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On Weakly Complete Universal Enveloping Algebras: A Poincaré-Birkhoff-Witt Theorem

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Abstract. The Poincaré-Birkhoff-Witt Theorem deals with the structure and universal property of the universal enveloping algebra $U(L)$ of a Lie algebra L, e.g., over $\mathbb{R}$ or $\mathbb{C}$. K. H. Hofmann and L. Kramer (HK) [*On weakly complete group algebras of Compact Groups*, J. Lie Theory 30 (2020) 407–426] recently introduced the weakly complete universal enveloping algebra $\mathbf{U}(\mathfrak{g})$ of a profinite-dimensional topological Lie algebra $\mathfrak{g}$. Here it is shown that the classical universal enveloping algebra $U(|\mathfrak{g}|)$ of the abstract Lie algebra underlying $\mathfrak{g}$ is a dense subalgebra of $\mathbf{U}(\mathfrak{g})$, algebraically generated by $\mathfrak{g} \subseteq \mathbf{U}(\mathfrak{g})$. It is further shown that, inspite of $\mathbf{U}$ being a left adjoint functor, it nevertheless preserves projective limits in the form $\mathbf{U}(\lim_{\mathfrak{i}} \mathfrak{g}/\mathfrak{i}) \cong \lim_{\mathfrak{i}} \mathbf{U}(\mathfrak{g}/\mathfrak{i})$, for profinite-dimensional Lie algebras $\mathfrak{g}$ represented as projective limits of their finite-dimensional quotients. The required theory is presented in an appendix which is of independent interest.

In a natural way, a weakly complete enveloping algebra $\mathbf{U}(\mathfrak{g})$ is a weakly complete symmetric Hopf algebra with a Lie subalgebra $\mathbb{P}(\mathbf{U}(\mathfrak{g}))$ of *primitive* elements containing $\mathfrak{g}$ (indeed properly if $\mathfrak{g} \neq \{0\}$), and with a nontrivial multiplicative pro-Lie group $\mathbb{G}(\mathbf{U}(\mathfrak{g}))$ of *grouplike* units, having $\mathbb{P}(\mathbf{U}(\mathfrak{g}))$ as its Lie algebra – in contrast with the classical Poincaré-Birhoff-Witt environment of $U(L)$, thus providing a new aspect of Lie's Third Fundamental Theorem: Indeed a canonical pro-Lie subgroup $\Gamma^*(\mathfrak{g})$ of $\mathbb{G}(\mathbf{U}(\mathfrak{g}))$ is identified whose Lie algebra is naturally isomorphic to $\mathfrak{g}$. The structure of $\mathbf{U}(\mathfrak{g})$ is described in detail for $\dim \mathfrak{g} = 1$. The primitive and grouplike components and their mutual relationship are evaluated precisely.

In (HK), cited above, and in the work of R. Dahmen and K. H. Hofmann [*The pro-Lie group aspect of weakly complete algebras and weakly complete group Hopf algebras*, J. Lie Theory 29 (2019) 413–455] the real weakly complete group Hopf algebra $\mathbb{R}[G]$ of a compact group G was described. In particular, the set $\mathbb{P}(\mathbb{R}[G]))$ of primitive elements of $\mathbb{R}[G]$ was identified as the Lie algebra $\mathfrak{g}$ of G. It is now shown that for any compact group G with Lie algebra $\mathfrak{g}$ there is a natural morphism of weakly complete symmetric Hopf algebras $\omega_{\mathfrak{g}}\colon \mathbf{U}(\mathfrak{g}) \to \mathbb{R}[G]$, implementing the identity on $\mathfrak{g}$ and inducing a morphism of pro-Lie groups $\Gamma^*(G) \to \mathbb{G}(\mathbb{R}[G]) \cong G$: yet another aspect of Sophus Lie's Third Fundamental Theorem!

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Key Words: Associative algebra, Lie algebra, universal enveloping algebra, weakly complete vector space, projective limit, pro-Lie group, profinite-dimensional Lie algebra, power series algebra, symmetric Hopf algebra, primitive element, grouplike element, Poincaré-Birkhoff-Witt theorem.

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1. The Weakly Complete Enveloping Algebra of a Profinite-Dimensional Lie Algebra

In [7] we have initiated the theory of *weakly complete universal enveloping algebras over* $\mathbb{K}$ hoping that in some fashion this concept would resemble the classical universal enveloping algebra of a Lie algebra such as it is presented in the famous Poincaré-Birkhoff-Witt-Theorem (see e.g. [1], Chap. 1, Paragraph 2, n° 7, Théorème 1., p. 30). While this was not exactly the case, we shall discuss now how close we come to that theorem.

So we let $\mathbb{K}$ denote one of the topological fields $\mathbb{R}$ or $\mathbb{C}$. For a topological Lie algebra $\mathfrak{g}$ over $\mathbb{K}$ we let $\mathcal{I}(\mathfrak{g})$ denote the filter basis of all closed ideals $\mathfrak{i} \subseteq \mathfrak{g}$ such that $\dim \mathfrak{g}/\mathfrak{i} < \infty$.

Definition 1.1. A topological Lie algebra $\mathfrak{g}$ over $\mathbb{K}$ is called *profinite-dimensional* if $\mathfrak{g} = \lim_{\mathfrak{i}\in\mathcal{I}(\mathfrak{g})} \mathfrak{g}/\mathfrak{i}$. Let $\mathcal{WL}$ denote the category of profinite-dimensional Lie algebras (over $\mathbb{K}$) and continuous Lie algebra morphisms between them.

Notice that by its definition every profinite-dimensional Lie algebra is weakly complete. A comment following Theorem 3.12 of [2] exhibits an example of a weakly complete $\mathbb{K}$-Lie algebra which is not a profinite-dimensional Lie algebra.

Let $\mathcal{WA}$ denote the category of weakly complete associative unital algebras over $\mathbb{K}$. However, instead of considering the full category of weakly complete Lie algebras over $\mathbb{K}$, in the following we consider $\mathcal{WL}$, the category of profinite-dimensional Lie algebras over $\mathbb{K}$ and continuous $\mathbb{K}$-Lie algebra morphisms. The reason for this restriction is Theorem 7.1 stating that every weakly complete unital $\mathbb{K}$-algebra is the projective limit of its finite-dimensional quotient algebras. This implies at once the following

Proposition 1.2. *Let A be any weakly complete unital $\mathbb{K}$-algebra and A_{Lie} the weakly complete Lie algebra obtained by considering on the weakly complete vector space A the Lie algebra obtained with the Lie bracket $[x,y] = xy - yx$. Then A_{Lie} is profinite-dimensional.*

The functor which associates with a weakly complete associative algebra A the profinite-dimensional Lie algebra A_{Lie} is called the *underlying Lie algebra functor.*

For the complete proof of the following existence theorem, we shall invoke a considerable portion of a bulk category theoretical arguments. We shall collect these in an appendix, since, firstly, they reach far beyond the current application and, secondly, their full presentation might have led the reader astray from the present line of thought had we presented them at this point.

Theorem 1.3. (The Existence Theorem of $\mathbf{U}$) *The underlying Lie algebra functor $A \mapsto A_{\mathrm{Lie}}$ from $\mathcal{WA}$ to $\mathcal{WL}$ has a left adjoint $\mathbf{U}\colon \mathcal{WL} \to \mathcal{WA}$.*
The front adjunction $\lambda_{\mathfrak{g}}\colon \mathfrak{g} \to \mathbf{U}(\mathfrak{g})_{\mathrm{Lie}}$ is an embedding of profinite-dimensional Lie algebras.

Proof. The category $\mathcal{WL}$ is complete. (Exercise. Cf. Theorem A3.48 of [11], p. 819.) The "Solution Set Condition" (of Definition A3.58 in [11], p. 824) holds. (Exercise: Cf. the proof Lemma 3.58 of [11], p. 91.) Hence $\mathbf{U}$ exists by the Adjoint Functor Existence Theorem (i.e., Theorem A3.60 of [11], p. 825).

The assertion about $\lambda_{\mathfrak{g}}$ being an embedding follows from Proposition 7.20 (ii) in the Appendix. ■

In other words, each profinite-dimensional Lie algebra $\mathfrak{g}$ may be considered as a closed Lie subalgebra of $\mathbf{U}(\mathfrak{g})_{\text{Lie}}$ with the property that each continuous Lie algebra morphism $f\colon \mathfrak{g} \to A_{\text{Lie}}$ for some weakly complete associative unital algebra A extends uniquely to a $\mathcal{WA}$-morphism $f'\colon \mathbf{U}(\mathfrak{g}) \to A$.

$$\begin{array}{ccccc}
 & \mathcal{WL} & & & \mathcal{WA} \\
\hline
\mathfrak{g} & \xrightarrow{\lambda_{\mathfrak{g}}} & \mathbf{U}(\mathfrak{g})_{\text{Lie}} & & \mathbf{U}(\mathfrak{g}) \\
{\scriptstyle \forall f}\big\downarrow & & \big\downarrow{\scriptstyle (f')_{\text{Lie}}} & & \big\downarrow{\scriptstyle \exists! f'} \\
A_{\text{Lie}} & \xrightarrow[\text{id}]{} & A_{\text{Lie}} & & A.
\end{array}$$

If necessary we shall write $\mathbf{U}_{\mathbb{K}}$ instead of $\mathbf{U}$ whenever the ground field should be emphasized.

Definition 1.4. For each profinite-dimensional $\mathbb{K}$-Lie algebra, we shall call $\mathbf{U}_{\mathbb{K}}(\mathfrak{g})$ *the weakly complete enveloping algebra* of $\mathfrak{g}$ (over $\mathbb{K}$).

Remark 1.5. For every profinite-dimensional Lie algebra $\mathfrak{g}$, a morphism $f\colon \mathfrak{g} \to \mathbb{K}$, $f(x) = 0$, according to the definition of $\mathbf{U}_{\mathbb{K}}(\mathfrak{g})$ induces a natural $\mathcal{WA}$-morphism $\alpha_{\mathfrak{g}}\colon \mathbf{U}_{\mathbb{K}}(\mathfrak{g}) \to \mathbb{K}$ such that $\alpha_{\mathfrak{g}}(\mathfrak{g}) = \{0\}$ and that $\alpha_{\mathfrak{g}} \circ \iota_{\mathbf{U}(\mathfrak{g})} = \text{id}_{\mathbb{K}}$. ■

The retraction $\alpha_{\mathfrak{g}}$ is also called the *augmentation* of $\mathbf{U}_{\mathbb{K}}(\mathfrak{g})$.

In the Appendix we shall also introduce for any weakly complete vector space W its weakly complete tensor algebra $\mathbf{T}(W)$ (cf. paragraph (**C**) preceding Proposition 7.20 and Theorem 8.1) and show that $\mathbf{U}(\mathfrak{g})$ is a quotient algebra of $\mathbf{T}(|\mathfrak{g}|)$ if $|\mathfrak{g}|$ is the underlying weakly complete vector space underlying $\mathfrak{g}$. There is a commutative diagram

$$\begin{array}{ccc}
\mathbf{T}(|\mathfrak{g}|) & \xrightarrow{\alpha_{|\mathfrak{g}|}} & \mathbb{K} \\
{\scriptstyle f'}\big\downarrow & & \big\downarrow{\scriptstyle =} \\
\mathbf{U}_{\mathbb{K}}(\mathfrak{g}) & \xrightarrow[\alpha_{\mathfrak{g}}]{} & \mathbb{K}.
\end{array}$$

Moreover, we have the following corollary to our existence theorem:

Corollary 1.6. *For any profinite-dimensional Lie algebra $\mathfrak{g}$, the unital associative subalgebra $\langle\mathfrak{g}\rangle$ generated algebraically in $\mathbf{U}(\mathfrak{g})$ by $\mathfrak{g}$ is dense in $\mathbf{U}(\mathfrak{g})$.*

Proof. The assertion follows from Proposition 7.19 in the Appendix. ■

Of course we would like to have a better insight into the structure of the algebra $\langle\mathfrak{g}\rangle$. This information we provide in the following section and thereby close the gap between the concepts of the weakly complete enveloping algebra and the classical universal enveloping abstract algebra dealt with in the Poincaré-Birkhoff-Witt Theorem.

2. The abstract enveloping algebra $U(L)$ of a Lie Algebra L

We briefly recall that the functor which assigns to a unital $\mathbb{K}$-algebra X the underlying Lie algebra $X_{\rm Lie}$ (with the underlying $\mathbb{K}$ vector space of X as vector space structure endowed with the bracket operation $(x,y)\mapsto [x,y]:=xy-yx$ as Lie bracket) has a left adjoint functor U which assigns to a Lie algebra L a unital associative algebra $U(L)$ and a natural Lie algebra morphism $\rho_L\colon L\to U(L)_{\rm Lie}$ such that for each Lie algebra morphim $f\colon L\to X_{\rm Lie}$ for a unital algebra X there is a unique morphism of unital algebras $f'\colon U(L)\to X$ such that $f=f'_{\rm Lie}\circ\rho_L$. The algebra $U(L)$ is called the *universal enveloping algebra of* L. A large body of text book literature is available on it. A prominent result is the Poincaré-Birkhoff-Witt Theorem on the structure of $U(L)$ which implies in particular that $\rho_L\colon L\to U(L)_{\rm Lie}$ is injective.

From the Theorem of Poincaré, Birkhoff and Witt it is known that ρ_L is injective. One may therefore assume that $L\subseteq U(L)$ such that ρ_L is the inclusion function. (See [1] or [3].) In this parlance the universal property reads as follows:

For each unital algebra A, each Lie algebra morphism $f\colon L\to A_{\rm Lie}$ extends uniquely to an algebra morphism $f'\colon U(L)\to A$.

Also from the Theorem of Poincaré, Birkhoff and Witt we know that

$U(L)$ is the unital algebra generated by L, i.e., $U(L)=\langle L\rangle$.

In the present section we shall now denote by $|\mathfrak{g}|$ the abstract Lie algebra underlying the profinite-dimensional Lie algebra $\mathfrak{g}$. Then the main result of this section will be a complete clarification of the relation of the weakly complete enveloping algebra $\mathbf{U}(\mathfrak{g})$ of a profinite-dimensional Lie algebra $\mathfrak{g}$ and the universal enveloping algebra $U(|\mathfrak{g}|)$ of $|\mathfrak{g}|$.

Lemma 2.1. *For a profinite-dimensional Lie algebra $\mathfrak{g}$ there is a natural morphism $\varepsilon_{\mathfrak{g}}\colon U(|\mathfrak{g}|)\to|\,\mathbf{U}(\mathfrak{g})|$ of unital algebras such that*

(i) *the following diagram is commutative:*

$$\begin{array}{ccc} |\mathfrak{g}| & \xrightarrow{\ \rm incl\ } & U(|\mathfrak{g}|)\\ {\scriptstyle {\rm id}_{|\mathfrak{g}|}}\big\downarrow & & \big\downarrow{\scriptstyle \varepsilon_{\mathfrak{g}}}\\ |\mathfrak{g}| & \xrightarrow[|\,{\rm incl}\,|]{} & |\,\mathbf{U}(\mathfrak{g})|. \end{array}$$

(ii) *The image of $\varepsilon_{\mathfrak{g}}$ is dense in $\mathbf{U}(\mathfrak{g})$.*

(iii) *The morphism $\varepsilon_{\mathfrak{g}}$ is injective if $\mathfrak{g}$ is finite-dimensional.*

Proof. (i) The claim is a direct consequence of the universal property of the functor U.

(ii) We have ${\rm im}(\varepsilon_{\mathfrak{g}})=\varepsilon_{\mathfrak{g}}(U(|\mathfrak{g}|))=\varepsilon_{\mathfrak{g}}(\langle|\mathfrak{g}|\rangle=\langle\varepsilon_{\mathfrak{g}}(\mathfrak{g})\rangle=\langle|\mathfrak{g}|\rangle$ in $|\,\mathbf{U}(\mathfrak{g})|$. From Corollary 1.6 we know that $\langle|\mathfrak{g}|\rangle$ is dense in $\mathbf{U}(\mathfrak{g})$.

(iii) If $\mathfrak{g}$ is finite-dimensional, then every finite-dimensional Lie algebra representation $\rho\colon\mathfrak{g}\to A_{\rm Lie}={\rm End}(V)_{\rm Lie}$ for a finite-dimensional vector space V extends to an associative representation $\rho'\colon\ \mathbf{U}(\mathfrak{g})\to{\rm End}(V)$. Then $\rho\circ\varepsilon_{\mathfrak{g}}\colon U(\mathfrak{g})\to{\rm End}(V)$ is an extension to an associative representation of $U(\mathfrak{g})$ which is unique. By Harish-Chandra's Lemma (see Dixmier [3], 2.5.7), the extensions of associative representations of $U(\mathfrak{g})$ of finite-dimensional Lie algebra representations of $\mathfrak{g}$ separate the points of $U(\mathfrak{g})$ and so the claim follows. ∎

The remainder of this section now is devoted to removing the restriction to finite-dimensionality in Lemma 2.1(iii). That is, we want to show

Lemma 2.2. *For a profinite-dimensional Lie algebra* $\mathfrak{g}$*, the algebra morphism* $\varepsilon_{\mathfrak{g}}\colon U(|\mathfrak{g}|) \to |\,\mathbf{U}(\mathfrak{g})|$ *is injective.*

The proof will occupy the remainder of this section. We shall resort to the existing literature on $U(L)$ such as [1] or [3]. We are given the profinite-dimensional Lie algebra $\mathfrak{g}$ and we write $L := |\mathfrak{g}|$ for the underlying Lie algebra. So $L \subseteq U(L)_{\rm Lie}$. Let B be a totally ordered basis of L. We begin by recalling the following basic fact from the Poincaré-Birkhoff-Witt Theorem (see [1], Corollary 3, Section 7 of Paragraph 2):

(PBW) $\widetilde{B}{:=}\{b_1b_2\cdots b_m|1{\le}m,\ b_1,\dots,b_m\in B,\ b_1{\le}b_2{\le}\cdots{\le}b_m\}$ is a basis of $U(L)$.

Now assume that $0\neq u\in U(L)$. Then there is a finite subset $F\subseteq B$ such that $u\in\operatorname{span}(\widetilde{F})$ for

$$\widetilde{F}=\{b_1b_2\cdots b_m|1\le m,\quad b_1,\dots,b_m\in F,\quad b_1\le b_2\le\cdots\le b_m\}\subseteq\widetilde{B}.$$

Lemma 2.3. *There is a closed ideal* J *of* $\mathfrak{g}$ *so that* $J\cap\operatorname{span}F=\{0\}$.

Proof. The vector space $V=\operatorname{span}F$ is finite-dimensional. Let C be the boundary of a compact 0-neighborhood in V. Then $V=\mathbb{K}$.

Returning at this point to the fact that $\mathfrak{g}$ is a profinite-dimensional Lie algebra, we conclude that there is a filterbasis $\mathcal{I}$ of closed ideals I of $\mathfrak{g}$ such that $\dim\mathfrak{g}/I<\infty$ for $I\in\mathcal{I}$ such that $\mathfrak{g}\cong\lim_{I\in\mathcal{I}}\mathfrak{g}/I$. In particular, $\bigcap\mathcal{I}=\{0\}$. Therefore $\bigcap_{I\in\mathcal{I}}(C\cap I)=C\cap\{0\}=\emptyset$. Since C is compact, the filter basis $\{C\cap I : I\in\mathcal{I}\}$ with empty intersection consists of compact sets and therefore must contain the empty set. Thus there is a $J\in\mathcal{I}$ such that $C\cap J=\emptyset$. In fact, we have $J\cap\operatorname{span}F=\{0\}$. Indeed, suppose there were a nonzero $t\in\mathbb{K}$ and a $c\in C$ such that $t\in J$, then $c=t^{-1}(t)\in t^{-1}J=J$ which is impossible. ∎

Lemma 2.4. *Assume that there is an ideal* J *of* L *such that* $J\cap\operatorname{span}F=\{0\}$. *Then the image of* u *under the morphism* $U(L)\to U(L/J)$ *is nonzero.*

Proof. We choose a finite dimensional vector subspace H of L containing F such that $L=H\oplus J$. Let E be a totally ordered basis for H such that the order of E extends that of F, choose a totally ordered basis D of J and make sure that $E\cup D$ has a total order extending the orders of E and D, thus yielding a totally ordered basis of L.

We consider the quotient morphism of Lie algebras $q_J\colon L\to L/J$. Then q_J maps E bijectively onto a basis $E'=q_J(E)$ of L/J. Let

$$\widetilde{E}=\{b_1\cdots b_m|\quad m\ge 1,b_1,\dots,b_m\in E\}.$$

Then $U(q_J)\colon U(L)\to U(L/J)$ maps $\widetilde{E}$ bijectively onto a basis $\widetilde{E'}$ of $U(L/J)$ by (PBW). If we now write $u=\sum_{S\in\widetilde{E}}c_SS$ with $c_S\in\mathbb{K}$, then

$$U(q_J)(u)=\textstyle\sum_{q_J(S)\in\widetilde{E'}}c_Sq_J(S)\neq 0,$$

since $\widetilde{E'}$ is a basis of $U(L/J)$. ∎

As the kernel of the quotient map $U(L) \to U(L/J)$ is $U(L)J$, the claim of the preceding lemma may be expressed equivalently in the form $u \notin U(L)J$.

