Journal of Lie Theory
Volume **33** (2023) 329–360

Hodge Operators and Exceptional Isomorphisms between Unitary Groups

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Communicated by K.-H. Neeb

Abstract. We give a generalization of the Hodge operator to spaces (V, h) endowed with a hermitian or symmetric bilinear form h over arbitrary fields, including the characteristic two case. Suitable exterior powers of V become free modules over an algebra K defined using such an operator. This leads to several exceptional homomorphisms from unitary groups (with respect to h) into groups of semi-similitudes with respect to a suitable form over some subfield of K. The algebra K depends on h; it is a composition algebra unless h is symmetric and the characteristic is two.

Mathematics Subject Classification: 20G15, 20E32, 20G20, 20G40, 22C05, 11E39, 11E57.

Key Words: Hermitian form, symmetric bilinear form, exterior product, Pfaffian form, Hodge operator, exceptional isomorphism, composition algebra, quaternion algebra.

Introduction

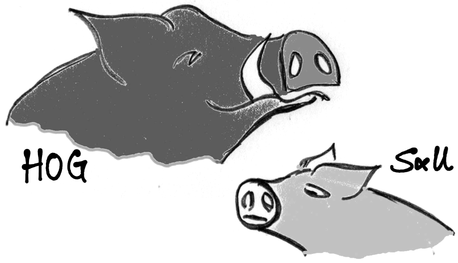
In many branches of mathematics, attempts at a systematic study (or even a classification) of certain objects lead to the consideration of families of examples constructed from given data (such as a ground field, dimension, an invariant form, etc.). With suitably defined categories, these constructions may even be considered as functors. The reader may think of classical groups (data comprise a field, a dimension, and an invariant form), or Dynkin diagrams (data comprise a letter indicating a construction of a graph, and a natural number giving the number of vertices in that graph), or split forms of simple Lie algebras (data comprise a field, and a Dynkin diagram), or finite simple groups of Lie type (data again comprise a field, and a Dynkin diagram), or classical polar spaces (data comprise a field, a dimension, and a polarity of the projective space of that dimension over that field).

While these functors usually assign non-isomorphic structures to different data sets, it often happens that unexpected isomorphisms occur for “small” data sets. If such an exceptional isomorphism occurs, it usually induces an exceptional situation in a related field, or can be explained and understood by such a situation.

In the present paper, we are interested in exceptional isomorphisms between certain classical groups. In this context, the simple groups in question are usually obtained by forming first the commutator group and then its quotient by the center, while the natural habitats of classical groups are inside linear groups. For that reason, it is sometimes better to study not only isomorphisms but also homomorphisms.

ISSN 0949–5932 / \$2.50

We show that certain homomorphisms between classical groups can be explained by variants of the Hodge operator. Some of the isomorphisms discussed here are not easily accessible by other methods, so we think our considerations are worth the effort. We have tried to keep the presentation reasonably self-contained, in particular, we avoid the use of elaborate results from Lie theory or the theory of algebraic groups. Applications to real Lie groups are made explicit in several examples (see 2.14, 3.10, 4.3, 4.4, 5.9, 5.10, and 7.8); applications to finite groups are explicit in 5.14, and 7.9.



The present paper is dedicated to our academic teacher, mentor, co-author, and friend, Karl Heinrich Hofmann. We express our sincere gratitude for his guidance of our own journey into mathematics. Karl Heinrich Hofmann is not only a traveler between different parts of the earth, but also between different parts of mathematics. We owe him thanks for many souvenirs from these travels. Last but not least, Karl Heinrich Hofmann is a mediator between the sometimes dry and arduous realm of mathematics and the visual arts. We are impressed by his artful cartoons; in particular, the series of poster cartoons illustrating and advertising the mathematical colloquium at Darmstadt. For that reason, we attempt a cartoon in his spirit, showing the hog (impersonating the Hodge operator) and a sow (whose name in the German spelling nicely corresponds to the name $\mathrm{S}\alpha\mathrm{U}_3(\mathbb{H},g)$ given by Jacques Tits to one of the main protagonists in the realm of exceptional isomorphisms between simple Lie groups (see 2.13 and 2.14 below).

1. The Hodge operator

1.1 Notation. We consider a field $\boldsymbol{F}$ of arbitrary characteristic and a field automorphism $\sigma\colon x \mapsto \bar{x}$ whose square is the identity. Let $\boldsymbol{R}$ denote the fixed field of σ. So either $\boldsymbol{F} = \boldsymbol{R}$ or $\boldsymbol{F}|\boldsymbol{R}$ is a Galois extension of degree 2 (by Artin's Theorem, see [24, VI, Th. 1.8]). In any case we call $N_{\boldsymbol{F}|\boldsymbol{R}}\colon \boldsymbol{F} \to \boldsymbol{R}\colon x \mapsto \bar{x}x$ the *norm form* of $\boldsymbol{F}|\boldsymbol{R}$. Note that $N_{\boldsymbol{F}|\boldsymbol{R}}(\boldsymbol{F})$ is just the set $\boldsymbol{F}^\square := \left\{x^2 \,\middle|\, x \in \boldsymbol{F}\right\}$ of squares if $\sigma = \mathrm{id}$.

A map $\lambda\colon V \to W$ between vector space over $\boldsymbol{F}$ is called $\boldsymbol{F}$*-semilinear* if λ is additive and there exists an automorphism φ of $\boldsymbol{F}$ (called the companion of λ) such that $\lambda(sv) = \varphi(s)\lambda(v)$ holds for each $s \in \boldsymbol{F}$ and each $v \in V$. If we want to state explicitly that the companion is φ, we speak of an $\boldsymbol{F}$-φ-semilinear map. (In the context of real and complex analysis, one encounters $\mathbb{C}$-$^{-}$-semilinear maps under the name $\mathbb{C}$-antilinear maps.)

Let (V,h) be an n-dimensional non-degenerate σ-hermitian space over $\boldsymbol{F}$; i.e., a vector space V over F with a bi-additive map $h\colon V \times V \to F$ such that $h(v,ws) = h(v,w)s$ and $h(w,v) = \overline{h(v,w)}$ hold for all $v,w \in V$ and each $s \in F$. (In particular, the map h is $\boldsymbol{F}$-linear in the right variable, and $\boldsymbol{F}$-σ-semilinear in the left variable.)

If $\sigma = \mathrm{id}$ then the form h is a symmetric bilinear form. Such forms occur as polarizations of quadratic forms; i.e., as $f(x,y) := q(x+y) - q(x) - q(y)$ for a quadratic form q. (Some authors refer to f as the polar form associated with q.) For any quadratic form in characteristic two, the polarization is an alternating form.

If $\sigma = \mathrm{id}$ and $\operatorname{char} \boldsymbol{F} = 2$ we also assume that h is diagonalizable, i.e. that there exists an orthogonal basis. We note that such a form will not occur as polarization of a quadratic form (unless it is zero). In particular, the orthogonal group with respect to a *bilinear* form on $\mathbb{F}_{2^e}^n$ differs from the group usually denoted by $\mathrm{O}^\varepsilon(n, 2^e)$. (If $\operatorname{char} \boldsymbol{F} \neq 2$ or $\sigma \neq \mathrm{id}$ then diagonalizability is no extra condition, cf. [8, §8, p. 15] or [27, I.3.4].) A vector $v \in V \smallsetminus \{0\}$ is called *isotropic* (with respect to h) if $h(v,v) = 0$; the form h is called isotropic if there exists an isotropic vector.

As we are going to extend the field $\boldsymbol{F}$ to a non-commutative composition algebra (containing $\boldsymbol{F}$ as a non-central subalgebra) over $\boldsymbol{R}$, we have to be precise: elements of $\boldsymbol{F}^n$ will be considered as columns, with scalars acting from the right, and matrices acting from the left. However, elements of $\boldsymbol{R}$ (in particular, signs such as $(-1)^m$) will also occur on the left.

As $\dim V$ is assumed to be finite, our assumption that h be not degenerate is equivalent to the fact that we obtain an $\boldsymbol{F}$-σ-semilinear isomorphism onto the dual space $V^\vee$, namely

$$h^\vee \colon V \to V^\vee \colon v \mapsto h(v, -)$$

see e.g. [21, Ch. I, §2]. Consider now the exterior algebra $\bigwedge V$, cf. [19, VI 9]. We note that $\bigwedge$ is a functor on vector spaces and semilinear maps, cf. 1.6 below. Moreover, there is a natural isomorphism $(\bigwedge V)^\vee \cong \bigwedge(V^\vee)$, so we may write unambiguously $\bigwedge V^\vee$. Explicitly, we have $\langle f_1 \wedge \cdots \wedge f_\ell, w_1 \wedge \cdots \wedge w_\ell \rangle = \det(\langle f_i, w_j \rangle)$ for $f_i \in V^\vee$ and $w_j \in V$, see [27, I.5.6] or [1] [4, § 8, Thm. 1, p. 102]; here $\langle \cdot, \cdot \rangle$ denotes the natural pairing between a vector space and its dual. Applying the functor $\bigwedge$ to $h^\vee \colon V \to V^\vee$, we obtain $\bigwedge h^\vee \colon \bigwedge V \to \bigwedge V^\vee$; we interpret this as a hermitian form $\bigwedge h$ on the exterior algebra $\bigwedge V$. Using the explicit formula above, we find

$$\textstyle\bigwedge h(v_1 \wedge \cdots \wedge v_\ell, w_1 \wedge \cdots \wedge w_\ell) = \bigwedge^\ell h(v_1 \wedge \cdots \wedge v_\ell, w_1 \wedge \cdots \wedge w_\ell) = \det(h(v_i, w_j)).$$

1.2 The Pfaffian form. The exterior algebra comes with a natural $\mathbb{Z}$-grading and $\bigwedge^n V$ is 1-dimensional. We fix an $\boldsymbol{F}$-linear isomorphism $b \colon \bigwedge^n V \to \boldsymbol{F}$. For each positive integer $\ell \leq n$, the map b then induces an $\boldsymbol{F}$-linear isomorphism $\mathrm{Pf} \colon \bigwedge^{n-\ell} V \to \bigwedge^\ell V^\vee$ given by

$$\mathrm{Pf}(v_1 \wedge \cdots \wedge v_{n-\ell})(w_1 \wedge \cdots \wedge w_\ell) = b(v_1 \wedge \cdots \wedge v_{n-\ell} \wedge w_1 \wedge \cdots \wedge w_\ell)\,.$$

This is the *Pfaffian form*, see [19, VI 10 Problems 23–28, VIII 12 Problem 42]. The resulting bilinear map Pf on $\bigwedge V$ is "graded symmetric",

$$\mathrm{Pf}(v_1 \wedge \cdots \wedge v_{n-\ell}, w_1 \wedge \cdots \wedge w_\ell) = (-1)^{(n-\ell)\ell}\, \mathrm{Pf}(w_1 \wedge \cdots \wedge w_\ell, v_1 \wedge \cdots \wedge v_{n-\ell})\,.$$

If $n = 2\ell$ then the bilinear form Pf is isotropic on $\bigwedge^\ell V$; in fact, we then have $\mathrm{Pf}(v_1 \wedge \cdots \wedge v_\ell, v_1 \wedge \cdots \wedge v_\ell) = 0$ for each basic ℓ-vector $v_1 \wedge \cdots \wedge v_\ell$.

[1] The treatment in [4] is quite different from that in later editions [6], [7].

1.3 Remark. For $n = 4$ and $\ell = 2$ we are dealing with the space $\bigwedge^2 \boldsymbol{F}^4$ that carries the Klein quadric. The *quadratic form* Pq defining the Klein quadric is also referred to as a Pfaffian form (cf. [12] and [29] where this form is denoted by q), and Pf is the symmetric bilinear form obtained as polarization of that quadratic form Pq. This is no source of confusion as long as $\operatorname{char} \boldsymbol{F} \neq 2$. However, the polar form Pf carries less information than the quadratic form Pq if $\operatorname{char} \boldsymbol{F} = 2$.

If one interprets the elements of $\bigwedge^2 \boldsymbol{F}^4$ as alternating matrices then there exists a scalar $s \in \boldsymbol{F}^\times$ such that $\mathrm{Pq}(X)^2 = s \det X$ holds for each $X \in \bigwedge^2 \boldsymbol{F}^4$, cf. [5, § 5 no. 2, Prop. 2, p. 84]; the scalar s reflects the choice of basis underlying that interpretation. See [31, 12.14] for an interpretation of Pq in terms of the exterior algebra.

1.4 The Hodge operator. We now consider the composite

$$J := \mathrm{Pf}^{-1} \circ \textstyle\bigwedge h \colon \ \bigwedge^\ell V \xrightarrow[\cong]{\bigwedge h} \bigwedge^\ell V^\vee \xrightarrow[\cong]{\mathrm{Pf}^{-1}} \bigwedge^{n-\ell} V \,.$$

This σ-linear isomorphism is the *Hodge operator*. It depends, of course, on h and on b but not on the choice of basis. See, e.g., [26, pp. 21–31] for a discussion of the Hodge operator for euclidean spaces over $\mathbb{R}$.

1.5 Explicit computation. Suppose that $v_1, \ldots, v_n$ is an orthogonal[2] basis of V. For $\bigwedge^\ell V$ we use the basis vectors $v_{i_1} \wedge \cdots \wedge v_{i_\ell}$ with ascending $i_1 < \cdots < i_\ell \leq n$. Then $\bigwedge^\ell h(v_1 \wedge \cdots \wedge v_\ell, -)$ is a linear form on $\bigwedge^\ell V$ which annihilates each one of those basis vectors, except for $v_1 \wedge \cdots \wedge v_\ell$; in fact

$$\textstyle\bigwedge^\ell h(v_1 \wedge \cdots \wedge v_\ell, v_1 \wedge \cdots \wedge v_\ell) = h(v_1, v_1) \cdots h(v_\ell, v_\ell) \,.$$

In other words: $\bigwedge^\ell h$ is again diagonalizable, and so is $\bigwedge h$. It then also follows that both $\bigwedge^\ell h \colon \bigwedge^\ell V \times \bigwedge^\ell V \to \boldsymbol{F}$ and $\bigwedge h \colon \bigwedge V \times \bigwedge V \to \boldsymbol{F}$ are not degenerate. The linear form $\mathrm{Pf}(v_{\ell+1} \wedge \cdots \wedge v_n)$ annihilates the same collection of basis ℓ-vectors, and

$$\mathrm{Pf}(v_{\ell+1} \wedge \cdots \wedge v_n, v_1 \wedge \cdots \wedge v_\ell) = (-1)^{(n-\ell)\ell}\, b(v_1 \wedge \cdots \wedge v_n) \,.$$

Therefore

$$J(v_1 \wedge \cdots \wedge v_\ell) \;=\; v_{\ell+1} \wedge \cdots \wedge v_n \, \frac{h(v_1, v_1) \cdots h(v_\ell, v_\ell)}{b(v_1 \wedge \cdots \wedge v_n)} \, (-1)^{(n-\ell)\ell} \,.$$

Similarly, we compute

$$\begin{aligned} J(v_{\ell+1} \wedge \cdots \wedge v_n) &= v_1 \wedge \cdots \wedge v_\ell \, \frac{h(v_{\ell+1}, v_{\ell+1}) \cdots h(v_n, v_n)}{b(v_{\ell+1} \wedge \cdots \wedge v_n \wedge v_1 \wedge \cdots \wedge v_\ell)} \, (-1)^{(n-\ell)\ell} \\ &= v_1 \wedge \cdots \wedge v_\ell \, \frac{h(v_{\ell+1}, v_{\ell+1}) \cdots h(v_n, v_n)}{b(v_1 \wedge \cdots \wedge v_n)} \,. \end{aligned}$$

Note that these formulae are correct only if $v_1, \ldots, v_n$ is an orthogonal basis, and cannot be used if $v_1 \wedge \cdots \wedge v_\ell$ corresponds to a subspace U of V such that $h|_{U \times U}$ is degenerate.

[2] If $\operatorname{char} \boldsymbol{F} = 2$ and $\sigma = \mathrm{id}$ then an orthogonal basis exists by our diagonalizability assumption.

1.6 Functoriality. It is well known that $\bigotimes^k\colon V \mapsto \bigotimes^k V$ and $\bigwedge^k\colon V \mapsto \bigwedge^k V$ are functors in the category of vector spaces over $\boldsymbol{F}$ and $\boldsymbol{F}$-linear maps. We need the fact that these are indeed functors in the category of vector spaces over $\boldsymbol{F}$ and $\boldsymbol{F}$-*semi* linear maps. Since the pertinent facts seem less well known, we take the liberty to expand some of the details, in the following.

Let V, W be vector spaces over $\boldsymbol{F}$. An explicit construction of the tensor product $V \otimes W$ is obtained by factoring the free vector space $\boldsymbol{F}^{(V\times W)}$ with basis $V \times W$ over $\boldsymbol{F}$ modulo the subspace $R_{V,W}$ generated by all expressions of the form $(v,w)+(v,y)-(v,w+y)$, $(v,w)+(x,w)-(v+x,w)$, $(vc,w)-(v,w)c$ and $(v,wc)-(v,w)c$ where $v,x \in V$, $w,y \in W$ and $c \in \boldsymbol{F}$, see [19, p. 262]. As usual, we write $v\otimes w := (v,w)+R_{V,W}$. The tensor power $\bigotimes^k V$ can now be constructed inductively from $\bigotimes^0 V := \boldsymbol{F}$ and $\bigotimes^{k+1} V := V \otimes \bigotimes^k V$.

In order to extend the morphism part of $\bigotimes^k$ and $\bigwedge^k$ to the category of vector spaces over $\boldsymbol{F}$ and $\boldsymbol{F}$-*semi*linear maps we first consider semilinear maps $\lambda\colon V \to W$ and $\mu\colon X \to Y$ with the same companion field homomorphism $\varphi\colon F \to F$. Mapping $(v,x) \in V \times X$ to $(\lambda(v), \mu(x))$ extends to an $\boldsymbol{F}$-φ-semilinear map from $\boldsymbol{F}^{(V\times X)}$ to $\boldsymbol{F}^{(W\times Y)}$ which maps $R_{V,X}$ into $R_{W,Y}$. Thus $(\lambda\otimes\mu)(v\otimes x) := \lambda(v)\otimes\mu(x)$ defines an $\boldsymbol{F}$-φ-semilinear map $\lambda\otimes\mu\colon V\otimes X \to W \otimes Y$.

Proceeding inductively with $\mu = \bigotimes^{k-1}\lambda\colon \bigotimes^{k-1} V \to \bigotimes^{k-1} W$, we obtain $\boldsymbol{F}$-φ-semilinear maps

$$\bigotimes^k\lambda\colon \bigotimes^k V \to \bigotimes^k W\colon v_1\otimes\cdots\otimes v_k \mapsto \lambda(v_1)\otimes\cdots\otimes\lambda(v_k)$$

for each $k \in \mathbb{N}$.

Now consider $\boldsymbol{F}$-semilinear maps $\lambda\colon V \to W$ and $\mu\colon W \to Y$ with (possibly different) companion field homomorphisms φ and ψ, respectively. Then

$$\bigotimes^k\mu\left(\bigotimes^k\lambda(v_1\otimes\cdots\otimes v_k)\right) = \mu(\lambda(v_1))\otimes\cdots\otimes\mu(\lambda(v_k)) = \bigotimes^k(\mu\circ\lambda)(v_1\otimes\cdots\otimes v_k)$$

shows $\bigotimes^k\mu\circ\bigotimes^k\lambda = \bigotimes^k(\mu\circ\lambda)$, and functoriality of $\bigotimes^k$ is established. Combination of these functors yields a functor $\bigotimes$ with $\bigotimes V := \bigoplus_{k\in\mathbb{N}}\bigotimes^k V$ and $(\bigotimes\lambda)(\sum_{k\in\mathbb{N}} t_k) := \sum_{k\in\mathbb{N}}\bigotimes^k\lambda(t_k)$.

