{0\}$. The centralizer of an element $H \in \mathfrak{h}$ is therefore

$$\mathrm{Cen}_{\mathfrak{g}}(H) = \mathfrak{h} \oplus \sum_{\alpha(H)=0} L_\alpha.$$

Hence $\mathrm{Cen}_{\mathfrak{g}}(H) = \mathfrak{h}$ holds if and only if $\alpha(H) \neq 0$ for all positive real roots $\alpha$. Such elements $H$ are called *regular*.

**Lemma 5.** *Let* $\mathfrak{g}$ *be a compact semisimple Lie algebra*, *with a Cartan subalgebra* $\mathfrak{h}$ *and the corresponding decomposition*

$$\mathfrak{g} = \mathfrak{h} \oplus \sum_{\alpha \in \Phi^+} L_\alpha$$

*as above*, *and let* $\gamma \in \Phi^+$ *be a positive real root. Let* $H_\gamma \in \mathfrak{h}$ *be a nonzero vector orthogonal to* $\ker(\gamma)$. *Then*

$$\mathfrak{m}_\gamma = \mathbb{R} H_\gamma \oplus L_\gamma \cong \mathfrak{so}(3)$$

*is the Lie algebra generated by* $L_\gamma$.

*Proof.* We let $\mathfrak{m}_\gamma$ denote the Lie algebra generated by $L_\gamma$. The centralizer of $\ker(\gamma)$ is $\mathfrak{h} \oplus L_\gamma$, whence $\mathfrak{m}_\gamma \subseteq \mathfrak{h} \oplus L_\gamma$. Let $X \in L_\gamma$ be an element of norm $|X| = 1$. Then $X, \boldsymbol{i}X$ is an orthonormal basis for $L_\gamma$, and we put $Y = [X, \boldsymbol{i}X]$. Then

$$\langle X, Y \rangle = \langle [X, X], \boldsymbol{i}X \rangle = 0 = \langle X, [\boldsymbol{i}X, \boldsymbol{i}X] \rangle = \langle Y, \boldsymbol{i}X \rangle,$$

and thus $Y \in \mathfrak{h}$. For $H \in \mathfrak{h}$ we have

$$\langle H, Y \rangle = \langle [H, X], \boldsymbol{i}X \rangle = 2\pi \gamma(H) \langle \boldsymbol{i}X, \boldsymbol{i}X \rangle = -2\pi \gamma(H),$$

hence $Y$ is nonzero and orthogonal to $\ker(\gamma)$. Thus $H_\gamma = tY$ for some nonzero real $t$. Moreover, $\langle Y, Y\rangle = -2\pi\gamma(Y) < 0$. If we put $\varrho = 1/\sqrt{2\pi\gamma(Y)}$ and $U = \varrho X$, $V = \varrho \boldsymbol{i} X,\ W = \varrho^2 Y$, then

$$[U, V] = W, \quad [V, W] = U, \quad [W, U] = V,$$

and thus $\mathfrak{m}_\gamma \cong \mathfrak{so}(3)$. □

**Key Lemma 6.** *Let $\mathfrak{g}$ be a compact semisimple Lie algebra and let $A, B \in \mathfrak{g}$. Suppose that $A$ is orthogonal to some Cartan subalgebra. Then there exists a Cartan subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$ which is orthogonal both to $A$ and to $B$.*

*Proof.* Among all Cartan subalgebras $\mathfrak{h}$ orthogonal to $A$, we choose one for which the orthogonal projection $B_0$ of $B$ to $\mathfrak{h}$ has minimal length $r = |B_0|$. We claim that $r = 0$. Assume towards a contradiction that this is false. We decompose $\mathfrak{g}$ orthogonally as

$$\mathfrak{g} = \mathfrak{h} \oplus \sum_{\alpha\in\Phi^+} L_\alpha.$$

Accordingly we have $A = \sum_\alpha A_\alpha$ and $B = B_0 + \sum_\alpha B_\alpha$, with $A_\alpha, B_\alpha \in L_\alpha$. By assumption, $B_0 \neq 0$. Hence there is a positive real root $\gamma \in \Phi^+$ with $\gamma(B_0) \neq 0$. We decompose $B_0$ further in $\mathfrak{h}$ as an orthogonal sum $B_0 = B_{00} + H_\gamma$, where $\gamma(B_{00}) = 0$ and $H_\gamma \neq 0$. Then

$$\mathfrak{h} = \mathbb{R}H_\gamma \oplus \ker(\gamma)$$

and

$$\mathfrak{m}_\gamma = \mathbb{R}H_\gamma \oplus L_\gamma \cong \mathfrak{so}(3)$$

by Lemma 5. In the 3-dimensional Lie algebra $\mathfrak{m}_\gamma \cong \mathfrak{so}(3)$ there is a 1-dimensional subspace $V \subseteq \mathfrak{m}_\gamma$ which is orthogonal to $H_\gamma$ and to $A_\gamma$. The adjoint representation of SO(3) on its Lie algebra $\mathfrak{so}(3)$ is transitive on the 1-dimensional subspaces. Hence there is an element $g \in G$ of the form $g = \exp(\mathrm{ad}(Z))$, for some $Z \in \mathfrak{m}_\gamma$, with $g(H_\gamma) \in V$. Moreover, $g$ fixes $\ker(\gamma)$ pointwise. The Cartan subalgebra $\mathfrak{h}' = g(\mathfrak{h}) = V \oplus \ker(\gamma)$ is then orthogonal to $A$. The projection of $B$ to $\mathfrak{h}'$ is $B_{00}$ and has therefore strictly smaller length than $B_0$. This is a contradiction. □