Now we recall that $q_J\colon L \to L/J \subseteq U(L/J)$ is in fact the underlying Lie algebra morphism $|p_J|$ of a quotient morphism $p_J\colon \mathfrak{g} \to \mathfrak{g}/J$ of profinite Lie algebras and that q_J extends uniquely to an algebra morphism $U(|p_J|)\colon U(L) \to U(L/J)$ with kernel $U(L)J$. Then from Lemma 2.4 we know that $U(|p_J|)(u)$ is nonzero in $U(L/J)$ and from Lemma 2.1 we infer that $\varepsilon_{\mathfrak{g}/J}$ is injective. The commutative diagram

$$\begin{array}{ccccc}
|\mathfrak{g}| & \xrightarrow{\text{incl}} & U(|\mathfrak{g}|) & \xrightarrow{\varepsilon_\mathfrak{g}} & |\,\mathbf{U}(\mathfrak{g})| \\
{\scriptstyle |p_J|}\downarrow & & {\scriptstyle U(|p_J|)}\downarrow & & \downarrow{\scriptstyle |\,\mathbf{U}(p_J)|} \\
|\mathfrak{g}/J| & \xrightarrow{\text{incl}} & U(|\mathfrak{g}/J|) & \xrightarrow{\varepsilon_{\mathfrak{g}/J}} & |\,\mathbf{U}(\mathfrak{g}/J)|
\end{array}$$

then shows that $\varepsilon_\mathfrak{g}(u) \neq 0$. Therefore, since $u \in U(L) \setminus \{0\}$ was arbitrary, $\varepsilon_\mathfrak{g}$ is injective, leaving the elements of $L = |\mathfrak{g}|$ fixed. This completes the proof of Lemma 2.2. Thus $U(|\mathfrak{g}|)$ may be considered as a subalgebra of $|\,\mathbf{U}(\mathfrak{g})|$, containing $|\mathfrak{g}| \subseteq |\,\mathbf{U}(\mathfrak{g})|$.

This may be rephrased in the following Theorem which summarizes our efforts to elucidate the close relation between $U(|\mathfrak{g}|)$ and $\mathbf{U}(\mathfrak{g})$:

Theorem 2.5. (The Relation of $U(-)$ and $\mathbf{U}(-)$) *For any profinite-dimensional real or complex Lie algebra $\mathfrak{g}$ considered as a closed Lie subalgebra of $\mathbf{U}(\mathfrak{g})_{\text{Lie}}$, the associative unital subalgebra $\langle\mathfrak{g}\rangle$ generated algebraically by $\mathfrak{g}$ in $\mathbf{U}(\mathfrak{g})$ is naturally isomorphic to $U(|\mathfrak{g}|)$ (under an isomorphism fixing the elements of $\mathfrak{g}$) and is dense in $\mathbf{U}(\mathfrak{g})$.*

In a slightly careless sense we may memorize this as saying:

For a profinite-dimensional Lie algebra $\mathfrak{g}$, the weakly complete topological enveloping algebra $\mathbf{U}(\mathfrak{g})$ is "a completion of $U(|\mathfrak{g}|)$", and we have

$$\mathfrak{g} \subseteq \langle\mathfrak{g}\rangle = U(\mathfrak{g}) \subseteq \overline{U(\mathfrak{g})} = \mathbf{U}(\mathfrak{g}).$$

3. The projective limit preservation of the weakly complete enveloping functor U

Since every weakly complete unital algebra is a strict projective limit of all finite-dimensional quotient algebras, it will now turn out to be sufficient to test the universal property of the functor $\mathbf{U}$ only for *finite-dimensional* unital associative algebras:

Proposition 3.1. *Assume that the profinite-dimensional Lie algebra $\mathfrak{g}$ is contained functorially in a weakly complete unital algebra $\mathbf{V}(\mathfrak{g})$ such that for each* finite-dimensional *unital algebra A and each morphism of profinite-dimensional Lie algebras $f\colon \mathfrak{g} \to A_{\text{Lie}}$ there is a unique morphism of weakly complete unital algebras $f'\colon \mathbf{V}(\mathfrak{g}) \to A$ extending f. Then $\mathbf{V}(\mathfrak{g}) \cong \mathbf{U}(\mathfrak{g})$ naturally.*

Proof. We apply the Density and Adjunction Theorem 7.5 in the Appendix with $\mathcal{A}$ as the category of weakly complete associative unital algebras, and $\mathcal{B}$ as the category of profinite-dimensional Lie algebras with the full subcategory $\mathcal{B}_d$ of finite-dimensional Lie algebras which is topologically dense in $\mathcal{B}$. Then, by hypothesis, the function $\mathbf{V}\colon \operatorname{ob}\mathcal{B} \to \operatorname{ob}\mathcal{A}$ is conditionally left adjoint to the functor $(\cdot)_{\rm Lie}\colon \mathcal{A} \to \mathcal{B}$ which maps an associative algebra to the Lie algebra with the Lie bracket $[a,b] = ab - ba$ (see Definition 7.3). Then by Theorem 7.5, $\mathbf{V}$ is naturally isomorphic to the left adjoint $\mathbf{U}$ of $(\cdot)_{\rm Lie}$. ■

Perhaps more deeply we shall see now that, while as a left-adjoint functor, $\mathbf{U}$ preserves colimits, it also preserves certain limits, namely, the projective limits $\mathfrak{g} = \lim_{\mathfrak{i}\in\mathcal{I}(\mathfrak{g})} \mathfrak{g}/\mathfrak{i}$ of Definition 1.1. Indeed in Theorem 7.23 in the Appendix we show:

Theorem 3.2. ($\mathbf{U}$ preserves some projective limits) *For a profinite-dimensional Lie algebra $\mathfrak{g}$ with its filter basis $\mathcal{I}(\mathfrak{g})$ of cofinite-dimensional ideals $\mathfrak{i}$ we have*

$$\mathfrak{g} \cong \lim_{\mathfrak{i}\in\mathcal{I}(\mathfrak{g})} \mathfrak{g}/\mathfrak{i} \text{ in } \mathcal{WL} \text{ and } \mathbf{U}(\mathfrak{g}) \cong \lim_{\mathfrak{i}\in\mathcal{I}(\mathfrak{g})} \mathbf{U}(\mathfrak{g}/\mathfrak{i}) \text{ in } \mathcal{WA}.$$

The argument in the Appendix shows, that while the assertion of the theorem is natural and easy to absorb, its proof is deeper than one would expect initially.

4. The weakly complete universal enveloping algebra as a Hopf algebra

We now address the important aspect of enveloping algebras from their beginning, namely, the fact that they are symmetric Hopf algebras. For some of the proofs in this section we refer to our predecessor paper [7].

Proposition 4.1. *The universal enveloping functor $\mathbf{U}$ is multiplicative, that is, there is a natural isomorphism $\mathbf{U}(\mathfrak{g}_1 \times \mathfrak{g}_2) \to \mathbf{U}(\mathfrak{g}_1) \otimes_{\mathcal{W}} \mathbf{U}(\mathfrak{g}_2)$.*

For a proof see [7], Proposition 6.3.

Lemma 4.2. *For any weakly complete unital algebra A, the vector space morphism $\Delta_A\colon A \to A \otimes_{\mathcal{W}} A$, $\Delta_A(a) = a \otimes 1 + 1 \otimes a$ is a morphism of weakly complete Lie algebras $A_{\rm Lie} \to (A \otimes_W A)_{\rm Lie}$.*

Cf. [7], Lemma 6.4.

Recall the natural morphism $\lambda_{\mathfrak{g}}\colon \mathfrak{g} \to \mathbf{U}(\mathfrak{g})_{\rm Lie}$ which we consider as an inclusion morphism. By Lemma 4.2,

$$p_{\mathfrak{g}} = \delta_{\mathbf{U}(\mathfrak{g})} \circ \lambda_{\mathfrak{g}}\colon \mathfrak{g} \to (\mathbf{U}(\mathfrak{g}) \otimes_{\mathcal{W}} \mathbf{U}(\mathfrak{g}))_{\rm Lie}$$

is a morphism of weakly complete Lie algebras. By the universal property of $\mathbf{U}$, $p_{\mathfrak{g}}$ yields a unique natural morphism of weakly complete associative unital algebras $\gamma_{\mathfrak{g}}\colon \mathbf{U}(\mathfrak{g}) \to \mathbf{U}(\mathfrak{g}) \otimes_{\mathcal{W}} \mathbf{U}(\mathfrak{g})$ such that $p_{\mathfrak{g}} = (\gamma_{\mathfrak{g}})_{\rm Lie} \circ \lambda_{\mathfrak{g}}$. Recall the augmentation $\alpha_{\mathfrak{g}}\colon \mathbf{U}_{\mathbb{K}}(\mathfrak{g}) \to \mathbb{K}$ (see Remark 1.5) and the inclusion morphism $\iota_{\mathbf{U}(\mathfrak{g})}\colon \mathbb{K} \to \mathbf{U}_{\mathbb{K}}(\mathfrak{g})$ (see Remark 8.4). Accordingly, we have an idempotent endomorphism

$$\iota_{\mathbf{U}_K(\mathfrak{g})} \circ \alpha_{\mathfrak{g}}\colon \mathbf{U}_{\mathbb{K}}(\mathfrak{g}) \to \mathbf{U}_{\mathbb{K}}(\mathfrak{g}).$$

Further, the augmentation $\alpha_{\mathfrak{g}}$ acts as coidentity, and the function

$$x \mapsto -x : \mathbf{U}(\mathfrak{g}) \to \mathbf{U}(\mathfrak{g})$$

as symmetry as is readily checked for $x \in \mathfrak{g}$, and $\mathfrak{g}$ generates $\mathbf{U}_{\mathbb{K}}(\mathfrak{g})$ as topological algebra by Corollary 1.6. Now we have

Proposition 4.3. ($\mathbf{U}(\mathfrak{g})$ as a Hopf algebra) (a) *Each weakly complete enveloping algebra* $\mathbf{U}(\mathfrak{g})$ *is a weakly complete symmetric Hopf algebra with the comultiplication* $\gamma_\mathfrak{g}$ *and the augmentation* $\alpha_\mathfrak{g}\colon \mathbf{U}(\mathfrak{g}) \to \mathbb{K}$ *as coidentity.*

(b) *If* $f\colon \mathfrak{g} \to \mathfrak{h}$ *is a morphism of profinite-dimensional Lie algebras, then the morphism* $\mathbf{U}_\mathbb{K}(f)\colon \mathbf{U}_K(\mathfrak{g}) \to \mathbf{U}_K(\mathfrak{h})$ *respects comultiplication, coidentity, and symmetry, that is,* $\mathbf{U}_\mathbb{K}(f)$ *is a morphism of symmetric Hopf algebras.*

Proof. For (a), see [7], Corollary 6.5.

For (b) we consider a morphism $f\colon \mathfrak{g} \to \mathfrak{h}$ a morphism of profinite-dimensional Lie algebras and for the functoriality of $\mathbf{U}$ (short for $\mathbf{U}_K(-)$) as regards to comultiplication $\gamma_\mathfrak{g}\colon \mathbf{U}(\mathfrak{g}) \to \mathbf{U}(\mathfrak{g}) \otimes \mathbf{U}(\mathfrak{g})$ we verify the commutativity of the following diagram (with $\otimes = \otimes_\mathcal{W}$)

$$\begin{array}{ccccc}
\mathbf{U}(\mathfrak{g}) & \xrightarrow{\gamma_\mathfrak{g}} & \mathbf{U}(\mathfrak{g}) \otimes \mathbf{U}(\mathfrak{g}) & \xrightarrow{\cong} & \mathbf{U}(\mathfrak{g} \times \mathfrak{g}) \\
{\scriptstyle \mathbf{U}(f)}\downarrow & & {\scriptstyle \mathbf{U}(f)\otimes\mathbf{U}(f)}\downarrow & & \downarrow{\scriptstyle \mathbf{U}(f\times f)} \\
\mathbf{U}(\mathfrak{h}) & \xrightarrow[\gamma_\mathfrak{h}]{} & \mathbf{U}(\mathfrak{h}) \otimes \mathbf{U}(\mathfrak{h}) & \xrightarrow[\cong]{} & \mathbf{U}(\mathfrak{h} \times \mathfrak{h}).
\end{array}$$

(See also Proposition 4.1.) Coidentity and symmetry are treated similarly. ∎

This proposition expresses the fact that $\mathbf{U}_\mathbb{K}$ is a functor from the category of profinite-dimensional Lie algebras to the category of weakly complete symmetric Hopf algebras. Its significance is emphasised by the fact that essential portions of the noteworthy theory of weakly complete symmetric Hopf algebras have meanwhile entered the textbook literature. (See [11], Appendix A3, Appendix A7, Chapter 3–Part 3.) We have collected some essential features in our Appendix such as Theorems 9.2 and 9.6.

Now we specialize these to the case of $A = \mathbf{U}_\mathbb{K}(\mathfrak{g})$. We use the common notation $\mathbb{R}_< = \{r \in \mathbb{R} : 0 < r\}$ and recall that $\mathfrak{g} \subseteq A$ and that $A^\times$ denotes the group of units of A. For the exponential function $\exp\colon A_{\mathrm{Lie}} \to A^\times$ as in Theorem 9.2 we define the closed subgroup

$$\Gamma^*(\mathfrak{g}) := \overline{\langle \exp \mathfrak{g} \rangle} \subseteq A^\times.$$

The following theorem now is a principal result in the theory of weakly complete enveloping algebras of profinite-dimensional real or complex Lie algebras.

Theorem 4.4. (The Weakly Complete Enveloping Hopf Algebra) *Let* $\mathfrak{g}$ *be a profinite-dimensional Lie algebra and* $\mathbf{U}(\mathfrak{g})$ *its weakly complete enveloping algebra containing* $\mathfrak{g}$ *according to* Theorem 2.5. *Then the following statements hold:*

(a) *The group of units* $\mathbf{U}(\mathfrak{g})^\times$ *is dense in* $\mathbf{U}(\mathfrak{g})$. *It is an almost connected pro-Lie group, connected in the case of* $\mathbb{K} = \mathbb{C}$. *The algebra* $\mathbf{U}(\mathfrak{g})$ *has an exponential function* $\exp\colon \mathbf{U}(\mathfrak{g})_{\mathrm{Lie}} \to \mathbf{U}(\mathfrak{g})^\times$. *The Lie algebra* $\mathfrak{L}(\mathbf{U}(\mathfrak{g})^\times)$ *of* $\mathbf{U}(\mathfrak{g})^\times$ *is (naturally isomorphic to)* $\mathbf{U}(\mathfrak{g})_{\mathrm{Lie}}$.

(b) *The pro-Lie algebra* $\mathbb{P}(\mathbf{U}(\mathfrak{g}))$ *is the Lie algebra of the pro-Lie group* $\mathbb{G}(\mathbf{U}(\mathfrak{g}))$ *of grouplike elements and the restriction and corestriction of* exp *is the exponential function for this group.*

(c) *The profinite-dimensional Lie algebra* $\mathbb{P}(\mathbf{U}(\mathfrak{g}))$ *contains*

$$\mathfrak{g} = \mathbb{P}(U(|\mathfrak{g}|)) = \mathbb{P}(\mathbf{U}(\mathfrak{g})) \cap U(|\mathfrak{g}|).$$

(d) *For* $\mathbb{K} = \mathbb{R}$*, the restriction and corestriction of* exp *yields the exponential function*

$$\exp_{\Gamma^*(\mathfrak{g})} : \mathfrak{g} = \mathfrak{L}(\Gamma^*(\mathfrak{g})) \to \Gamma^*(\mathfrak{g})$$

of that pro-Lie subgroup $\Gamma^*(\mathfrak{g})$ *of* $\mathbf{U}_{\mathbb{R}}(\mathfrak{g})^\times$ *whose Lie algebra is precisely* $\mathfrak{g}$*.*

(e) *Define the hyperplane ideal* $\mathbb{I}$ *as the kernel of the augmentation* $\alpha_{\mathfrak{g}}$*. Then we have*

(i) *for* $\mathbb{K} = \mathbb{R}$*:* $\exp(\mathbf{U}_{\mathbb{R}}(\mathfrak{g})) = (\mathbb{R}_{<}1) \oplus \mathbb{I}$*, an open half space,*

(ii) *for* $\mathbb{K} = \mathbb{C}$*:* $\exp(\mathbf{U}_{\mathbb{C}}(\mathfrak{g})) = \mathbf{U}_{\mathbb{C}}(\mathfrak{g}) \setminus \mathbb{I}$*.*

Proof. For the proofs of (a) and (b) see Theorem 9.2 in the Appendix. The proof of (c) follows from [7], Theorem 3.4 and Theorem 2.5. Cf. also [11], Theorem A3.102 and its proof for $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. The proof of (d) must verify that $\mathfrak{L}(\overline{\langle \mathfrak{g} \rangle}) \subseteq \mathfrak{g}$. This conclusion we derive from [10], Corollary 4.22 and its proof. The proof of (e) follows from Theorem 9.6 in the Appendix. ■

Here are some immediate consequences:

Corollary 4.5. *The weakly complete enveloping algebra* $\mathbf{U}(\mathfrak{g})$ *of a nonzero profinite-dimensional weakly complete Lie algebra* $\mathfrak{g}$ *has nontrivial grouplike elements contained in* $\mathbf{U}(\mathfrak{g})^\times$*. Specifically, there is a pro-Lie subgroup* $\Gamma^*(\mathfrak{g})$ *of grouplike elements whose Lie algebra is isomorphic to* $\mathfrak{g}$ *and whose exponential function is induced by that of* $\mathbf{U}(\mathfrak{g})$*.*

By contrast, on the purely algebraic side, the universal enveloping Hopf algebra $U(L)$ of a Lie algebra L shows no visible nontrivial grouplike elements while a nontrivial weakly complete enveloping algebra always does.

We shall see that even in the case of the smallest possible nonzero candidate $\mathfrak{g} = \mathbb{K}$, the space $\mathbb{P}(\mathbf{U}(\mathfrak{g}))$ is substantially larger than $\mathfrak{g}$ (see Theorem 5.10 below). In the classical setting of the discrete enveloping Hopf algebra in characteristic 0 we have $\mathbb{P}(U(L)) = L$: see e.g. [18], Theorem 5.4 on p. LA 3.10.

Corollary 4.6. *For any profinite-dimensional Lie algebra* $\mathfrak{g}$ *there is a pro-Lie group* G *whose Lie algebra* $\mathfrak{L}(G)$ *may be identified with* $\mathfrak{g}$*.*

Indeed the theorem provides a weakly complete unital algebra A with an exponential function $\exp_A$ such that $\exp_G : \mathfrak{L}(G) \to G$ may be identified with a restriction and corestriction of $\exp_A$.

This is indeed much more than what is historically known as *Sophus Lie's Third Fundamental Theorem.*

4.1. Lie's Third Fundamental Theorem for profinite-dimensional Lie algebras

It is worthwhile to elucidate the insight that our present context throws a new light on Lie's Third Fundamental Theorem. Therefore we recall the contemporary aspect of this background:

Theorem 4.7. (Sophus Lie's Third Principal Theorem) *For every profinite-dimensional real Lie algebra* $\mathfrak{g}$ *there is a simply connected pro-Lie group* $\Gamma(\mathfrak{g})$*, whose Lie algebra* $\mathfrak{L}(\Gamma(\mathfrak{g}))$ *is (isomorphic to)* $\mathfrak{g}$*. For any pro-Lie group* G *with Lie algebra* $\mathfrak{g}$ *there is a quotient morphism* $\alpha_{\mathfrak{g}}\colon\Gamma(\mathfrak{g})\to G$ *such that the following diagram commutes:*

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{=} & \mathfrak{g} \\ {\scriptstyle \exp_{\Gamma(\mathfrak{g})}}\downarrow & & \downarrow{\scriptstyle \exp_G} \\ \Gamma(\mathfrak{g}) & \xrightarrow[\alpha_{\mathfrak{g}}]{} & G. \end{array}$$

For a systematic proof see [9], or e.g. [10], Chapter 6, p. 249, see notably Theorem 6.4, p. 232. Our Theorem 4.7 is also cited in [11], Theorem A7.29. For the definition of simple connectivity see [11], Definition A2.6. Let us recall here that for an abelian $\mathfrak{g}$ (that is, a weakly complete real vector space), the underlying vector space of $\mathfrak{g}\cong\mathfrak{L}(\Gamma(\mathfrak{g}))$ is isomorphic to $\Gamma(\mathfrak{g})$ via $\exp_{\mathfrak{L}(\Gamma(\mathfrak{g}))}\colon\mathfrak{L}(\Gamma(\mathfrak{g}))\to\Gamma(\mathfrak{g})$.
Theorem 4.7 applies at once to $G=\Gamma^*(\mathfrak{g})$ as follows:

Corollary 4.8. *For each profinite-dimensional real Lie algebra* $\mathfrak{g}$ *there is a natural morphism* $\alpha_{\mathfrak{g}}\colon\Gamma(\mathfrak{g})\to\Gamma^*(\mathfrak{g})$ *such that the following diagram commutes:*

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{=} & \mathfrak{g} \\ {\scriptstyle \exp_{\Gamma_{\mathfrak{g}}}}\downarrow & & \downarrow{\scriptstyle \exp_{\Gamma^*(\mathfrak{g})}} \\ \Gamma(\mathfrak{g}) & \xrightarrow[\alpha_{\mathfrak{g}}]{} & \Gamma^*(\mathfrak{g}) \end{array}$$

The pro-Lie group $A^\times$ of units of a weakly complete unital associative algebra A has the property that finite-dimensional continuous representations separate the points, and so any pro-Lie group injected into such a group $A^\times$ shares this property. Consider the Lie algebra $\mathfrak{g}=\mathrm{sl}(2,\mathbb{R}))$ of the Lie group $G=\mathrm{SL}(2,\mathbb{R})$, and let $\widetilde{G}=\Gamma(\mathfrak{g})$ be the universal covering group of G. Every continuous linear representation of $\widetilde{G}$, however, factorizes through G (see [6], p.590, Example 16.1.8), and therefore $\Gamma(\mathfrak{g})$ cannot be injected into any group of the form $A^\times$. Hence for $\mathfrak{g}=\mathrm{sl}(2,\mathbb{R})$ the morphism $\alpha_{\mathfrak{g}}\colon\Gamma(\mathfrak{g})\to\Gamma^*(\mathfrak{g})$ cannot be injective and thus certainly cannot be an isomorphism.