Finally we consider the exterior algebra $\bigwedge V = \bigoplus_{k\in\mathbb{N}}\bigwedge^k V$ which is obtained as the quotient of $\bigotimes V$ modulo the two-sided ideal S_V generated by $\left\{v\otimes v \,\middle|\, v\in V\right\}$, cf. [19, p. 288]. We write $v_1\wedge\cdots\wedge v_k := (v_1\otimes\cdots\otimes v_k)+S_V$ as usual. For each $\boldsymbol{F}$-semilinear map $\lambda\colon V\to W$ we note that $\bigotimes\lambda(S_V)$ is contained in S_W. Thus $\bigwedge^k\lambda(v_1\wedge\cdots\wedge v_k) := \lambda(v_1)\wedge\cdots\wedge\lambda(v_k)$ is well defined. Note that we use the multiplication in $\bigotimes V$ only for the definition of S_V.

1.7 Examples. Assume $\dim V = n$ and consider an $\boldsymbol{F}$-semilinear endomorphism $\lambda\colon V\to V$ with companion field endomorphism φ. Choose a basis $v_1,\ldots,v_n$ for V; then λ is the composite $\lambda = \lambda'\circ\hat\varphi$ of some linear map $\lambda'\colon V\to V$ with the map $\hat\varphi\colon \sum_{k=1}^n v_k x_k \mapsto \sum_{k=1}^n v_k\varphi(x_k)$. It is well known (cf. [24, Thm. 4.11] or [7, III § 8]) that $\bigwedge^n\lambda'$ is just multiplication by $\det\lambda'$. For each $s\in\boldsymbol{F}$ we now have $(\bigwedge^n\hat\varphi)(v_1 s\wedge\cdots\wedge v_n) = (v_1\wedge\cdots\wedge v_n)\varphi(s)$ and thus

$$\begin{aligned}(\textstyle\bigwedge^n\lambda)\left((v_1\wedge\cdots\wedge v_n)s\right) &= (\textstyle\bigwedge^n\lambda'\circ\bigwedge^n\hat\varphi)(v_1 s\wedge\cdots\wedge v_n)\\ &= (v_1\wedge\cdots\wedge v_n)\,\varphi(s)\,(\det\lambda')\,.\end{aligned}$$

1.8 The unitary group. *The special unitary group* $\mathrm{SU}(V,h)$ *acts in a natural way both on* $\bigwedge V$ *and on* $\bigwedge V^\vee$ *and commutes with* $\bigwedge h\colon \bigwedge V \to \bigwedge V^\vee$ *because* $\bigwedge$ *is a functor. The group* $\mathrm{SU}(V,h)$ *also commutes with* $\mathrm{Pf}\colon \bigwedge^{n-\ell} V \to \bigwedge^{\ell} V^\vee$ *because its elements have determinant* 1. *Therefore it centralizes the Hodge operator* J. *More generally, the groups*

$$\mathrm{GU}(V,h) := \left\{\lambda \in \mathrm{GL}(V) \,\middle|\, \exists r_\lambda \in \boldsymbol{F}\,\forall v,w \in V\colon h\left(\lambda(v),\lambda(w)\right) = r_\lambda\, h(v,w)\right\} \text{ and}$$

$$\Gamma\mathrm{U}(V,h) := \left\{\lambda \in \Gamma\mathrm{L}(V) \,\middle|\, \exists r_\lambda \in \boldsymbol{F}\,\forall v,w \in V\colon h\left(\lambda(v),\lambda(w)\right) = r_\lambda\, \varphi_\lambda(h(v,w))\right\}$$

of similitudes and semi-similitudes, respectively, also act naturally on $\bigwedge V$*; here* φ_λ *denotes the companion field automorphism of* $\lambda \in \Gamma\mathrm{L}(V)$. *A general element* λ *of* $\Gamma\mathrm{U}(V,h)$ *will not centralize the Hodge operator but it will normalize the set* $\boldsymbol{F}\mathrm{id}\circ J$*; in fact, the conjugate* $\bigwedge^{n-\ell}\lambda \circ J \circ (\bigwedge^{\ell}\lambda)^{-1} = t_\lambda \mathrm{id}\circ J$, *with*

$$t_\lambda = \frac{\varphi_\lambda(\det\lambda')}{r_\lambda^\ell}\,\frac{b(v_1\wedge\cdots\wedge v_n)}{\varphi_\lambda(b(v_1\wedge\cdots\wedge v_n))}$$

where $v_1,\dots,v_n$ *is an orthonormal basis used to obtain a decomposition* $\lambda = \lambda'\circ\hat\varphi$ *as in* 1.7. *(This simplifies to* $t_\lambda = r_\lambda^{-\ell}\det\lambda$ *if* λ *is* $\boldsymbol{F}$*-linear.)*

Proof. Only the assertion about $\bigwedge^{n-\ell}\lambda \circ J \circ (\bigwedge^{\ell}\lambda)^{-1}$ remains to be verified. Let $v_1,\dots,v_n$ be the orthogonal basis that was used for the decomposition $\lambda = \lambda'\circ\hat\varphi$ as in 1.7, and put $w_i := \lambda(v_i)$. Then $w_1,\dots,w_n$ is an orthogonal basis, and $(\bigwedge^{\ell}\lambda)^{-1}(w_1\wedge\cdots\wedge w_\ell) = v_1\wedge\cdots\wedge v_\ell$. Using 1.5, we obtain that $\bigwedge^{n-\ell}\lambda \circ J \circ (\bigwedge^{\ell}\lambda)^{-1}$ maps $w_1\wedge\cdots\wedge w_\ell$ to

$$\begin{aligned}\left(\textstyle\bigwedge^{n-\ell}\lambda\circ J\right)(v_1\wedge\cdots\wedge v_\ell) &= \left(\textstyle\bigwedge^{n-\ell}\lambda\right)\left(v_{\ell+1}\wedge\cdots\wedge v_n\,\frac{h(v_1,v_1)\cdots h(v_\ell,v_\ell)}{b(v_1\wedge\cdots\wedge v_n)}\,(-1)^{(n-\ell)\ell}\right)\\ &= (w_{\ell+1}\wedge\cdots\wedge w_n)\;\varphi_\lambda\left(\frac{h(v_1,v_1)\cdots h(v_\ell,v_\ell)}{b(v_1\wedge\cdots\wedge v_n)}\right)\,(-1)^{(n-\ell)\ell}.\end{aligned}$$

Now $h(w_i,w_i) = h(\lambda(v_i),\lambda(v_i)) = r_\lambda\varphi_\lambda(h(v_i,v_i))$ yields $\varphi_\lambda(h(v_i,v_i)) = r_\lambda^{-1}h(w_i,w_i)$. From 1.7 we know $w_1\wedge\cdots\wedge w_n = (\bigwedge^{n}\lambda)(v_1\wedge\cdots\wedge v_n) = v_1\wedge\cdots\wedge v_n\det\lambda'$. So $\varphi_\lambda\big(b(v_1\wedge\cdots\wedge v_n)\big) = \varphi_\lambda\big(b(w_1\wedge\cdots\wedge w_n)(\det\lambda')^{-1}\big)$, and the assertion follows. ∎

1.9 Application in 3D. Let h be the standard bilinear form on $\mathbb{R}^3$; i.e., a form with an ortho*normal* basis v_1,v_2,v_3. Moreover, let $b(v_1\wedge v_2\wedge v_3) = -1$. Then $J(v_1) = -v_2\wedge v_3$, $J(v_2) = v_1\wedge v_3$, $J(v_3) = -v_1\wedge v_2$, see 1.5. In other words:

$$J\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}0 & -z & y\\ z & 0 & -x\\ -y & x & 0\end{pmatrix}$$

and this map is $\mathrm{SO}_3(\mathbb{R})$-equivariant, while $\lambda\circ J\circ\lambda^{-1} = -J$ holds for each element $\lambda\in \mathrm{O}(3,\mathbb{R})\setminus\mathrm{SO}_3(\mathbb{R})$.

Let $\gamma\colon\Omega\to\mathbb{R}^3\colon (x,y,z)^\intercal \mapsto (\gamma_1(x,y,z),\gamma_2(x,y,z),\gamma_3(x,y,z))^\intercal$ be a vector field defined on some domain $\Omega\subseteq\mathbb{R}^3$. Then the Hodge operator maps

$$\operatorname{curl}\gamma = (\partial_y\gamma_3-\partial_z\gamma_2, \partial_z\gamma_1-\partial_x\gamma_3, \partial_x\gamma_2-\partial_y\gamma_1)^\intercal$$

to the anti-symmetrization of the Jacobian $\operatorname{Jac}\gamma$;

indeed

$$\operatorname{Jac}\gamma - (\operatorname{Jac}\gamma)^{\intercal} = \begin{pmatrix} 0 & \partial_y\gamma_1 - \partial_x\gamma_2 & \partial_z\gamma_1 - \partial_x\gamma_3 \\ \partial_x\gamma_2 - \partial_y\gamma_1 & 0 & \partial_z\gamma_2 - \partial_y\gamma_3 \\ \partial_x\gamma_3 - \partial_z\gamma_1 & \partial_y\gamma_3 - \partial_z\gamma_2 & 0 \end{pmatrix} = J \begin{pmatrix} \partial_y\gamma_3 - \partial_z\gamma_2 \\ \partial_z\gamma_1 - \partial_x\gamma_3 \\ \partial_x\gamma_2 - \partial_y\gamma_1 \end{pmatrix}.$$

1.10 The square of the Hodge operator. *Let H be the Gram matrix of h with respect to the basis $v_1, \dots, v_n$. The square of J is a linear automorphism of $\bigwedge^\ell V$, and we find that*

$$J^2 = \delta_\ell \operatorname{id} \quad \text{where} \quad \delta_\ell := (-1)^{(n-\ell)\ell} \frac{\det(H)}{N_{\boldsymbol{F}|\boldsymbol{R}}(b(v_1 \wedge \cdots \wedge v_n))}.$$

Recall that $\det(H)$ depends on the choice of the basis; the invariant would be $\operatorname{disc}(h) \in \boldsymbol{F}^\times / N_{\boldsymbol{F}|\boldsymbol{R}}(\boldsymbol{F}^\times)$, the norm class of $\det(h(v_i, v_j))$. However, the whole expression depends only on h and b. Replacing the isomorphism $b\colon \bigwedge^n V \to \boldsymbol{F}$ changes J by a factor and J^2 by the norm of that factor. In particular, the isomorphism type of the algebra $\boldsymbol{K}_\ell$ introduced in 2.1 below does not depend on the choice of b. If b is chosen in such a way that $b(v_1 \wedge \cdots \wedge v_n) = 1$, then we arrive at the textbook formula $J^2 = (-1)^{(n-\ell)\ell} \det(H) \operatorname{id}$, cp. [36, 6.1].

1.11 Lemma. *For all $x, y \in \bigwedge^\ell V$ we have*

(1) $\operatorname{Pf}(J(x), y) = \bigwedge^\ell h(x, y)$,

(2) $\operatorname{Pf}(x, J(y)) = (-1)^{(n-\ell)\ell}\, \overline{\bigwedge^\ell h(x, y)}$,

(3) $\operatorname{Pf}(J(x), J(y)) = \delta_\ell\, \overline{\operatorname{Pf}(y, x)}$,

(4) $\bigwedge^\ell h(J(x), y) = \delta_\ell \operatorname{Pf}(x, y)$,

(5) $\bigwedge^\ell h(x, J(y)) = \delta_\ell\, \overline{\operatorname{Pf}(y, x)}$,

(6) $\bigwedge^\ell h(J(x), J(y)) = (-1)^{(n-\ell)\ell}\, \delta_\ell\, \overline{\bigwedge^\ell h(x, y)}$.

Proof. We compute $\operatorname{Pf}(J(x), -) = (\operatorname{Pf} \circ \operatorname{Pf}^{-1} \circ \bigwedge^\ell h)(x, -) = \bigwedge^\ell h(x, -)$ and $\bigwedge^\ell h(J(x), -) = (\bigwedge^\ell h \circ \operatorname{Pf}^{-1} \circ \bigwedge^\ell h)(x, -) = (\operatorname{Pf} \circ \operatorname{Pf}^{-1} \circ \bigwedge^\ell h \circ \operatorname{Pf}^{-1} \circ \bigwedge^\ell h)(x, -) = \operatorname{Pf}(J^2(x), -) = \delta_\ell \operatorname{Pf}(x, -)$. This gives the assertions (a) and (d), the rest follows from $\bigwedge^\ell h(y, x) = \overline{\bigwedge^\ell h(x, y)}$, $\operatorname{Pf}(y, x) = (-1)^{(n-\ell)\ell} \operatorname{Pf}(x, y)$ and $J^2 = \delta_\ell \operatorname{id}$. ∎

2. Composition algebras generated by Hodge operators

From now on, assume $n = 2\ell$. Then J gives an $\boldsymbol{F}$-σ-semilinear endomorphism

$$J\colon \textstyle\bigwedge^\ell V \to \bigwedge^\ell V.$$

We are going to use J to give $\bigwedge^\ell V$ the structure of a right module over an associative algebra of dimension $2 \dim_{\boldsymbol{R}} \boldsymbol{F}$ over $\boldsymbol{R}$. Note that $(-1)^{(n-\ell)\ell} = (-1)^\ell$ holds because $n = 2\ell$.

2.1 The algebra $\boldsymbol{K}_\ell$. Let $\delta_\ell := (-1)^\ell \frac{\det(H)}{N_{\boldsymbol{F}|\boldsymbol{R}}(b(v_1 \wedge \cdots \wedge v_n))}$ as in 1.10 and let $\boldsymbol{K}_\ell$ denote the $\boldsymbol{R}$-algebra consisting of all matrices of the form

$$x = \begin{pmatrix} x_0 & \delta_\ell\, \overline{x_1} \\ x_1 & \overline{x_0} \end{pmatrix} \in \boldsymbol{F}^{2\times 2}.$$

We identify $x_0 \in \boldsymbol{F}$ with the diagonal matrix $\begin{pmatrix} x_0 & 0 \\ 0 & \overline{x_0} \end{pmatrix}$ and put $\boldsymbol{j}_\ell := \begin{pmatrix} 0 & \delta_\ell \\ 1 & 0 \end{pmatrix}$.

Thus
$$\begin{pmatrix} x_0 & \delta_\ell\,\overline{x_1} \\ x_1 & \overline{x_0} \end{pmatrix} = x_0 + \boldsymbol{j}_\ell\, x_1\,.$$

We also have $\boldsymbol{j}_\ell^2 = \delta_\ell$ and $\boldsymbol{j}_\ell\, x = \overline{x}\,\boldsymbol{j}_\ell$ for each $x \in \boldsymbol{F}$. Note $\dim_{\boldsymbol{R}} \boldsymbol{K}_\ell = 2\,\dim_{\boldsymbol{R}} \boldsymbol{F}$. For $x_0, x_1 \in \boldsymbol{F}$ we put $\kappa(x_0 + \boldsymbol{j}_\ell\, x_1) := \overline{x_0} - \boldsymbol{j}_\ell\, x_1$. The map κ is called the *standard involution* of $\boldsymbol{K}_\ell$. We also write $\overline{x} := \kappa(x)$ for $x \in \boldsymbol{K}_\ell$. As κ extends the action of σ on $\boldsymbol{F} \cong \left\{ \left(\begin{smallmatrix} x_0 & 0 \\ 0 & \overline{x_0} \end{smallmatrix} \right) \,\middle|\, x_0 \in \boldsymbol{F} \right\}$, this should cause no confusion.

We write $\boldsymbol{F}^\times := \boldsymbol{F} \setminus \{0\}$ for the multiplicative group, $\boldsymbol{F}^\square := \left\{ s^2 \,\middle|\, s \in \boldsymbol{F} \right\}$, and $\boldsymbol{F}^\boxtimes := \boldsymbol{F}^\square \setminus \{0\}$. Then $\boldsymbol{F}^\times / \boldsymbol{F}^\boxtimes$ is the group of non-trivial square classes of $\boldsymbol{F}$; note that this is an elementary abelian 2-group.

2.2 Lemma. *The standard involution is an anti-automorphism of $\boldsymbol{K}_\ell$ satisfying $\kappa(x)\,x = x\,\kappa(x) = \det_{\boldsymbol{F}}(x_0 + \boldsymbol{j}_\ell\, x_1) = N_{\boldsymbol{F}|\boldsymbol{R}}(x_0) - \delta_\ell\, N_{\boldsymbol{F}|\boldsymbol{R}}(x_1) \in \boldsymbol{R}$ for every $x = x_0 + \boldsymbol{j}_\ell\, x_1 \in \boldsymbol{K}_\ell$. Thus the determinant gives a multiplicative quadratic form $\det\colon \boldsymbol{K}_\ell \to \boldsymbol{R}$. The corresponding polarization $f_{\det}$ is degenerate precisely if $\operatorname{char} \boldsymbol{F} = 2$ and $\sigma = \mathrm{id}$; indeed, it vanishes in that case. The quadratic form $\det$ is then degenerate if, and only if, the bilinear form h has discriminant $1 \in \boldsymbol{F}^\times / \boldsymbol{F}^\boxtimes$.*

Proof. Standard matrix computations suffice to verify that κ is indeed an anti-automorphism, with the properties as claimed. The value of the polar form at $(x_0 + \boldsymbol{j}_\ell\, x_1, y_0 + \boldsymbol{j}_\ell\, y_1)$ is obtained as $\det_{\boldsymbol{F}}(x_0 + y_0 + \boldsymbol{j}_\ell(x_1 + y_1)) - \det_{\boldsymbol{F}}(x_0 + \boldsymbol{j}_\ell\, x_1) - \det_{\boldsymbol{F}}(y_0 + \boldsymbol{j}_\ell\, y_1) = x_0\overline{y_0} + y_0\overline{x_0} - \delta_\ell(x_1\overline{y_1} + y_1\overline{x_1})$. Clearly, this polar form is zero if $\operatorname{char} \boldsymbol{F} = 2$ and $\sigma = \mathrm{id}$.

If $\sigma \neq \mathrm{id}$ we pick $s \in \boldsymbol{F} \setminus \boldsymbol{R}$ with $s + \overline{s} = 1$; then $s\overline{s} = s - s^2$. With respect to the basis $(1,0)$, $(s,0)$, $(0,1)$, $(0,s)$, the Gram matrix of the polar form becomes
$$\begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 2s\overline{s} & 0 & 0 \\ 0 & 0 & -2\delta_\ell & -\delta_\ell \\ 0 & 0 & -\delta_\ell & -2\delta_\ell s\overline{s} \end{pmatrix},$$
with determinant $\delta_\ell^2(4s\overline{s} - 1)^2 = \delta_\ell^2(1 - 2s)^2 \neq 0$.

If $\sigma = \mathrm{id}$ then the Gram matrix $\left(\begin{smallmatrix} 2 & 0 \\ 0 & -2\delta_\ell \end{smallmatrix} \right)$ for the polar form is nonsingular precisely if $\operatorname{char} \boldsymbol{F} \neq 2$.

Finally, assume $\sigma = \mathrm{id}$ and $\operatorname{char} \boldsymbol{F} \neq 2$. Then h has discriminant 1 precisely if δ_ℓ is a square. In that case, the quadratic form $\det_{\boldsymbol{F}}(x_0 + \boldsymbol{j}_\ell\, x_1) = x_0^2 - \delta_\ell x_1^2 = (x_0 - \sqrt{\delta_\ell}x_1)^2$ is isotropic, and then degenerate because its polar form is zero. ∎

2.3 Remarks. The isomorphism type of $\boldsymbol{K}_\ell$ as an $\boldsymbol{R}$-algebra does not depend on the choice of the orthogonal basis $v_1, \dots, v_n$, and does not depend on the choice of b; cp. the discussion following 1.10. In the sequel, we will choose the basis in such a way that the Gram matrix H has a convenient determinant, and then usually adapt b such that $b(v_1 \wedge \cdots \wedge v_n) = 1$; then $\delta_\ell = (-1)^\ell \det H$.

If the polar form $f_{\det}$ is non-degenerate then $\boldsymbol{K}_\ell$ is indeed a *composition algebra*, cf. [28, 1.5.1] or [17, §7.6]. In this case the standard involution κ is uniquely determined by the properties of being an $\boldsymbol{R}$-linear anti-automorphism of $\boldsymbol{K}_\ell$ and inducing $-\mathrm{id}$ on $\boldsymbol{R}^\perp$. As such, it is rightfully termed "standard". Note that $\boldsymbol{R}^\perp = \{x \in \boldsymbol{K}_\ell \setminus \boldsymbol{R} \mid x^2 \in \boldsymbol{R}\} \cup \{0\}$ if $\operatorname{char} \boldsymbol{R} \neq 2$, and $\boldsymbol{R}^\perp = \{x \in \boldsymbol{K}_\ell \mid x^2 \in \boldsymbol{R}\}$ if $\operatorname{char} \boldsymbol{R} = 2$. The norm form of the composition algebra $\boldsymbol{K}_\ell$ is a Pfister form; it is the orthogonal sum of the norm form $N_{\boldsymbol{F}|\boldsymbol{R}}$ and its scalar multiple $-\delta_\ell N_{\boldsymbol{F}|\boldsymbol{R}}$.