**Corollary 7.** *Let $\mathfrak{g}$ be a compact semisimple Lie algebra and let $A, B \in \mathfrak{g}$. Then $A^\perp \cap B^\perp$ contains a Cartan subalgebra $\mathfrak{h}$.*

*Proof.* We apply Key Lemma 6 to 0 and $A$ to obtain a Cartan subalgebra which is orthogonal to $A$. Another application of Key Lemma 6 to $A$ and $B$ then yields a Cartan subalgebra $\mathfrak{h}$ which is orthogonal to both $A$ and $B$. □

*Proof of Theorem 1.* Let $\mathfrak{h}$ be a Cartan subalgebra which is orthogonal to $A$ and to $B$ and let $X \in \mathfrak{h}$ be a regular element. Then $\mathfrak{h} = \mathrm{Cen}_{\mathfrak{g}}(X)$ and thus $A, B \in [X, \mathfrak{g}]$ by Lemma 3. □

The proofs above made strong use of the assumption that $\mathfrak{g}$ is a compact semisimple Lie algebra. The fact that the Killing form is (negative) definite was used at several places in order to find orthogonal decompositions. Nevertheless, I conjecture that a similar result holds for other semisimple Lie algebras.

## Further remarks and an open problem

We close with some remarks and an open problem. Suppose that $\mathfrak{h}$ is a Cartan subalgebra in the compact Lie algebra $\mathfrak{g}$. If we pick nonzero elements $Z_\alpha \in L_\alpha$, for every positive root $\alpha$, and if we put $Z = \sum_{\alpha\in\Phi^+} Z_\alpha$, then $\mathfrak{h} \cap \mathrm{Cen}_{\mathfrak{g}}(Z) = 0$. Since $\mathrm{Cen}_{\mathfrak{g}}(Z)$ contains a Cartan subalgebra $\mathfrak{h}'$, this shows that there exists a Cartan subalgebra $\mathfrak{h}'$ which intersects $\mathfrak{h}$ trivially. However, one can do better. The following is shown in [Malkoun and Nahlus 2017].

**Theorem 8** (Malkoun–Nahlus). *Let $\mathfrak{h}$ be a Cartan subalgebra in a compact semisimple Lie algebra $\mathfrak{g}$. Then there exists a Cartan subalgebra $\mathfrak{h}' \subseteq \mathfrak{h}^\perp$.*

We reproduce the beautiful proof from [Malkoun and Nahlus 2017].

*Proof.* We may assume that $\mathfrak{g} \neq 0$. Let $w$ be a Coxeter element in the Weyl group $W = N/T$, where $T$ is the maximal torus corresponding to $\mathfrak{h}$, and $N \subseteq G$ is the normalizer of $T$. Then $W$ acts as a finite reflection group on $\mathfrak{h}$, and 1 is not an eigenvalue of $w$ in this action, see [Humphreys 1990, Section 3.16]. We choose $X \in \mathfrak{g}$ with $w = \exp(\mathrm{ad}(X))T$ and we claim that every Cartan subalgebra $\mathfrak{h}'$ containing $X$ is orthogonal to $\mathfrak{h}$. The linear endomorphism $\exp(\mathrm{ad}(X)) - \mathrm{id}_{\mathfrak{g}}$ of $\mathfrak{g}$ maps $\mathfrak{h}$ onto $\mathfrak{h}$, and

$$\exp(\mathrm{ad}(X)) - \mathrm{id}_{\mathfrak{g}} = \sum_{k=1}^{\infty} \frac{1}{k!}\,\mathrm{ad}(X)^k = \mathrm{ad}(X) \sum_{k=1}^{\infty} \frac{1}{k!}\,\mathrm{ad}(X)^{k-1}.$$

In particular, $\mathrm{ad}(X)(\mathfrak{g}) \supseteq \mathfrak{h}$. Thus $\mathrm{Cen}_{\mathfrak{g}}(X) \subseteq \mathfrak{h}^\perp$ by Lemma 3. □

Christoph Böhm has explained to me the following remarkable result.