5. The Abelian case

For an abelian Lie algebra $\mathfrak{g}$, the weakly complete unital algebra $\mathbf{U}(\mathfrak{g})$ is commutative. In various special aspects we considered this situation in [2], Lemmas 3.4, 3.5, and 3.10ff., and in [7], Section 5 and Example 6.2. We now return to the commutative situation more systematically and discuss the structure of $\mathbf{U}_{\mathbb{K}}(\mathfrak{g})$ completely for $\dim\mathfrak{g}=1$, and derive consequences for the abelian case in general.

5.1. The power series algebra

A first and simplest step is the discussion of the power series algebra. We recall the notation $\mathbb{N}=\{1,2,3,\dots\}$ and $\mathbb{N}_0=\{0,1,2,3\dots\}$. The set $\mathbb{N}_0$ is a semiring for addition and multiplication (that is, an addition and multiplicative commutative monoid with distributivity).

Lemma 5.1. *The weakly complete vector space* $\mathbb{S}:=\mathbb{K}^{\mathbb{N}_0}$ *supports a monoid multiplication called convolution as follows: Let* $\mathbf{a}=(a_k)_{k\in\mathbb{N}_0}\in\mathbb{S}$ *and* $\mathbf{b}=(b_m)_{m\in\mathbb{N}_0}\in\mathbb{S}$.

Then
$$\mathbf{a} * \mathbf{b} = \Big(\sum_{k+m=n} a_k b_m\Big)_{n\in\mathbb{N}_0}. \tag{1}$$

With pointwise addition and convolution, $\mathbb{S} = (\mathbb{S}, +, *)$ *is a weakly complete topological algebra.*

Proof. The verification is an exercise: We write $X = (0,1,0,0,\dots)$ and observe

$$\begin{array}{rcl} X & = & (0,1,0,0,0,\dots) \\ X^2 & = & (0,0,1,0,0,\dots) \\ X^3 & = & (0,0,0,1,0,\dots) \\ \vdots & & \vdots \end{array}$$

Then $\mathbf{a} * X = (a_n)_{n\in\mathbb{N}} * X = a_0X + a_1X^2 + a_2X^3 + \cdots$.
The projective limit representation $\mathbb{S} \cong \lim_{n\in\mathbb{N}} \frac{\mathbb{K}[X]}{X^{n+1}\mathbb{K}[X]}$ completes the proof. ∎

Accordingly, $\mathbb{S}$ is called the *power series algebra in one variable*, usually written as $\mathbb{K}[[X]]$.

It is useful to recall that in the category $\mathcal{W}$ of weakly complete $\mathbb{K}$-vector spaces, for any pair of sets $\mathbf{X}$ and $\mathbf{Y}$ we have a natural isomorphism

$$\mathbb{K}^{\mathbf{X}} \otimes_{\mathcal{W}} \mathbb{K}^{\mathbf{Y}} \to \mathbb{K}^{\mathbf{X}\times\mathbf{Y}}$$

induced by the bijection $(a_x)_{x\in\mathbf{X}} \otimes (b_y)_{y\in\mathbf{Y}}) \mapsto (a_xb_y)_{(x,y)\in\mathbf{X}\times\mathbf{Y}}$.
We have a multiplication $\mu\colon \mathbb{S}\otimes_{\mathcal{W}}\mathbb{S} \to \mathbb{S}$ according to

$$\mu(\mathbf{a}\otimes\mathbf{b}) = \Big(\sum_{k+m=n} a_k b_m\Big)_{n\in\mathbb{N}_0}.$$

We write $\mathbf{1} := (1,0,0,\dots)$ and set

$$X_1 := X\otimes\mathbf{1} = (0,1,0,\dots)\otimes(1,0,0,\dots) \in \mathbb{S}\otimes_{\mathcal{W}}\mathbb{S},$$

and
$$X_2 := \mathbf{1}\otimes X = (1,0,0,\dots)\otimes(0,1,0,\dots) \in \mathbb{S}\otimes_{\mathcal{W}}\mathbb{S}.$$

So we obtain $X_1X_2 = X_2X_1$ in $\mathbb{S}\otimes_{\mathcal{W}}\mathbb{S}$ and compute

$$\Big(\sum_{m\in\mathbb{N}_0} a_m X_1^m\Big)\Big(\sum_{n\in\mathbb{N}_0} b_n X_2^n\Big) = \sum_{m,n\in\mathbb{N}_0} a_m b_n X_1^m X_2^n$$

for $\mathbf{a} = (a_m)_{m\in\mathbb{N}_0}$ and $\mathbf{b} = (b_n)_{n\in\mathbb{N}_0}$ in $\mathbb{S}$. Thus

$$\mathbb{S}\otimes_{\mathcal{W}}\mathbb{S} = \Big\{\sum\nolimits_{(m,n)\in\mathbb{N}_0\times\mathbb{N}_0} c_{mn}X_1^mX_2^n : c_{mn}\in\mathbb{K}\Big\}$$

is the ring of power series in two commuting variables. Write $\mathbf{a} = \sum_{n\in\mathbb{N}_0} a_nX^n \in \mathbb{S}$ and note $\mathbf{a}\otimes\mathbf{1} = \sum_{m\in\mathbb{N}_0} a_mX_1^m$ and $\mathbf{1}\otimes\mathbf{a} = \sum_{n\in\mathbb{N}_0} a_nX_2^n$ in $\mathbb{S}\otimes_{\mathcal{W}}\mathbb{S}$. We then have two morphisms of vector spaces $\Delta,\gamma\colon \mathbb{S}\to\mathbb{S}\otimes_{\mathcal{W}}\mathbb{S}$ as follows:

For $\mathbf{a} = \sum_{n\in\mathbb{N}_0} a_nX^n$

$$\Delta(\mathbf{a}) := \mathbf{a}\otimes\mathbf{1} + \mathbf{1}\otimes\mathbf{a} = \sum_{m,n\in\mathbb{N}_0} (a_mX_1^m + a_nX_2^n) \tag{2}$$

and
$$\gamma(\mathbf{a}) := \gamma\Big(\sum_{n\in\mathbb{N}_0} a_nX^n\Big) = \sum_{n\in\mathbb{N}_0} a_n(X_1+X_2)^n, \tag{3}$$

where γ is in fact a morphism of weakly complete algebras.

Also, there is an identity $\epsilon\colon \mathbb{K} \to \mathbb{S}$, $\epsilon(t) = t\mathbf{1}$. and a coidentity (or *augmentation* $\kappa\colon \mathbb{S} \to \mathbb{K}$ given by $\kappa(\mathbf{a}) = \kappa((a_n)_{n\in\mathbb{N}_0}) := a_0$, and a symmetry $\sigma\colon \mathbb{S} \to \mathbb{S}$ given by $\sigma(X) := -X$, that is $\sigma(\mathbf{a}) = \sigma((a_n)_{n\in\mathbb{N}_0}) := ((-1)^n a_n)_{n\in\mathbb{N}}$. The following diagram is commutative:

$$\begin{array}{ccc}
\mathbb{S} \otimes_{\mathcal{W}} \mathbb{S} & \xrightarrow{\sigma\otimes \mathrm{id}} & \mathbb{S} \otimes \mathbb{S} \\
\gamma \uparrow & & \downarrow \mu \\
\mathbb{S} & \xrightarrow{\kappa\circ\epsilon} & \mathbb{S}.
\end{array}$$

Thus $\mathbb{S}$ is a weakly complete commutative symmetric Hopf algebra.

Let us discuss its *primitive* and *grouplike* elements:

An element $\mathbf{a} = (a_n)_{n\in\mathbb{N}_0} \in \mathbb{S}$ is *primitive* if and only if $\Delta(\mathbf{a}) = \gamma(\mathbf{a})$, that is, by (2) and (3), if and only if

$$\sum_{m,n\in\mathbb{N}_0} (a_m X_1^m + a_n X_2^n) = \sum_{n\in\mathbb{N}_0} a_n (X_1 + X_2)^n, \tag{4}$$

if and only if $n \neq 1 \implies a_n = 0$ if and only if $\mathbf{a} = tX$ for some $t \in \mathbb{K}$. Thus

(PR) $$\mathbb{P}(\mathbb{S}) = \mathbb{K}X.$$

An element $\mathbf{a}$ is *grouplike* if and only if it is nonzero and satisfies $\gamma(\mathbf{a}) = \mathbf{a} \otimes \mathbf{a}$ if and only if

$$\sum_{n\in N_0} a_n (X_1 + X_2)^n = \sum_{m,n\in\mathbb{N}_0} a_m X_1^m a_n X_2^n, \tag{5}$$

which is the case if and only if $$(\forall n \in \mathbb{N}_0)\ \ a_n = \frac{1}{n!}. \tag{6}$$

Thus

(GR) $$\mathbb{G}(\mathbb{S}) = \exp \mathbb{K}X = \begin{cases} \exp \mathbb{R}X, & \text{if } \mathbb{K} = \mathbb{R}, \\ (\exp \mathbb{R}X)(\exp \mathbb{R}iX), & \text{if } \mathbb{K} = \mathbb{C}. \end{cases}$$

This confirms the general result in [2], Theorem 6.15.

It is now urgent that we precisely describe the *exponential function* $\exp\colon \mathbb{S} \to \mathbb{S}^\times$:

For any field $\mathbb{K}$ we write $\mathbb{K}^\times$ for $\mathbb{K} \setminus \{0\}$. We set $\mathbf{I} := \{\mathbf{a} : a_0 = 0\}$. Then $\mathbf{I}$ is the maximal ideal of $\mathbb{S}$ with $\mathbb{S}/\mathbf{I} \cong \mathbb{K}$. Notice that we have $\mathbb{S} = \mathbb{K}\mathbf{1} \oplus \mathbf{I}$ and $\exp(t\mathbf{1} + x) = (\exp t)(\exp x)$ for $t \in \mathbb{K}$ and $x \in \mathbf{I}$. In particular,

$$\exp \mathbb{S} = (\exp \mathbb{K}) \exp(\mathbf{1} + \mathbf{I}), \text{ where } \exp \mathbb{K} = \begin{cases} \mathbb{R}_< = \{r \in \mathbb{R} : 0 < r\} & \text{if } \mathbb{K} = \mathbb{R}, \\ \mathbb{C}^\times & \text{if } \mathbb{K} = \mathbb{C}. \end{cases}$$

Lemma 5.2. (i) *For $\mathbf{a} = (a_n)_{n\in\mathbb{N}} \in \mathbb{S}$ we have $\mathbf{a} \in \mathbb{S}^\times$ if and only if $a_0 \neq 0$ and so $\mathbb{S}^\times = \mathbb{S} \setminus \mathbf{I}$.*

(ii) *The function $\exp\colon \mathbb{K} \to \mathbb{K}^\times$ maps $\mathbb{R}$ bijectively onto $\mathbb{R}_<$ if $\mathbb{K} = \mathbb{R}$ and $\mathbb{C}$ surjectively onto $\mathbb{C}^\times$ with kernel $2\pi i\mathbb{Z}$ if $\mathbb{K} = \mathbb{C}$.*

The function $\exp\colon \mathbf{I} \to \mathbf{1} + \mathbf{I}$ is bijective with inverse $\log\colon \mathbf{1} + \mathbf{I} \to \mathbf{I}$.

(iii) $\exp\colon \mathbb{S} = \mathbb{K}\mathbf{1} \oplus \mathbf{I} \to \mathbb{S}^\times = \mathbb{K}^\times \times (1 + \mathbf{I})$ *is injective for $\mathbb{K} = \mathbb{R}$ and surjective for $\mathbb{K} = \mathbb{C}$.*

(iv) *The function $\exp\colon \mathbb{P}(\mathbb{S}) \to \mathbb{G}(\mathbb{S})$ is bijective, the inverse function being the logarithm $\log$.*

Proof. (i) If $t := a_0 \neq 0$ then $\mathbf{a} = t(\mathbf{1} - Y)$ where $Y = (0, -a_1/t, a_2/t, \dots)$ and $(\mathbf{1} - Y)^{-1} = \mathbf{1} + Y + Y^2 + \cdots$. So $\mathbf{a}$ is invertible. On the other hand, if $a_0 = 0$, then $\mathbf{ab} = (0, b_0, \dots) = (0, \dots)$ and so $\mathbf{a}$ fails to be invertible.

(ii) and (iii) were shown above. (iv) is immediate from the preceding, since we have $\mathbb{P}(\mathbb{S}) = \mathbb{K}X$ and $\mathbb{G}(\mathbb{S}) = \exp(\mathbb{P}(\mathbb{S}))$. ■

We summarize our results on the power series algebra in one variable:

Proposition 5.3. (The power series algebra $\mathbb{K}[[X]]$) (i) *The weakly complete power series algebra* $\mathbb{S} := \mathbb{K}[[X]] = (\mathbb{K}^{\mathbb{N}_0}, +, *)$ *is a singly generated weakly complete symmetric Hopf algebra generated by the element* X.

(ii) $\mathbb{S}$ *is a local weakly complete algebra with maximal ideal*

$$\mathbf{I} = \Big\{ \sum_{n=1}^{\infty} a_n X^n : \mathbf{a}{=}(0, a_1, a_2, \dots){\in}\mathbb{K}^{\mathbb{N}_0} \Big\} \quad \textit{and} \quad \mathbb{S}^\times{=}\mathbb{S} \setminus \mathbf{I}.$$

Further, $\mathbb{S} = \mathbb{K}\mathbf{1} \oplus \mathbf{I}$ *and* $(\forall t \in \mathbb{K}, x \in \mathbf{I})\ \exp(t\mathbf{1} + x) = e^t \exp x$.

(iii) $\exp\colon \mathbf{I} \to \mathbf{1} + \mathbf{I}$ *has the inverse* log *and therefore implements an isomorphism of pro-Lie groups* $(\mathbf{I}, +) \cong (\mathbf{1} + \mathbf{I},\,)$

(iv) *The additive group* $\mathbb{P}(\mathbb{S})$ *of primitive elements is* $\mathbb{K}\cdot X$, *the multiplicative group* $\mathbb{G}(A)$ *of grouplike elements is* $(\exp \mathbb{K}X, *)$.

Recall that on $\mathbb{K}$ (with $\mathbb{T} = \mathbb{R}/\mathbb{Z}$) we have

$$\exp \mathbb{K} = \begin{cases} \mathbb{R}_< = \{r \in \mathbb{R} : 0 < r\} \cong (\mathbb{R}, +), \text{ if } \mathbb{K} = \mathbb{R}, \text{ and} \\ \mathbb{C}^\times \cong (\mathbb{R} \oplus \mathbb{T}, +), \text{ if } \mathbb{K} = \mathbb{C}. \end{cases}$$

By Proposition 5.3 we have the following isomorphisms of abelian pro-Lie groups

$$(\mathbb{S}, +) = \mathbb{K}1 \oplus \mathbf{I} \cong (\mathbb{K}, +) \times (\mathbf{I}, +) \cong (\mathbb{K}, +)^{\mathbb{N}_0},$$

and

$$(\mathbb{S}^\times,\,){=}(\mathbb{S} \setminus \mathbf{I},\,){=}(\mathbb{K}^\times,\,)(\mathbf{1} + \mathbf{I},\,){\cong}(\mathbb{K}^\times,\,) \times (\mathbb{K}, +)^{\mathbb{N}},$$

$$(e^{\mathbb{K}},\,){\cong} \begin{cases} (\mathbb{R}_<,\,), \text{ if } \mathbb{K}{=}\mathbb{R}, \\ (\mathbb{C}^\times, \cdot), \text{ if } \mathbb{K}{=}\mathbb{C}. \end{cases}$$

Note also that on the level of primitive and grouplike elements we have simply

$$\mathbb{P}(\mathbb{S}) = \mathbb{K}X \cong (\mathbb{K}, +), \quad \mathbb{G}(\mathbb{S}) = \exp(\mathbb{K}X) \cong \begin{cases} (\mathbb{R}, +), & \mathbb{K} = \mathbb{R}, \\ \mathbb{R} \times \mathbb{T}, & \mathbb{K} = \mathbb{C}. \end{cases}.$$

Here one should keep in mind the example of the power series algebra over $\mathbb{K}$:

$$\mathbb{S} = \mathbb{K}[[X]] \quad (\cong \mathbb{K}^{\mathbb{N}_0} \quad \text{in } \mathcal{W}).$$

5.2. The universal monothetic algebra

We know from [2] that there is a singly generated universal weakly complete algebra $\mathbb{K}\langle X\rangle = \mathbf{U}_{\mathbb{K}}(\mathfrak{g})$, where $\mathfrak{g} = \mathbb{K}$ is the one-dimensional Lie algebra. At his point we shall also discuss the weakly complete symmetric Hopf algebra structure of $\mathbb{K}\langle X\rangle$.

For the following we refer to [2], Corollary 3.3ff. The defining fact of $\mathbb{K}\langle X\rangle$ is the following universal property:

(UP) *For each weakly complete unital algebra A in $\mathcal{WA}$ and each element $a \in A$ there is a unique $\mathcal{WA}$-morphism $\phi\colon \mathbb{K}\langle X\rangle \to A$ such that $\phi(X) = a$.*

We shall see that the internal structure of $\mathbb{K}\langle X\rangle$ is more complicated overall than one might expect initially.

It is clear that without loss of generality we may assume that A is abelian. By Theorem 7.1, we have $A \cong \lim_{I\in\mathcal{J}(A)} A/I$ and so (UP) holds if and only if it holds for all finite-dimensional commutative algebras A. However, this universal property is satisfied exactly by the weakly complete algebra $\lim_{J\in\mathbf{I}(\mathbb{K}[x])} \mathbb{K}[x]/J$ for the polynomial ring $\mathbb{K}[x]$ in one variable x over $\mathbb{K}$ and the filter basis of all of its ideals $\mathbf{I}(\mathbb{K}[x])$. Since $\mathbb{K}[x]$ is a principal ideal domain, every $J \in \mathbf{I}(\mathbb{K}[x])$ is of the form $J = (f) = f\mathbb{K}[x]$ for some polynomial $f \in \mathbb{K}[x]$. We may assume

$$\mathbb{K}\langle X\rangle = \lim_{J\in\mathbf{I}(\mathbb{K}[x])} \frac{\mathbb{K}[x]}{J} = \lim_{f\in\mathbb{K}[x]} \frac{\mathbb{K}[x]}{(f)} \subseteq \prod_{f\in\mathbb{K}[x]} \frac{\mathbb{K}[x]}{f\mathbb{K}[x]}, \tag{7}$$

generated by $X := (x + f\mathbb{K}[x])_{f\in\mathbb{K}[x]} \in \mathbb{K}\langle X\rangle$. (See also [2], Lemma 3.4.)

We let $\mathfrak{P} = \mathfrak{P}_{\mathbb{K}}$ denote the set of the irreducible polynomials p over $\mathbb{K}$ with leading coefficient 1 from the polynomial ring $\mathbb{K}[x]$. Then $f \in \mathbb{K}[x]$, by the Chinese Remainder Theorem, is of the form

$$f = t\prod\nolimits_{p\in\mathfrak{P}} p^{k_p} \quad \text{some } t \in \mathbb{K} \text{ and } (k_p)_{p\in\mathfrak{P}} \in {\mathbb{N}_0}^{(\mathbb{N}_0)},$$

where ${\mathbb{N}_0}^{(\mathbb{N}_0)}$ denotes the set of all families of nonnegative integers vanishing with the exception of indices p from a finite subset of $\mathfrak{P}$. Then for all $f \in \mathbb{K}[x]$ we have

$$\frac{\mathbb{K}[x]}{(f)} \cong \prod_{p\in\mathfrak{P}} \frac{\mathbb{K}[x]}{(p^{k_p})}. \tag{8}$$

Now $\{\mathbb{K}[x]/p^n\mathbb{K}[x] : n \in \mathbb{N}\}$ is a projective system and we introduce the notation

$$\mathbb{S}_p := \lim_{n\in\mathbb{N}} \frac{\mathbb{K}[x]}{p^n\mathbb{K}[x]} \tag{9}$$

which is is a weakly complete commutative algebra generated by

$$X_p := (x + p^n\mathbb{K}[x])_{n\in\mathbb{N}} \in \lim_{n\in\mathbb{N}} \frac{\mathbb{K}[x]}{p^n\mathbb{K}[x]}.$$

In $\mathbb{S}_p$ we have a maximal ideal $\mathbf{I}_p := X_p\mathbb{S}_p$ so that $\mathbb{S}_p/\mathbf{I}_p \cong \mathbb{K}$ and $\mathbb{S}_p = \mathbb{K}\mathbf{1} \oplus \mathbf{I}_p$.