If $f_{\det}$ is degenerate we have $\kappa = \mathrm{id}$ and $\det(x) = x\overline{x} = x^2$. These inseparable cases will be discussed in [22] in greater detail.

Note that $\boldsymbol{K}_\ell$ contains non-invertible elements different from 0 if, and only if, the factor δ_ℓ is a norm; i.e., if $(-1)^\ell$ lies in $N_{\boldsymbol{F}|\boldsymbol{R}}(\boldsymbol{F})$. In that case, we call the algebra $\boldsymbol{K}_\ell$ *split*; we may (and will) normalize $\delta_\ell = 1$ then. The case where $\boldsymbol{K}_\ell$ is split will be discussed in Section 3 below. In particular, the existence of idempotents or of nilpotent elements in $\boldsymbol{K}_\ell \smallsetminus \{0,1\}$ leads to the existence of certain $\mathrm{SU}(V,h)$-submodules in $\bigwedge^\ell V$, see 3.3 below.

If δ_ℓ is not a norm then $\boldsymbol{K}_\ell|\boldsymbol{F}$ is either a quadratic extension (for $\sigma = \mathrm{id}$; that extension is inseparable if $\operatorname{char}\boldsymbol{F} = 2$) or a quaternion division algebra over $\boldsymbol{R}$ (for $\sigma \neq \mathrm{id}$). See Section 7 below.

2.4 Proposition. *Via conjugation in* $\mathrm{End}_{\boldsymbol{R}}(\bigwedge^\ell V)$*, every similitude of* h *induces an automorphism of the* $\boldsymbol{R}$*-subalgebra* M_ℓ *generated by* J *and* $\boldsymbol{F}\,\mathrm{id}$ *in* $\mathrm{End}_{\boldsymbol{R}}(\bigwedge^\ell V)$.

Proof. Let γ be a similitude with multiplier r. Then $\gamma \circ J \circ \gamma^{-1} = J r^{-\ell}\det\gamma$. Using a matrix A describing γ and the Gram matrix H for h with respect to the same basis, we note that $\overline{A}^{\intercal} H A = rH$, and infer $N_{\boldsymbol{F}|\boldsymbol{R}}(\det A) = r^{2\ell}$. So $N_{\boldsymbol{F}|\boldsymbol{R}}(r^{-\ell}\det\gamma) = 1$, and a straightforward calculation yields that conjugation by γ yields an algebra automorphism, as claimed. ■

Obviously, the algebra M_ℓ is anti-isomorphic to $\boldsymbol{K}_\ell$, and then in fact isomorphic because $\boldsymbol{K}_\ell$ admits an anti-automorphism.

2.5 Definition. For $v \in \bigwedge^\ell V$ we put $v\,\boldsymbol{j}_\ell := J(v)$.

In this way, $\bigwedge^\ell V$ becomes a $\mathrm{SU}(V,h)$-$\boldsymbol{K}_\ell$-bimodule, i.e., it becomes a right module over $\boldsymbol{K}_\ell$ and $\mathrm{SU}(V,h)$ acts $\boldsymbol{K}_\ell$-linearly from the left. Choosing an orthogonal basis $v_1,\dots,v_n$ for V with a fixed ordering we obtain a basis B for $\bigwedge^\ell V$ consisting of all $v_{j_1}\wedge\cdots\wedge v_{j_\ell}$ where $(j_1,\dots,j_\ell)$ is an increasing sequence of length ℓ in $\{1,\dots,n\}$. The sequences with $j_1 = 1$ form a subset B_1 of B, and J maps each element of B_1 to one of $B \smallsetminus B_1$. Moreover, the set B_1 forms a basis for the $\boldsymbol{K}_\ell$-module $\bigwedge^\ell V$, showing that the latter is a free module.

2.6 The α-hermitian form. We define $g\colon \bigwedge^\ell V \times \bigwedge^\ell V \to \boldsymbol{K}_\ell$ by

$$g(u,v) := \textstyle\bigwedge^\ell h(u,v) + \bigwedge^\ell h(u,v\boldsymbol{j}_\ell)\,\boldsymbol{j}_\ell^{-1} = \bigwedge^\ell h(u,v) + \boldsymbol{j}_\ell\,(-1)^\ell\,\mathrm{Pf}(u,v)\,;$$

see 1.11 for the description on the right hand side. This expression is $\boldsymbol{F}$-linear in the right argument (i.e., $g(u,vs) = g(u,v)s$), and $\boldsymbol{F}$-σ-semilinear in the left argument (in the sense that $g(us,v) = \overline{s}g(u,v)$ for all $s \in \boldsymbol{F}$ and all $u,v \in \bigwedge^\ell V$). Moreover,

$$\begin{aligned} g(u,v\boldsymbol{j}_\ell) &= \textstyle\bigwedge^\ell h(u,v\boldsymbol{j}_\ell) + \bigwedge^\ell h(u,(v\boldsymbol{j}_\ell)\boldsymbol{j}_\ell)\,\boldsymbol{j}_\ell^{-1} \\ &= \textstyle\bigwedge^\ell h(u,v\boldsymbol{j}_\ell) + \bigwedge^\ell h(u,v)\,\delta_\ell\,\boldsymbol{j}_\ell^{-1} g(u,v)\,\boldsymbol{j}_\ell, \end{aligned}$$

so g is indeed $\boldsymbol{K}_\ell$-linear in the right argument. Let α denote the involution

$$\alpha(x_0 + \boldsymbol{j}_\ell x_1) := \overline{x_0} + (-1)^\ell\,\boldsymbol{j}_\ell\,x_1\,.$$

Then α is an $\boldsymbol{R}$-linear anti-automorphism of $\boldsymbol{K}_\ell$, with $\alpha(x) = \overline{x}$ for $x \in \boldsymbol{F}$ and $\alpha(\boldsymbol{j}_\ell) = (-1)^\ell\,\boldsymbol{j}_\ell$. Using the fact that Pf is graded symmetric we find $g(v,u) = \alpha(g(u,v))$. This shows that g is an α-hermitian form on $\bigwedge^\ell V$.

Note that $\boldsymbol{K}_\ell$ is not a field, in general: we need the more general concept of hermitian forms over rings (e.g., see [21]).

2.7 Proposition. *The form g is diagonalizable.*

Proof. In fact, for any orthogonal basis $v_1, \dots, v_n$ of V, consider the vector $w_\gamma := v_1 \wedge v_{\gamma_2} \wedge \dots \wedge v_{\gamma_\ell}$ for each increasing sequence $\gamma = (\gamma_2, \dots, \gamma_\ell)$. Then the w_γ form a basis for the free $\boldsymbol{K}_\ell$-module $\bigwedge^\ell V$. By 1.5, we have $\bigwedge^\ell h(w_\gamma, w_\beta) = 0$ if $\gamma \neq \beta$, and $\bigwedge^\ell h(w_\gamma, w_\gamma) = h(v_1, v_1) h(v_{\gamma_2}, v_{\gamma_2}) \dots h(v_{\gamma_\ell}, v_{\gamma_\ell})$ for each admissible γ.

As every w_γ is a wedge product starting with the same vector v_1, we obviously have $\mathrm{Pf}(w_\gamma, w_\beta) = 0$, and $g(w_\gamma, w_\beta) = \bigwedge^\ell h(w_\gamma, w_\beta)$. ∎

2.8 Remarks. The involution α coincides with the standard involution κ if $\operatorname{char} \boldsymbol{F} = 2$ or if ℓ is odd. If $\sigma \neq \mathrm{id}$ and ℓ is even we pick any $a \in \boldsymbol{F}^\times$ with $\bar{a} = -a$ and obtain $\alpha(x) = a^{-1} \kappa(x) a$ for all $x \in \boldsymbol{K}_\ell$. If $\sigma = \mathrm{id}$ and ℓ is even then $\alpha = \mathrm{id}$. The idea of (re-)constructing the hermitian form g dates back to [16, p. 266], cf. [30, 4.4] and [3, 2.9].

2.9 Lemma.

(a) *For $X, Y \in \bigwedge^\ell V$ we have*
$$g(X,Y) = 0 \iff \mathrm{Pf}(X,Y) = 0 = \textstyle\bigwedge^\ell h(X,Y).$$
In particular, the form g is not degenerate (*since h is not degenerate, see* 1.5).

(b) *If $\bigwedge^\ell h$ is anisotropic then g is anisotropic, too.*

(c) *If h is isotropic then both $\bigwedge^\ell h$ and g are isotropic, as well.*

Proof. Comparing coefficients in $\boldsymbol{K}_\ell = \boldsymbol{F} \oplus \boldsymbol{j}_\ell \boldsymbol{F}$ we obtain the first assertion. The second one follows immediately.

The form $\bigwedge^1 h$ coincides with h. For $n = \dim V$ the form $\bigwedge^n h$ on $\bigwedge^n V \cong \boldsymbol{F}^1$ will be anisotropic, regardless of what the (non-degenerate) form h may be.

So consider $2 \le \ell < n$, and assume that h is isotropic. Then V is the orthogonal sum of an anisotropic space and a non-trivial split space S, see [27, III, (1.1), p. 56]. In the split space S, we find x, y such that $h(x,x) = 0$ and $h(x,y) = 1$, see [27, I, (6.3), p. 56].

If $\operatorname{char} \boldsymbol{F} \neq 2$, we may also assume $h(y,y) = 0$. We then put $v_1 := x + y$ and $v_2 := x - y$, and extend to an orthogonal basis $v_1, v_2, \dots, v_n$ for V.

If $\operatorname{char} \boldsymbol{F} = 2$, there are two possibilities: either the restriction of h to S is alternating, or it is possible to choose $x, y \in S$ such that $h(x,x) = 0 \neq h(y,y)$ and $h(x,y) = 1$. In the first case, we find $u \in S^\perp \smallsetminus \{0\}$ because the (diagonalizable and non-degenerate) form h is not alternating. In the second case, we just take $u = 0$. We now put $v_1 := y + u$ and $v_2 := x h(v_1, v - y_1)$. Then $h(v_2, v_2) = h(v_1, v_1) \neq 0$, and $h(v_1, v_2) = 0$. So there exists an orthogonal basis $v_1, v_2, \dots, v_n$ for V.

If $\ell = 2$, put $w_1 := v_1 \wedge v_3$ and $w_2 := v_2 \wedge v_3$. If $\ell > 2$, put $w_1 := v_1 \wedge v_3 \wedge \dots \wedge v_{\ell+1}$ and $w_2 := v_2 \wedge v_3 \wedge \dots \wedge v_{\ell+1}$. Then from $\bigwedge^\ell h(w_1, w_1) = -\bigwedge^\ell h(w_2, w_2)$ we obtain $\bigwedge^\ell h(w_1 + w_2, w_1 + w_2) = 0$, so $\bigwedge^\ell h$ is isotropic.

Finally, recall from 1.2 that $\mathrm{Pf}(w_1 + w_2, w_1 + w_2) = 0$, so the form g is isotropic, as well. ∎

2.10 Remarks. It may happen that h is anisotropic but $\bigwedge^\ell h$ becomes isotropic. For instance, this happens whenever we start with an anisotropic bilinear form h on a vector space of dimension 4 over a p-adic field because there are no anisotropic quadratic forms in more than four variables over such a field (see [23, VI 2.2]).

For more explicit examples, take h_1 and h_2 as the polarizations of the quadratic forms $x_0^2 - 2x_1^2 + 3x_2^2 - 6x_3^2$ and $x_0^2 + 2x_1^2 + 10x_3^2 - 5x_4^2$, respectively, over $\mathbb{Q}$. Both forms are anisotropic; the discriminant of h_1 is a square while that of h_2 is not. Both $\bigwedge^2 h_1$ and $\bigwedge^2 h_2$ are isotropic. For h_2, the corresponding form g is isotropic (over $\boldsymbol{K}_2 \cong \mathbb{Q}(\sqrt{-1})$), as well, see 8.1 below.

2.11 The homomorphism. From the definition of g it is clear that $\mathrm{SU}(V,h)$ preserves g and that $\Gamma\mathrm{U}(V,h)$ acts by semi-similitudes of g, see 1.8. Thus we have, for $\dim(V) = n = 2\ell$, constructed a homomorphism $\eta_\ell\colon \Gamma\mathrm{U}(V,h) \to \Gamma\mathrm{U}(\bigwedge^\ell V, g)$ which restricts to

$$\eta_\ell|_{\mathrm{SU}(V,h)}\colon \mathrm{SU}(V,h) \to \mathrm{U}(\textstyle\bigwedge^\ell V, g)\,.$$

Of course, we have to read $\Gamma\mathrm{U}(V,h)$ as $\Gamma\mathrm{O}(V,h)$ if $\sigma = \mathrm{id}$, and $\Gamma\mathrm{U}(\bigwedge^\ell V, g)$ as $\Gamma\mathrm{O}(\bigwedge^\ell V, g)$ if $\sigma = \mathrm{id}$ and ℓ is even.

2.12 Lemma. *The kernel of η_ℓ consists of all scalar multiples $s\,\mathrm{id}$ of the identity where $\bar{s}s = 1 = s^\ell$. The center of the image $\eta_\ell(\mathrm{SU}(V,h))$ has order at most 2 because $1 = \det(s\,\mathrm{id})$ already implies $s^n = s^{2\ell} = 1$.*

Proof. Via $(v_1 \wedge \cdots \wedge v_\ell)\boldsymbol{F} \mapsto v_1\boldsymbol{F} + \cdots + v_\ell\boldsymbol{F}$ we identify $\mathrm{Gr}_{1,\boldsymbol{F}}(\bigwedge^\ell V)$ with $\mathrm{Gr}_{\ell,\boldsymbol{F}}(V)$. If some $A \in \Gamma\mathrm{L}(V)$ with companion μ_A fixes each element of $\mathrm{Gr}_{\ell,\boldsymbol{F}}(V)$ then it also fixes each element of $\mathrm{Gr}_{1,\boldsymbol{F}}(V)$ because these are obtained as intersections of ℓ-dimensional spaces. As $\boldsymbol{F}$ is commutative, we find (by evaluating A on a basis $b_1,\dots,b_n$ and at $b_1 + \cdots + b_n t$ for $t \in \boldsymbol{F}$, see [2, III.1, Prop. 3(b), p. 43]) that there exists $s \in \boldsymbol{F}$ such that $A = s\,\mathrm{id}$ and $ts = s\mu_A(t)$. In particular, A is linear. Now A is trivial on $\bigwedge^\ell V$ precisely if $s^\ell = 1$. The rest is clear. ∎

2.13 Example: $\mathrm{S}\alpha\mathrm{U}_n(\mathbb{H})$. Let $\mathbb{H} = \mathbb{R} + i\mathbb{R} + j\mathbb{R} + k\mathbb{R}$ be the (essentially unique) quaternion division algebra over $\mathbb{R}$. Then $\mathbb{C} = \mathbb{R} + i\mathbb{R}$ is the field of complex numbers, and we obtain $\mathbb{H} = \mathbb{C} + j\mathbb{C}$. Composing the standard involution $\kappa\colon a \mapsto \bar{a}$ with the inner automorphism $a \mapsto i^{-1}ai = -iai$, we obtain the involutory anti-automorphism $\alpha\colon x + jy \mapsto \bar{x} + jy$ (here $x, y \in \mathbb{C}$). On $\mathbb{H}^n$, we now define the α-hermitian form $f\colon \mathbb{H}^n \times \mathbb{H}^n \to \mathbb{H}$ by

$$f\big((a_1,\dots,a_n),(b_1,\dots b_n)\big) := \alpha(a_1)\,b_1 + \cdots + \alpha(a_n)\,b_n\,.$$

The corresponding unitary group is denoted by $\mathrm{S}\alpha\mathrm{U}_n(\mathbb{H}, f)$, its Lie algebra by $\mathsf{D}_n^{\mathbb{H}}$ (in the notation of Tits [32]; Helgason [14] denotes the corresponding Lie algebras by $\mathfrak{so}^*(2n)$). We also note that the form f can be replaced by the skew-hermitian form if, without changing the unitary group (but changing the involution back to the standard one).

2.14 Example. An interesting special case of our construction occurs if $\boldsymbol{F} = \mathbb{C}$, $\boldsymbol{R} = \mathbb{R}$, $\ell = 2$, and h is a hermitian form of Witt index 1 on $\mathbb{C}^4$. The algebra $\boldsymbol{K}$ is then isomorphic to $\mathbb{H}$, the involution α has a 3-dimensional space of fixed points, and is thus a conjugate of the involution used in 2.13. Our result 2.11 thus provides a covering $\mathrm{SU}_4(\mathbb{C},1) \to \mathrm{S}\alpha\mathrm{U}_3(\mathbb{H}, g)$ corresponding to the isomorphism from the

Lie algebra $\mathsf{A}_3^{\mathbb{C},1}$ onto $\mathsf{D}_3^{\mathbb{H}}$ (in the notation of Tits [32]; Helgason [14] denotes the corresponding Lie algebras by $\mathfrak{su}(3,1)$ and $\mathfrak{so}^*(6)$, respectively).

The group $\mathrm{U}_4(\mathbb{C},1)$ is a semidirect product of its commutator group $\mathrm{SU}_4(\mathbb{C},1)$ with a group isomorphic to $\mathbb{R}/\mathbb{Z}$. With respect to a basis v_1, v_2, v_3, v_4 with $h(v_1,v_1) = h(v_2,v_2) = h(v_3,v_3) = 1$ and $h(v_4,v_4) = -1$, we consider the linear maps γ_s defined by $\gamma_s(v_1) = v_1$, $\gamma_s(v_2) = v_2$, $\gamma_s(v_3) = v_3$, and $\gamma_s(v_4) = v_4 s$, with $s\bar{s} = 1$. Then $\left\{\gamma_s \,\middle|\, s \in \mathbb{C}, s\bar{s} = 1\right\}$ is a complement for $\mathrm{SU}_4(\mathbb{C},1)$ in $\mathrm{U}_4(\mathbb{C},1)$. It turns out that $\eta_2(\gamma_s)$ is an $\mathbb{H}$-semilinear map with companion $\varphi_s\colon x + \boldsymbol{j}y \mapsto x + \boldsymbol{j}sy$ (see 2.4), fixing each vector in the $\mathbb{H}$-basis $v_1 \wedge v_2$, $v_1 \wedge v_3$, $v_2 \wedge v_3$.

We also remark that the $\mathbb{C}$-semilinear map λ with companion σ and fixing each v_k induces the $\mathbb{H}$-semilinear map $\eta_2(\lambda)$ with companion $\varphi\colon x + \boldsymbol{j}y \mapsto \bar{x} + \boldsymbol{j}\bar{y}$ fixing each one of $v_1 \wedge v_2$, $v_1 \wedge v_3$, $v_2 \wedge v_3$.

3. The split cases

We will call $\boldsymbol{K}_\ell$ *split* whenever it contains divisors of zero. This extends the established terminology for composition algebras. Recall that $\boldsymbol{K}_\ell$ is split precisely if δ_ℓ is a norm: $\delta_\ell = \bar{s}s$ for some $s \in \boldsymbol{F}^\times$ (and this happens precisely if h has discriminant $(-1)^\ell$).