**Theorem 9.** *The orthogonal Lie algebras $\mathfrak{so}(m)$, for $m \geq 3$, can be decomposed as orthogonal direct sums of Cartan subalgebras.*

*Proof.* The rank of $\mathfrak{so}(m)$ is $r = \lfloor \frac{1}{2}m \rfloor$, and the dimension of $\mathfrak{so}(m)$ is $n = \frac{1}{2}m(m-1)$. We let $e_1, \ldots, e_m$ denote the standard basis of $\mathbb{R}^m$, and we put $X_{i,j} = e_i e_j^T - e_j e_i^T$. Then the $X_{i,j}$ with $i < j$ form an orthonormal basis of $\mathfrak{so}(m)$. Moreover, two distinct basis elements $X_{i,j}$, $X_{k,\ell}$ commute if and only if $\{i, j\} \cap \{k, \ell\} = \varnothing$. The standard Cartan subalgebra for $\mathfrak{so}(m)$ is spanned by $X_{1,2}, X_{3,4}, \ldots, X_{2r-1,2r}$. The claim follows if we can partition the set $\mathcal{T}_m$ of all two-element subsets of $\{1, \ldots, m\}$ into $n/r$ subsets consisting of $r$ pairwise disjoint two-element subsets. The latter is possible by the scheduling algorithm for round robin tournaments.

An explicit construction of such a partition of $\mathcal{T}_m$ can be described as follows, see [van Lint and Wilson 1992, Example 36.2]. For odd $m \geq 3$ put

$$M_k = \{\{i, j\} \mid i < j \text{ and } i + j \equiv 2k \pmod m\},$$

for $k = 1, \ldots, m$. The $M_k$ partition $\mathcal{T}_m$ into $m$ subsets of cardinality $\frac{1}{2}(m-1)$, each consisting of pairwise disjoint two-element subsets. From this we obtain also such a partition of $\mathcal{T}_{m+1}$ by putting $M'_k = M_k \cup \{\{k, m+1\}\}$. □

We cannot expect such a result for general compact semisimple Lie algebras. For example, the compact semisimple Lie algebra $\mathfrak{g} = \mathfrak{so}(5) \oplus \mathfrak{so}(3)$ has dimension 13, hence such a decomposition cannot exist. The following question is thus very natural.

**Problem 10.** Which compact semisimple Lie algebras $\mathfrak{g}$ can be decomposed as an orthogonal sum of Cartan subalgebras?

The monograph [Kostrikin and Tiep 1994] is devoted to the complex version of this problem.

For the Lie algebras $\mathfrak{su}(m)$, Problem 10 can be rephrased as follows, using the Veronese embedding of $\mathbb{C}\mathrm{P}^{m-1}$. To each unit vector $u \in \mathbb{C}^m$ we may assign the selfadjoint projector

$$P(u) = uu^*,$$

where $*$ denotes the conjugate transpose, and its traceless part

$$P_0(u) = uu^* - \frac{1}{m}\,\mathrm{id}_{\mathbb{C}^m}\,.$$

We note that $P(uz) = P(u)$ holds for all complex numbers $z$ with $|z| = 1$. Suppose that $u_1, \ldots, u_m$ is an orthonormal basis of $\mathbb{C}^m$. Then the projectors $P(u_1), \ldots, P(u_m)$ commute, and the matrices $\boldsymbol{i}\,P_0(u_1), \ldots, \boldsymbol{i}\,P_0(u_m)$ span a Cartan subalgebra $\mathfrak{h}$ in $\mathfrak{su}(m)$. Conversely, the Cartan subalgebra $\mathfrak{h}$ determines the set of subspaces $u_1\mathbb{C}, \ldots, u_m\mathbb{C}$ uniquely, since these are the fixed points of the maximal torus $T \subseteq \mathrm{PSU}(m)$ with Lie algebra $\mathfrak{h}$ in its action on the complex projective space $\mathbb{C}\mathrm{P}^{m-1}$. Hence $\mathfrak{h}$ determines the orthonormal basis $u_1, \ldots, u_m$ up to a permutation of vectors, and up to multiplication of the basis vectors by complex numbers of norm 1.

The Killing form for $\mathfrak{su}(m)$ is given by $\langle X, Y\rangle = 2m\,\mathrm{tr}(XY)$. The Cartan subalgebras $\mathfrak{h}$ and $\mathfrak{h}'$ provided by two orthonormal bases $u_1, \ldots, u_m, v_1, \ldots, v_m$ are thus orthogonal if and only if

$$|\langle u_k, v_\ell\rangle|^2 = \frac{1}{m}$$

holds for all $k, \ell$. In this case, the two bases are called *mutually unbiased.* Such bases were considered in quantum mechanics by J. Schwinger [1960]. The construction of mutually unbiased bases has interesting connections to finite geometry, see [Kantor

2012; 2017; Thas 2016; 2018]. It is an open problem in which dimensions $m$ there exist $m+1$ pairwise mutually unbiased orthonormal bases. They are known to exist if $m$ is a prime power [Wootters and Fields 1989; Klappenecker and Rötteler 2004]. As we have seen, this question is equivalent to the existence of an orthogonal decomposition of $\mathfrak{su}(m)$ into Cartan subalgebras. There is a related problem about maximal abelian subalgebras in operator theory, see [Haagerup 1997]. It is presently an open problem if $\mathfrak{su}(6)$ admits an orthogonal decomposition into seven Cartan subalgebras.

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