We conclude
$$\mathbb{K}\langle X\rangle = \prod_{p\in\mathfrak{P}_{\mathbb{K}}} \mathbb{S}_p, \quad X = (X_p)_{p\in\mathfrak{P}_{\mathbb{K}}}. \tag{10}$$

In the interest of brevity again, we shall also write $\mathbf{S}$ in the place of $\mathbb{K}\langle X\rangle$. For easy reference we summarize the preceding discussion in the following lemma:

Lemma 5.4. (i) *We have $\mathbf{S} = \prod_{p\in\mathfrak{P}_{\mathbb{K}}} \mathbb{S}_p$. For each $p \in \mathfrak{P}$ the algebra $\mathbb{S}_p$ is generated algebraically and topologically by X_p, and $\mathbf{S}$ is generated algebraically and topologically by $X = (X_p)_{p\in\mathfrak{P}} \in \mathbf{S}$.*

(ii) *For each weakly complete algebra A and each $a \in A$ there is a unique $\mathcal{WA}$-morphism $\phi_a\colon \mathbf{S} \to A$ such that $\phi_a(X) = a$.*

The remainder of this section is devoted to a clarification of the structure of $\mathbb{S}_p$ defined in (3).

Lemma 5.5. *If $p \in \mathfrak{P}$ is of degree 1, then*

$$\mathbb{S}_p \cong \mathbb{K}[[X_p]]. \tag{11}$$

Proof. If $c \in \mathbb{K}$ and $p = x-c$, abbreviate $R := \mathbb{K}[x] \cong \mathbb{K}[x-c]$. Then $x \mapsto x-c$ induces an automorphism of R and $\mathbb{S}_p = \lim_{n\in\mathbb{N}} \frac{R}{(x-c)^n R} \cong \lim_{n\in\mathbb{N}} \frac{R}{x^n R} \cong \mathbb{K}[[X_p]]$. ∎

Since for $\mathbb{K} = \mathbb{C}$ every $p \in \mathfrak{P}_\mathbb{C}$ is of degree 1, we know that in this case $\mathbb{S}_p \cong \mathbb{C}[[X_p]]$ for all $p \in \mathbb{P}$.

Now we assume $\mathbb{K} = \mathbb{R} \subseteq \mathbb{C}$. Then there are two cases:

(a) $p = x - r$ for $r \in \mathbb{R}$. Then $\quad \mathbb{S}_p \cong \mathbb{R}[[X_p]]. \quad (12)$

(b) There is a $c \in \mathbb{C} \setminus \mathbb{R}$ such that

$$p(x) = (x-c)(x-\overline{c}) = x^2 - (c+\overline{c})x + c\overline{c} = x^2 - 2\mathrm{Re}(c)x + |c|^2, \quad p \in \mathfrak{P}_\mathbb{R}.$$

In this case we write $p_1 = x - c$, Im $c > 0$, and $p_2 = x - \overline{c}$, $p_n \in \mathfrak{P}_\mathbb{C}$, $n = 1,2$.

Lemma 5.6. *In the case of* (b) *above, the real algebra $\frac{\mathbb{R}[x]}{p^n\mathbb{R}[x]}$ is isomorphic to the real algebra underlying $\frac{\mathbb{C}[x]}{p_1^n\mathbb{C}[x]}$.*

Proof. We use the abbreviations $R_n = \frac{\mathbb{R}[x]}{p^n\mathbb{R}[x]}$, $C_n = \frac{\mathbb{C}[x]}{p^n\mathbb{C}[x]}$, and $C_{kn} = \frac{\mathbb{C}[x]}{p_k^n\mathbb{C}[x]}$, $k = 1,2$. By the Chinese Remainder Theorem we have an isomorphism

$$\rho\colon C_n \to C_{1n} \times C_{2n} \tag{13}$$

such that $\rho(u + p^n\mathbb{C}[x]) = (u + p_1^n\mathbb{C}[x], u + p_2^n\mathbb{C}[x])$. The real algebra underlying the right hand side of (13) has an involution σ defined by

$$\sigma(u + p_1^n\mathbb{C}[x], v + p_2^n\mathbb{C}[x]) = (\overline{v} + p_1^n\mathbb{C}[x], \overline{u} + p_2^n\mathbb{C}[x]), \tag{14}$$

such that the elements of the real fixed point algebra F of σ are those elements $(u + p_1^n\mathbb{C}[x], \overline{u} + p_2^n\mathbb{C}[x])$ with $u \in \mathbb{C}[x]$. In consequence the restriction of the projection $C_{1n} \times C_{2n} \to C_{1n}$ to F is an isomorphism. Thus $F \cong C_{1n}$ as real algebra. In particular, $\dim_\mathbb{R} F = \dim_\mathbb{R} C_{1n} = 2n = \dim_\mathbb{R} R_n$. Thus the injection $R_n \to F$ via ρ is in fact surjective. Hence $\frac{\mathbb{R}[x]}{p^n\mathbb{R}[x]} = R_n \cong F = \frac{\mathbb{C}[x]}{p_1^n\mathbb{C}[x]}$ as real algebras. ∎

As a consequence we conclude that

$$\mathbb{S}_p = \lim_{n\in\mathbb{N}} \frac{\mathbb{R}[x]}{p^n\mathbb{R}[x]} \cong \lim_{n\in\mathbb{N}} \frac{\mathbb{C}[x]}{p_1^n\mathbb{C}[x]}$$

and thus $\quad \mathbb{S}_p \cong \mathbb{C}[[X_{p_1}]]$ as real algebras. $\quad (15)$

For $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, there is an injection $p \mapsto c_p : \mathfrak{P}_\mathbb{K} \to \mathbb{C}$, where

$$p(x) = \begin{cases} x - c_p, & \text{if } \deg(p) = 1, \\ x - c_{p_1},\ \mathrm{Im}\, c_{p_1} > 0, & \text{if } \deg(p) = 2. \end{cases} \tag{16}$$

If $\mathbb{K}=\mathbb{C}$, then we are in the first case and c_p ranges through all of $\mathbb{C}$. If $\mathbb{K}=\mathbb{R}$, then both cases occur, and in the first case c_p ranges through $\mathbb{K}=\mathbb{R}$ and in the second case c_{p_1} ranges through the open upper complex half-plane.

The different cases now sum up to the following statement:

Lemma 5.7.

$$\mathbb{K}\langle X\rangle=\prod_{p\in\mathfrak{P}_\mathbb{K}}\mathbb{S}_p\cong\begin{cases}\prod_{p\in\mathfrak{P}_\mathbb{R},\deg p=1}\mathbb{R}[[X_p]]\times\prod_{p\in\mathfrak{P}_\mathbb{R},\deg p=2}\mathbb{C}[[X_{p_1}]], & \textit{if } \mathbb{K}=\mathbb{R},\\ \prod_{p\in\mathfrak{P}_\mathbb{C}}\mathbb{C}[[X_p]], & \textit{if } \mathbb{K}=\mathbb{C},\end{cases}\tag{17}$$

where all algebras in the top line are real algebras.

For $p\in\mathfrak{P}_\mathbb{K}$ we write

$$\mathbb{K}_p=\begin{cases}\mathbb{R}, & \text{if } \mathbb{K}=\mathbb{R} \text{ and } \deg p=1,\\ \mathbb{C}, & \text{if either } \mathbb{K}=\mathbb{R} \text{ and } \deg p=2, \text{ or } \mathbb{K}=\mathbb{C}.\end{cases}\tag{18}$$

Moreover, elements in $\mathbb{S}_p$ with $p\in\mathfrak{P}_\mathbb{K}$ we denote by

$$\mathbf{a}_p=\sum_{n\in\mathbb{N}_0}a_{np}{X_p}^n,\quad a_{np}\in\mathbb{K}_p.$$

Finally we have

$$\mathbf{S}:=\mathbb{K}\langle X\rangle=\prod_{p\in\mathfrak{P}_\mathbb{K}}\mathbb{S}_p=\{\mathfrak{a}:=(\mathbf{a}_p)_{p\in\mathfrak{P}_\mathbb{K}}:\mathbf{a}_p\in\mathbb{S}_p\},\tag{19}$$

a weakly complete commutative symmetric Hopf algebra, with componentwise operations and co-operations. By Lemma 5.4(i), the weakly complete algebra $\mathbf{S}$ is the algebraically and topologically singly generated weakly complete algebra with generator

$$X:=(X_p)_{p\in\mathfrak{P}_\mathbb{K}}\in\prod_{p\in\mathfrak{P}_\mathbb{K}}\mathbb{S}_p,\tag{20}$$

For each $p\in\mathfrak{P}_\mathbb{K}$ we define $\mathbf{I}_p\subseteq\mathbb{S}_p$ to be the maximal ideal of $\mathbb{S}_p$, where $\exp\colon\mathbf{I}_p\to 1+\mathbf{I}_p$ has the inverse $\log\colon 1+\mathbf{I}_p\to\mathbf{I}_p$. In view of Lemma 5.2 we observe the following fact:

Remark 5.8. The exponential function $\exp\colon\mathbf{S}\to\mathbf{S}^\times$ is given componentwise for $\mathfrak{a}=(\mathbf{a}_p)_{p\in\mathfrak{P}}$ as $\exp\mathfrak{a}=(\exp\mathbf{a}_p)_{p\in\mathfrak{P}}$.

The exponential function $\exp\colon\mathbb{S}_p\to\mathbb{S}_p$ is surjective if either $\mathbb{K}=\mathbb{R}$ and $\deg p=2$, or $\mathbb{K}=\mathbb{C}$, and it is injective if $\mathbb{K}=\mathbb{R}$ and $\deg p=1$. ∎

We now aim to discuss the restriction of the exponential function to the set of primitive elements. First recall that on $\mathbf{S}=\mathbb{K}\langle X\rangle$ we have the operations

$$\mathfrak{a}+\mathfrak{b}=(\mathbf{a}_p+\mathbf{b}_p)_{p\in\mathfrak{P}} \text{ and } \mathfrak{a}*\mathfrak{b}=(\mathbf{a}_p*\mathbf{b}_p)_{p\in\mathfrak{P}}=\Big(\sum_{n\in\mathbb{N}_0}\Big(\sum\nolimits_{k+m=n}a_{kp}b_{mp}\Big){X_p}^n\Big)_{p\in\mathfrak{P}}.$$

In $\mathbf{S}\otimes_\mathcal{W}\mathbf{S}$, with $\mathfrak{P}=\mathfrak{P}_\mathbb{K}$ we write

$$X_{1p}:=(X_p)_{p\in\mathfrak{P}}\otimes\mathbf{1}=(X_p\otimes\mathbf{1})_{p\in\mathfrak{P}}\in\mathbf{S}\otimes_\mathcal{W}\mathbf{S},$$

and

$$X_{2p}:=\mathbf{1}\otimes(X_p)_{p\in\mathfrak{P}}=(\mathbf{1}\otimes X_p)_{p\in\mathfrak{P}}\in\mathbf{S}\otimes_\mathcal{W}\mathbf{S},$$

Again we may consider $\mathbf{S} \otimes_{\mathcal{W}} \mathbf{S}$ as weakly complete power series algebra with commuting variables X_{1p} and X_{2p} with p ranging through $\mathfrak{P}$.

The identity and coidentity $\epsilon\colon \mathbb{K} \to \mathbf{S}$, $\kappa\colon \mathbf{S} \to \mathbb{K}$ (augmentation), and symmetry $\sigma\colon \mathbf{S} \to \mathbf{S}$ are straightforward from the respective operations in $\mathbb{S}$, but let us also consider the diagonal vector space morphism Δ and the algebra comultiplication γ:

$$\Delta, \gamma : \mathbf{S} \to \mathbf{S} \otimes_{\mathcal{W}} \mathbf{S} \quad \text{defined as follows:}$$

For $\mathfrak{a} = \sum_{(n,p)\in\mathbb{N}_0\times\mathfrak{P}} a_{np} {X_p}^n$ we have

$$\Delta(\mathfrak{a}) := \mathfrak{a} \otimes \mathbf{1} + \mathbf{1} \otimes \mathfrak{a} = \sum_{m,n\in\mathbb{N}_0, f,g\in\mathfrak{P}} (a_{mp}{X_{1p}}^m + a_{ng}{X_{2g}}^n) \tag{21}$$

and

$$\gamma(\mathfrak{a}) = \sum_{n\in\mathbb{N}_0, p\in\mathfrak{P}} a_{np}(X_{1p} + X_{2p})^n. \tag{22}$$

We have the commutative diagram

$$\begin{array}{ccc} \mathbf{S} \otimes_{\mathcal{W}} \mathbf{S} & \xrightarrow{\sigma\otimes_{\mathcal{W}}\mathrm{id}} & \mathbf{S} \otimes_{\mathcal{W}} \mathbf{S} \\ \gamma\Big\uparrow & & \Big\downarrow\mu \\ \mathbf{S} & \xrightarrow{\kappa\circ\epsilon} & \mathbf{S} \end{array}$$

identifying $\mathbf{S}$ as weakly complete symmetric Hopf algebra allowing us now to turn to the determination of the primitive and grouplike elements of $\mathbf{S}$. Indeed an element $\mathfrak{a}$ is *primitive* in $\mathbf{S}$ if $\gamma(\mathfrak{a}) = \Delta(\mathfrak{a})$, that is, if and only if

$$\sum_{m,n\in\mathbb{N}_0,\ p\in\mathfrak{P}} (a_{mp}{X_{1p}}^m + a_{nf}{X_{2p}}^n) = \sum_{n\in\mathbb{N}_0,\ p\in\mathfrak{P}} a_{np}(X_{1p} + X_{2p})^n$$

if and only if $\qquad (\forall p \in \mathfrak{P})\, n \neq 1 \implies a_{np} = 0$

if and only if $\qquad (\forall p \in \mathfrak{P})(\exists t_p \in \mathbb{K}_p)\, \mathfrak{a} = (t_p X_p)_{p\in\mathfrak{P}}.$

Thus we have

(PR) $$\mathbb{P}(\mathbf{S}) = \prod_{p\in\mathfrak{P}_{\mathbb{K}}} \mathbb{K}_p \cdot X_p.$$

On the other hand, an element $\mathfrak{a}$ is *grouplike* if it is nonzero and satisfies $\gamma(\mathfrak{a}) = \mathfrak{a}\otimes\mathfrak{a}$, that is,

$$\sum_{n\in\mathbb{N}_0, p\in\mathfrak{P}} a_{np}(X_{1p} + X_{2p})^n = \sum_{m,n\in\mathbb{N}_0, p\in\mathfrak{P}} a_{m,p}X_1^m a_{ng}X_2^n,$$

which is the case if and only if $(\forall (n,p) \in \mathbb{N}_0 \times \mathfrak{P})\ a_{np} = \frac{1}{n!}$. Thus we have

(GR) $$\mathbb{G}(\mathbf{S}) = \prod_{p\in\mathfrak{P}_{\mathbb{K}}} \exp(\mathbb{K}_p X_p).$$

Definition 5.9. The algebraically and topologically singly generated universal weakly complete algebra $\mathbf{S} = \mathbb{K}\langle X\rangle$ is called the *universal monothetic algebra.*

Lemma 5.4(i) justifies the name. Recall from Proposition 5.3 that for each $p \in \mathfrak{P}_{\mathbb{K}}$ the algebra $\mathbb{S}_p$ is a local weakly complete algebra with a maximal ideal $\mathbf{I}_p$ and that ${\mathbb{S}_p}^\times = \mathbb{K}_p^\times(\mathbb{S}_p \setminus \mathbf{I}_p)$ where $\mathbb{K}_p$ was defined in (18). Now we are prepared to summarize the structure theorem for $\mathbf{S}$:

Theorem 5.10. (The universal monothetic algebra $\mathbb{K}\langle X\rangle$)
(i) *The universal monothetic algebra*

$$\mathbf{S} := \mathbb{K}\langle X\rangle = \prod_{p\in\mathfrak{P}_\mathbb{K}} \mathbb{S}_p$$

is a weakly complete symmetric Hopf algebra generated by the element $X = (X_p)_{p\in\mathfrak{P}_\mathbb{K}}$.
(ii) *The group of units* $\mathbf{S}^\times$ *is dense in* $\mathbf{S}$, *where*

$$\mathbf{S}^\times = \prod_{p\in\mathfrak{P}_\mathbb{K}} {\mathbb{S}_p}^\times = \Big\{\Big(\sum_{n\in\mathbb{N}_0} a_{np}{X_p}^n\Big)_{p\in\mathfrak{P}_\mathbb{K}} : a_{np}\in\mathbb{K}_p \ \ and\ \ a_{0,p}\neq 0\Big\} = \prod_{p\in\mathfrak{P}_\mathbb{K}} \mathbb{K}_p^\times(\mathbb{S}_p\backslash\mathbf{I}_p).$$

(iii) *The exponential function* $\exp\colon \mathbf{S}\to\mathbf{S}^\times$ *operates componentwise on* $\prod_{p\in\mathfrak{P}_\mathbb{K}}\mathbb{S}_p$ *and induces an isomorphism of of topological groups* $\prod_{p\in\mathfrak{P}_K}\mathbf{I}_p \to \prod_{p\in\mathfrak{P}_\mathbb{K}}(1+\mathbf{I}_p)$, *whose inverse is given by the componentwise logarithm.*
(iv) *The additive group* $\mathbb{P}(\mathbf{S})$ *of primitive elements is*

$$\mathbb{P}(\mathbf{S}) = \prod_{p\in\mathfrak{P}_\mathbb{K}} \mathbb{K}_p\cdot X_p \subseteq \prod_{p\in\mathfrak{P}_\mathbb{K}} \mathbb{S}_p.$$

In particular, the element X *is primitive. The image of* $\mathfrak{g}$ *in* $\mathbb{K}\langle X\rangle$ *is* $\mathbb{K}X$, *i.e.* $\mathfrak{g}\subset\mathbb{P}(\mathbf{U}(\mathfrak{g}))$.
Let $\mathbb{T}=\mathbb{R}/\mathbb{Z}$ *denote the additive circle group again and* $\mathfrak{c}$ *the cardinality* $2^{\aleph_0}$ *of the continuum. The multiplicative group* $\mathbb{G}(\mathbf{S})$ *of grouplike elements is*

$$\mathbb{G}(\mathbf{S}) = \prod_{p\in\mathfrak{P}_\mathbb{K}} \exp(\mathbb{K}_pX_p) \cong \begin{Bmatrix} \mathbb{R}^{\mathfrak{P}_\mathbb{R}}\oplus\mathbb{T}^{\{p\in\mathfrak{P}_\mathbb{R}:\ \deg p=2\}}, \textit{ if } \mathbb{K}=\mathbb{R}, \\ (\mathbb{R}\oplus\mathbb{T})^{\mathfrak{P}_\mathbb{C}}, \textit{ if } \mathbb{K}=\mathbb{C} \end{Bmatrix} \cong (\mathbb{C}^\times)^\mathfrak{c}.$$

The exponential function $\exp\colon\mathbb{P}(\mathbf{S})\to\mathbb{G}(\mathbf{S})$ *is a quotient morphism of topological abelian groups onto its image.*

In particular we derive the following (with $\mathfrak{c}=2^{\aleph_0}$):

Corollary 5.11. *The abelian pro-Lie group* $\mathbb{G}(\mathbb{K}\langle X\rangle)$ *is connected and, for* $\mathbb{K}=\mathbb{R}$, *is isomorphic to* $\mathbb{R}^\mathfrak{c}\oplus\mathbb{T}^\mathfrak{c}\cong(\mathbb{R}\times\mathbb{T})^\mathfrak{c}\cong(\mathbb{C}^\times)^\mathfrak{c}$.

We do not know whether in general the pro-Lie group $\mathbb{G}(\mathbf{U}(\mathfrak{g}))$ is connected. Theorem 5.10 (iv)) shows that $\mathfrak{g}\subseteq A := \mathbf{U}_\mathbb{K}(\mathfrak{g})$ is considerably smaller than $\mathbb{P}(A)$. The discrepancy between $\mathfrak{g}$ and $\mathbb{P}(A)$ arises in the detailed description of the universal monothetic algebra $\mathbb{K}\langle X\rangle$. The origin of this complication is the Galois theory of the polynomial ring $\mathbb{K}[x]$.

5.3. Comments

We discussed extensively the "smallest possible" nontrivial weakly complete enveloping algebra $\mathbf{U}_\mathbb{K}(\mathfrak{g})$, namely, the one arising for $\dim\mathfrak{g}=1$. Any *abelian* profinite-dimensional Lie algebra $\mathfrak{g}$ is isomorphic to $\mathbb{K}^J$ for some set J. We have a pair of adjoint functors between the category $\mathcal{W}$ of weakly complete vector spaces over $\mathbb{K}$ and the category $\mathcal{WAC}$ of weakly complete commutative unital algebras, namely, the functor $A\mapsto|A| : \mathcal{WAC}\to\mathcal{W}$ assigning to a weakly complete commutative algebra its underlying weakly complete vector space and $\mathfrak{g}\mapsto\mathbf{U}_\mathbb{K}(\mathfrak{g}) : \mathcal{W}\to\mathcal{WAC}$, the restriction of the universal enveloping functor. Then $\mathbf{U}_{\mathbb{K}}|\mathcal{W}$ is left adjoint to $|\cdot|$, and therefore it preserves colimits. For a finite set J of n elements we note that in the category $\mathcal{W}$ we have $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_n$ with $\mathfrak{g}_k \cong \mathbb{K}$ for $k = 1, \ldots, n$ and so $\mathfrak{g}$ is the coproduct of n cofactors of dimension 1. Accordingly, in the category $\mathcal{WAC}$ we observe

$$\mathbf{U}_{\mathbb{K}}(\mathfrak{g}) \cong \coprod_{j=1}^{n} \mathbf{U}_{\mathbb{K}}(\mathfrak{g}_j), \quad \mathbf{U}_K(\mathfrak{g}_j) \cong \mathbf{S} \cong \mathbb{S}^{\mathfrak{P}_{\mathbb{K}}}. \tag{23}$$

To the extent that finite coproducts in the category $\mathcal{WAC}$ are understood, one knows $\mathbf{U}_{\mathbb{K}}(\mathfrak{g})$ for finite dimensional abelian Lie algebras $\mathfrak{g}$.