In that case, we may assume $\delta = 1$ without loss of generality. In fact, if we replace our isomorphism $b\colon \bigwedge^n V \to \boldsymbol{F}$ by sb then the Hodge operator J is replaced by $J s^{-1}\colon X \mapsto J(X)s^{-1}$, and we have $(J s^{-1})^2 = \mathrm{id}$. So the subalgebra M_ℓ of $\mathrm{End}_{\boldsymbol{R}}(\bigwedge^\ell V)$ remains the same (see 2.4), while the algebra $\boldsymbol{K}_\ell$ is replaced by its conjugate
$$\begin{pmatrix} 1 & 0 \\ 0 & s \end{pmatrix}^{-1} \boldsymbol{K}_\ell \begin{pmatrix} 1 & 0 \\ 0 & s \end{pmatrix} = \left\{ \begin{pmatrix} y_0 & \overline{y_1} \\ y_1 & \overline{y_0} \end{pmatrix} \,\middle|\, y_0, y_1 \in \boldsymbol{F} \right\}.$$

3.1 Convention. For the rest of this section, we will always assume that we have normalized b in such a way that $J^2 = \mathrm{id}$ whenever $\boldsymbol{K}_\ell$ is split (so $\delta_\ell = 1$). Then $z := 1 + \boldsymbol{j}_\ell \in \boldsymbol{K}_\ell$ satisfies $z^2 = 2z$. Thus z is nilpotent if $\mathrm{char}\,\boldsymbol{F} = 2$, and $p := \frac{1}{2}z$ is an idempotent if $\mathrm{char}\,\boldsymbol{F} \neq 2$.

3.2 Lemma. *If $\boldsymbol{K}_\ell$ is split then we have one of the following:*

(a) $\sigma \neq \mathrm{id}$*: then* $\dim_{\boldsymbol{R}} \boldsymbol{K}_\ell = 4$ *and* $\boldsymbol{K}_\ell \cong \boldsymbol{R}^{2\times 2}$ *is the split quaternion algebra over* $\boldsymbol{R}$*. On* $\boldsymbol{R}^{2\times 2}$ *the standard involution is given by* $\kappa\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \left(\begin{smallmatrix} d & -b \\ -c & a \end{smallmatrix}\right)$.

(b) $\sigma = \mathrm{id}$ *and* $\mathrm{char}\,\boldsymbol{F} \neq 2$*: then* $\boldsymbol{K}_\ell \cong \boldsymbol{F} \times \boldsymbol{F} \cong \boldsymbol{F}[X]/(X^2 - 1)$ *is the two-dimensional split composition algebra over* $\boldsymbol{F}$*. On* $\boldsymbol{F} \times \boldsymbol{F}$ *the standard involution is* $\kappa(a,b) = (b,a)$.

(c) $\sigma = \mathrm{id}$ *and* $\mathrm{char}\,\boldsymbol{F} = 2$*: then* $\boldsymbol{K}_\ell \cong \boldsymbol{F}[X]/(X^2)$ *is a local ring, and the standard involution is the identity.*

Proof. See [28, 1.8] for the first two cases. The last case is checked directly. ■

3.3 Lemma. *Let* $W := \bigwedge^\ell V$*, and assume that* $\boldsymbol{K}_\ell$ *is split.*

(a) *If* $p \in \boldsymbol{K}_\ell \setminus \{0,1\}$ *is an idempotent then* $\bar{p} = 1 - p$ *and* W *decomposes as the direct sum* $\bigwedge^\ell V = Wp \oplus W\bar{p}$ *of* $\mathrm{SU}(V,h)$*-modules* Wp *and* $W\bar{p} = W(1-p)$.

(b) *If* $\sigma \neq \mathrm{id}$ *then* $\boldsymbol{K}_\ell \cong \boldsymbol{R}^{2\times 2}$ *contains exactly one conjugacy class of idempotents apart from* 0 *and* 1*; in* $\boldsymbol{K}_\ell$ *that class is represented by* $\left(\begin{smallmatrix} u & \bar{u} \\ u & \bar{u} \end{smallmatrix}\right)$ *with* $1 = \bar{u} + u$*. In particular, the* $\mathrm{SU}(V,h)$*-modules* Wp *and* $W\bar{p}$ *are isomorphic.*

(c) *If $\sigma = \mathrm{id}$ and $\operatorname{char} \boldsymbol{F} \neq 2$ then $\boldsymbol{K}_\ell \cong \boldsymbol{F} \times \boldsymbol{F} = \boldsymbol{R} \times \boldsymbol{R}$ contains precisely two idempotents apart from 0 and 1, corresponding to $(1,0)$ and $(0,1)$, respectively. No element of $\boldsymbol{K}_\ell \smallsetminus \{0\}$ is nilpotent, and the two non-trivial idempotents of $\boldsymbol{K}_\ell$ are $\frac{1}{2}(1+\boldsymbol{j}_\ell)$ and $\frac{1}{2}(1-\boldsymbol{j}_\ell)$.*

If ℓ is odd then the involution α coincides with the standard involution κ and interchanges the two idempotents. If ℓ is even then $\alpha = \mathrm{id}$.

For any $\boldsymbol{K}_\ell$-basis B for W we define $\psi_B \colon \sum_{b\in B} b\, x_b \mapsto \sum_{b \in B} b\, \overline{x_b}$. Then ψ_B is a $\boldsymbol{K}_\ell$-semilinear bijection of W (with companion automorphism κ) which gives an isomorphism of $\mathrm{SU}(V,h)$-modules from Wp onto $\overline{Wp} := \psi_B(Wp) = W\overline{p}$.

(d) *If $\sigma = \mathrm{id}$ and $\operatorname{char} \boldsymbol{F} = 2$ then $\boldsymbol{K}_\ell \cong \boldsymbol{F}[X]/(X^2)$ does not contain any idempotents apart from 0 and 1. The maximal ideal in $\boldsymbol{K}_\ell$ is generated by the nilpotent element $z = 1+\boldsymbol{j}_\ell$. The submodule Wz and the quotient W/Wz are isomorphic via $\rho_z \colon w + Wz \mapsto wz$.*

Proof. Assume that $p \in \boldsymbol{K}_\ell \smallsetminus \{0,1\}$ is an idempotent. Then $p^2 = p$ implies $0 = p^2 - p = p(1-p)$, and $0 = \det_{\boldsymbol{F}} p = \overline{p}p$ follows. Then

$$(p+\overline{p})^2 = p^2 + 2\overline{p}p + \overline{p}^2 = p + \overline{p}$$

yields that $p+\overline{p}$ is an idempotent in the field $\boldsymbol{F}$. Since $p + \overline{p} = 0$ would imply $p = p^2 = -\overline{p}p = 0$ we infer $p+\overline{p} = 1$ and $\overline{p} = 1-p$. Thus $W = Wp + W\overline{p}$, and the sum is direct because $vp = w\overline{p}$ implies $w = (v+w)p \in Wp$ and $w\overline{p} \in Wp\overline{p} = \{0\}$. The summands are $\mathrm{SU}(V,h)$-modules because $\boldsymbol{K}_\ell$ centralizes $\mathrm{SU}(V,h)$.

Let $p, q \in \boldsymbol{K}_\ell \smallsetminus \{0,1\}$ be idempotents, and let $\varphi \colon \boldsymbol{R}^{2\times 2} \to \boldsymbol{K}_\ell$ be an isomorphism of algebras. Then both p and q are conjugates of $\varphi\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)$; in particular, there exists $a \in \boldsymbol{K}_\ell^\times$ with $q = a^{-1}pa$. Now the map $w \mapsto wa$ is $\mathrm{SU}(V,h)$-equivariant and induces a module isomorphism from $Wp = Wa^{-1}p$ onto $Wpa = Wa^{-1}pa = Wq$. Thus assertion (b) is established.

Now assume $\sigma = \mathrm{id}$ but $\operatorname{char} \boldsymbol{F} \neq 2$. The idempotents in $\boldsymbol{F} \times \boldsymbol{F}$ are easy to find and it is clear that there are no nilpotent elements apart from 0. As W is a *free* $\boldsymbol{K}_\ell$-module (see 2.5) it is clear that B and ψ_B exist with the required properties.

Finally, we assume $\sigma = \mathrm{id}$ but $\operatorname{char} \boldsymbol{F} = 2$. The assertion about idempotents in $\boldsymbol{F}[X]/(X^2)$ follows since $a^2 + \boldsymbol{F}X^2 \ni a^2 + b^2X^2 = (a+bX)^2 \in (a+bX) + \boldsymbol{F}X^2$ implies $b = 0$ and $a^2 = a \in \boldsymbol{F}$.

For the last assertion, it suffices to show that the kernel of ρ_z is precisely Wz. The inclusion $Wz \subseteq \ker \rho_z$ is clear from $z^2 = 0$. In order to see the reverse inclusion, pick an orthogonal basis $v_1, \dots, v_{2\ell}$ for V. Let W_1 be the subspace of W spanned by those ℓ-vectors $v_{t_1} \wedge \cdots \wedge v_{t_\ell}$ with $1 \in \{t_1, \dots, t_\ell\}$ and let W^1 denote the span of those with $1 \notin \{t_1, \dots, t_\ell\}$. Then $W = W_1 \oplus W^1$ and J (i.e., right multiplication by $\boldsymbol{j}_\ell$) induces an isomorphism from W_1 onto W^1. Right multiplication with z induces the operator $\mathrm{id} + J$ which has rank at least $\frac{1}{2}\dim W$ because its restriction to W_1 is injective. Thus $\dim\ker\rho_z \le \frac{1}{2}\dim W \le \dim Wz \le \dim\ker\rho_z$ yields $\ker\rho_z = Wz$ as required. ■

3.4 Remark. In $\boldsymbol{R}^{2\times 2}$, the nilpotent elements apart from 0 form a single conjugacy class, represented by $\left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right)$. The conjugacy class of idempotents (different from 1 and 0) is represented by $\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)$. If $\sigma \neq \mathrm{id}$ and $\boldsymbol{K}_\ell$ is split, the corresponding classes in $\boldsymbol{K}_\ell$ are represented by any $\left(\begin{smallmatrix} v & \overline{v} \\ v & \overline{v} \end{smallmatrix}\right)$ with $v \in \boldsymbol{K}_\ell \smallsetminus \{0\}$ such that $v + \overline{v} = 0$ (for a nilpotent element) or $v + \overline{v} = 1$ (for an idempotent).

3.5 Lemma. *Assume $\sigma = \mathrm{id}$, $\operatorname{char} \boldsymbol{F} \neq 2$ and that $\boldsymbol{K}_\ell$ splits so that there exist idempotents p and $\overline{p} = 1 - p$ in $\boldsymbol{K}_\ell \cong \boldsymbol{F} \times \boldsymbol{F}$.*

(a) *In any case the restriction of g to the subspace Wp is a multiple of $\bigwedge^\ell h$: we have $g(Xp, Yp) = \bigwedge^\ell h(Xp, Yp)\, 2p$ for all $X, Y \in W$.*

(b) *If ℓ is even then Wp and $W(1-p)$ are orthogonal with respect to g, and the restrictions of g to Wp and $W(1-p)$, respectively, are not degenerate.*

(c) *If ℓ is odd then the restrictions of g and of $\bigwedge^\ell h$ to Wp are trivial.*

Proof. According to 3.3(c), we may (up to an application of the standard involution) assume $p = \frac{1}{2}(1 + \boldsymbol{j}_\ell)$. We compute

$$p\boldsymbol{j}_\ell = \tfrac{1}{2}(1 + \boldsymbol{j}_\ell)\boldsymbol{j}_\ell = \tfrac{1}{2}(\boldsymbol{j}_\ell + \boldsymbol{j}_\ell^2) = \tfrac{1}{2}(\boldsymbol{j}_\ell + 1) = p.$$

This yields, as claimed,

$$\begin{aligned} g(Xp, Yp) &= \textstyle\bigwedge^\ell h(Xp, Yp) + \bigwedge^\ell h(Xp, Yp\boldsymbol{j}_\ell)\boldsymbol{j}_\ell^{-1} \\ &= \textstyle\bigwedge^\ell h(Xp, Yp)(1 + \boldsymbol{j}_\ell^{-1}) = \bigwedge^\ell h(Xp, Yp)\, 2p. \end{aligned}$$

If ℓ is even then $\alpha = \mathrm{id}$ and we compute

$$g(Xp, Y(1-p)) = p\, g(X, Y)\, (1-p) = g(X, Y)\, (p - p^2) = 0$$

for arbitrary $X, Y \in W$. Thus $Wp \perp W(1-p)$, as claimed. Using 2.9, it is easy to see that both the restrictions of g to Wp and to $W(1-p)$ are not degenerate.

If ℓ is odd then α interchanges p with $1 - p$ and

$$g(Xp, Yp) = \alpha(p) g(X, Y) p = g(X, Y)\, (1-p)p = 0$$

holds for all $X, Y \in W$. Thus the restriction of g to Wp is trivial, and so is the restriction of $\bigwedge^\ell h$. ∎

3.6 Lemma. *Assume $\sigma = \mathrm{id}$, $\operatorname{char} \boldsymbol{F} = 2$ and that $\boldsymbol{K}_\ell$ splits so that there exists a nilpotent element $z \in \boldsymbol{K}_\ell \smallsetminus \{0\}$. Then the restriction of g to the subspace Wz is trivial.*

Proof. We note $\alpha = \mathrm{id}$ and then compute $g(Xz, Yz) = g(X, Y)\, z^2 = 0$. ∎

3.7 Reduction of the form. Assume that $\boldsymbol{K}_\ell$ is split (with $\boldsymbol{j}_\ell^2 = 1$), and consider $z = 1 + \boldsymbol{j}_\ell$. The value of $\alpha(z)$ depends on ℓ; we find $\alpha(z) = z$ if ℓ is even and $\alpha(z) = 1 - \boldsymbol{j}_\ell$ if ℓ is odd. In the latter case, we thus have $\alpha(z)z = 0$.

In order to understand the restriction of the form g to the submodule Wz we introduce some notation, as follows. If ℓ is odd, $\sigma \neq \mathrm{id}$, and $\operatorname{char} \boldsymbol{F} \neq 2$ we pick $a_\ell \in \boldsymbol{F} \smallsetminus \{0\}$ such that $\overline{a_\ell} = -a_\ell$. In all other cases, we take $a_\ell := 1$. Using the $\boldsymbol{R}$-linear map

$$r_\ell \colon \boldsymbol{K}_\ell \to \boldsymbol{R} \colon x + \boldsymbol{j}_\ell y \mapsto \left(x + (-1)^\ell y + (-1)^\ell \left(\overline{x + (-1)^\ell y}\right)\right) a_\ell$$

we then find

$$\begin{aligned} &\alpha(z)(x + \boldsymbol{j}_\ell\, y)z = \left(1 + (-1)^\ell \boldsymbol{j}_\ell\right) (x + \boldsymbol{j}_\ell\, y)\, (1 + \boldsymbol{j}_\ell) \\ &= \left((x + (-1)^\ell y) + (-1)^\ell \overline{(x + (-1)^\ell y)}\right) (1 + \boldsymbol{j}_\ell) = r_\ell(x + \boldsymbol{j}_\ell\, y)\ a_\ell^{-1} z\,. \end{aligned}$$

Putting $g^o(Xz, Yz) := r_\ell\big(g(X,Y)\big)$ we obtain a $(-1)^\ell$-symmetric $\boldsymbol{R}$-bilinear form g^o on Wz. Thus there are homomorphisms

$$\eta^o_{2k}\colon \mathrm{SU}(V,h) \to \mathrm{O}(Wz, g^o) \quad \text{or} \quad \eta^o_{2k+1}\colon \mathrm{SU}(V,h) \to \mathrm{Sp}(Wz, g^o)\,,$$

respectively, according to the parity of $\ell \in \{2k, 2k+1\}$. ∎

Note that the $\boldsymbol{R}$-bilinear form g^o on Wz is zero if $\sigma = \mathrm{id}$ and either ℓ is odd or $\mathrm{char}\, F = 2$. This fits well with the fact (see 3.5(c) and 3.6, cp. also 1.11(f)) that even the restriction of the $\boldsymbol{K}_\ell$-sesquilinear form g to Wz is trivial in those cases.

3.8 Lemma. *If the* $\mathrm{SU}(V,h)$*-module* Wz *has a complement in* W *(in particular, if* $\sigma \neq \mathrm{id}$ *or* $\mathrm{char}\,\boldsymbol{F} \neq 2$*, see* 3.3(b), 3.3(c)*) then that complement is isomorphic to* Wz *and* $\ker \eta^o_\ell = \ker \eta_\ell$*. If there is no complementary* $\mathrm{SU}(V,h)$*-module to* Wz *then* $\sigma = \mathrm{id}$ *and* $\mathrm{char}\,\boldsymbol{F} = 2$ *and* W/Wz *is isomorphic to* Wz*, see* 3.3(d)*. In this case, the group* $\ker \eta^o_\ell / \ker \eta_\ell$ *is an elementary abelian* 2*-group because it is isomorphic to a subgroup of* $\mathrm{Hom}(Wz, Wz)$*.* ∎

3.9 Remarks. The description in 3.7 is somewhat complicated because ℓ is arbitrary (and may be odd). If one studies the Hodge operator in order to understand exceptional isomorphisms between classical groups then the values $\ell \in \{1,2\}$ give the most interesting examples.

For $\ell = 2$, we will see in 5.3, 5.7, and 5.13 below that the split cases lead to interesting and intimate connections with norm forms on composition algebras.

If $\sigma = \mathrm{id}$ and $\mathrm{char}\,\boldsymbol{F} = 2$ then the precise structure of $\mathrm{SU}(V,h) = \mathrm{SO}(V,h)$ and $\ker \eta^o_\ell / \ker \eta_\ell$ depends on the defect of the form h. Details will be given in [22]. Note that a complementary module for Wz may also exist in that case; for instance, this will happen if $\mathrm{SO}(V,h)$ is trivial (cf. [22, 2.4]).

3.10 Examples: Symmetric bilinear forms on $\mathbb{R}^4$. Let $\boldsymbol{F} = \mathbb{R}$ and $\sigma = \mathrm{id}$. We consider the symmetric bilinear forms of Witt index 0 and 2, respectively, on $V = \mathbb{R}^4$. In both cases, the form has discriminant 1. So $\boldsymbol{K}_2$ splits, we have $\boldsymbol{K}_2 \cong \mathbb{R} \times \mathbb{R}$, and 3.7 applies.

If h has Witt index 0 then $\mathrm{SO}(\mathbb{R}^4, h)$ is the compact form of type $\mathsf{D}_2 = \mathsf{A}_1 \times \mathsf{A}_1$; denoted by $\mathsf{D}_2^{\mathbb{R},0}$ in [32]. The homomorphisms obtained by the actions on Wz and on $W\bar{z}$ are the projections onto the two direct factors in $\mathrm{SO}(\mathbb{R}^4,h)/\langle -\mathrm{id}\rangle \cong \mathrm{SO}(\mathbb{R}^3, h_0) \times \mathrm{SO}(\mathbb{R}^3, h_0)$, where h_0 is the (essentially unique) symmetric bilinear form of Witt index 0 on $\mathbb{R}^3$.

If h has Witt index 2 then the connected component $\mathrm{EO}(\mathbb{R}^4,h)$ of $\mathrm{SO}(\mathbb{R}^4,h)$ is the split real form of type D_2; denoted by $\mathsf{D}_2^{\mathbb{R},2}$ in [32]. (See Section 7 below for a general definition of the groups $\mathrm{EO}(V,h)$ and $\mathrm{EU}(V,h)$.) The homomorphisms now obtained by the actions on Wz and on $W\bar{z}$ are the projections onto the two direct factors in $\mathrm{SO}(\mathbb{R}^4,h)/\langle -\mathrm{id}\rangle \cong \mathrm{SO}(\mathbb{R}^3,h_1) \times \mathrm{SO}(\mathbb{R}^3,h_1)$, where h_1 is the (essentially unique) symmetric bilinear form of Witt index 1 on $\mathbb{R}^3$.

We have $\mathrm{O}(\mathbb{R}^4,h) \cong \mathrm{C}_2^2 \ltimes \mathrm{EO}(\mathbb{R}^4,h)$, where C_2 is a cyclic group of order 2. More explicitly, assume that v_1, v_2, v_3, v_4 is an orthogonal basis with $h(v_1,v_1) = 1 = h(v_2,v_2)$ and $h(v_3,v_3) = -1 = h(v_4,v_4)$.

Consider the matrices

$$B := \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad C := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad T := \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

The complement for $\mathrm{EO}(\mathbb{R}^4, h)$ in $\mathrm{O}(\mathbb{R}^4, h)$ is generated by the involutions β, γ described by B and C, respectively. Both $\eta_2^o(\beta)$ and $\eta_2^o(\gamma)$ interchange Wz with $W\bar{z}$; their product $\eta_2^o(\beta\gamma)$ generates a complement for $\mathrm{EO}(Wz, g^o) \cong \mathrm{PSL}_2\mathbb{R}$ in $\mathrm{SO}(Wz, g^o) \cong \mathrm{PGL}_2\mathbb{R}$. The matrix T describes a similitude τ with $r_\tau = -1$, and $\eta_2^o(\gamma)$ generates a complement for $\mathrm{SO}(Wz, g^o)$ in $\mathrm{O}(Wz, g^o)$.

4. The smallest case

It may come as a surprise that the case $\ell = 1$ (and $n = 2$) is by no means trivial, but leads to interesting isomorphisms of groups. First of all, we note that $\bigwedge^1 V$ is naturally identified with V itself, see [7, II § 7.1], and $\bigwedge^1 h$ coincides with h.

Choose an orthogonal basis v_1, v_2 with respect to h in V, abbreviate $c_1 := h(v_1, v_1)$ and $c_2 := h(v_2, v_2)$, and define $b \colon \bigwedge^2 V \to \boldsymbol{F}$ by $b(v_1 \wedge v_2) = 1$. Then 1.5 specializes to

$$J(v_1) = v_2 \, \frac{-h(v_1, v_1)}{b(v_1 \wedge v_2)} = -v_2 \, c_1, \quad \text{and} \quad J(v_2) = v_1 \, \frac{h(v_2, v_2)}{b(v_1 \wedge v_2)} = v_1 \, c_2.$$

Thus $J(v_1 x_1 + v_2 x_2) = v_1 c_2 \overline{x_2} - v_2 c_1 \overline{x_1}$; in coordinates with respect to the basis v_1, v_2, we have

$$h \colon \boldsymbol{F}^2 \times \boldsymbol{F}^2 \to \boldsymbol{F} \colon \left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \right) \mapsto (\overline{x_1}, \overline{x_2}) \begin{pmatrix} c_1 & 0 \\ 0 & c_2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = c_1 \, \overline{x_1} \, y_1 + c_2 \, \overline{x_2} \, y_2$$

and
$$J \colon \boldsymbol{F}^2 \to \boldsymbol{F}^2 \colon \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} 0 & c_2 \\ -c_1 & 0 \end{pmatrix} \begin{pmatrix} \overline{x_1} \\ \overline{x_2} \end{pmatrix} = \begin{pmatrix} c_2 \overline{x_2} \\ -c_1 \overline{x_1} \end{pmatrix}.$$

Identifying $x_1 + \boldsymbol{j} x_2 \in \boldsymbol{K}_1$ with $v_1(x_1 + \boldsymbol{j} x_2) = v_1 x_1 - v_2 c_1 x_2 \in V$ and using $\bigwedge^1 h = h$ we compute $g(X, Y) = \overline{X} c_1 Y$, see 2.7. Straightforward verification yields

4.1 Lemma. *The centralizer of J in the ring $\boldsymbol{F}^{2\times 2}$ of $\boldsymbol{F}$-linear endomorphisms of V is $\boldsymbol{L} := \left\{ \begin{pmatrix} a & -\bar{b} c_2 \\ b c_1 & \bar{a} \end{pmatrix} \,\middle|\, a, b \in \boldsymbol{F} \right\}$. The $\boldsymbol{R}$-algebras $\boldsymbol{L}$ and $\boldsymbol{K}_1$ are isomorphic; the norm on $\boldsymbol{L}$ is given by the determinant. The group $\mathrm{SO}(V, h)$, or $\mathrm{SU}(V, h)$, respectively, coincides with $\mathbb{S}_{\boldsymbol{L}} := \boldsymbol{L} \cap \mathrm{SL}_2(\boldsymbol{F})$.* ■

Note that $\boldsymbol{K}_1$ is split precisely if $-c_1 c_2$ is a norm, see 1.10 and 2.1. In that case, we may (upon scaling of the form and one of the basis vectors) assume $c_1 = 1$ and $c_2 = -1$; then $1 + \boldsymbol{j}$ is a divisor of zero in $\boldsymbol{K}_1$.

4.2 Theorem. *Assume $\ell = 1$ and $n = 2$.*

(a) *If $\boldsymbol{K}_1$ is not split and $\sigma = \mathrm{id}$ then $\boldsymbol{L} = \boldsymbol{K}_1$. We obtain a quadratic field extension $\boldsymbol{K}_1 | \boldsymbol{F}$, and $\mathrm{SO}(V, h)$ is the norm one group $\mathbb{S}_{\boldsymbol{K}_1}$ in $\boldsymbol{K}_1$. Note that $\mathbb{S}_{\boldsymbol{K}_1} = \{1\}$ if $\boldsymbol{K}_1 | \boldsymbol{F}$ is inseparable.*

(b) *If $\boldsymbol{K}_1$ is not split and $\sigma \neq \mathrm{id}$ then $\boldsymbol{K}_1$ is a quaternion field over $\boldsymbol{R}$. We obtain $\boldsymbol{L} \cong \boldsymbol{K}_1$ and $\mathrm{SU}(V, h) = \mathbb{S}_{\boldsymbol{L}} \cong \mathbb{S}_{\boldsymbol{K}_1}$.*

Note that $\mathrm{SU}(V,h)$ *and* $\mathbb{S}_{\boldsymbol{K}_1}$ *do not coincide in the ring of* $\boldsymbol{R}$*-linear endomorphisms of* V*; the elements of these two groups may be interpreted as left and right multiplications, respectively, by elements of norm one in* $\boldsymbol{K}_1$*.*

(c) *If* $\boldsymbol{K}_1$ *is split,* $\operatorname{char}\boldsymbol{F} \neq 2$*, and* $\sigma = \mathrm{id}$ *then* $\boldsymbol{L} = \boldsymbol{K}_1 \cong \boldsymbol{F}\times\boldsymbol{F}$*. Thus* $\mathrm{SO}(V,h) = \mathbb{S}_{\boldsymbol{L}} \cong \mathbb{S}_{\boldsymbol{F}\times\boldsymbol{F}} = \left\{(x,x^{-1}) \;\middle|\; x\in\boldsymbol{F}^\times\right\} \cong \boldsymbol{F}^\times$*.*

(d) *If* $\boldsymbol{K}_1$ *is split,* $\operatorname{char}\boldsymbol{F} = 2$*, and* $\sigma = \mathrm{id}$ *then* $\boldsymbol{L} = \boldsymbol{K}_1 \cong \boldsymbol{F}[X]/(X^2)$*. Thus* $\mathrm{SO}(V,h) = \mathbb{S}_{\boldsymbol{K}_1} \cong \mathbb{S}_{\boldsymbol{F}[X]/(X^2)} = \left\{1+bX+(X^2) \;\middle|\; b\in\boldsymbol{F}\right\}$ *is isomorphic to the* additive *group of* $\boldsymbol{F}$*.*

(e) *If* $\boldsymbol{K}_1$ *is split and* $\sigma\neq\mathrm{id}$ *then we have* $\boldsymbol{K}_1 \cong \boldsymbol{R}^{2\times 2}$*. Thus we can conclude* $\mathrm{SU}(V,h) = \mathbb{S}_{\boldsymbol{L}} \cong \mathbb{S}_{\boldsymbol{R}^{2\times2}} = \mathrm{SL}_2(\boldsymbol{R})$ *in this case.* ■

4.3 Examples: Symmetric bilinear forms on $\mathbb{R}^2$ and on $\mathbb{C}^2$. We consider $\boldsymbol{F} = \mathbb{R}$ and $\sigma = \mathrm{id}$. Essentially (i.e., up to similarity), there are two non-degenerate symmetric bilinear forms on $\mathbb{R}^2$: An anisotropic form h_+, and a hyperbolic form h_-.

The discriminant of h_+ is 1, so $\delta_1 = -1$ and $\boldsymbol{j} = \left(\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right)$ and $\boldsymbol{K}_1 \cong \mathbb{C}$ (which comes as no surprise). Note that $\boldsymbol{L} = \boldsymbol{K}_1$. We obtain

$$\mathrm{SO}(\mathbb{R}^2,h_+) = \mathbb{S}_{\boldsymbol{L}} \cong \mathbb{S}_{\mathbb{C}} = \left\{c\in\mathbb{C} \;\middle|\; c\,\bar{c} = 1\right\}.$$

In particular, this group is homeomorphic to a circle. As h_+ is (positive or negative) definite, every similitude of h_+ has positive multiplier. Thus the multiplier is a square, and $\mathrm{GO}(\mathbb{R}^2,h_+) = \mathbb{R}^\times\mathrm{O}(\mathbb{R}^2,h_+)$. Via conjugation, the scaling factors act trivially on $M_\ell \cong \boldsymbol{K}_1 \cong \mathbb{C}$ (see 2.4). Each $\gamma\in\mathrm{O}(\mathbb{R}^2,h_+)\smallsetminus\mathrm{SO}(\mathbb{R}^2,h_+)$ has determinant 1, and induces complex conjugation on $\mathbb{C}$.

The discriminant of h_- is -1, we obtain $\delta_1 = 1$, $\boldsymbol{j} = \left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$ and

$$\boldsymbol{K}_1 = \left\{\begin{pmatrix}a&b\\b&a\end{pmatrix} \;\middle|\; a,b\in\mathbb{R}\right\} \cong \mathbb{R}\times\mathbb{R}$$

in this case; an explicit isomorphism from $\mathbb{R}\times\mathbb{R}$ onto $\boldsymbol{K}_1$ is given by

$$(x,y)\mapsto \frac{1}{2}\begin{pmatrix}x&x\\x&x\end{pmatrix} + \frac{1}{2}\begin{pmatrix}y&-y\\-y&y\end{pmatrix}.$$

Note that $\boldsymbol{L} = \boldsymbol{K}_1$, again. The isomorphism just stated maps $\mathrm{SO}(\mathbb{R}^2,h_-) = \mathbb{S}_{\boldsymbol{L}}$ onto $\left\{(a,a^{-1}) \;\middle|\; a\in\mathbb{R}^\times\right\}\subset\mathbb{R}\times\mathbb{R}$.

We may assume that the Gram matrix (with respect to the standard basis) is $H = \left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$. Then $v_1 := \left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)$, $v_2 := \left(\begin{smallmatrix}1\\-1\end{smallmatrix}\right)$ is an orthogonal basis; we have $h(v_1,v_1) = 2$ and $h(v_2,v_2) = -2$. Taking $b\colon \bigwedge^2\mathbb{R}^2\to\mathbb{R}$ with $b(v_1\wedge v_2) = -2$, we obtain $J(v_1) = v_2$ and $J(v_2) = v_1$. Thus the matrix $\left(\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right)$ describes J with respect to the standard basis.

In standard coordinates, we now obtain $\mathrm{SO}(\mathbb{R}^2,h_-) = \left\{\left(\begin{smallmatrix}a&0\\0&a^{-1}\end{smallmatrix}\right) \;\middle|\; a\in\mathbb{R}^\times\right\}$. The orthogonal group is the union of $\mathrm{SO}(\mathbb{R}^2,h_-)$ and the coset $\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)\mathrm{SO}(\mathbb{R}^2,h_-)$. Note that conjugation by $\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$ maps J to $-J$. (Using the isomorphism above, this corresponds to swapping $(1,-1)$ with $(-1,1)$ in $\mathbb{R}\times\mathbb{R}$.) For each $a\in\mathbb{R}^\times$, the matrix $S_a := \left(\begin{smallmatrix}1&0\\0&a\end{smallmatrix}\right)$ describes a similitude with multiplier a. These similitudes centralize J.

On $\mathbb{C}^2$, there is essentially only one non-degenerate symmetric bilinear form. That form is hyperbolic, the arguments for the form h_- above remain valid *verbatim.*

Actually, that reasoning holds for every hyperbolic form on a two-dimensional vector space over any field $\boldsymbol{F}$ with $\operatorname{char} \boldsymbol{F} \neq 2$.

4.4 Examples: Hermitian forms on $\mathbb{C}^2$. We consider $\boldsymbol{F} = \mathbb{C}$, and take complex conjugation for σ. Essentially, there are two non-degenerate σ-hermitian forms on $\mathbb{C}^2$: an anisotropic form h_+ and a hyperbolic form h_-.

The discriminant of h_+ is 1, so $\delta_1 = -1$ and $\boldsymbol{j} = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. The algebra $\boldsymbol{K}_1$ is not split, so $\boldsymbol{K}_1 \cong \mathbb{H}$, the skew field of Hamilton's quaternions. We obtain $\mathrm{SU}(\mathbb{C}^2, h_+) = \mathbb{S}_{\boldsymbol{L}} \cong \mathbb{S}_{\mathbb{H}}$. In particular, this group is homeomorphic to a 3-sphere. We may assume that the Gram matrix of h_+ is $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Then $J\binom{x}{y} = \binom{-\bar{y}}{\bar{x}}$, and the group $\mathrm{U}(\mathbb{C}^2, h_+)$ is the semi-direct product of $\mathrm{SU}(\mathbb{C}^2, h_+)$ and $\left\{\left(\begin{smallmatrix} 1 & 0 \\ 0 & u \end{smallmatrix}\right) \,\middle|\, u \in \mathbb{C}, u\bar{u} = 1\right\}$. In M_1, conjugation by $\left(\begin{smallmatrix} 1 & 0 \\ 0 & u \end{smallmatrix}\right)$ maps J to $u\,\mathrm{id} \circ J \colon \binom{x}{y} \mapsto \binom{-u\bar{y}}{u\bar{x}}$; see 1.8. As h_+ is a (positive or negative) definite form, each similitude has positive multiplier. Therefore, each multiplier is a norm, and $\Gamma\mathrm{U}(\mathbb{C}^2, h_+) = \mathbb{C}^\times \mathrm{SU}(\mathbb{C}^2, h_+)$. Conjugation by $c\,\mathrm{id}$ maps J to $c/\bar{c}\,J$, and induces on $M_1 \cong \mathbb{H}$ an inner automorphism (namely, conjugation by c) that fixes $\mathbb{C}\,\mathrm{id}$ pointwise and multiplies each element of $\mathbb{C}\,\mathrm{id} \circ J$ by the factor $c/\bar{c} \in \mathbb{S}_{\mathbb{C}}$. (Note that, by Hilbert's Theorem 90 (see [17, 4.31]), each element of $\mathbb{S}_{\mathbb{C}}$ is obtained in this way, as a quotient $c/\bar{c}$ for some $c \in \mathbb{C}^\times$.)

The discriminant of h_- is -1, so $\delta_1 = 1$ and $\boldsymbol{j} = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ in this case. The algebra $\boldsymbol{K}_1 = \left\{\left(\begin{smallmatrix} x & \bar{y} \\ y & \bar{x} \end{smallmatrix}\right) \,\middle|\, x, y \in \mathbb{C}\right\}$ is then isomorphic to $\mathbb{R}^{2\times 2}$; in fact, an explicit isomorphism is obtained by $\mathbb{R}$-linear extension of $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \mapsto \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} i & 0 \\ 0 & -i \end{smallmatrix}\right) \mapsto \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, $\boldsymbol{j} = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \mapsto \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, and $\left(\begin{smallmatrix} 0 & -i \\ i & 0 \end{smallmatrix}\right) \mapsto \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. We find that $\mathrm{SU}(\mathbb{C}^2, h_-)$ is isomorphic to $\mathbb{S}_{\boldsymbol{K}_1} \cong \mathbb{S}_{\mathbb{R}^{2\times 2}} = \mathrm{SL}_2(\mathbb{R})$. We may assume that the Gram matrix is $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. Then $\mathrm{U}(\mathbb{C}^2, h_-)$ is the semi-direct product of $\mathrm{SU}(\mathbb{C}^2, h_-)$ with $\left\{\left(\begin{smallmatrix} 1 & 0 \\ 0 & u \end{smallmatrix}\right) \,\middle|\, u \in \mathbb{C}, u\bar{u} = 1\right\}$. As above (in the anisotropic case), conjugation by the matrix $\left(\begin{smallmatrix} 1 & 0 \\ 0 & u \end{smallmatrix}\right)$ maps J to $u\,\mathrm{id} \circ J \colon \binom{x}{y} \mapsto \binom{-u\bar{y}}{u\bar{x}}$.

Translated into the algebra $\mathbb{R}^{2\times 2}$ (via the explicit isomorphism above) the isomorphism for $u = \cos(t) + i\sin(t)$ is obtained as conjugation by $\left(\begin{smallmatrix} \cos(t/2) & -\sin(t/2) \\ \sin(t/2) & \cos(t/2) \end{smallmatrix}\right)$.

For each similitude $\gamma \in \mathrm{GU}(\mathbb{C}^2, h_-)$, the multiplier μ_γ lies in $\mathbb{R}$ because we have $\left\{h(v,v) \,\middle|\, v \in \mathbb{C}^2\right\} \subseteq \mathbb{R}$. So every possible multiplier is obtained from some $\gamma \in S := \mathbb{C}^\times \mathrm{id} \cup \mathbb{C}^\times \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, and $\Gamma\mathrm{U}(\mathbb{C}^2, h_-)$ is the semidirect product of $\mathrm{U}(\mathbb{C}^2, h_-)$ with S. The operator $J \in M_1$ is centralized by $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, and conjugation of J by $c\,\mathrm{id}$ yields $(c/\bar{c}\,\mathrm{id}) \circ J$.

5. Norm forms of composition algebras

We concentrate on the case $\ell = 2$ and $n = 2\ell = 4$ from now on. Note that the factors $(-1)^\ell$ become irrelevant. We normalize $b(v_1 \wedge v_2 \wedge v_3 \wedge v_4) = 1$ and then simplify notation, writing

$$\boldsymbol{K} := \boldsymbol{K}_2, \quad \delta := \delta_2 = \det(H), \quad \boldsymbol{j} := \boldsymbol{j}_2 \quad \text{and} \quad \eta := \eta_2;$$

here H is the Gram matrix describing h with respect to the orthogonal basis v_1, v_2, v_3, v_4, and $\eta = \eta_2$ is the homomorphism constructed in Section 2.

In this section we study the split case; our aim is to understand $\eta^o(\mathrm{SU}(V,h))$ where η^o is as in 3.7. Recall that $\boldsymbol{K}$ splits precisely if $\delta = \det(H)$ is a norm, i.e., if the form h has discriminant $\operatorname{disc} h = 1 \in \boldsymbol{F}^\times / N_{\boldsymbol{F}|\boldsymbol{R}}(\boldsymbol{F}^\times)$. This excludes one of the three possible values for the Witt index; in fact, Witt index 1 is impossible in the split case:

5.1 Lemma. *If* $\operatorname{disc} h = 1$ *and* h *has positive Witt index then* h *has Witt index* 2.

Proof. The Witt index is at most 2 because the form is not degenerate.

Assume first that $\operatorname{char} \boldsymbol{F} \neq 2$ or that $\sigma \neq \mathrm{id}$; then h is trace-valued. If the Witt index is positive then there exists a hyperbolic pair, i.e. $w_1, w_2 \in V$ with $h(w_1, w_1) = 0 = h(w_2, w_2)$ and $h(w_1, w_2) = 1$. We pick an orthogonal basis w_3, w_4 for $\{w_1, w_2\}^\perp$ and obtain $-h(w_3, w_3)h(w_4, w_4) \in \operatorname{disc} h = N_{\boldsymbol{F}|\boldsymbol{R}}(\boldsymbol{F}^\times)$. Thus there exists $c \in \boldsymbol{F}$ with $-h(w_3, w_3)h(w_4, w_4) = \bar{c}c$, the vector $w_3\, c + w_4\, h(w_3, w_3) \in \{w_1, w_2\}^\perp$ is isotropic, and the Witt index is at least 2.

There remains the case $\operatorname{char} \boldsymbol{F} = 2$ and $\sigma = \mathrm{id}$. We assume that there exists an isotropic vector $w_1 \in V \setminus \{0\}$. Then there exists w_2 with $h(w_1, w_2) = 1$ but it will in general not be possible to achieve $h(w_2, w_2) = 0$. Now $\{w_1, w_2\}^\perp$ is a vector space complement to $w_1\boldsymbol{F} + w_2\boldsymbol{F}$. As the Witt index is at most 2, that complement contains w_3 with $h(w_3, w_3) \neq 0$. Thus we find an orthogonal basis w_3, w_4 for $\{w_1, w_2\}^\perp$ and an isotropic vector in $\{w_1, w_2\}^\perp$ as before. ∎

5.2 Remarks. As a partial converse to 5.1 we note that any two non-degenerate σ-hermitian forms of Witt index 2 on $\boldsymbol{F}^4$ are isometric; the discriminant is 1, and the corresponding Hodge operator yields a split algebra $\boldsymbol{K}$.