Let now $\mathfrak{g} = \mathbb{K}^J$ in $\mathcal{W}$ for some nonempty set J. If J is finite, then $\mathfrak{g}$ is a finite coproduct and the dual of a finite product. If J is infinite, then let J_{fin} denote the directed family of finite subsets $F \subseteq J$ and recall

$$\mathfrak{g} = \mathbb{K}^J \cong \lim_{F \in J_{\text{fin}}} \mathbb{K}^F.$$

We may then apply Theorem 3.2 and deduce

$$\mathbf{U}_{\mathbb{K}}(\mathfrak{g}) \cong \lim_{F \in J_{\text{fin}}} \mathbf{U}_{\mathbb{K}}(\mathbb{K}^F), \tag{24}$$

where $\mathbf{U}_{\mathbb{K}}(\mathbb{K}^F)$ is known by (23) if we know finite coproducts in the category $\mathcal{WAC}$.

In Theorem 8.7 in the appendix we argued that for any profinite-dimensional Lie algebra $\mathfrak{g}$ with the underlying weakly complete vector space $|\mathfrak{g}|$ there is a canononical quotient morphism of weakly complete unital algebras $q_{\mathfrak{g}} \colon \mathbf{T}(|\mathfrak{g}|) \to \mathbf{U}(\mathfrak{g})$ from the weakly complete tensor algebra $\mathbf{T}(|\mathfrak{g}|)$ onto $\mathbf{U}(\mathfrak{g})$. In the present situation of abelian Lie algebras we may write $|\mathfrak{g}| = \mathfrak{g}$ and conclude that for each set J and for $\mathfrak{g} = \mathbb{K}^J$ we have a natural quotient morphism of weakly complete algebras

$$q_{\mathfrak{g}} \colon \mathbf{T}_{\mathbb{K}}(\mathfrak{g}) \to \mathbf{U}_{\mathbb{K}}(\mathfrak{g}) \tag{25}$$

whose kernel is the closed ideal generated in $\mathbf{T}_{\mathbb{K}}(\mathfrak{g})$ by the elements $xy - yx$, $x, y \in \mathfrak{g} \subseteq \mathbf{T}(\mathfrak{g})$. The structure theory of $\mathbf{S}$ for the case $\mathfrak{g} = \mathbb{K}$ shows that the presentation (25) conceals more than it reveals. Indeed, the universal property of $\mathbf{U}_{\mathbb{K}}(\mathbb{K}^J)$ yields for $\mathfrak{g} = \mathbb{K}^J$ a surjective morphism of weakly complete unital algebras

$$A := \mathbf{U}_{\mathbb{K}}(\mathfrak{g}) = \mathbf{U}_{\mathbb{K}}(\mathbb{K}^J) \to \mathbf{U}_{\mathbb{K}}(\mathbb{K})^J \cong \mathbf{S}^J \quad (\cong \mathbb{K}^{\mathbb{N}_0 \times \mathfrak{P}_{\mathbb{K}} \times J} \text{ in } \mathcal{W}). \tag{26}$$

Since $\mathbb{P}(\mathbf{S}^J) = \mathbb{P}(\mathbf{S})^J \cong \mathbb{K}^{\mathfrak{P} \times J}$ the quotient morphism in (26) shows that the vector space of primitive elements of $\mathbf{S}^J$ is considerably larger than $\mathfrak{g} = \mathbb{K}^J$. This information indicates that for the weakly complete symmetric Hopf algebra $A := \mathbf{U}(\mathfrak{g})$, the subspace $\mathbb{P}(A)$ of primitive elements is likely to be large by comparison with $\mathfrak{g}$. Clearly $\mathbb{G}(\mathbf{S}^J)$ is a *connected* abelian group whose structure is known by Corollary 5.11. The simplest example along this line is the power series Hopf algebra $\mathbb{K}[[X]]$ (cf. Proposition 5.3). Accordingly, one expects the group $\mathbb{G}(A)$ to be considerable.

However, some caution is in order:

Example 5.12. (i) Let $A = \mathbb{R}[\hat{\mathbb{Q}}]$ be the real group algebra of $\hat{\mathbb{Q}}$, the universal solenoidal compact abelian group. Then $\mathbb{G}(A) = \hat{\mathbb{Q}} \subseteq A$, while $\mathbb{P}(A) = \mathfrak{L}(\hat{\mathbb{Q}}) \cong \mathbb{R}$. Then $\exp_{\hat{\mathbb{Q}}} \colon \mathbb{P}(A) \to \mathbb{G}(A)$ is a morphism of locally compact abelian groups with a dense image, but it is *not* surjective.

(ii) If we take $A = \mathbb{R}[\mathbb{Z}_p]$ for a prime number p, where $\mathbb{Z}_p$ is the additive group of the p-adic integers, then $\mathbb{G}(A) \cong \mathbb{Z}_p$ and $\mathbb{P}(A) = \mathfrak{L}(\mathbb{G}(A)) = \{0\}$, since $\mathbb{Z}_p$ is totally disconnected. The exponential map $\exp_A \colon \mathbb{P}(A) \to \mathbb{G}(A)$ is the zero morphism.

These examples show that even on the abelian level, the weakly complete symmetric Hopf algebra structure of the weakly complete enveloping algebras and that of the weakly complete group algebras behave rather differently. Yet they are related in a natural way as we shall observe in the following section.

6. Enveloping algebras versus group algebras

The class of compact groups and their Lie algebras are distinguished domains for which the relationship between weakly complete enveloping algebras and weakly complete group algebras is particularly lucid. Hence we focus on these classes.

6.1. The case of compact groups

A particularly appropriate situation is that of a *compact* topological group G. Our level of information regarding the associated *real* group algebra is particularly advanced in that situation. Indeed recall that for a compact group we may naturally identify G with the group of grouplike elements of $\mathbb{R}[G]$ (cf. [2], Theorems 8.7, 8.9 and 8.12), and we may further identify $\mathfrak{g} := \mathfrak{L}(G)$ with the pro-Lie algebra $\mathbb{P}(\mathbb{R}[G])$ of primitive elements. (Cf. also Theorem 9.2 in the Appendix.) We may also assume that the Lie algebra $\mathfrak{g}$ of G is contained in the set $\mathbb{P}(\mathbf{U}_\mathbb{R}(\mathfrak{g}))$ of primitive elements of $\mathbf{U}_\mathbb{R}(\mathfrak{g})$.

Theorem 6.1. (i) *Let G be a compact group and $\mathfrak{g}$ its Lie algebra. Then there is a natural morphism of weakly complete algebras $\omega_G \colon \mathbf{U}_\mathbb{R}(\mathfrak{g}) \to \mathbb{R}[G]$ fixing the elements of $\mathfrak{g}$ elementwise.*

(ii) *The image of ω_G is the closed subalgebra $\mathbb{R}[G_0]$ of $\mathbb{R}[G]$.*

(iii) *The pro-Lie group $\mathbb{G}(\mathbf{U}_\mathbb{R}(\mathfrak{g}))$ is mapped into $G_0 = \mathbb{G}(\mathbb{R}[G_0]) \subseteq \mathbb{R}[G]$. The connected pro-Lie group $\mathbb{G}(\mathbf{U}_\mathbb{R}(\mathfrak{g}))_0$ maps epimorphically to G_0 and $\mathbb{P}(\mathbf{U}_\mathbb{R}(\mathfrak{g}))$ maps surjectively onto $\mathbb{P}(\mathbb{R}[G]) = \mathfrak{g}$.*

Proof. (i) follows at once from the universal property of $\mathbf{U}$.

(ii) As a morphism of weakly complete Hopf algebras, ω_G has a closed image which is generated as a weakly complete subalgebra by $\mathfrak{g}$ which is $\mathbb{R}[G_0]$ by Corollary 3.3(ii) of [7].

(iii) The morphism ω_G of weakly complete Hopf algebras maps grouplike elements to grouplike elements, whence we have the commutative diagram

$$\begin{array}{ccc} \mathfrak{g} \subseteq \mathbb{P}(\mathbf{U}_\mathbb{R}(\mathfrak{g})) & \xrightarrow{\mathbb{P}(\omega_G)} & \mathbb{P}(\mathbb{R}(G)) = \mathfrak{g} \\ {\scriptstyle \exp_{\mathbb{G}(\mathbf{U}_\mathbb{R}(\mathfrak{g}))}} \downarrow & & \downarrow {\scriptstyle \exp_G} \\ \mathbb{G}(\mathbf{U}_\mathbb{R}(\mathfrak{g})) & \xrightarrow[\mathbb{G}(\omega_G)]{} & \mathbb{G}(\mathbb{R}[G]) = G. \end{array}$$

Since $\mathbb{P}(\omega_G)$ is a retraction and the image of $\exp_G$ topologically generates G_0, the image of $\mathbb{G}(\omega_G) \circ \exp_{\mathbf{U}_\mathbb{R}(\mathfrak{g})}$ topologically generates G_0. Since the image of the exponential function of the pro-Lie group $\mathbb{G}(\mathbf{U}_\mathbb{R}(\mathfrak{L}(G)))$ generates topologically its identity component, $\mathbb{G}(\omega_G)$ maps this identity component onto G_0.

Since $\mathfrak{g} \subseteq \mathbb{P}(\mathbf{U}_{\mathbb{R}}(\mathfrak{g}))$, and since also any morphism of Hopf algebras maps a primitive element onto a primitive element we know $\omega_G(\mathbb{P}(\mathbf{U}_{\mathbb{R}}(\mathfrak{g}))) = \mathbb{P}(\mathbb{R}[G])$. ∎

The following overview of the situation may be helpful:

$$
(D)\qquad
\begin{array}{ccccccc}
 & & & & & & \mathbb{R}[G] \\
 & & & & & & \Big| \\
\mathbf{U}_{\mathbb{R}}(\mathfrak{g}) & \xrightarrow{\omega_G,\text{onto}} & \mathbb{R}[G_0] & & & & \Big| \\
\Big| & & \Big| & & & & \Big| \\
\mathbb{G}(\mathbf{U}_{\mathbb{R}}(\mathfrak{g})) & \longrightarrow & \Big| & & \longrightarrow & & \mathbb{G}(\mathbb{R}[G]) = G \\
\Big| & & \Big| & & & & \Big| \\
\mathbb{G}(\mathbf{U}_{\mathbb{R}}(\mathfrak{g}))_0 & \xrightarrow{\text{onto}} & \mathbb{G}(\mathbb{R}[G_0]){=}G_0 & & = & & G_0 \\
{\scriptstyle \exp_{\mathbb{G}(\mathbf{U}_{\mathbb{R}}(\mathfrak{g}))}}\Big\uparrow & & \Big\uparrow{\scriptstyle \exp_{\mathbb{G}(\mathbb{R}[G])}} & & = & & \Big\uparrow{\scriptstyle \exp_G} \\
\mathfrak{g}{\subseteq}\mathbb{P}(\mathbf{U}_{\mathbb{R}}(\mathfrak{g})) & \xrightarrow[\text{retract}]{} & \mathbb{P}(\mathbb{R}[G_0]){=}\mathbb{P}(\mathbb{R}[G]) & & = & & \mathfrak{g}.
\end{array}
$$

Example 6.2. Let $\mathfrak{g}$ be a compact semisimple Lie algebra. Then $\mathfrak{g} = \mathfrak{L}(G)$ for the compact projective group $G = \mathrm{Pr}(\mathfrak{g})$. In this case, $G = \Gamma(\mathfrak{g})$, and we have a commutative diagram

$$
(D_1)\qquad
\begin{array}{ccc}
\mathbf{U}(\mathfrak{g}) & \xrightarrow{\omega_G,\, surjective} & \mathbb{R}[G] \\
\Big| & & \Big| \\
\mathbb{G}(\mathbf{U}(\mathfrak{g}))_0 & \xrightarrow{\text{retract}} & G = \mathbb{G}(\mathbb{R}[G]) \\
{\scriptstyle \exp_{\mathbb{G}}}\Big\uparrow & & \Big\uparrow{\scriptstyle \exp_G} \\
\mathfrak{g} \subseteq \mathbb{P}(\mathbf{U}(\mathfrak{g})) & \xrightarrow[\text{retract}]{} & \mathfrak{g}.
\end{array}
$$

We do not precisely know what $\mathbb{G}(\mathbf{U}(\mathfrak{g}))$ and $\mathbb{P}(\mathbf{U}(\mathfrak{g}))$ are even if $\mathfrak{g} = \mathrm{so}(3)$ in which case $\Gamma^*(\mathfrak{g}) \cong \mathrm{SU}(2)$. Still, in this case $\exp_{\Gamma(\mathfrak{g})}\colon \mathfrak{g} \to \Gamma(\mathfrak{g})$ is surjective (cf. [11], Theorems 6.30, 9.19(ii) and Theorem 9.32(ii)).

The group $\mathbb{G}(\mathbf{U}(\mathfrak{g}))$ of grouplike elements of $\mathbf{U}(\mathfrak{g})$ is a semidirect product of some unknown closed normal subgroup N by G. From the content of Diagram (D_1) we do not know anything about N.

The following example is the opposite to the preceding one:

Example 6.3. Let $\mathfrak{g} = \mathbb{R}^X$ for some set X. Then $G = \mathrm{Pr}(\mathfrak{g}) = (\hat{\mathbb{Q}})^X$ and $\Gamma(\mathfrak{g}) = \Gamma^*(G) = \mathbb{R}^X$.

In our discussion of abelian profinite-dimensional Lie algebras $\mathfrak{g}$ we have obtained more information on $\mathbf{U}(\mathfrak{g})$. Here we have our standard diagram:

$$
(D_2)\qquad
\begin{array}{ccc}
\mathbf{U}(\mathfrak{g}) & \xrightarrow{\omega_{\mathfrak{g}},\text{surjective}} & \mathbb{R}[G]{\subset}\mathbb{C}^{\mathbb{Q}^{(X)}} \\
\Big| & & \Big| \\
\mathbb{G}(\mathbf{U}(\mathfrak{g}))_0 & \xrightarrow{\text{onto}} & G = (\hat{\mathbb{Q}})^X \\
{\scriptstyle \exp_{\mathbb{G}}}\Big\uparrow & & \Big\uparrow{\scriptstyle \exp_G} \\
\mathfrak{g} \subseteq \mathbb{P}(\mathbf{U}(\mathfrak{g})) & \xrightarrow[\text{retract}]{} & \mathfrak{g}.
\end{array}
$$

Recall that a finite-dimensional real Lie algebra $\mathfrak{g}$ is called "compact" if it is isomorphic to the Lie algebra of a compact group (apologetically defined in [11] Definition 6.1 in that fashion). We now expand this definition to read as follows:

Definition 6.4. A Lie algebra is called *compact* if it is profinite-dimensional and is isomorphic to the Lie algebra of a compact group.

We know a real Lie algebra to be *compact* if and only if there exists a set X and a family $\mathcal{S}$ of compact simple Lie algebras $\mathfrak{s}$ such that $\mathfrak{g} \cong \mathbb{R}^X \times \prod \mathcal{S}$, where we wrote $\prod \mathcal{S}$ for $\prod_{\mathfrak{s}\in\mathcal{S}} \mathfrak{s}$. Now from [11], Theorem 9.76 we obtain the following statement:

Theorem 6.5. (Sophus Lie's Third Principal Theorem for Compact Lie Algebras) *For every compact real Lie algebra* $\mathfrak{g}$ *there is a projective connected compact group* $\mathrm{Pr}(\mathfrak{g})$ *whose Lie algebra* $\mathfrak{L}(\mathrm{Pr}(\mathfrak{g}))$ *is (isomorphic to)* $\mathfrak{g}$*.*

Every compact connected group G with $\mathfrak{L}(G) \cong \mathfrak{g}$ is a quotient of $\mathrm{Pr}(\mathfrak{g})$ modulo some central 0-dimensional subgroup. For details see [11], discussion following Lemma 9.72, notably Theorem 9.76 and Theorem 9.76bis. For the abelian case see [11], Theorem 8.78ff. Notice that for a compact Lie algebra $\mathfrak{g}$ the projective compact connected group $\mathrm{Pr}(\mathfrak{g})$ is simply connected if and only if $\mathfrak{g}$ is semisimple. By contrast, if $\mathfrak{g} = \mathbb{R}^X$ for some set then $\mathrm{Pr}(\mathfrak{g}) = (\hat{\mathbb{Q}})^X$ (see [11], Proposition 8.81), a compact connected abelian group that fails to be simply connected while $\pi_1(\mathrm{Pr}(\mathfrak{g})) = \{0\}$ (see [11], Theorem 8.62).

It is very important here to distinguish between the prosimply connected pro-Lie group $\Gamma(\mathfrak{g})$ and, in the case of a compact Lie algebra $\mathfrak{g}$, the projective compact group $\mathrm{Pr}(\mathfrak{g})$.

The present concept of weakly complete enveloping algebras now belongs to the circle of ideas of Lie's Third Fundamental Theorem.

Let $\mathfrak{g}$ be a profinite-dimensional Lie algebra over $\mathbb{R}$. By Theorem 4.4 above, $\Gamma^*(\mathfrak{g}) \subseteq \mathbb{G}(\mathbf{U}(\mathfrak{g}))$ is a pro-Lie group whose Lie algebra is $\mathfrak{g}$ and the exponential function $\exp\colon \mathbf{U}(\mathfrak{g})_{\mathrm{Lie}} \to \mathbf{U}(\mathfrak{g})^\times$ of $\mathbf{U}(\mathfrak{g})$ induces the exponential function

$$\exp : \mathfrak{L}(\Gamma^*(\mathfrak{g})) = \mathfrak{g} \to \Gamma^*(\mathfrak{g}).$$

If $G = G_0 = \Gamma(\mathfrak{g})$ embeds into its weakly complete group algebra $\mathbb{R}[G]$, then the diagram (D) above shows that $\alpha_{\mathfrak{g}}$ is an isomorphism.

We summarize for $\mathbb{K} = \mathbb{R}$, recalling that we consider $\mathfrak{g}$ as a Lie subalgebra of $\mathbb{P}(\mathbf{U}(\mathfrak{g})) \subseteq \mathbf{U}(\mathfrak{g})$. Indeed, in the context of Lie's Third Fundamental Theorem, there are two basic pro-Lie groups $\Gamma(\mathfrak{g})$ and $\Gamma^*(\mathfrak{g})$ attached, and, in the case of a compact profinite-dimensional Lie algebra $\mathfrak{g}$, a third one, $\mathrm{Pr}(\mathfrak{g})$, and for these we have:

Theorem 6.6. *Let* $\mathfrak{g}$ *be a profinite-dimensional real Lie algebra. Then the pro-Lie group of grouplike elements in the weakly complete enveloping algebra* $\mathbf{U}(\mathfrak{g})$ *contains the pro-Lie group* $\Gamma^*(\mathfrak{g})$*, having* $\mathfrak{g}$ *as Lie algebra with the exponential function* $\exp\colon \mathfrak{g} \to \Gamma^*(\mathfrak{g})$ *induced by the exponential function of* $\mathbf{U}(\mathfrak{g})$*. There is a natural quotient morphism* $\alpha_{\mathfrak{g}}\colon \Gamma(\mathfrak{g}) \to \Gamma^*(\mathfrak{g})$*. If the natural morphism* $\Gamma(\mathfrak{g}) \to \mathbb{R}[\Gamma(\mathfrak{g})]$ *of* $\Gamma(\mathfrak{g})$ *into its weakly complete group algebra is an embedding as is the case if* $\mathfrak{g}$ *is a compact Lie algebra, then* $\alpha_{\mathfrak{g}}$ *is an isomorphism, and in the latter case, there is a natural injective morphism* $\Gamma(\mathfrak{g}) \to \mathrm{Pr}(\mathfrak{g})$ *with dense image.*

7. Appendix: The category theoretical background

For a category $\mathcal{TA}$ of topological algebraic structures – in the simplest case the category $\mathcal{W}$ of weakly complete vector spaces, and for the category $\mathcal{WA}$ of weakly complete associative unital algebras, we shall repeatedly discuss an adjoint pair of functors $R\colon \mathcal{WA} \to \mathcal{TA}$ and $L\colon \mathcal{TA} \to \mathcal{WA}$. As an example on the simplest level, in the case of $\mathcal{TA} = \mathcal{W}$, for a weakly complete algebra A, the $\mathcal{W}$-object $R(A)$ will simply be the weakly complete vector space underlying A, while for a weakly complete vector space W, the weakly complete algebra $L(W)$ will be the weakly complete tensor algebra of W in the category $\mathcal{W}$.

7.1. Limits and topologically dense subcategories

Since the category $\mathcal{WA}$ of weakly complete associative unital algebras is at the focus of our considerations, let us point to one important property of the objects in this category, which was expressed in Appendix 7 of [11], Theorem A7.34.