Note that over any commutative field $\boldsymbol{F}$ with $\operatorname{char} \boldsymbol{F} \neq 2$, the norm forms of quaternion algebras are characterized (up to similitude) among the quadratic forms in four variables by the fact that they have discriminant 1; see [20, 3.1.2] for an explicit proof of this known result. So 5.3 below applies if the role of h is played by the polarization of such a norm form.

5.3 A quaternion algebra. *Assume* $\operatorname{char} \boldsymbol{F} \neq 2$ *and* $\sigma = \mathrm{id}$. *If* $\boldsymbol{K}$ *splits then* h *is similar to the polarization* f_N *of the norm* N *of some quaternion algebra* $\boldsymbol{H}$ *over* $\boldsymbol{F}$.

Moreover, the form g^o *on* Wp *is similar to the restriction of* f_N *to the space* $\mathrm{Pu}(\boldsymbol{H}) := \{x \in \boldsymbol{H} \setminus \boldsymbol{F} \mid x^2 \in \boldsymbol{F}\} \cup \{0\}$ *of pure quaternions.*

The group of similitudes of f_N *is well understood, it is generated by the standard involution together with left and right multiplications by invertible elements of the quaternion algebra*[3]. *The representation on* $W = \bigwedge^2 V$ *is the sum of two representations that are quasi-equivalent to that on the space of pure quaternions, cf.* [12, Sect. 6]. *In particular, we have* $\eta_2^o(\mathrm{GO}(V,h)) = \boldsymbol{F}^\times \mathrm{SO}(\mathrm{Pu}(\boldsymbol{H}), f_N|_{\mathrm{Pu}(\boldsymbol{H})})$ *and* $\eta_2^o(\mathrm{SO}(V,h)) = \mathrm{SO}(\mathrm{Pu}(\boldsymbol{H}), f_N|_{\mathrm{Pu}(\boldsymbol{H})})$.

Proof. The case where h has Witt index 2 is covered by the observation that the norm form of the split quaternion algebra over $\boldsymbol{F}$ is the unique non-degenerate quadratic form of Witt index 2 in 4 variables. If h is anisotropic then a suitable scalar multiple of h is the polarization of the norm of a quaternion field because $\operatorname{disc} h = 1$; see [20, 3.1.2].

[3] See [21, V (4.2.4), p. 266] for the general result, or [3, 5.2] for quaternion fields, or [34, 4.5.17] for arbitrary quaternion algebras with $\operatorname{char} \boldsymbol{F} \neq 2$

In any case, we may (upon scaling) assume that $2h$ is the polarization of the norm N of $\boldsymbol{H}$. There exists an orthogonal basis $v_1 := 1, v_2, v_3, v_4$ such that $v_4 = v_2 v_3$ and that v_2, v_3, v_4 span $\mathrm{Pu}(\boldsymbol{H})$. Thus $h(v_4, v_4) = N(v_4) = N(v_2)\,N(v_3) = h(v_2, v_2)\,h(v_3, v_3)$. Now the sequence $X_1 := (v_1 \wedge v_2)z$, $X_2 := (v_1 \wedge v_3)z$, $X_3 := (v_1 \wedge v_4)z$ forms an orthogonal basis for g^o on Wz such that $g^o(X_1, X_1) = 2\,h(v_2, v_2)$, $g^o(X_2, X_2) = 2\,h(v_3, v_3)$, and $g^o(X_3, X_3) = 2\,h(v_2, v_2)\,h(v_3, v_3)$. Thus g^o is isometric to the restriction of $2h$ to $\mathrm{Pu}(\boldsymbol{H})$. ∎

5.4 Remark. We have excluded the characteristic 2 case in 5.3. Indeed, the situation becomes more complicated, see [22].

5.5 An octonion algebra. We consider $\sigma \neq \mathrm{id}$ and $\mathrm{char}\,\boldsymbol{F} \neq 2$ and assume that $\boldsymbol{K}$ splits; so $\boldsymbol{K} \cong \boldsymbol{R}^{2\times 2}$. Replacing h by a suitable scalar multiple, we may assume that there is an orthogonal basis v_1, v_2, v_3, v_4 such that $h(v_1, v_1) = 1$ and $h(v_4, v_4) = h(v_2, v_2)\,h(v_3, v_3)$; this last condition can be satisfied because h has discriminant 1. Then we may regard the $\boldsymbol{R}$-linear span $\boldsymbol{H} := v_1\boldsymbol{R} \oplus v_2\boldsymbol{R} \oplus v_3\boldsymbol{R} \oplus v_4\boldsymbol{R}$ as a quaternion algebra with norm $N_{\boldsymbol{H}}(x) = h(x, x)$. Choosing any $q \in \boldsymbol{F}^\times$ with $\sigma(q) = -q$ we obtain $V = \boldsymbol{H} \oplus \boldsymbol{H}q$ and that the quadratic form $N(v) := h(v, v)$ is the norm of the octonion algebra $\boldsymbol{C}$ obtained as the q^2-double of $\boldsymbol{H}$; cf. [28, 1.5.3] or [17, p. 444 f]. Finally, we identify $\boldsymbol{F}$ with the subalgebra $v_1\boldsymbol{F} = v_1\boldsymbol{R} \oplus v_1 q\boldsymbol{R}$ of $\boldsymbol{C}$ and note $(v_1\boldsymbol{F})^\perp = v_2\boldsymbol{F} \oplus v_3\boldsymbol{F} \oplus v_4\boldsymbol{F}$.

5.6 Examples. The quaternion algebra $\boldsymbol{H}$ in 5.5 need not be isomorphic to the split quaternion algebra $\boldsymbol{K}$. In general, the norm form of *every* quaternion field B over $\boldsymbol{R}$ has discriminant 1, and extending that form to a hermitian form on the tensor product $V := \boldsymbol{F} \otimes_{\boldsymbol{R}} B$ we obtain a case where $\boldsymbol{H} \cong B$ is non-split, and not isomorphic to $\boldsymbol{K}$.

For instance, consider $\boldsymbol{F} = \mathbb{C}$ and $\boldsymbol{R} = \mathbb{R}$. The (essentially unique) positive definite hermitian form on $\mathbb{C}^4$ has discriminant 1; then the restriction of the form h to $\boldsymbol{H}$ is positive definite, and the quaternion algebra $\boldsymbol{H}$ is isomorphic to the quaternion field $\mathbb{H}$ (and is, in particular, not split in this case). The hermitian form of Witt index 2 also has discriminant 1, but the quaternion algebra $\boldsymbol{H}$ is split in that case (and then isomorphic to $\mathbb{R}^{2\times 2} \cong \boldsymbol{K}$).

5.7 Theorem. *Assume $\sigma \neq \mathrm{id}$, $\mathrm{char}\,\boldsymbol{F} \neq 2$ and that $\boldsymbol{K}$ splits. Let f_N denote the polarization of the norm of the octonion algebra $\boldsymbol{C}$ introduced in* 5.5. *Then the form g^o on $Wz = Wp$ is equivalent to the restriction of f_N to $\boldsymbol{F}^\perp$ in that octonion algebra. So η_2^o may be interpreted as a homomorphism $\eta_2^o \colon \mathrm{SU}(V, h) \to \mathrm{O}(\boldsymbol{F}^\perp, N|_{\boldsymbol{F}^\perp})$.*

Proof. Let $c_k := h(v_k, v_k)$. We normalize b such that $b(v_1 \wedge v_2 \wedge v_3 \wedge v_4) = 1$. Straightforward computation yields the values of g^o on the basis

$$\begin{aligned} X_1 &:= (v_1 \wedge v_2)z, & X_2 &:= (v_1 \wedge v_3)z, & X_3 &:= (v_1 \wedge v_4)z, \\ X_4 &:= (v_1 q \wedge v_2)z, & X_5 &:= (v_1 q \wedge v_3)z, & X_6 &:= (v_1 q \wedge v_4)z; \end{aligned}$$

it turns out that this is an orthogonal basis with

$$\begin{aligned} g^o(X_1, X_1) &= 2c_2, & g^o(X_2, X_2) &= 2c_3, & g^o(X_3, X_3) &= 2c_4, \\ g^o(X_4, X_4) &= -2q^2 c_2, & g^o(X_5, X_5) &= -2q^2 c_3, & g^o(X_6, X_6) &= -2q^2 c_4. \end{aligned}$$

So g^o is equivalent to the polarization of $N|_{\boldsymbol{F}^\perp}$. ∎

5.8 Remark. Depending on the form h (and not only on the ground field $\boldsymbol{F}$), the image $\eta_2^o(\mathrm{SU}(V,h))$ may coincide with $\mathrm{SO}(Wp, g^o) \cong \mathrm{SO}(\boldsymbol{F}^\perp, f_{N|_{\boldsymbol{F}^\perp}})$, or may form a *proper* subgroup. See 5.9, 5.10, and 5.14 below for examples.

As $\mathrm{SO}(Wp, g^o)$ is generated by its half-turns (cf. 8.2 below) this means that in some cases elements of $\mathrm{SU}(V,h)$ will induce all these half-turns, while in some other cases they will not.

5.9 Example: Compact groups. On $\mathbb{C}^4$ the standard (positive definite) hermitian form h_0 defines the compact real form $\mathrm{SU}_4(\mathbb{C})$ of the simply connected complex Lie group of type A_3. The discriminant of h_0 is 1, the quadratic form $h_0(v,v)$ is the norm of *the* quaternion field $\mathbb{H}$ (over the reals), and $\boldsymbol{K}$ splits. The octonion algebra in 5.5 is the octonion field $\mathbb{O}$, and the form $N|_{\mathbb{C}^\perp}$ is a positive definite quadratic form on $\mathbb{R}^6$ defining the compact form $\mathrm{SO}_6(\mathbb{R})$ of the complex Lie group of type D_3. Of course, types A_3 and D_3 are identical because the Coxeter diagrams are the same. The homomorphism η_2^o induces a two-sheeted covering from $\mathrm{SU}_4(\mathbb{C})$ onto $\mathrm{SO}_6(\mathbb{R})$.

5.10 Example: Non-compact groups. Let h be a hermitian form of Witt index 2 on $\mathbb{C}^4$. Then $\operatorname{disc} h = 1$ and $\boldsymbol{K}$ is split. The quadratic form $N(v) := h(v,v)$ is the norm form of the split quaternion algebra $\mathbb{R}^{2\times 2}$, and the octonion algebra constructed in 5.5 is split, as well. The norm form of that octonion algebra has the (maximal possible) Witt index 4 but its restriction to $\mathbb{C}^\perp$ only has index 2. In fact, a 3-dimensional positive definite subspace in $\mathbb{C}^\perp$ would sum up with $\mathbb{C}$ to form a 5-dimensional positive definite subspace in the split octonion algebra, which is impossible. Thus η_2^o yields a homomorphism from $\mathrm{SU}_4(\mathbb{C},2)$ to $\mathrm{SO}_6(\mathbb{R},2)$. This homomorphism is not surjective; in fact $\mathrm{SU}_4(\mathbb{C},2)$ and its image are connected but $\mathrm{SO}_6(\mathbb{R},2)$ is disconnected (see [15] or [25]; in fact, the commutator group of $\mathrm{SO}_6(\mathbb{R},2)$ coincides with the kernel of the spinor norm, and has index 2 in $\mathrm{SO}_6(\mathbb{R},2)$ because there are two square classes; see [9, §8, p. 54], [11, Th. 9.7, p. 77]). The tangent object of that morphism of Lie groups is the isomorphism between the real forms of types $\mathsf{A}_3^{\mathbb{R},2}$ and $\mathsf{D}_3^{\mathbb{R},2}$, respectively.

5.11 An octonion algebra in characteristic two. We consider $\sigma \neq \mathrm{id}$ again, but assume $\operatorname{char} \boldsymbol{F} = 2$ now. The quadratic form $N(v) := h(v,v)$ is not degenerate; in fact, its polarization $f_N(v,w) = h(v,w) + h(w,v) = h(v,w) + \overline{h(v,w)}$ is not degenerate (cf. [30, 4.3]).

Pick $u \in F$ with $u + \bar{u} = 1$, choose an orthogonal basis $v_1, \dots, v_4$ in V, and put $c_k := N(v_k)$ for $1 \le k \le 4$. Passing to a suitable scalar multiple of h, we may assume $c_1 = 1$. As h has discriminant 1, we may then further assume $c_4 = c_2 c_3$.

Let $\boldsymbol{C}_1$ be the $\boldsymbol{R}$-linear span of v_1 and $v_1 u + v_2$. Then the restriction $N|_{\boldsymbol{C}_1}$ is equivalent to the norm form of the two-dimensional composition algebra $\boldsymbol{R}[X]/(X^2 - X + r)$ with $r := \bar{u}u + c_2$. The spaces

$$\boldsymbol{C}_2 := v_2\boldsymbol{R} + (v_1 c_2 + v_2 u)\boldsymbol{R},\ \boldsymbol{C}_3 := v_3\boldsymbol{R} + (v_3 u + v_4)\boldsymbol{R},\ \boldsymbol{C}_4 := v_4\boldsymbol{R} + (v_3 c_2 + v_4 u)\boldsymbol{R}$$

yield an orthogonal decomposition $V = \boldsymbol{C}_1 \oplus \boldsymbol{C}_2 \oplus \boldsymbol{C}_3 \oplus \boldsymbol{C}_4$. For each $k \le 4$, the restriction $N|_{\boldsymbol{C}_k}$ is equivalent to $c_k N|_{\boldsymbol{C}_1}$. In particular, the restriction $N|_{\boldsymbol{C}_1 + \boldsymbol{C}_2}$ is equivalent to the norm of the quaternion algebra $\boldsymbol{H}$ obtained as the c_2-double of $\boldsymbol{R}[X]/(X^2 - X + r)$, and N itself is equivalent to the norm of the octonion algebra $\boldsymbol{C}$ obtained as the c_3-double of $\boldsymbol{H}$.

5.12 Remark. Clearly, the restriction $N|_{v_1\boldsymbol{F}}$ of the norm in $\boldsymbol{C}$ to the subspace $v_1\boldsymbol{F}$ is isometric to the norm form $N_{\boldsymbol{F}|\boldsymbol{R}}$. Indeed, one may choose the multiplication (of $\boldsymbol{C}$) on V in such a way that $v_1\boldsymbol{F}$ forms a subalgebra isomorphic to $\boldsymbol{F}$. Note that the restriction $N|_{\boldsymbol{C}_1}$ need not be isometric to the norm form $N_{\boldsymbol{F}|\boldsymbol{R}}$. In fact, the form $N|_{\boldsymbol{C}_1}$ may be isotropic; for instance, this happens if $c_2 = \bar{u}u$. ■

We now use the quadratic form Pq that gives rise to the Klein quadric, see 1.3: Up to a scalar, we have $\mathrm{Pq}(X)^2 = \det X$ for each $X \in \bigwedge^2 \boldsymbol{F}^4$, and this determines Pq (again, up to a scalar) because square roots are unique (if existent) in characteristic 2.

5.13 Theorem. *Assume $\sigma \neq \mathrm{id}$, $\mathrm{char}\,\boldsymbol{F} = 2$ and that $\boldsymbol{K}$ splits. Let f_N denote the polarization of the norm N of the octonion algebra $\boldsymbol{C}$ introduced in* 5.11*. Then the restriction of the quadratic form* Pq *to Wz is similar to the restriction of the norm N to $\boldsymbol{F}^\perp$ in that octonion algebra. In particular, the form g^o on Wz is a non-trivial scalar multiple of the restriction of f_N to $\boldsymbol{F}^\perp$.*
Thus η_2^o may be interpreted as a homomorphism $\eta_2^o\colon \mathrm{SU}(V,h) \to \mathrm{O}(\boldsymbol{F}^\perp, N|_{\boldsymbol{F}^\perp})$.

Proof. Notation and normalizations are as in 5.11. For Wz, we use the $\boldsymbol{R}$-basis

$$(v_1 \wedge v_2)z, \quad (v_1 \wedge v_3)z, \quad (v_1 \wedge v_4)z, \quad (v_1u \wedge v_2)z, \quad (v_1u \wedge v_3)z, \quad (v_1u \wedge v_4)z,$$

and claim that $\boldsymbol{R}$-linear extension of $\psi((v_1 \wedge v_k)z) := v_k$ and $\psi((v_1u \wedge v_k)z) := v_ku$ (for $k \in \{2,3,4\}$, respectively) gives an isometry from $\mathrm{Pq}|_{Wz}$ to $N|_{\boldsymbol{F}^\perp}$.

The values of Pq are $\mathrm{Pq}((v_1 \wedge v_2)z) = c_2$, $\mathrm{Pq}((v_1 \wedge v_3)z) = c_3$, $\mathrm{Pq}((v_1 \wedge v_4)z) = c_2c_3$, $\mathrm{Pq}((v_1u \wedge v_2)z) = \bar{u}uc_2$, $\mathrm{Pq}((v_1u \wedge v_3)z) = \bar{u}uc_3$, and $\mathrm{Pq}((v_1u \wedge v_4)z) = \bar{u}uc_2c_3$. Computing $f_{\mathrm{Pq}}(Xz, Yz) = \mathrm{Pq}(Xz + Yz) - \mathrm{Pq}(Xz) - \mathrm{Pq}(Yz)$ for Xz, Yz in the basis we see that ψ is an isometry, as claimed.

Using 3.7 we compute $g^o(Xz, Yz) = \bigwedge^2 h(X,Y) + \overline{\bigwedge^2 h(X,Y)} = f_{\mathrm{Pq}}(Xz, Yz)$ for arbitrary members X, Y in the basis. ■

5.14 Example: Finite groups. Let $\boldsymbol{F}$ be a finite field of square order e^2. Then $\boldsymbol{F} \cong \mathbb{F}_{e^2}$, there is a unique involution (namely, $\sigma\colon x \mapsto x^e$) in $\mathrm{Aut}(\boldsymbol{F})$, and the algebra $\boldsymbol{K}_2$ is split (as is every finite quaternion algebra).

The (essentially unique) non-degenerate σ-hermitian form $h\colon \boldsymbol{F}^4 \times \boldsymbol{F}^4 \to \boldsymbol{F}$ has Witt index 2, and the special unitary group $\mathrm{SU}(4,e^2) := \mathrm{SU}(\boldsymbol{F}^4, h)$ has order $|\mathrm{SU}(4,e^2)| = e^6\,(e^2-1)(e^3+1)(e^4-1)$.

The norm form on the (necessarily split) octonion algebra $\boldsymbol{C}$ over $\boldsymbol{R}$ has maximal Witt index. Constructing that algebra by repeated doubling, starting with the separable extension $\boldsymbol{F}|\boldsymbol{R}$, we find that the norm form on $\boldsymbol{F}^\perp \leq \boldsymbol{C}$ is a quadratic form of Witt index 2. The corresponding orthogonal group is usually denoted by $\mathrm{O}^-(6,e)$; it has order $|\mathrm{O}^-(6,e)| = 2\,e^6\,(e^2-1)(e^3+1)(e^4-1)$, cf. [11, 9.11, 14.48].

As $\mathrm{SU}(4,e^2)$ is a perfect group (cf. [11, 11.22]), the image $\eta_2^o(\mathrm{SU}(4,e^2))$ is contained in the commutator group $\Omega^-(6,e)$ of $\mathrm{O}^-(6,e)$. The order of $\Omega^-(6,e)$ is $\frac{1}{4}|\mathrm{O}^-(6,e)|$ if e is odd, and it is $\frac{1}{2}|\mathrm{O}^-(6,e)|$ if e is even, cf. [11, 6.28, 9.7, 9.11, 14.49]. In any case, the map η_2^o found in 5.7 and 5.13 is a homomorphism $\eta_2^o\colon \mathrm{SU}(4,e^2) \to \Omega^-(6,e)$.

(a) If e is odd then the kernel of η_2^o has order 2, see 2.12. So η_2^o yields an isomorphism from $\mathrm{SU}(4,e^2)/\langle -\mathrm{id}\rangle$ onto $\Omega^-(6,e)$.

(b) If e is even then η_2^o is injective by 2.12, and η_2^o yields an isomorphism from $\mathrm{SU}(4,e^2)$ onto $\Omega^-(6,e)$.

See also Taylor [31, p. 198] for a proof of $\mathrm{SU}(4,e^2) \cong \Omega^-(6,e)$ for even e.

6. Geometry

We continue to assume $\ell = 2$ and $n = 2\ell = 4$. After Section 5 it remains to consider the case where $\delta = \delta_2$ is not a norm; so $\boldsymbol{K} = \boldsymbol{K}_2$ is a division algebra.

The Klein quadric $\{X\boldsymbol{F} \mid X \in \bigwedge^2 V \setminus \{0\}, \mathrm{Pq}(X) = 0\}$ provides a model for the space of lines in the projective space $\mathrm{P}(V)$ (see [29, Sect. 3] or [31, Ch. 12]); one maps $u\boldsymbol{F} \oplus v\boldsymbol{F}$ to $(u \wedge v)\boldsymbol{F}$. Recall that points on that quadric represent confluent lines if, and only if, they are orthogonal with respect to Pf, the polarization of Pq. There are two types of maximal totally isotropic subspaces, corresponding to line pencils (i.e., points) and line spaces of planes in $\mathrm{P}(V)$, respectively. Every semi-similitude of Pf thus induces either a collineation or a duality of $\mathrm{P}(V)$.

6.1 Lemma. *Consider a vector space U of dimension $d \geq 2$ over $\boldsymbol{F}$, and let $f \colon U \times U \to \boldsymbol{F}$ be a non-degenerate diagonalizable σ-hermitian form. Two collineations from $\mathrm{P}(U)$ onto some projective space $\mathcal{P}$ are equal if they agree on the complement $\mathcal{R}_U$ of the set $\{v\boldsymbol{F} \mid v \in U, f(v,v) = 0\}$.*

Proof. Let $v_1, \dots, v_d$ be an orthogonal basis with respect to the form f.

Assume first that $\boldsymbol{F}$ has more than two elements. Each line meeting $\mathcal{R}_U$ then contains at least 2 points of $\mathcal{R}_U$. For any point $v\boldsymbol{F} \in \mathrm{P}(U) \setminus \mathcal{R}_U$ there are at least two lines joining $v\boldsymbol{F}$ with points in $\{v_1\boldsymbol{F}, \dots, v_d\boldsymbol{F}\}$. As each of these lines contains at least two points of $\mathcal{R}_U$, the images of $v\boldsymbol{F}$ under the considered collineations are uniquely determined by the images of points in $\mathcal{R}_U$.

It remains to study the case where $\boldsymbol{F} = \mathbb{F}_2$. Then $\sigma = \mathrm{id}$, and $f(x,y) = \sum_{k=1}^d x_k y_k$ for $x = \sum_{k=1}^d v_k x_k$ and $y = \sum_{k=1}^d v_k y_k$. Therefore, we have

$$x\mathbb{F}_2 \in \mathcal{R}_U \iff \sum_{k=1}^{d} x_k^2 \neq 0 \iff \sum_{k=1}^{d} x_k \neq 0 .$$

This means that $\mathrm{P}(U) \setminus \mathcal{R}_U$ is a hyperplane in $\mathrm{P}(U)$, and the assertion of the lemma follows from the fact that projective collineations are determined by restrictions to affine spaces (obtained by removing hyperplanes). ∎

6.2 Theorem. *The Hodge operator induces the polarity $\pi_h \colon U \leftrightarrow U^{\perp_h}$ on $\mathrm{P}(V)$.*

Proof. From 1.11 we know that J is a semi-similitude of Pf. Like any semi-similitude of the Pfaffian form Pf, the map J induces either a collineation or a duality of the projective space $\mathrm{P}(V)$. Let v_1, v_2, v_3, v_4 be an orthogonal basis with respect to h. The $\boldsymbol{F}$-subspaces M and $J(M)$ spanned by $\{v_1 \wedge v_2,\, v_1 \wedge v_3,\, v_1 \wedge v_4\}$ and by $\{v_3 \wedge v_4,\, v_2 \wedge v_4,\, v_2 \wedge v_3\}$, respectively, are complementary maximally Pq-isotropic subspaces of $\bigwedge^2 V$. One of these corresponds to a point, the other to a plane in $\mathrm{P}(V)$. (Cf. also [33, 7.2].) Since J interchanges the two it induces a duality (and not a collineation), which is a polarity because $J^2 \in \boldsymbol{F}\,\mathrm{id}$.

Recall from 1.5 that $J(v_k\wedge v_m)$ corresponds to the line $\{v_k,v_m\}^\perp=\pi_h(v_k\boldsymbol{F}+v_m\boldsymbol{F})$.

We will apply 6.1 now, with restrictions of h to various subspaces U playing the role of f. Assume first that either $\operatorname{char}\boldsymbol{F}\neq 2$ or $\sigma\neq\mathrm{id}$. Put $U=v_k^\perp$ for some $k\le 4$. For each point $v\boldsymbol{F}\in\mathrm{P}(U)\smallsetminus\mathcal{R}_U$, we then know that it is possible to extend the orthogonal system v_k,v_m to an orthogonal basis containing v and see that J and π_h agree on the line corresponding to $(v_k\wedge v)\boldsymbol{F}$. From 6.1 we now infer that J and π_h agree on the points of the projective plane $\mathrm{P}(U)=\mathrm{P}(v_k^\perp)$. As this is true for each k, the assertion of the theorem follows.

Now consider the case where $\operatorname{char}\boldsymbol{F}=2$ and $\sigma=\mathrm{id}$. Fix $k,m\le 4$ and choose k',m' such that $\{k,m,k',m'\}=\{1,2,3,4\}$. For any $x=v_kx_k+v_mx_m\in\{v_{k'},v_{m'}\}^\perp$ we find that $y:=v_kx_mh(v_m,v_m)-v_mx_kh(v_k,v_k)$ satisfies $h(x,y)=0$ and $h(y,y)=h(v_k,v_k)h(v_m,v_m)h(x,x)\in h(x,x)\boldsymbol{F}^\times$. So $v_{k'},x,y,v_{m'}$ forms an orthogonal basis whenever $h(x,x)\neq 0$. This means that J maps the line $v_{k'}\boldsymbol{F}+x\boldsymbol{F}$ to the line $y\boldsymbol{F}+v_{m'}\boldsymbol{F}=\pi_h(v_{k'}\boldsymbol{F}+x\boldsymbol{F})$, and J coincides with π_h on $\mathrm{P}(v_k\boldsymbol{F}+v_m\boldsymbol{F})\smallsetminus\mathcal{R}_{v_k\boldsymbol{F}+v_m\boldsymbol{F}}$. The solutions $x\in v_k\boldsymbol{F}+v_m\boldsymbol{F}$ for $h(x,x)=0$ form a subspace of dimension at most one (here we use $\operatorname{char}\boldsymbol{F}=2$ and $\sigma=\mathrm{id}$, and also that the restriction of h to the line is diagonalizable). Thus $\mathrm{P}(v_k\boldsymbol{F}+v_m\boldsymbol{F})\cap\mathcal{R}_{v_k\boldsymbol{F}+v_m\boldsymbol{F}}$ contains at most one point, and J coincides with π_h on $\mathrm{P}(v_k\boldsymbol{F}+v_m\boldsymbol{F})$. Now the two polarities coincide on each point row of the six lines of form $v_k\boldsymbol{F}+v_m\boldsymbol{F}$, and the assertion of the theorem follows also in this case. ∎

The *absolute points* of the polarity π_h are the one-dimensional subspaces of V that are isotropic with respect to h. We write Abs_h for the set of absolute points. Analogously, we write Abs_g for the set of absolute points of the polarity $\pi_g\colon C\mapsto X^{\perp_g}$ of the projective plane $\mathrm{P}_{\boldsymbol{K}}(\bigwedge^2V)$.

6.3 Proposition. *The map* $\lambda\colon \mathrm{Gr}_{2,\boldsymbol{F}}(V)\to\mathrm{Gr}_{1,\boldsymbol{K}}(\bigwedge^2V)\colon u\boldsymbol{F}\oplus v\boldsymbol{F}\mapsto(u\wedge v)\boldsymbol{K}$ *is well defined, and we have the following:*

(a) *If the restriction of* h *to* $L\in\mathrm{Gr}_{2,\boldsymbol{F}}(V)$ *is non-degenerate then* $\lambda(L)$ *has precisely two preimages under* λ*, namely,* L *and* $L^{\perp_h}$*. In this case, the Pfaffian* Pf *has non-degenerate restriction to* $\lambda(L)$*, and the* $\boldsymbol{K}$*-space* $\lambda(L)$ *is not isotropic with respect to* g*.*

(b) *If the restriction of* h *to* $L\in\mathrm{Gr}_{2,\boldsymbol{F}}(V)$ *is degenerate then* $\lambda(L)$ *has more than two preimages. In fact, the preimages of* L *are the elements of the pencil of tangents* $\mathrm{Gr}_{2,\boldsymbol{F}}(A^{\perp_h})$ *to the absolute point* $A:=L\cap L^{\perp_h}\in\mathrm{Abs}_h$*; that set is in bijection with the projective line over* $\boldsymbol{F}$*. In this case, the* $\boldsymbol{K}$*-space* $\lambda(L)$ *is isotropic with respect to* g*; i.e., it belongs to the set* Abs_g *of absolute points of the polarity induced by* g *on the projective space over* $\boldsymbol{K}$*.*

(c) *For each* $Z\in\bigwedge^2V$ *with* $g(Z,Z)=0$ *there exist* $z,w\in V$ *such that* $Z=z\wedge w$ *and* $h(z,z)=0=h(z,w)$*. In particular, we have* $\mathrm{Abs}_g\subset\lambda\left(\mathrm{Gr}_{2,\boldsymbol{F}}(V)\right)$*.*

Proof. Let $L=u\boldsymbol{F}\oplus v\boldsymbol{F}$. If $L\cap L^{\perp_h}=\{0\}$ then Pf is non-degenerate on $\lambda(L)$ because

$$\begin{aligned}\mathrm{Pf}(u\wedge v,u\wedge v)&=\mathrm{Pf}((u\wedge v)\boldsymbol{j},(u\wedge v)\boldsymbol{j})=0\\&\neq{\textstyle\bigwedge^2}h(u\wedge v,u\wedge v)=\mathrm{Pf}(u\wedge v,(u\wedge v)\boldsymbol{j}).\end{aligned}$$

Thus $\lambda(L)$ meets the Klein quadric in two points; these are $(u\wedge v)\boldsymbol{F}$ and $(u\wedge v)\boldsymbol{j}\boldsymbol{F}$.

If $L \cap L^{\perp_h} \neq \{0\}$ we may assume $h(u,v) = 0 = h(v,v)$. For

$$X \in (ux_u + v\boldsymbol{F}) \wedge (uy_u + v\boldsymbol{F})$$

we obtain
$$\textstyle\bigwedge^2 h\,(X,X)) = \det \begin{pmatrix} x_u^2 & x_u y_u \\ y_u x_u & y_u^2 \end{pmatrix} = 0.$$

Thus $g(\lambda(L) \times \lambda(L)) = \{0\}$ follows from 2.9 since $\mathrm{Pf}(u \wedge v, u \wedge v) = 0$.

For the last assertion, we note that $g(Z,Z) = 0$ means $0 = \mathrm{Pf}(Z,Z)$ (whence $Z = u \wedge v$ for some $u,v \in V$) and then

$$0 = \textstyle\bigwedge^2 h(Z,Z) = \bigwedge^2 h(u \wedge v, u \wedge v) = \det \begin{pmatrix} h(u,u) & h(u,v) \\ h(v,u) & h(v,v) \end{pmatrix}.$$

So the restriction of h to $u\boldsymbol{F} + v\boldsymbol{F}$ is degenerate, and we find z, w as required. ∎

7. The range of our homomorphism in the non-split case

We consider the case where $\ell = 2$ and $\delta = \delta_2$ is not a norm. Moreover, we assume that the form h is isotropic (see Section 8 for the anisotropic case). From 5.1 and 5.2 we know that these assumptions are equivalent to the assumption that the Witt index is precisely 1. Again, our aim is to understand $\eta(\mathrm{SU}(V,h))$ where $\eta = \eta_2$ is the homomorphism constructed in Section 2.

7.1 Notation. An Eichler (or Siegel) transformation (cf. [13, p. 214 f]) of (V,h) is given as

$$\Sigma_{z,w,p} \colon V \to V \colon x \mapsto x + z\,h(w,x) - (w + z\,p)\,h(z,x)$$

where $z, w \in V$ satisfy $h(z,z) = 0 = h(z,w)$ and $\sigma(p) + p = h(w,w)$. The special case $w = 0$ leads to an isotropic transvection $\Sigma_{z,0,p}$, with $\sigma(p) = -p$.

Conversely, every transvection in $\mathrm{U}(V,h)$ is of the form $\Sigma_{z,0,p}$ with $h(z,z) = 0$ and $\sigma(p) = -p$ (see, for instance, [11, p. 94]). Note that only the trivial isotropic transvection exists if $\sigma = \mathrm{id}$ and $\mathrm{char}\,\boldsymbol{F} \neq 2$. This is the reason why we also have to study the more general Eichler transformations. If $\mathrm{char}\,\boldsymbol{F} = 2$ and $\sigma = \mathrm{id}$ then there do exist isotropic transvections.

By $\mathrm{EO}(V,h) \le \mathrm{O}(V,h)$ and $\mathrm{EU}(V,h) \le \mathrm{U}(V,h)$ we denote the subgroups generated by all Eichler transformations. Note that $\mathrm{EU}(V,h)$ is generated by its isotropic transvections (see [13, 6.3.1]), except if $V = \mathbb{F}_4^3$. (However, that group is not of interest here because 3 is odd.)

7.2 Lemma. *If $\sigma \neq \mathrm{id}$ and $\mathrm{char}\,\boldsymbol{F} \neq 2$ then the image $\eta(\mathrm{SU}(V,h))$ in $\mathrm{U}(\bigwedge^2 V, g)$ contains each isotropic transvection.*

Proof. The transvections in $\mathrm{U}(\bigwedge^2 V, g)$ are of the form

$$\Sigma_{Z,0,p} \colon \textstyle\bigwedge^2 V \to \bigwedge^2 V \colon X \mapsto X - Zp\,g(Z,X)$$

where $Z \in \bigwedge^2 V$ and $p \in \boldsymbol{K}$ satisfy $g(Z,Z) = 0 = \alpha(p) + p$. Our assumption $\mathrm{char}\,\boldsymbol{F} \neq 2$ implies that $\alpha(p) + p = 0$ is equivalent to $p \in \boldsymbol{F}$ and $\sigma(p) + p = 0$. There exist $z, w \in V$ with $h(z,z) = 0 = h(z,w)$ such that $Z = z \wedge w$, cf. 6.3.

For fixed $z \in V$ the alternating bilinear map $(x,y) \mapsto (x\,h(z,y) - y\,h(z,x)) \wedge z$ has its range in $z^{\perp_h} \wedge z$. If $h(z,z) = 0$ then for each $w \in z^\perp \setminus z\boldsymbol{F}$ the latter set is contained in $(z \wedge w)\boldsymbol{K} = Z\boldsymbol{K}$, cf.6.3. Thus we obtain an $\boldsymbol{F}$-linear map

$$\lambda_z \colon \bigwedge^2 V \to Z\boldsymbol{K} \colon x \wedge y \mapsto (x\,h(z,y) - y\,h(z,x)) \wedge z\,.$$

For $q \in \boldsymbol{F}$ with $\sigma(q) = -q$, the image $\eta(\Sigma_{z,0,q}) \in \mathrm{U}(\bigwedge^2 V, g)$ is the linear extension of $(x \wedge y) \mapsto (x \wedge y) + \lambda_z(x \wedge y)\,q$. This is a transvection because $h(z,z) = 0$ yields $\lambda_z(z \wedge w) = 0$, and it is, of course, in the unitary group. Thus $\eta(\Sigma_{z,0,q})$ is of the form $\Sigma_{z\wedge w,0,p'} = \Sigma_{Z,0,p'}$ with $p' \in \{z \in \boldsymbol{K} \mid \alpha(z) = -z\}$. Since $p\boldsymbol{R} = p'\boldsymbol{R}$ we may achieve $\eta\left(\Sigma_{z,0,q}\right) = \Sigma_{Z,0,p}$, as required. ■

7.3 Remark. If $\operatorname{char}\boldsymbol{F} = 2$ then $\alpha(p) + p = 0$ is equivalent to $p \in \boldsymbol{R} + \boldsymbol{j}\boldsymbol{F}$.

7.4 Theorem. *Assume* $\operatorname{char}\boldsymbol{F} \neq 2$, $\sigma \neq \mathrm{id}$ *and that* $\boldsymbol{K}$ *is not split. Then the image of* $\mathrm{SU}(V,h)$ *under* η *in* $\mathrm{U}(\bigwedge^2 V, g)$ *is the group* $\mathrm{EU}(\bigwedge^2 V, g)$ *generated by all isotropic transvections.*

Proof. From 7.2 we know that $\eta(\mathrm{SU}(V,h))$ contains $\mathrm{EU}(\bigwedge^2 V, g)$. On the other hand, the image (like $\mathrm{SU}(V,h)$ itself, cf. [13, 6.4.26][4]) is a perfect group, and the quotient $\mathrm{U}(\bigwedge^2 V, g)/\mathrm{EU}(\bigwedge^2 V, g)$ is abelian ([35], cf. [13, 6.4.52]). Thus $\eta(\mathrm{SU}(V,h))$ coincides with $\mathrm{EU}(\bigwedge^2 V, g)$. ■

We cannot expect a result similar to 7.4 in the characteristic two case, see 7.3. It remains to study the case where $\sigma = \mathrm{id}$; then $\alpha = \mathrm{id}$, as well, and we are dealing with a homomorphism $\eta \colon \mathrm{SO}(V,h) \to \mathrm{O}(\bigwedge^2 V, g)$. We must consider Eichler transformations now. We may safely ignore the characteristic two case, cf. [22].

7.5 Lemma. *If* $\sigma = \mathrm{id}$, $\operatorname{char}\boldsymbol{F} \neq 2$ *and* h *is isotropic then* $\eta(\mathrm{SO}(V,h))$ *acts transitively on the set* Abs_g *of absolute points of the polarity* $\pi_g \colon U \leftrightarrow U^{\perp_g}$ *on* $\mathrm{P}(\bigwedge^2 V)$.

Proof. Using $\operatorname{char} F \neq 2$ and the fact that the isotropic bilinear form h secures the existence of a hyperbolic pair in V, we find an orthogonal basis v_1, v_2, v_3, v_4 for V such that $v_1 + v_2$ is isotropic. Passing to a scalar multiple of h we may assume $h(v_1,v_1) = 1$; then $h(v_2,v_2) = -1$. We abbreviate $c := h(v_3,v_3)$, normalize $b(v_1 \wedge v_2 \wedge v_3 \wedge v_4) = 1$ and compute $J\left((v_1+v_2) \wedge v_3\right) = (v_1+v_2) \wedge v_4(-c)$.

From 6.3 we know that every absolute point of π_g is of the form $(u \wedge w)\boldsymbol{K}$ with u, w in V such that $h(u,u) = 0 = h(u,w)$. We may assume $u = v_1 + v_2$ because the group $\mathrm{SO}(V,h)$ acts transitively on the set of all one-dimensional h-isotropic subspaces of V (by Witt's Theorem).

So it suffices to remark that $(v_1+v_2) \wedge (v_1+v_2)^{\perp_g} = (v_1+v_2)\boldsymbol{K}$; this follows from $((v_1+v_2) \wedge v_3)\,(r + \boldsymbol{j}_2 s) = (v_1+v_2) \wedge (v_3 r - v_4 c s)$ and the observation $(v_1+v_2)^{\perp_g} = (v_1+v_2)\boldsymbol{F} + v_3\boldsymbol{F} + v_4\boldsymbol{F}$. ■

7.6 Theorem. *Assume* $\sigma = \mathrm{id}$, $\operatorname{char}\boldsymbol{F} \neq 2$ *and that* $\boldsymbol{K}$ *is not split. Then* $\eta(\mathrm{SO}(V,h))$ *contains* $\mathrm{EO}(\bigwedge^2 V, g)$.