Theorem 7.1. *For every weakly complete unital topological* $\mathbb{K}$*-algebra* A*, the set* $\mathcal{J}(A)$ *of closed two-sided ideals* I *with finite-dimensional quotient algebras* A/I *is a filterbasis converging to* 0 *in* A*, and* A *is (naturally isomorphic to) the projective limit* $\lim_{I\in\mathcal{J}(A)} A/I$ *of these* finite-dimensional *unital quotient algebras.*

This theorem says that any weakly complete unital associative algebra "is approximated by finite-dimensional $\mathbb{K}$-algebras". Let us briefly recall our approach to projective limits in a category $\mathcal{A}$. Each directed set J is a category with the elements of J as objects and for each pair (j,k) satisfying $j \le k$ an arrow (J-morphism) $k \to j$. A *projective (or inverse) system* is a functor $J \to \mathcal{A}$, usually written $j \mapsto A_j$ and $(k \to j) \mapsto (f_{jk}\colon A_k \to A_j)$. The *projective limit of this system* is an object $\lim_{j\in J} A_j$ together with a family of morphisms $f_k\colon \lim_{j\in J} A_j \to A_k$, $k \in J$ such that $f_k = f_{kn} f_n$ for all arrows $n \to k$. The limit has the universal property that for any system of morphisms $\phi_k\colon A \to A_k$ of $\mathcal{A}$-morphisms satisfying $\phi_k = f_{kn}\phi_n$ for all arrows $n \to k$ there is a *unique* morphism $\phi\colon A \to \lim_{j\in J} A_j$ satisfying $\phi_k = f_k\phi$ for all $k \in J$. The morphisms f_k are called *limit morphisms.* (For the example of the category of compact groups see e.g. [11], Definitions 1.25 and 1.27, or see Chapter 1 of [10]. For the general concept of a limit see [11], Definition A3.41, or go to MacLane's general source book [15].) We have already seen a concrete example of a projective limit in Theorem 7.1. In fact, that example was particular insofar the limit morphisms f_k were all quotient morphisms. To mathematicians working on the topological algebra of locally compact groups, projective limits are utterly familiar by the Theorem of Yamabe saying that

> *every locally compact topological group* G *with the identity component* G_0 *is a projective limit of Lie groups provided that* G/G_0 *is compact.*

(See the classic of 1955 by Montgomery and Zippin [16].)

In particular, this says that every connected locally compact group is approximated by connected Lie groups. Therefore we need to pinpoint in functorial terms what important theorems like these say on the principle of "approximating complicated topological algebraic structures" by simpler ones.

Topologists like to use the concept of a *net* on a set X generalizing that of a sequence [14]: A net $(x_j)_{j\in J}$ is a function $j \mapsto x_j : J \to X$ for a directed poset J. If X is a topological space and Y a subset of X such that for every $x \in X$ there is a net $(y_j)_{j\in J}$ of elements in Y such that $x = \lim_{j\in J} y_j$, then we say that Y is dense in X.

So let us now look at a category $\mathcal{B}$ with a subcategory $\mathcal{B}_d$.

Definition 7.2. We call $\mathcal{B}_d$ *topologically dense* in $\mathcal{B}$ if it is a full subcategory of $\mathcal{B}$ such that for each object B in $\mathcal{B}$ there is a directed set J and some projective system

$$\{f_{jk} : B_k \to B_j; (j,k) \in J \times J, j \leq k\}$$

of morphisms in $\mathcal{B}_d$ such that in $\mathcal{B}$ the object B is (isomorphic to) the projective limit $\lim_{j\in J} B_j$ of this system with suitable limit morphisms

$$B \cong \lim_{k\in J} B_k \xrightarrow{q_j} B_j\,,\ j \in J.$$

As an example we have seen in Theorem 7.1 that the full subcategory of finite-dimensional unital algebras $\mathcal{WA}_d$ is topologically dense in the category of weakly complete unital algebras $\mathcal{WA}$. In the same spirit, by Peter and Weyl, the category of compact Lie groups is topologically dense in the category of compact groups and continuous group morphisms (see [11],Corollary 2.43). In [11] this Density Theorem is exploited widely.

We owe our readers an explanation of our choice of terminology of a *topologically dense subcategory* which, as we have argued intuitively, is indeed close to the geometric idea of a dense subspace in a topological space. The necessity of a comment arises from the fact that in category theoretical circles, the choice of the terminology of a "dense subcategory of a category" is half a century old or older as can be seen from MacLane's standard text of 1971, where the terminology is introduced close to the end of the book [15] on pp. 241, 242, 243. However, that generation of ground breaking category theoreticians had a distinct leaning towards examples supplied by combinatorics and algebra. Therefore, in their eyes, a category D is, firstly, dense in a category C if every object of C is a COLIMIT of a subsystem of objects from D.

We would accordingly suggest to call their approach an approach to CODENSITY. However, secondly, their formation of colimits is NOT RESTRICTED TO DIRECTED SYSTEMS (in the way we insist to use projective limits when we (truly!) use limits). As a consequence in their terminology, in the category of sets a category consisting of one singleton object is codense in the whole category, and the category consisting of the object $\mathbb{Z}^2$ is codense in the category of abelian groups. So dualizing their approach via Pontryagin would yield that the category consisting of the single object of the traditional torus $\mathbb{T}^2$ would be dense in the category of all compact abelian groups. At any rate, this predicament causes us to set off our own terminology of "topologically dense subcategories."

7.2. Density and adjunction

For the class of objects of a category $\mathcal{A}$ we write $\mathrm{ob}(\mathcal{A})$. Let $L_o\colon \mathrm{ob}(\mathcal{A}) \to \mathrm{ob}(\mathcal{B})$ be a function and $R\colon \mathcal{B} \to \mathcal{A}$ a functor and assume that $\mathcal{B}$ has a subcategory $\mathcal{B}_d$.

Definition 7.3. We say that L_o is *conditionally left adjoint to R with respect to a subcategory $\mathcal{B}_d$ of $\mathcal{B}$* if for each $A \in \mathrm{ob}(\mathcal{A})$ there is an $\mathcal{A}$-morphism $\eta_A\colon A \to RL_o(A)$ such that for each $B \in \mathrm{ob}(\mathcal{B}_d)$ and each morphism $f\colon A \to R(B)$ in $\mathcal{A}$ there is a unique morphism $f'\colon L_o(A) \to B$ in $\mathcal{B}$ such that $f = R(f') \circ \eta_A$.

A special case illustrates this technical concept:

Remark 7.4. If L_o is conditionally left adjoint to R with respect to $\mathcal{B}$ itself (in place of $\mathcal{B}_d$), then L_o is the restriction to the objects of a functor $L\colon \mathcal{A} \to \mathcal{B}$ which is left adjoint to the functor R.

Proof. [11], Theorem A3.28. ■

But now we show that the much weaker condition in Definition 7.3 suffices frequently for L_o to extend to a left adjoint of R.

Theorem 7.5. (The Density and Adjunction Theorem) *Assume that $\mathcal{A}$ and $\mathcal{B}$ are two categories and that $\mathcal{B}$ has a topologically dense subcategory $\mathcal{B}_d$. Further assume that*

$$\begin{array}{llllll} L_o & : & \mathrm{ob}(\mathcal{A}) & \to & \mathrm{ob}(\mathcal{B}) & \textit{is a function and} \\ R & : & \mathcal{B} & \to & \mathcal{A} & \textit{is a functor.} \end{array}$$

Then the following conditions are equivalent:

(a) *L_o is conditionally left adjoint to R with respect to $\mathcal{B}_d$, and R preserves projective limits.*

(b) *L_o extends to a left adjoint L of R.*

Proof. For (b) $\Rightarrow$ (a) we refer to Remark 7.4 and to [11], Theorem A3.52, saying that right adjoints are continuous, that is, preserve all limits.

Now we prove (a) $\Rightarrow$ (b): In view of Remark 7.4 it suffices to show that L_o is conditionally left adjoint to R with respect to $\mathcal{B}$ (in place of merely to $\mathcal{B}_d$). So assume now that A and B are objects of $\mathcal{A}$ and $\mathcal{B}$, respectively, and that $f\colon A \to R(B)$ is a morphism in $\mathcal{A}$. Then since $\mathcal{B}_d$ is topologically dense in $\mathcal{B}$ we know that there exists a projective system

$$\{f_{jk}\colon B_k \to B_j;\ (j,k) \in J \times J,\ j \le k\} \text{ of morphisms in } \mathcal{B}_d$$

for some directed set J in $\mathcal{B}_d$ such that

$$B = \lim_{j\in J} B_j. \tag{27}$$

Then we obtain a projective system

$$\{R(f_{jk})\colon R(B_k) \to R(B_j);\ (j,k) \in J \times J,\ j \le k\} \text{ of morphisms in } \mathcal{A}$$

for our directed set J. Since L_o is conditionally adjoint to R with respect to $\mathcal{B}_d$, for each $j \in J$, then there is a unique $\mathcal{B}$-morphism $(Rf_j \circ f)'\colon L_oA \to B_j$ such that

$$Rf_j \circ f = R\big((Rf_j \circ f)'\big) \circ \eta_A. \tag{28}$$

We claim that for $j \le k$ in J we have

$$(Rf_j \circ f)' = f_{jk} \circ (Rf_k \circ f)'. \tag{29}$$

For a proof of this claim, we recall from (2) that $(Rf_k \circ f)'$ is the *unique* $\mathcal{B}$-morphism for which $R\big((Rf_k \circ f)'\big) \circ \eta_A = Rf_k \circ f$. Now

$$\begin{aligned} R\big((f_{jk} \circ (Rf_k \circ f)'\big) \circ \eta_A &= Rf_{jk} \circ R\big((Rf_k \circ f)'\big) \circ \eta_A && \text{(since } R \text{ is a functor)} \\ &= Rf_{jk} \circ Rf_k \circ f && \text{(by (28))} \\ &= R(f_{jk} \circ f_k) \circ f = Rf_j \circ f && \text{(since } R \text{ is a functor)} \\ &= R\big(R(f_j \circ f)'\big) \circ \eta_A && \text{(by (28)).} \end{aligned}$$

By the uniqueness in the definition of $(Rf_j \circ f)'$ this proves the Claim.

By the universal property of the limit, there is a unique $\mathcal{B}$-morphism $f'\colon L_oA \to B$ so that

$$(\forall j \in J)\,(Rf_j \circ f)' = f_j \circ f'. \tag{30}$$

Consequently, since $R(Rf_j \circ f)' \circ \eta_A = Rf_j \circ f$ by (28), we have

$$(\forall j \in J)\, Rf_j \circ f = Rf_j \circ (Rf' \circ \eta_A). \tag{31}$$

By (a) we know that we may write $RB = \lim_{j\in J} RB_j$ with $Rf_j\colon RB \to RB_j$ as limit morphisms. By the uniqueness in the universal property of the limit (as specified in great generality in [11], Definition A3.41), from (31) we conclude

$$(\forall f\colon A \to RB)(\exists! f'\colon L_oA \to B) \quad f = Rf' \circ \eta_A. \tag{32}$$

This completes the proof of (b). ■

7.3. An application: The weak completion of a $\mathbb{K}$-vectorspace

As an example, consider the functor $W \to W_{\rm u}\colon \mathcal{W} \to \mathcal{V}$ which assigns to a weakly complete vector space W the underlying $\mathbb{K}$-vector space $W_{\rm u}$. This functor has a left adjoint $L\colon \mathcal{V} \to \mathcal{W}$ characterized by the usual universal property recognized in the usual diagram:

$$\begin{array}{ccccc}
\mathcal{V} & & & & \mathcal{W} \\
\hline
V & \xrightarrow{\epsilon_V} & L(V)_{\rm u} & & L(V) \\
{\scriptstyle\forall f}\big\downarrow & & \big\downarrow{\scriptstyle L(f')_{\rm u}} & & \big\downarrow{\scriptstyle\exists! f'} \\
W_{\rm u} & \xrightarrow{\rm id} & W_{\rm u} & & W.
\end{array}$$

The function $f \mapsto f' : \mathcal{V}(V, W_{\rm u}) \to \mathcal{W}(L(V), W)$ is a natural bijection.

Proposition 7.6. *For a $\mathbb{K}$-vector space V we have*

$$L(V) = (V^*)_{\rm u}{}^* \quad \textit{and} \quad (\forall \omega \in V^*)\, \epsilon_V(v)(\omega) = \omega(v)).$$

Note: In a loose fashion we might write $L(V) = V^{**}$ and say:

The weak completion of a $\mathbb{K}$-vector space V is its bidual V^{**}.

Proof. First we test the universal property of $L(V)$ for $W \in \mathrm{ob}(\mathcal{W})$ with $\dim W < \infty$. Then the natural morphism $\epsilon_W\colon W \to W^{**}$ is an isomorphism and for $f\colon V \to W$ we have a commutative diagram

$$\begin{array}{ccc}
V & \xrightarrow{\epsilon_V} & V^{**} \\
{\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle f^{**}} \\
W & \xrightarrow[\epsilon_W]{} & W^{**}.
\end{array}$$

Any $\mathcal{V}$-morphism $f\colon V \to W$ yields a unique morphism $f' = \epsilon_W^{-1} \circ f^{**}\colon V^{**} \to W$. The equation $f = (f')_{\rm u} \circ \epsilon_V$ is now clear. Thus the function $V \mapsto (V^*)_{\rm u}{}^* : \mathrm{ob}\,\mathcal{V} \to \mathrm{ob}\,\mathcal{W}$ has the universal property of a conditional left adjoint of the functor $W \mapsto W_{\rm u}$ with respect to the topologically dense subcategory $\mathcal{W}_d$ of $\mathcal{W}$ consisting of all finite-dimensional vector spaces. So Theorem 7.5 applies and proves that $V \mapsto (V^*)_{\rm u}{}^*$ is left adjoint to $W \mapsto W_{\rm u}$. ■

We note that the necessity of invoking Theorem 7.5 indicates that the proof is not entirely trivial. For $\mathbb{K} = \mathbb{R}$ we have seen that V^* for $V \in \mathrm{ob}(\mathcal{V})$ is naturally isomorphic to the Pontryagin dual $\hat{V} = \mathrm{Hom}_{\mathrm{continuous}}(V, \mathbb{T})$ with $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ when V is endowed with the finest locally convex topology. (See [11], Theorem A7.10). If, for a $W \in \mathrm{ob}(\mathcal{W})$ we let W_f denote the underlying vector space of W endowed with its finest locally convex topology. Then we have

$$L(V) = (V^*)_{\mathrm{u}}{}^* \cong (\hat{V}_f)\hat{}\,.$$

If A is any abelian topological group and $\hat{A}_d$ is its character group endowed with the discrete topology, then the compact "bidual" $\alpha(A) := (\hat{A}_d)\hat{}$ together with natural continuous morphism $A \to \alpha(A)$ is the so called *almost periodic compactification* of A. We mention this here in order to exhibit the analogy between the weak completion and the almost periodic compactification.

7.4. Strict density and the preservation of projective limits

We continue with categories $\mathcal{A}$ and $\mathcal{B}$ having topologically dense subcategories $\mathcal{A}_d$, respectively, $\mathcal{B}_d$, and we consider a pair of adjoint functors $L\colon \mathcal{A} \to \mathcal{B}$ and $R\colon \mathcal{B} \to \mathcal{A}$ between them. Thus we have the following situation

Strict Density. *For each object A of $\mathcal{A}$ we have some family $q_j\colon A \to A_j$, $j \in Q(A)$ of morphisms with A_j in $\mathcal{A}_d$ with a directed set $Q(A)$ of indices, together with a projective system in $\mathcal{A}_d$, say, $q_{jk}\colon A_k \to A_j$ for $j \le k$ in $Q(A)$ such that $q_j = q_{jk} \circ q_k$ for $j \le k$, giving us a unique isomorphism $q_A\colon A \to \lim_{j\in Q(A)} A_j$ such that*

$$\begin{array}{ccc} A & \xrightarrow{q_A} & \lim_{k\in Q(A)} A_k \\ {\scriptstyle q_j}\downarrow & & \downarrow{\scriptstyle \rho_j} \\ A_j & \xrightarrow[q_{jj}=id]{} & A_j \end{array}$$

commutes for each $j \in Q(A)$ for the limit morphisms ρ_j.

The universal property of the limit will now provide us with the existence of a crucial morphism

$$\phi_A\colon L(A) \to \lim_{j\in Q(A)} L(A_j) \tag{33}$$

We shall investigate this situation in more detail in the remainder of the chapter.

Lemma 7.7. *If $L\colon \mathcal{A} \to \mathcal{B}$ is any functor into a complete category $\mathcal{B}$, then $\{L(q_{jk})\colon L(A_k) \to L(A_j);\ (j,k) \in Q(A) \times Q(A),\ j \le k\}$ is a projective system in $\mathcal{B}$, which has a limit*

$$L^{\#}(A) := \lim_{j\in Q(A)} L(A_j)$$

and which provides a morphism $\phi_A\colon L(A) \to L^{\#}(A)$ such that

$$\begin{array}{ccc} L(A) & \xrightarrow{\phi_A} & L^{\#}(A) \\ {\scriptstyle Lq_j}\downarrow & & \downarrow{\scriptstyle \rho_j} \\ L(A_j) & \xrightarrow[=]{} & L(A_j) \end{array} \tag{34}$$

commutes for all $j \in Q(A)$ for the limit morphisms ρ_j.

If $\mathcal{A}$, for example, is the category $\mathcal{W}$ of weakly complete vector spaces V over $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, then a vector subspace E of V is called a *cofinite-dimensional* vector subspace of V if $\dim V/E < \infty$. Now each V defines naturally the filter basis $J(V)$ of cofinite-dimensional closed vector subspaces W such that $V \cong \lim_{W\in J(V)} V/W$ where V/W ranges through the finite-dimensional quotient spaces of V so that the subcategory $\mathcal{W}_{\rm fin}$ of all finite-dimensional vector spaces is topologically dense in $\mathcal{W}$. The left adjoint functor $L\colon \mathcal{A} \to \mathcal{B}$ preserves colimits. But it would be interesting to know whether it preserves also at least some of the significant limits in the contexts that are of interest to us. For instance: If $\mathcal{A} = \mathcal{W}$, the category of weakly complete $\mathbb{K}$-vector spaces: does then L under certain circumstances preserve projective limits in $\mathcal{W}$ such as $\lim_{W\in J(V)} V/W$? That is: Is the natural morphism $\phi_V\colon L(V) \to \lim_{W\in J(V)} L(V/W)$ an isomorphism for certain categories $\mathcal{B}$?

In the example of the category $\mathcal{W}$ of weakly complete vector spaces each object V gave rise to the projective system

$$\{f_{UW}\colon V/W \to V/U; U, W \in J(V), W \subseteq U\},$$

whose limit was naturally isomorphic to V. Here the filter basis $J(V)$ converges to 0 in the topological space underlying V. If F is any finite-dimensional $\mathbb{K}$-vector space, then the filter base of vector subspaces $\{F \cap W : W \in J(F)\}$ of F converges to zero in F. Now, since $\dim_{\mathbb{K}} F < \infty$, there is some member $W_F \in J(V)$ such that $W \subseteq W_F$ implies $F \cap W = \{0\}$. In terms of quotient morphisms of V this can be expressed as follows:

Each object V of $\mathcal{W}$ has a canonical projective system of quotient maps $q_W : V \to V/W$, $W \in J(V)$ and $\dim V/W < \infty$ such that $V \cong \lim_{W\in J(V)} V/W$, and that for each morphism $f\colon V \to F$ into a finite-dimensional vector space F we have $\ker f \in J(V)$ so that for every $W \in J(V)$ with $W \subseteq \ker f$ the morphism f factors through $q_W\colon V \to V/W$. In other words, there is an index $W_f \in J(V)$ such that for every $W \in J(V)$ such that $W \subseteq W_f$ there is a morphism $p_W\colon V/W \to F$ such that $f = p_W \circ q_W$, as in the following commutative diagram:

$$\begin{array}{ccc} V & \xrightarrow{=} & A \\ {\scriptstyle q_W}\downarrow & & \downarrow{\scriptstyle f} \\ V/W & \xrightarrow[p_W]{} & F. \end{array}$$

Following this example we formulate the following definition:

Definition 7.8. For an object A of $\mathcal{A}$, a projective system

(PS) $$\{q_{jk}\colon A_k \to A_j : (j,k) \in Q(A) \times Q(A), \quad j \le k\}$$

in $\mathcal{A}$ will be called *appropriate for* A if $A \cong \lim_{j\in Q(A)} A_j$ with limit morphisms $q_j\colon A \to A_j$ such that the following conditions are satisfied at least for a cofinal set of indices j in $Q(A)$:

(i) For every $\mathcal{A}$-morphism $f\colon A \to F$ into an $\mathcal{A}_d$-object F there is a $j_0 \in Q(A)$ such that for all $j \ge j_0$ there is an $\mathcal{A}$-morphism $p_j\colon A_j \to F$ such that $f = p_j \circ q_j$.

(ii) The limit morphisms $q_j\colon A \to A_j$ are epic.

If for an object A of $\mathcal{A}$ there is an appropriate projective system $\{q_{jk}\}$, then we say that A is *appropriately representable*.

Definition 7.9. A subcategory $\mathcal{A}_d$ of a category $\mathcal{A}$ is called *strictly dense in* $\mathcal{A}$ if each object A in $\mathcal{A}$ is appropriately representable by a projective system (PS) such that all A_j are objects from $\mathcal{A}_d$.

Note that in Condition 7.8(i) the factorisation $f = p_j \circ q_j$ is depicted by the commuting diagram

$$\begin{array}{ccc} A & \xrightarrow{=} & A \\ {\scriptstyle q_j}\downarrow & & \downarrow{\scriptstyle f} \\ A_j & \xrightarrow[p_j]{} & F. \end{array}$$

Moreover, condition (ii) is certainly satified if the morphisms $q_j\colon A \to A_j$ are quotient maps as in the case $\mathcal{A} = \mathcal{W}$ that we used as motivation above. Accordingly, we observe that in the cateory $\mathcal{W}$ of weakly complete $\mathbb{K}$-vector spaces, the subcategory of finite-dimensional vector spaces is strictly dense.

A more sophisticated example is the category of pro-Lie groups, in which the subcategory of Lie groups is strictly dense.