[4] The exceptions in [13, 6.4.26] are groups over finite fields. Apart from symplectic groups and the orthogonal group with respect to a quadratic form on $\mathbb{F}_2^4$ (all of which are of no interest in this paper), there is a unitary group on $\mathbb{F}_4^2$ (which is not of interest here, as we assume $\ell = 2$ now).

Proof. Recall from 5.1 that h has Witt index 1.

We show first that Eichler transformations in $\mathrm{SO}(V,h)$ are mapped to Eichler transformations in $\mathrm{EO}(\bigwedge^2 V, g)$. Consider linearly independent $z, w \in V$ with $h(z,z) = 0 = h(z,w)$. As $X := z\boldsymbol{F} + w\boldsymbol{F}$ is not totally isotropic, we have $h(w,w) \neq 0$, and $X^{\perp_h} \cap X = z\boldsymbol{F}$. Choose $v \in X^{\perp_h} \smallsetminus z\boldsymbol{F}$ and then $u \in \{w,v\}^{\perp_h}$ such that $h(z,u) = 1$. Then z, w, v, u are linearly independent, in fact, the Gram matrix for g with respect to this basis is

$$\begin{pmatrix} 0 & 0 & -2p \\ 0 & -1 & 0 \\ -2p & 0 & 0 \end{pmatrix}.$$

Put $p := \frac{1}{2} h(w,w)$. Evaluating at the $\boldsymbol{K}$-basis $z \wedge w$, $z \wedge u$, $w \wedge u$ for $\bigwedge^2 V$ we obtain $\eta(\Sigma_{z,w,p}) = \Sigma_{z\wedge w, z\wedge u, -\frac{1}{2}}$.

Now consider an Eichler transformation $\tau \in \mathrm{EO}(\bigwedge^2 V, g)$. By 7.5 we know that $\eta(\mathrm{SO}(V,h))$ acts transitively on the set Abs_g of absolute points of the polarity π_g. Therefore, we may assume $\tau = \Sigma_{Z,W,q}$ for $Z = z \wedge w$. Now $g(Z,W) = 0$ implies $W = Za + Uc$ for $U = z \wedge u$ and some $a, c \in \boldsymbol{K}$. We find $\Sigma_{Z,W,q} = \Sigma_{Z,Uc,q}$ for $q = \frac{1}{2} g(Uc, Uc) = \frac{c^2}{2} g(U,U)$. Note also that $\Sigma_{Z,Uc,q} = \Sigma_{Zc,U,p}$ with $q = c^2 p$. ∎

7.7 Remark. If $\sigma = \mathrm{id}$ and the discriminant of h is not 1 then $\boldsymbol{K}$ is a commutative field, and the group $\mathrm{EO}(\bigwedge^2 V, g)$ coincides (see [11, 9.7]) with the kernel of the spinor norm $\Theta\colon \mathrm{SO}(\bigwedge^2 V, g) \to \boldsymbol{K}^\times / \boldsymbol{K}^{\boxtimes}$ where $\boldsymbol{K}^{\boxtimes}$ denotes the group of squares in $\boldsymbol{K}^\times$. We do not explicitly determine the range of η here.

7.8 Example: The Lorentz group. Consider $\boldsymbol{R} = \mathbb{R}$ and the (essentially, up to a choice of basis, unique) symmetric bilinear form h of Witt index 1 on $\mathbb{R}^4$; in fact, this is (up to multiplying the form by -1, which does not affect the corresponding group of isometries) the form used to describe special relativity in physics, the group $\mathrm{O}(\mathbb{R}^4, h)$ is known as the *Lorentz group*. The group $\mathrm{SO}(\mathbb{R}^4, h)$ is not connected; its connected component is $\mathrm{EO}(\mathbb{R}^4, h)$; this subgroup may be interpreted as the subgroup of $\mathrm{SO}(\mathbb{R}^4, h)$ leaving invariant the direction of the light cone. We have a direct product $\mathrm{SO}(\mathbb{R}^4, h) = \langle -\mathrm{id} \rangle\, \mathrm{EO}(\mathbb{R}^4, h)$.

The discriminant of h is -1, so $\boldsymbol{K}$ is not split. Then $\boldsymbol{K} \cong \mathbb{C}$, the unique quadratic extension field of $\mathbb{R}$. The form g is now a non-degenerate symmetric bilinear form (see 2.6) on $W \cong \mathbb{C}^3$, and uniquely determined (up to a choice of basis); on $\mathbb{C}^3$, such a form $\tilde{g}$ is given as the sum of the squares of the coordinates. From 7.6 we know that the image of $\mathrm{SO}(\mathbb{R}^4, h)$ under η contains $\mathrm{EO}(W, g) \cong \mathrm{EO}(\mathbb{C}^3, \tilde{g})$, and is contained in $\mathrm{O}(W, g)$. Note that $\eta(\mathrm{SO}(\mathbb{R}^4, h)) = \eta(\mathrm{EO}(\mathbb{R}^4, h))$; the kernel $\langle -\mathrm{id} \rangle$ is a complement to $\mathrm{EO}(\mathbb{R}^4, h)$ in $\mathrm{SO}(\mathbb{R}^4, h)$.

Moreover, one knows that $\mathrm{EO}(\mathbb{C}^3, \tilde{g})$ coincides with $\mathrm{SO}(\mathbb{C}^3, \tilde{g})$; either by a connectivity argument, or by the more general argument from 7.7. The isomorphism from $\mathrm{EO}(\mathbb{R}^4, h)/\langle -\mathrm{id} \rangle \cong \mathrm{EO}(\mathbb{R}^4, h)$ onto $\mathrm{SO}(\mathbb{C}^3, \tilde{g})$ that we obtain here is known as the isomorphism between the simple Lie algebras of types $\mathsf{D}_2^{\mathbb{R},1}$ and B_1, in the notation of Tits [32]. Note also that $\mathrm{EO}(\mathbb{C}^3, \tilde{g}) \cong \mathrm{PSL}_2\mathbb{C}$. This is a special case of a general result; for real and complex Lie algebras it reflects $\mathsf{B}_1^{\mathbb{R},1} \cong \mathsf{A}_1^{\mathbb{R}}$ and $\mathsf{B}_1 \cong \mathsf{A}_1$.

7.9 Examples: Orthogonal groups over finite fields of odd order. Let $\boldsymbol{F} = \mathbb{F}_e$ be the finite field of order e, and consider a diagonalizable σ-hermitian form h on $\boldsymbol{F}^4$. If $\sigma \neq \mathrm{id}$ then h has discriminant one, and $\boldsymbol{K}$ is split. So assume $\sigma = \mathrm{id}$, and that the discriminant is not a square. Ignoring the inseparable cases, we assume $\operatorname{char} \boldsymbol{F} \neq 2$. Then the assumption that $\boldsymbol{K}$ is non-split yields that h is not the hyperbolic form on $\boldsymbol{F}^4$, but equivalent to the form known as q_4^- (cp. [31, pp. 139 ff]). In other words, there exists $\delta \in \boldsymbol{F} \smallsetminus \boldsymbol{F}^\square$ such that we may assume that (with respect to the standard basis $e_1, \ldots, e_4$) the Gram matrix of h is

$$T = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -\delta \end{pmatrix}.$$

We have $\boldsymbol{F}(\sqrt{\delta}) \cong \mathbb{F}_{e^2}$. The norm form of the quadratic field extension $\mathbb{F}_{e^2}|\mathbb{F}_e$ is surjective: for each $s \in \boldsymbol{F}^\times$, there exist $x, y \in \boldsymbol{F}$ such that $s = x^2 - \delta y^2$. In standard coordinates, the matrix

$$\begin{pmatrix} 0 & s & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & x & \delta y \\ 0 & 0 & y & x \end{pmatrix}$$

now describes a similitude γ_s with multiplier $r_{\gamma_s} = s$ and $\det \gamma_s = -s^2$.

With respect to the basis $v_1 := e_1 + e_2$, $v_2 := e_1 - e_2$, $v_3 := e_3$, $v_4 := e_4$, the Gram matrix for h is

$$H = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -\delta \end{pmatrix}.$$

Using b with $b(v_1 \wedge v_2 \wedge v_3 \wedge v_4) = 2$, we obtain $\boldsymbol{j}^2 = \delta$, and $\boldsymbol{K} \cong \boldsymbol{F}(\sqrt{\delta})$. In $M_2 \cong \boldsymbol{K}$ (see 2.4), conjugation by γ_s maps J to

$$\tfrac{\det \gamma_s}{r_{\gamma_s}^2} \mathrm{id} \circ J = \tfrac{-s^2}{s^2} \mathrm{id} \circ J = -J.$$

So conjugation by γ_s induces the generator of the Galois group of the extension $\boldsymbol{K}|\boldsymbol{F}$.

The group $\mathrm{O}(\boldsymbol{F}^4, h) \cong \mathrm{O}^-(4, e)$ has order $2e^2(e^2+1)(e^2-1)$, we have of course $|\mathrm{SO}(\boldsymbol{F}^4, h)| = e^2(e^2+1)(e^2-1)$. The group $\mathrm{EO}(\boldsymbol{F}^4, h)$ has index 2 in $\mathrm{SO}(\boldsymbol{F}^4, h)$, the element $-\mathrm{id}$ generates a complement to $\mathrm{EO}(\boldsymbol{F}^4, h)$ in $\mathrm{SO}(\boldsymbol{F}^4, h)$, see [11, 9.7, 9.8]. So $\eta(\mathrm{SO}(\boldsymbol{F}^4, h)) = \eta(\mathrm{EO}(\boldsymbol{F}^4, h))$ is isomorphic to $\mathrm{EO}(\boldsymbol{F}^4, h)$, and contains $\mathrm{EO}(W, g)$. The latter group has index 2 in $\mathrm{SO}(W, g) \cong \mathrm{PGL}_2\boldsymbol{K}$, see [31, 11.8].

So
$$|\mathrm{EO}(W, g)| = (e^2+1)e^2(e^2-1)/2 = |\mathrm{EO}(\boldsymbol{F}^4, h)|,$$
and
$$\mathrm{EO}(\boldsymbol{F}^4, h) \cong \mathrm{EO}(W, g) \cong \mathrm{PSL}_2\mathbb{F}_{e^2}$$

follows. We note that
$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

describes an element $\rho \in \mathrm{O}(\boldsymbol{F}^4, h) \setminus \mathrm{SO}(\boldsymbol{F}^4, h)$, and therefore induces an element $\eta(\rho) \in \mathrm{O}(W, g) \setminus \mathrm{SO}(W, g)$. So we have found that $\eta(\mathrm{O}(\boldsymbol{F}^4, h))$ contains $\mathrm{EO}(W, g)$ as a subgroup of index 2, but does not contain $\mathrm{SO}(W, g)$.

We remark that the group $\mathrm{EO}(V, h)$ is often denoted $\mathrm{SO}^+(V, h)$.

8. Anisotropic forms

It remains to study the case where the form h is anisotropic. Recall from 2.10 that $\bigwedge^\ell h$ may become isotropic. Also, the form g may be isotropic, see 8.1 below.

We concentrate on the case $\sigma = \mathrm{id}$ in this section; the hermitian case is left open in the present paper. In the case where $\mathrm{char}\, \boldsymbol{F} \neq 2$, the isometry groups of anisotropic (quadratic) forms are notoriously difficult; in particular, we do not have the tool of Eichler transformations at our disposal. It also happens that $\mathrm{SO}(\bigwedge V, g)$ has large normal subgroups with large index ([1, Ch. 5, § 3], [18], [13, 6.4.55], cp. also [3]), see 8.3 below.

If $\mathrm{char}\, \boldsymbol{F} = 2$ then the isometry group of h is contained in the isometry group of the anisotropic quadratic diagonal form (over $\boldsymbol{R}$) mapping v to $h(v, v)$; these groups are trivial. The split cases have been treated in 5.3, 5.7, and 5.13; except for the case where $\mathrm{char}\, \boldsymbol{F} = 2$ and $\sigma = \mathrm{id}$. Those latter cases are treated in [22]. In the present section, we therefore concentrate on the non-split case, and assume $\mathrm{char}\, \boldsymbol{F} \neq 2$.

We show that the homomorphism

$$\eta_2 \colon \mathrm{SO}(V, h) = \mathrm{SO}(\boldsymbol{F}^4, h) \to \mathrm{SO}(\textstyle\bigwedge^2 V, g) \cong \mathrm{SO}(\boldsymbol{K}^3, g)$$

need not be surjective if $\boldsymbol{K}$ is not split and h is anisotropic.

Over the field of real numbers, anisotropic bilinear forms are positively or negatively definite ones, and the discriminant is 1 if the dimension is even. As this leads to split cases, the real case does not play any role in the present section.

8.1 Example. Let $\boldsymbol{F} = \mathbb{Q}$ and consider the bilinear form h on $\mathbb{Q}^4$ given by

$$h(x_1, x_2, x_3, x_4) = x_1^2 + 2\, x_2^2 + 10\, x_3^2 - 5\, x_4^2 \,.$$

The discriminant of this form is the square class of -1. Thus the algebra $\boldsymbol{K}_2$ is not split; in fact it is isomorphic to $\mathbb{Q}(i)$ where i is a square root of -1.

Moreover, the form h is anisotropic. In fact, if this were not the case we could pick a minimal isotropic vector z with integer entries. Considering the equation $h(z, z) = 0$ first modulo 5, and using the fact that the squares in $\mathbb{Z}/5\mathbb{Z}$ are 0, 1, and $4 \equiv -1$, we see $z_1, z_2 \in 5\mathbb{Z}$. Considering the equation modulo 25, we then see $z_3, z_4 \in 25\mathbb{Z}$. So z has to lie in $5\mathbb{Z}^4$, contradicting our minimality assumption.

The form g is isotropic; e.g., for $w := (v_1 \wedge v_3)10 - (v_1 \wedge v_4)(10 + \boldsymbol{j})$ we have $g(w, w) = 0$, cp. 2.7. Thus we have an explicit example of an anisotropic form h such that $\boldsymbol{K}_2$ is not split and g is isotropic.

Starting from the standard basis e_1, e_2, e_3, e_4 for $\mathbb{Q}^4$ we get the basis $e_1 \wedge e_2$, $e_1 \wedge e_3$, $e_1 \wedge e_4$ for $\bigwedge^2 \mathbb{Q}^4$ considered as a vector space over $\boldsymbol{K}_2 = \mathbb{Q}(i)$ where $\boldsymbol{j} = 10i$ maps $e_1 \wedge e_2$, $e_1 \wedge e_3$, and $e_1 \wedge e_4$ to $2\, e_3 \wedge e_4$, $10\, e_4 \wedge e_2$, and $-5\, e_2 \wedge e_3$, respectively.

The group $\mathrm{O}(V, h)$ does not act transitively on the $\mathrm{SO}(\bigwedge^2 V, g)$-orbit

$$g^{\leftarrow}(2) := \{ X\boldsymbol{K} \mid X \in \textstyle\bigwedge^2 V,\ g(X, X) = 2 \} \,.$$

In order to prove this, we note that the restrictions of h to the two subspaces $L_1 := (1,0,0,0)^\intercal \boldsymbol{F} \oplus (0,1,0,0)^\intercal \boldsymbol{F}$ and $L_2 := (0,0,1,0)^\intercal \boldsymbol{F} \oplus (1/2,0,0,1/10)^\intercal \boldsymbol{F}$ both have discriminant 2, and $\lambda(L_1)$ and $\lambda(L_2)$ both belong to $g^{\leftarrow}(2)$. From 6.3 we know that $\lambda(L) = \lambda(L_k)$ means $L \in \{L_k, L_k^{\perp_h}\}$ because the restriction of h to L_k is not degenerate. As L_2 is not isometric[5] to any $L \in \{L_1, L_1^{\perp_h}\}$, we find that there is no element of $\mathrm{O}(V,h)$ mapping $\lambda(L_1)$ to $\lambda(L_2)$.

8.2 Lemma. [8, II §6, 1), p. 49]. *If* $\dim X \geq 3$ *and* f *is a non-degenerate symmetric bilinear form on* X *then* $\mathrm{SO}(X,f)$ *is generated by the half-turns, i.e. the involutions whose space of fixed points has codimension* 2 *in* X. ■

8.3 Theorem. *The subgroup* $H \leq \mathrm{SO}(\bigwedge^2 V, g)$ *generated by all half-turns around axes that meet the Klein quadric is a normal subgroup of* $\Gamma\mathrm{O}(\bigwedge^2 V, g)$, *and this subgroup* H *is contained in the image* $\eta(\mathrm{SO}(V,h))$. *In general, both* H *and* $\eta(\mathrm{SO}(V,h))$ *are proper subgroups of* $\mathrm{SO}(\bigwedge^2 V, g)$.

Proof. Each half-turn is determined by its axis; thus half-turns are conjugates precisely if their axes are in the same orbit under the group of similitudes. In order to prove normality of H in $\Gamma\mathrm{O}(\bigwedge^2 V, g)$, it remains to note that the Klein quadric is invariant under the action of the group of semi-similitudes because $\mathrm{Pf}(X,X) = 0$ characterizes the points $X\boldsymbol{F}$ on the Klein quadric, and $\mathrm{Pf}(X,X) = 0 \iff g(X,X) \in \boldsymbol{F}$ holds for $X \in \bigwedge^2 V$ by the construction of the form g, see 2.6.

Now let $X \in \bigwedge^2 V$ such that $X\boldsymbol{K}$ meets the Klein quadric. Then there exists $Y \in X\boldsymbol{K}$ such that $\mathrm{Pf}(Y,Y) = 0$, and there are $v_1, v_2 \in V$ such that $Y = v_1 \wedge v_2$.

Since h is anisotropic we may assume that $h(v_1,v_2) = 0$, and we can extend this to an orthogonal basis v_1, v_2, v_3, v_4 for V. Then $Y_1 := v_1 \wedge v_2$, $Y_2 := v_1 \wedge v_3$, $Y_3 := v_1 \wedge v_4$ form a $\boldsymbol{K}$-basis for $\bigwedge^2 V$. The linear map ρ_1 mapping v_1 to $-v_1$ and fixing v_2, v_3, v_4 is a hyperplane reflection in $\mathrm{O}(V,h)$. We see immediately that $\eta(\rho_1) = \eta(-\rho_1)$ is a $\boldsymbol{K}$-semilinear map with companion automorphism κ, fixing $Y_1\boldsymbol{j}$, $Y_2\boldsymbol{j}$ and $Y_3\boldsymbol{j}$. Interchanging the roles of v_1 and v_2 we obtain ρ_2 such that $\eta(\rho_2 \circ \rho_1)$ is the $\boldsymbol{K}$-linear map fixing Y_1 and inducing $-\mathrm{id}$ on $Y_2\boldsymbol{K} \oplus Y_3\boldsymbol{K}$. Thus $\eta(\rho_2 \circ \rho_1)$ is one of the involutions in $\mathrm{SO}(\bigwedge^2 V, g)$. In this way we obtain all those half-turns in $\mathrm{SO}(\bigwedge^2 V, g)$ whose axis is spanned by some $X \in \{u \wedge v \mid u, v \in V\} \smallsetminus \{0\}$, i.e., those with axes that meet the Klein quadric.

Finally, we recall from 8.1 that it may happen that $\eta(\mathrm{SO}(V,h))$ does not act transitively on some $\mathrm{SO}(\bigwedge^2 V, g)$-orbit. In such cases, we clearly have

$$H \leq \eta(\mathrm{SO}(V,h)) \lneqq \mathrm{SO}(\textstyle\bigwedge^2 V, g).$$ ■

In the situation of 8.3, it remains as an open question whether there are examples where $H \lneqq \eta(\mathrm{SO}(V,h))$.

Acknowledgements. In her diploma thesis [10] Ulrike Fiedler worked out several examples explicitly, and spotted various slips in a draft of the present paper. We thank the anonymous referee for several helpful comments and for requiring some clarifications.

Funded by the Deutsche Forschungsgemeinschaft through a Polish-German *Beethoven* grant KR1668/11, and under Germany's Excellence Strategy EXC 2044-390685587, Mathematics Münster: Dynamics-Geometry-Structure.

[5] The restrictions of h to L_1 and L_2 are similar, but not isometric; e.g., $5 \notin \{h(v,v) \mid v \in L_2\}$.

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Received August 27, 2022
and in final form September 27, 2022