However, the most relevant example for us is the category $\mathcal{WA}$ of weakly complete associative unital algebras in which the full subcategory $\mathcal{WA}_{\text{fin}}$ of finite-dimensional $\mathbb{K}$-algebras is strictly dense by Theorem 7.1.

For the applications of the next theorem it is useful to first recall the following general lemma:

Lemma 7.10. *Left adjoint functors preserve epics.*

Proof. By [11], Theorem A3.52 a left adjoint L preserves colimits. A morphism $e\colon A_1 \to A_2$ is an epic if and only if

$$\begin{array}{ccc} A_1 & \xrightarrow{e} & A_2 \\ {\scriptstyle e}\downarrow & & \downarrow{\scriptstyle \mathrm{id}_{A_2}} \\ A_2 & \xrightarrow[\mathrm{id}_{A_2}]{} & A_2 \end{array}$$

is a pushout. A pushout is a colimit (cf. [11] EA3.27), epimorphisms and pushouts are dual to monomorphisms and pullbacks; the latter are defined in [11] Definition A3.9 and Definition A3.43(ii), respectively. ∎

Now we apply Definition 7.8 to provide circumstances in which the morphism (33) is an epimorphism. In order to simplify the language of our notation we introduce the following definition.

Definition 7.11. A pair of categories $\mathcal{A}$ and $\mathcal{B}$ shall be called a *suitable pair of categories* if

(i) $\mathcal{A}$ posesses a strictly dense subcategory $\mathcal{A}_d$,

(ii) $\mathcal{B}$ posseses a topologically dense subcategory $\mathcal{B}_d$,

(iii) there is a pair of functors $L\colon \mathcal{A} \to \mathcal{B}$ and $R\colon \mathcal{B} \to \mathcal{A}$ such that L is left adjoint to R, and

(iv) R maps $\mathcal{B}_d$ into $\mathcal{A}_d$.

Theorem 7.12. *Let $\mathcal{A}$ and $\mathcal{B}$ be a suitable pair of categories. Assume that the object A of $\mathcal{A}$ is appropriately representable in the form $A = \lim_{j\in Q(A)} A_j$ for an appropriate projective system*

$$\{q_{jk}\colon A_k \to A_j : (j,k) \in Q(A)\times Q(A), j \le k\}.$$

Then the following statements hold:

(a) $$\{L(q_{jk})\colon L(A_k) \to L(A_j) : (j,k) \in Q(A)\times Q(A), \quad j \le k\}$$

is appropriate for $L^{\#}(A) := \lim_{j\in J} L(A_j)$ in $\mathcal{B}$.

(b) *The morphism* $$\phi_A\colon L(A) \to L^{\#}(A) \tag{35}$$ *is an epimorphism.*

Proof. For proving (b) let $\alpha, \beta\colon L^{\#}(A) \to B$ be $\mathcal{B}$-morphisms such that $\alpha\circ\phi_A = \beta\circ\phi_A$. We must show that $\alpha = \beta$. We shall first argue that we may assume that B is in $\mathcal{B}_d$. Since $\mathcal{B}_d$ is topologically dense in $\mathcal{B}$ by 7.11(ii), there is a projective system

$$\{r_{mn}\colon B_n \to B_n;\ (m,n) \in Q(B)\times Q(B),\ m \le n\}$$

in $\mathcal{B}_d$ such that $B = \lim_{m\in Q(B)} B_m$ with limit morphisms $r_m\colon B \to B_m$ such that $r_m = r_{mn}\circ r_n$ for $m \le n$. Then for each $m \in Q(B)$ we have morphisms $r_m\circ\alpha,\ r_m\circ\beta\colon L^{\#}(A) \to B_m$ such that

$$r_m\circ\alpha\circ\phi_A = r_m\circ\beta\circ\phi_A. \tag{36}$$

If we can show that for all m we have $r_m\circ\alpha = r_m\circ\beta\colon L^{\#}(A) \to B_m$, then by the uniquenes of the universal property of the limit this will show $\alpha = \beta$, and we shall be done. So from now on we shall assume that B is in $\mathcal{B}_d$.

(a) For a proof of (a) we shall have to prove that the projective system

$$\{L(q_{jk})\colon L(A_k) \to L(A_j) : (j,k) \in Q(A)\times Q(A), \quad j \le k\}$$

with the limit morphisms $\rho_j\colon L^{\#}(A) \to L(A_j)$ is appropriate, that is, for each morphism $\mathbf{f}\colon L^{\#}(A) \to B$ for an object B in $\mathcal{B}_d$, and all sufficiently large j there will be morphisms $\mathbf{p}_j\colon L(A_j) \to B$ such that $\mathbf{p}_j\circ\rho_j = \mathbf{f} : L^{\#}(A) \to B$ with the limit morphisms and $\rho_j\colon L^{\#}(A) \to L(A_j)$.

So let B be a $\mathcal{B}_d$-object and $\mathbf{f}\colon L^{\#}(A) \to B$ a $\mathcal{B}$-morphism. We define $f := \mathbf{f}\circ\phi_A$. Then $R(f)\colon RL(A) \to R(B)$ is an $\mathcal{A}$-morphism into an $\mathcal{A}_d$ object $R(B)$ since R maps $\mathcal{B}_d$ into $\mathcal{A}_d$. By the hypothesis, that A is appropriately represented in the form $A = \lim_{j\in Q(A)} A_j$, the morphism

$$A \xrightarrow{\eta_A} RL(A) \xrightarrow{R(f)} R(B)$$

is a morphism from A to an $\mathcal{A}_d$ object for which we find a $j_0 \in Q(A)$ such that for all $j \in Q(A)$ with $j_0 \le j$ there is a $p_j\colon A_j \to R(B)$ such that $R(f)\circ\eta_A = p_j\circ q_j$. By the universal property of the left adjoint L there is a unique $\mathcal{B}$-morphism $p_j'\colon L(A_j) \to B$ such that $p_j = R(p_j')\circ\eta_{A_j}$. The diagram on the following page illustrates the situation.

We claim that ${p_j}'\circ\rho_j = \mathbf{f} : L^{\#}(A) \to B$ in the right half of the diagram. For a proof of this claim we invoke the functoriality of the limit in the following lemma, which we consider well understood.

$$\begin{array}{ccc|ccc}
 & \mathcal{A} & & & \mathcal{B} & \\
\hline
A & \xrightarrow{\eta_A} & RL(A) & L(A) & \xrightarrow{\phi_A} & L^{\#}(A) \\
{\scriptstyle q_j}\downarrow & & \downarrow{\scriptstyle RL(q_j)} & \downarrow{\scriptstyle L(q_j)} & & \downarrow{\scriptstyle \rho_j} \\
A_j & \xrightarrow{\eta_{A_j}} & RL(A_j) & L(A_j) & \xrightarrow{=} & L(A_j) \\
{\scriptstyle p_j}\downarrow & & \downarrow{\scriptstyle R(p'_j)} & \downarrow{\scriptstyle p'_j} & & \downarrow{\scriptstyle p'_j} \\
R(B) & \xrightarrow[=]{} & R(B) & B & \xrightarrow[=]{} & B.
\end{array}$$

$$R(p'_j) \circ RL(q_j) = R(f) \qquad p'_j \circ L(q_j) = f.$$

Lemma 7.13. *Let* $\xi_j : \lim_{k\in J} X_k \to X_j$ *and* $\omega_j : \lim_{k\in J} Y_k \to Y_j$ *the corresponding limit cones of two projective limits and assume that there is a compatible family* $\phi_j \colon X_j \to Y_j$ *of morphism such that for all* $j \le k$ *the diagrams*

$$\begin{array}{ccc}
Y_k & \xleftarrow{\omega_{jk}} & Y_j \\
{\scriptstyle \phi_k}\downarrow & & \downarrow{\scriptstyle \phi_j} \\
X_k & \xleftarrow[\xi_{jk}]{} & X_j
\end{array}$$

commute. Then there is a unique morphism $\phi\colon \lim_k X_k \to \lim_k Y_k$ *such that* $\omega_j \circ \phi = \phi_j \circ \xi_j$ *for all* $j \in J$*, i.e., the following diagram commutes*

$$\begin{array}{ccc}
\lim_k X_k & \xrightarrow{\phi} & \lim_k Y_k \\
{\scriptstyle \xi_j}\downarrow & & \downarrow{\scriptstyle \omega_j} \\
X_j & \xrightarrow[\phi_j]{} & Y_j.
\end{array}$$

Now we apply this lemma to the special case that the Y_j arise as a a constant projective diagram with $Y_j = Y$ and $\omega_{jk} = \mathrm{id}_Y$ for all $j \le k$ in J, and $\lim_k Y_k = Y$ with $\omega_k = \mathrm{id}_Y$ for all $k \in J$. Then $\phi\colon \lim_k X_k \to Y$ agrees with $\phi_j \circ \xi_j$ for all j, that is

$$\begin{array}{ccc}
\lim_k X_k & \xrightarrow{\phi} & Y \\
\downarrow{\scriptstyle \xi_j} & & \downarrow{\scriptstyle \mathrm{id}_Y} \\
X_j & \xrightarrow[\phi_j]{} & Y
\end{array} \tag{37}$$

commutes for all $j \in J$. This we apply with $J = \{j \in Q(A) : j_0 \le j\}$, $X_j = L(A_j)$, $Y = Y_j = B$, $\xi_j = \rho_j \colon L^{\#}(A) = \lim_k X_k \to X_j = L(A_j)$, $\phi_j = p'_j \colon L(A_j) = X_j \to Y = B$, $\phi = \mathbf{f} \colon L^{\#}(A) \to B = Y$. Then the commuting of (37) yields exactly $p'_j \circ \rho_j = \mathbf{f}$ for $j_0 \le j$ as asserted. So for all $j \in Q(A)$, $j_0 \le j$, the morphisms $p'_j \colon L(A_j) \to B$ are the required morphisms $\mathbf{p}_j \colon L(A_j) \to B$.

(b) We finally prove that ϕ_A is an epic. By (b) there is a j_0 such that for all $j \ge j_0$ there exist $\mathcal{B}$-morphisms $\alpha_j \colon L(A_j) \to B$ and $\beta_j \colon L(A_j) \to B$ such that $\alpha = \alpha_j \rho_j$ and $\beta = \beta_j \rho_j$:

$$\begin{array}{ccc}
L^{\#}(A) & \xrightarrow{\rho_j} & L(A_j) \\
{\scriptstyle \alpha}\downarrow\downarrow{\scriptstyle \beta} & & {\scriptstyle \alpha_j}\downarrow\downarrow{\scriptstyle \beta_j} \\
B & \xrightarrow[\mathrm{id}_B]{} & B.
\end{array} \tag{38}$$

Then we must show that $$(\forall j \ge j_0) \ \ \alpha_j = \beta_j. \tag{39}$$

Now by the Definition of ϕ_A we have

$$(\forall j \in J) \ \ \rho_j \phi_A = L(q_j). \tag{40}$$

We consider the following diagram

$$\begin{array}{ccc} L(A) & \xrightarrow{\mathrm{id}_{L(A)}} & L(A) \\ {\scriptstyle \phi_A}\big\downarrow & & \big\downarrow{\scriptstyle L(q_j)} \\ L^{\#}(A) & \xrightarrow{\rho_j} & L(A_j) \\ {\scriptstyle \alpha}\big\downarrow\big\downarrow{\scriptstyle \beta} & & {\scriptstyle \alpha_j}\big\downarrow\big\downarrow{\scriptstyle \beta_j} \\ B & \xrightarrow[\mathrm{id}_B]{} & B. \end{array} \tag{41}$$

The top square commutes by (40) for all $j \in J$. The two bottom squares commute for each α and β and all $j \geq j_0$ by (38). Accordingly, the outside rectangles commute for both α and β. The left vertical edges $\alpha\phi_A = \beta\phi_A$ agree by assumption on α and β. So for each $j \geq j_0$ we compute

$$\alpha_j L(q_j) = \mathrm{id}_B \, \alpha\phi_A \, \mathrm{id}_{L(A)}^{-1} = \mathrm{id}_B \, \beta\phi_a \, \mathrm{id}_{L(A)}^{-1} = \beta_j L(q_j). \tag{42}$$

The morphisms $q_j \colon A \to A_j$ are epic by Definition 7.8(ii). Then Lemma 7.10 shows that the morphisms $L(q_j) \colon L(A) \to L(A_j)$ are all epic. Now (41) implies that $\alpha_j = \beta_j$ for all $j \geq j_0$. So (39) is proved and this is what we had to show. ■

Notice that Theorem 7.12 does not assert that the objects $L(A_j)$ are (even cofinally) objects of the topologically dense subcategory $\mathcal{B}_d$. In fact, in the applications, which we aim for, this is not the case. It is nevertheless assumed by Definition 7.11(ii) that every object B of $\mathcal{B}$ is a projective limit of objects from $\mathcal{B}_d$.

Lemma 7.14. *Let $\mathcal{A}$ and $\mathcal{B}$ be a suitable pair of categories. Assume that the object A of $\mathcal{A}$ is appropriately representable as $A = \lim_{j \in Q(A)} A_j$. Abbreviate $\lim_{j \in Q(A)} L(A_j)$ by $L^{\#}(A)$ and define an $\mathcal{A}$-morphism $\eta_A^{\#} \colon A \to R(L^{\#}(A))$ by $\eta_A^{\#} = R(\phi_A) \circ \eta_A$. Then for each object $B \in \mathcal{B}_d$ and each $\mathcal{A}$-morphism $f \colon A \to RB$ there is a unique $\mathcal{B}$-morphism $f^{\#} \colon L^{\#}(A) \to B$ such that $f = R(f^{\#}) \circ \eta_A^{\#}$.*

Proof. By Definition 7.11(iv), the functor R maps $\mathcal{B}_d$ into $\mathcal{A}_d$. So RB is in $\mathcal{A}_d$. By Definition 7.11(i), the subcategory $\mathcal{A}_d$ is strictly dense in $\mathcal{A}$. Since A is appropriately representable, there indeed exists a projective system

$$\{q_{jk} \colon A_k \to A_j : (j,k) \in Q(A) \times Q(a), \quad j \leq k\}$$

which is appropriate for A. So there is a $j_0 \in Q(A)$ such that for all j with $j_0 \leq j$ there are $\mathcal{A}_d$ morphisms $p_j \colon A_j \to R(B)$ such that $f = p_j \circ q_j$. Since L is left adjoint to R, there are unique $\mathcal{B}$-morphisms $f' \colon LA \to B$ and $(p_j)' \colon LA_j \to B$ such that $f = Rf' \circ \eta_A$ and $p_j = R(p_j)' \circ \eta_A$. Now from $f = p_j \circ q_j$, by [11], Proposition A3.33 we deduce

$$f' = (p_j)' \circ Lq_j. \tag{43}$$

The fill-in morphism $\phi_A \colon LA \to L^{\#}A$ of Lemma 7.7 satisfies $Lq_j = \rho_j \circ \phi_A$ with the limit morphism $\rho_j \colon L^{\#}A \to LA_j$. We set $f_j^{\#} = (p_j)' \circ \rho_j$.

If $k \in Q(A)$ satisfies $j \leq k$, then we have a commutative diagram

$$\begin{array}{ccc} L^{\#}A & \xrightarrow[=]{} & L^{\#}A \\ {\scriptstyle \rho_k}\downarrow & & \downarrow{\scriptstyle \rho_j} \\ L(A_k) & \xrightarrow{L(q_{jk})} & L(A_j) \\ {\scriptstyle (p_k)'}\downarrow & & \downarrow{\scriptstyle (p_j)'} \\ B & \xrightarrow{=} & B \end{array}$$

with $f_j^{\#} = (p_j)' \circ \rho_j = (p_k)' \circ \rho_k = f_k^{\#}$. That is, for a cofinal subset $Q_c(A) \subseteq Q(A)$ of $Q(A)$ the function $k \mapsto f_k^{\#} \colon Q_c(A) \to \mathcal{B}(L^{\#}(A), B)$ is constant. Hence we have a unique morphism $f^{\#} \colon L^{\#}A \to B$ such that $f^{\#} = f_j^{\#}$ for all sufficiently large j such that (34) and (43) imply

$$f^{\#} \circ \phi_A = (p_j)' \circ \rho_j \circ \phi_A = (p_j)' \circ Lq_j = f' \tag{44}$$

for all sufficiently large j, and thus

$$R(f^{\#}) \circ \eta_A^{\#} = R(f^{\#}) \circ R(\phi_A) \circ \eta_A = R(f') \circ \eta_A = f.$$

If $f^* \colon L^{\#}A \to B$ is a $\mathcal{B}$-morphism such that $f = R(f^*) \circ \eta_A^{\#} = R(f^*) \circ R(\phi_A) \circ \eta_A$, then $f^* \circ \phi_A = f'$ by the uniqueness in determining f'. Since also $f^{\#} \circ \phi_A = f'$ by (44) above, we may conclude $f^* = f^{\#}$, since ϕ_A is an epimorphism by Theorem 7.12. This completes the proof of Lemma 7.14. ■

As a corollary of the epimorphism Theorem 7.12 we now have the following main result, in which it happens that a left adjoint functor L preserves, in addition to all colimits, also certain limits. In its formulation we retain the notation of Lemma 7.7 and Definition 7.8.

Theorem 7.15. *Let $\mathcal{A}$ and $\mathcal{B}$ be a suitable pair of categories and assume that the projective system*

$$\{q_{jk} \colon A_k \to A_j : (j,k) \in Q(A) \times Q(A), \quad j \leq k\}$$

is appropriate for A. Then the morphism

$$\phi_A \colon L(\lim_{j \in Q(A)} A_j) \to \lim_{j \in Q(A)} L(A_j) \tag{45}$$

is an isomorphism.

Proof. From Lemma 7.14 and Theorem 7.5 it now follows that $L^{\#}$ extends to a functor $L^{\#} \colon \mathcal{A} \to \mathcal{B}$ which is left adjoint to R. Thus L and $L^{\#}$ are naturally isomorphic functors. Then there is a commutative diagram of natural functions

$$\begin{array}{ccc} \mathcal{B}(L^{\#}(A), B) & \xrightarrow[\beta_{AB}]{} & \mathcal{B}(L(A), B) \\ {\scriptstyle \alpha_{AB}^{\#}}\downarrow & & \downarrow{\scriptstyle \alpha_{AB}} \\ \mathcal{A}(A, R(B)) & \xrightarrow{=} & \mathcal{A}(A, R(B)), \end{array} \tag{46}$$

(a) $\beta_{AB}(h) = h \circ \phi_A$ for $h \colon L^{\#}(A) \to B$,

(b) $\alpha_{AB}^{\#}(h) = R(h \circ \phi_A) \circ \eta_A$, for $h \colon LA \to B$,

(c) $\alpha_{AB}(h) = R(h) \circ \eta_A$, for $h \colon L(A) \to B$.

The bijectivity of α_{AB} expresses the fact that L is left adjoint to R, and likewise the bijectivity of $\alpha^{\#}_{AB}$ is now secured since we proved that $L^{\#}$ is left adjoint to R. the commutativity of the diagram (46) then shows the bijectivity of β_{AB} which in turn proves that ϕ_A is an isomorphism. This completes the proof. ∎

Since the right adjoint $R\colon \mathcal{B} \to \mathcal{A}$ preserve limits, the following corollary is immediate:

Corollary 7.16. *Under the hypotheses of* Theorem 7.15*, for each* $A = \lim_{j\in\mathbb{Q}(A)} A_j$*, the* $\mathcal{A}$*-morphism* $R(\phi_A)\colon RL(A) = RL(\lim_{j\in Q(A)} A_j) \to \lim_{j\in Q(A)} RL(A_j)$ *is an isomorphism.*

If $r_j\colon L^{\#}(A) := \lim_k L(A_k) \to L(A_j)$, $j \in Q(A)$ denotes the limit morphisms, the situation is illustrated by the following diagram:

$$\begin{array}{ccccccc}
 & & \mathcal{A} & & & & \mathcal{B} \\
\hline
A & \xrightarrow{\eta_A} & RL(A) & \xrightarrow{R(\phi_A)} & \lim_k RL(A_k) & & L(A) \\
{\scriptstyle\forall q_j}\downarrow & & \downarrow{\scriptstyle RL(q_j')} & & \downarrow{\scriptstyle R(r_j)} & & \downarrow{\scriptstyle \exists! q_j'} \\
A_j & \xrightarrow[=]{} & A_j & \xrightarrow[\eta_{A_j}]{} & RL(A_j) & & L(A_j)
\end{array}$$

Corollary 7.17. *Assume the hypotheses of* Theorem 7.15*, and, in addition, that for all objects* $A \in \mathrm{ob}(\mathcal{A}_d)$ *the front adjunction* $\eta_A\colon A \to RL(A)$ *is monic. Then it is monic for all objects* $A \in \mathrm{ob}(\mathcal{A})$ *in* $\mathcal{A}$*.*

Proof. Let $\alpha, \beta\colon X \to A$ be morphisms such that $\eta_A\alpha = \eta_A\beta$. Then for $j \in Q(A)$ we have $\eta_{A_j} q_j\alpha = \eta_{A_j} RL(q_j')\eta_A\alpha = \eta_{A_j} RL(q_j')\eta_A\beta = \eta_{A_j} q_j\beta$. Since η_{A_j} is monic, we have

$$(\forall j \in Q(A)) \quad q_j\alpha = q_j\beta.$$

Since $A = \lim_{j\in Q(A)} A_j$, the uniqueness of the morphism in the universal property of the limit (see. e.g. [11], Definition A3.41) implies $\alpha = \beta$. ∎

7.5. Application: $\mathcal{B} = \mathcal{WA}$.

Our main target category for various left adjoint functors is the category $\mathcal{WA}$ of weakly complete unital algebras over $\mathbb{K} = \mathbb{R}, \mathbb{C}$.

Proposition 7.18. *The subcategory* $\mathcal{WA}_d$ *of finite-dimensional associative unital* $\mathbb{K}$*-algebras is strictly dense in* $\mathcal{WA}$*.*

Proof. Each weakly complete algebra A has a filter base $Q(A)$ of closed two sided ideals I such tht A/I is a finite-dimensional $\mathbb{K}$ algebra, and the natural morphims $q_A\colon A \to \lim_{I\in Q(A)} A/I$ is an isomorphism according to Theorem 7.1. Thus by Definition 7.2, $\mathcal{WA}_d$ is topologically dense in $\mathcal{WA}$.

We need to verify the conditions of Definition 7.8. Let $f\colon A \to F$ be an $\mathcal{WA}$-morphism for a finite-dimensional algebra F and let $I = \ker f$. Then $A/I \cong \operatorname{im} f$ (see [11], Theorem A7.12(b)) and so, since $\dim \operatorname{im} f \le \dim F < \infty$, $I \in Q(A)$. Let $q_I\colon A \to A/I$ denote the quotient morphism and $p_I\colon A/I \to F$ the injective morphism induced by f. Then $f = p_I \circ q_I$. Hence condition (i) of Definition 7.8 is satisfied. Since the quotient morphisms $q_I\colon A\to A/I$ are surjective and therefore epimorphisms, condition (ii) is satisfied as well. This completes the proof. ∎

For observing first applications, we let $\mathcal{A}$ be a category $\mathcal{TA}$ of topological algebraic structures and $R\colon \mathcal{WA}\to\mathcal{TA}$ a limit preserving functor. We assume that R satisfies the Solution Set Condition (see [11] A3.58), as is the case in the examples we discuss below (cf. [2, 7]). Hence a left adjoint functor $L\colon \mathcal{TA}\to\mathcal{WA}$ exists (see [11], Theorem A3.60.)

Assume that $L\colon \mathcal{TA}\to\mathcal{WA}$ is left adjoint to R and that $\eta_X\colon X\to RL(X)$ is the front adjunction (see [11], Definition A3.37). Now $L(X)$ is a weakly complete unital algebra. In practically all examples of interest to us, for a weakly complete unital algebra A, the $\mathcal{TA}$-object $R(A)$ is a subset of A such as for instance $A^\times$ (if $\mathcal{TA}$ is a category of topological groups), or $A_{\rm Lie}$ (if $\mathcal{TA}$ is a category of topological Lie algebras), or the underlying topological space $|A|$ (if $\mathcal{TA}$ is a category of topological spaces).

In such a situation $\eta_X\colon X\to RL(X)$ is a function and we can consider its image $\eta_X(X)$ as a subset of $RL(X)$. Then $\langle\eta_X(X)\rangle$ denotes the smallest unital subalgebra containing $\eta_X(X)$ and $\overline{\langle\eta_X(X)\rangle}$ the smallest $\mathcal{WA}$ subobject of $L(X)$. Under these circumstances we define a function

$$L_o\colon \mathrm{ob}(\mathcal{TA})\to\mathrm{ob}(\mathcal{WA}) \text{ by } L_o(X)=\overline{\langle\eta_X(X)\rangle},$$

the smallest $\mathcal{WA}$-subobject for which the morphism $\eta_X\colon X\to LR(X)$ factors through the inclusion morphism $R(L_o(X))\to RL(X)$. Then one observes immediately that L_o is conditionally left adjoint to R with respect to $\mathcal{WA}$. (Cf. Definition 7.3.) However, here Remark 7.4 applies and shows that the containment $L_o(X)\subseteq L(X)$ is equality in all cases. Thus we have

Proposition 7.19. *Assume that $R\colon \mathcal{WA}\to\mathcal{TA}$ and $L\colon \mathcal{TA}\to\mathcal{WA}$ is a pair of adjoint functors where $\mathcal{TA}$ is a category of topological algebraic structures for which the front adjunctions $\eta_X\colon X\to RL(X)$ are functions whose image $\eta_X(X)$ is a subset of the weakly complete unital algebra $L(X)$. Then for each $X\in\mathrm{ob}(\mathcal{TA})$, the abstract unital algebra $\langle\eta_X(X)\rangle$ generated by the image of η_X is dense in the weakly complete unital algebra $L(X)$.*

Our immediate examples for the category $\mathcal{TA}$ are as follows:

(A) $\mathcal{TA}=\mathcal{PROGR}$, the category of pro-Lie groups, $RA=A^\times$ the group of units of the weakly complete algebra A. For the fact that $A^\times$ is a pro-Lie group see [2] or [11], Proposition A7.37. The left adjoint L is the *weakly complete group algebra* $G\mapsto\mathbb{K}[G]$ *over* $\mathbb{K}$. It was discussed in [2], [7], and [11] (mostly for $\mathbb{K}=\mathbb{R}$ and compact groups G).

A prominent subcategory of $\mathcal{PROGR}$ is the full subcategory $\mathcal{COMPGR}$ of compact groups for which the real weakly complete group algebra $\mathbb{R}[G]$ is particularly effective. See [2].

(B) $\mathcal{TA}=\mathcal{PROLIE}$, the category of profinite-dimensional Lie algebras over $\mathbb{K}$ (cf. [7], [8]). The functor $R\colon \mathcal{WA}\to\mathcal{PROLIE}$ associates with a weakly complete unital algebra A the profinite-dimensional Lie algebra $A_{\rm Lie}$ defined on the weakly complete underlying weakly complete $\mathbb{K}$-vector space endowed with the Lie algebra multiplication $[x,y]=xy-yx$. Then the left adjoint $L\colon \mathcal{PROLIE}\to\mathcal{WA}$ is the *weakly complete universal enveloping algebra* $\mathfrak{g} \mapsto \mathbf{U}_{\mathbb{K}}(\mathfrak{g})$ *over* $\mathbb{K}$ [7, 8] which we shall address again below.

(C) $\mathcal{TA} = \mathcal{W}$, the category of weakly complete $\mathbb{K}$-vector spaces. The functor $R\colon \mathcal{WA} \to \mathcal{W}$ associates with a weakly complete unital algebra A the underlying weakly complete topological $\mathbb{K}$-vector space $|A|$. The left adjoint $L\colon \mathcal{W} \to \mathcal{WA}$ of R is, as we shall discuss in the subsequent section, the functor which associates with any weakly complete vector space W the *weakly complete tensor algebra* $\mathbf{T}(W)$ *of* W *over* $\mathbb{K}$.

Proposition 7.20. (i) *In each of the categories* $\mathcal{TA} = \mathcal{COMPGR}$, $\mathcal{PROLIE}$, *and* $\mathcal{W}$, *a monomorphism* $f\colon X \to Y$ *induces an isomorphism* $X \to f(X)$ *onto the image, that is, an embedding in the respective category* $\mathcal{TA}$.

(ii) *The front adjunction* $\eta_X\colon X \to RL(X)$, *namely,*

$$G \to \mathbb{K}[G]^\times,\ \mathfrak{g} \to \mathbf{U}_K(\mathfrak{g})_{\rm Lie},\ \text{and}\ W \to |\,\mathbf{T}(W)|,$$

is an embedding in the respective category. That is, X *may be considered as a* $\mathcal{TA}$*-subobject of* $RL(X)$ *and a subset of* $L(X)$.

(iii) *If* $X \in \mathrm{ob}(\mathcal{TA})$ *and* $X \subset L(X)$ *as in* (ii) *above, then* $\langle X\rangle$, *the abstract unital algebra generated by* X *in* $L(X)$ *is dense in* $L(X)$.

Proof. Part (i) may be safely considered as an exercise; for the two categories $\mathcal{PROLIE}$ and $\mathcal{W}$ see also [11], Theorem A7.12. Part (ii) is then a consequence of Part (i), Corollary 7.17 and the following facts which secure that the front adjunction $\eta_X\colon X \to RL(X)$ is injective, hence monic for $X \in \mathrm{ob}(\mathcal{TA})$. Part(iii) follows from Proposition 7.19.

(a) Every compact Lie group has a faithful linear representation (see e.g. [11], Corollary 2.40).

(b) Every finite-dimensional Lie algebra over a field of characteristic 0 has a faithful linear representation (Ado's Theorem, see [1], Chap. 1, Paragraph 7, n°3, Théorème 3).

(c) It suffices to observe that the one-dimensional vector space $\mathbb{K}$ has a faithful linear representation, e.g.

$$c \mapsto \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix}.$$

■

We now secure the validity of the hypotheses of Theorem 7.15 for the examples **(A)**, **(B)**, and **(C)**.

Proposition 7.21. *For the functors* $L = \mathbb{K}[-]$, $\mathbf{U}_{\mathbb{K}}$, *and* $\mathbf{T}$ *the morphism* $\phi_A\colon A \to \lim_{j\in Q(A)} L(A_j)$ *is an isomorphism.*

Proof. We show that the hypothesis of Theorem 7.15 is satisfied in each of the three examples **(A)**, **(B)**, and **(C)**.

(A) $G \mapsto \mathbb{K}[G] : \mathcal{PROGR} \to \mathcal{WA}$, is left adjoint to $A \mapsto A^\times : \mathcal{WA} \to \mathcal{PROGR}$. We note that the subcategory $\mathcal{LIE}$ of Lie groups is strictly dense in $\mathcal{PROGR}$:

Each pro-Lie group G is the projective limit of its Lie group quotients G/N, $N \in \mathcal{N}(G)$, where $\mathcal{N}(G)$ denotes the filter basis of normal subgroups N of G such that G/N is a Lie group. If $f\colon G \to L$ is a morphism of G into a Lie group, let N be the kernel of f. Then f factors through the quotient morphism

$q\colon G \to G/N$ followed by an injection of Lie groups $G/N \to L$. Each quotient morphism $G \to G/N$ is an epimorphism. So

$$\{q_{MN} : G/N \to G/M : (M,N) \in \mathcal{N}(G) \times \mathcal{N}(G), N \subseteq M\}$$

is appropriate for G. The functor $A \mapsto A^\times : \mathcal{WA} \to \mathcal{PROGR}$ maps finite-dimensional algebras to Lie groups. So the hypothesis of Theorem 7.15 is satisfied and so

$$\phi_G\colon G \to \lim_{N\in\mathcal{N}(G)} \mathbb{K}[G/N]^\times$$

is an isomorphism. The cases **(B)** and **(C)** are equally simple and are left as an exercise. ∎

Theorem 7.15 now has immediately the following corollaries:

Theorem 7.22. *Each pro-Lie group G has an appropriate projective limit representation $G = \lim_{j\in J} G_j$ in terms of Lie groups. Therefore we have*

$$\mathbb{K}[G] \cong \lim_{j\in J} \mathbb{K}[G_j].$$

Theorem 7.23. *Each profinite-dimensional weakly complete Lie algebra has an appropriate projective limit representation $\lim_{j\in J} \mathfrak{g}_j$ in terms of finite-dimensional Lie algebras. Therefore*

$$\mathbf{U}_\mathbb{K}(\mathfrak{g}) \cong \lim_{j\in J} \mathbf{U}_\mathbb{K}(\mathfrak{g}_j).$$

Theorem 7.24. *If a weakly complete $\mathbb{K}$-vector space W is represented in terms of an appropriate projective limit representation $W = \lim_{j\in J} W_j$ in terms of finite-dimensional vector spaces. Therefore*

$$\mathbf{T}(W) \cong \lim_{j\in J} \mathbf{T}(W_j).$$

8. Appendix: The definition of the tensor algebra

In Paragraph **(C)** above we already introduced the tensor algebra of a weakly complete vector space. Let us now review this concept more systematically. So we let $\mathbb{K}$ again denote one of the topological fields $\mathbb{R}$ or $\mathbb{C}$, and $\mathcal{WA}$ the category of weakly complete associative unital algebras over $\mathbb{K}$.

Here is the definition of the tensor algebra via its universal property:

Theorem 8.1. (The Existence Theorem of $\mathbf{T}$) *The underlying weakly complete vector space functor $A \mapsto |A|$ from $\mathcal{WA}$ to $\mathcal{W}$ has a left adjoint $\mathbf{T}\colon \mathcal{W} \to \mathcal{WA}$. The front adjunction $\omega_V\colon V \to |\mathbf{T}(V)|$ is an embedding of topological vector spaces.*

Proof. The category $\mathcal{W}$ is complete (Exercise. Cf. Theorem A3.48 of [11], p. 819.). The "Solution Set Condition" (of Definition A3.58 in [11], p. 824) holds (Exercise: Cf. the proof Lemma 3.58 of [11].). Hence $\mathbf{T}$ exists by the Adjoint Functor Existence Theorem (i.e., Theorem A3.60 of [11], p. 825).
The assertion about ω_V follows from Proposition 7.20 (iii). ∎

In other words, each weakly complete vector space V may be considered as a weakly complete vector subspace of the weakly complete tensor algebra $\mathbf{T}(V)$ with the property that each continuous linar map $f\colon V \to |A|$ with some weakly complete associative unital algebra A and its underlying weakly complete vector space $|A|$ extends uniquely to a $\mathcal{WA}$-morphism $f'\colon \mathbf{T}(V) \to A$.

$$\begin{array}{ccc|c}
\multicolumn{3}{c}{\mathcal{W}} & \mathcal{WA} \\
\hline
V & \xrightarrow{\subseteq} & |\,\mathbf{T}(V)| & \mathbf{T}(V) \\
{\scriptstyle\forall f}\big\downarrow & & \big\downarrow{\scriptstyle |f'|} & \big\downarrow{\scriptstyle \exists! f'} \\
|A| & \xrightarrow[\mathrm{id}]{} & |A| & A.
\end{array}$$

If necessary we shall write $\mathbf{T}_{\mathbb{K}}$ instead of $\mathbf{T}$ whenever the ground field should be emphasized.

Definition 8.2. For each weakly complete $\mathbb{K}$-vector space V we shall call $\mathbf{T}_{\mathbb{K}}(V)$ *the weakly complete tensor algebra* of V (over $\mathbb{K}$).

We record what we already saw in Section 1 in Theorem 7.24:

Corollary 8.3. *If V is represented as a projective limit $\lim_{j\in J} V_j$ of finite-dimensional vector spaces, then $\mathbf{T}(V) \cong \lim_{j\in J} \mathbf{T}(V_j)$.*

Every unital associative algebra A has injective morphism $\iota_A\colon \mathbb{K} \to A$ given by $\iota_A(t) = t1$. In some circumstances, ι is a coretraction:

Remark 8.4. Note that for every weakly complete vector space V, the morphism $v \mapsto 0 : V \to \mathbb{K}$, according to the definition of $\mathbb{T}(V)$, induces a natural $\mathcal{W}$-morphism $\alpha_V\colon \mathbf{T}(V) \to \mathbb{K}$ such that $\alpha_V(V) = \{0\}$ and that $\alpha_V \circ \iota_{\mathbf{T}(V)} = \mathrm{id}_{\mathbb{K}}$. ∎

The retraction α_V is frequently called the *augmentation* of $\mathbf{T}(V)$.

Let us compare the weakly complete tensor algebra with the abstract tensor algebra $T(E)$ of a plain $\mathbb{K}$-vector space E. By the universal property of the abstract tensor product $T(|V|)$, the linear inclusion map $\xi_{|V|}\colon |V| \to T(|V|)$ extends to a unique morphism of unital algebras $j_{|V|}\colon T(|V|) \to |\,\mathbf{T}(V)|$ such that (47) commutes:

$$\begin{array}{ccc}
|V| & \xrightarrow{\xi_{|V|}} & T(|V|) \\
{\scriptstyle |\omega_V|}\big\downarrow & & \big\downarrow{\scriptstyle j_V} \\
|\,\mathbf{T}(V)| & \xrightarrow[\mathrm{id}]{} & |\,\mathbf{T}(V)|.
\end{array} \tag{47}$$

For a natural number m and a weakly complete vector space V, set $A_m = \bigotimes_{\mathcal{W}}^m V$. Then A_m is a weakly complete vector space. If m, and n are natural numbers, then there is a canonical continuous bilinear map

$$(a_m, a_n) \mapsto a_m a_n := a_m \otimes_{\mathcal{W}} a_n : A_m \times A_n \to A_{m+n},$$

where we have identified the naturally isomorphic weakly complete vector spaces $A_m \otimes_{\mathcal{W}} A_n$ and A_{m+n}. Set $A_0 = \mathbb{K}$. Then $\bigoplus_{m=0}^{\infty} A_m$ is a graded unital algebra D with the multiplication

$$(a_m)_{m\in\mathbb{N}_0}(b_m)_{m\in\mathbb{N}_0} = (\textstyle\sum_{j+k=m} a_j b_k)_{m\in\mathbb{N}_0},$$

dense in the weakly complete vector space $A(V) := \prod_{m=0}^{\infty} A_m$. By the definition of the unital algebra D, there is a unique injective morphism of unital algebrs $i_V\colon T(|V|) = \bigoplus_{m=0}^{\infty} \bigotimes_{\mathcal{V}}^m V \to |A(V)|$ so that we have a commutative diagram:

$$\begin{array}{ccc} |V| & \xrightarrow{\xi_{|V|}} & T(|V|) \\ {\scriptstyle \mathrm{id}}\downarrow & & \downarrow{\scriptstyle i_V} \\ V & \xrightarrow[\mathrm{incl}]{} & A(V). \end{array} \tag{48}$$

Now, multiplication in D is continuous w.r.t. the topology induced from $A(V)$ and therefore extends continuously to a multiplication on $A(V)$, making $A(V)$ a weakly complete unital algebra. There is an injective continuous linear map $\iota_V\colon V \to A(V)$ given by $\iota_V(v) = (0, v, 0, 0, \dots) \in \mathbb{K} \times A_1 \times A_2 \times \cdots = A(V)$ which by Theorem 8.1 yields a unique morphism of weakly complete unital algebras $\iota_V{}'\colon T(V) \to A(V)$ such that $\iota_V(v) = \iota_V{}'(\omega_V(v))$ for all $v \in V$, i.e. such that the following diagram commutes:

$$\begin{array}{ccccc} V & \xrightarrow{\omega_V} & |\,\mathbf{T}(V)| & & \mathbf{T}(V) \\ {\scriptstyle \iota_V}\downarrow & & \downarrow{\scriptstyle |\iota_V{}'|} & & \downarrow{\scriptstyle \iota_V{}'} \\ |A(V)| & \xrightarrow[\mathrm{id}]{} & |A(V)| & & A(V). \end{array} \tag{49}$$

We now have $j_V \circ \xi_{|V|} = |\omega_V|$ by (47), $i_V \circ \xi_{|V|} = |\iota_V|$ by (48), and $|\iota_V| = |\iota_V{}'| \circ |\omega_V|$ by (49). Therefore $i_V \circ \xi_{|V|} = |\iota_V{}'| \circ j_V \circ \xi_{|V|}$, and so the uniqueness in the universal property of $T(|V|)$ allows us to conclude

$$i_V = |\iota_V{}'| \circ j_V. \tag{50}$$

But i_V is injective, and so $j_V\colon T(|V|) \to |\,\mathbf{T}(V)|$ is injective.

Collecting the information we have collected we now arrive at the following insight:

Lemma 8.5. *For each weakly complete vector space V, the weakly complete unital algebra $\mathbf{T}(V)$ contains a copy of the algebraic tensor algebra $T(|V|) = \bigoplus_{m=0}^{\infty} \bigotimes^m |V|$ algebraically generated by $V \subseteq \mathbf{T}(V)$.*

Proof. The completion of the proof is now an exercise. ■

Theorem 8.6. (i) *For any weakly complete vector space V, the unital associative subalgebra $\langle V\rangle$ generated algebraically in $\mathbf{T}(V)$ by V is dense in $\mathbf{T}(V)$.*

(ii) *Moreover, $\langle V\rangle$ is algebraically isomorphic to the algebraic tensor algebra $T(|V|)$ generated by $|V|$.*

Proof. (i) The assertion was proved in Proposition 7.20(iii).

(ii) By the universal property of the algebraic tensor algebra $T(|V|)$ generated by $|V|$ there is a morphism $j_V\colon T(|V|) \to |\,\mathbf{T}(V)|$ (see (1) above) whose corestriction to its image is a morphism of unital algebras from $T(|V|)$ to $\langle V\rangle$ which is is injective by Lemma 8.5 and therefore is an isomorphism of unital algebras. ■

Let us now use the weakly complete tensor algebra to construct $\mathbf{U}_{\mathbb{K}}(\mathfrak{g})$ as a quotient of $T(|\mathfrak{g}|)$. In the classical theory of universal enveloping algebras of Lie algebras, the construction usually does proceed from the tensor algebra as an origin and progresses to the enveloping algebra as a quotient. In the world $\mathcal{W}$ of weakly complete vector spaces we proceeded systematically via universal properties using category theoretical standard methods. In this fashion we have developed the weakly complete tensor algebra and the weakly complete universal enveloping algebra separately albeit with unified methods. Now let us pause and bring the two together again using the principle of the universal property.

Let $\mathfrak{g}$ be a profinite-dimensional Lie algebra and $|\mathfrak{g}|$ the weakly complete vector space on which it is based. Let $\|\mathfrak{g}\|$ denote the underlying vector space and $\mathfrak{g}$ the underlying abstract Lie algebra. There is a quotient morphism of unital algebras $p_{|\mathfrak{g}|}\colon T(\|\mathfrak{g}\|) \to U(\mathfrak{g})$ well known form the apparatus of the Poincaré-Birkhoff-Witt-Theorem where we may consider $\|\mathfrak{g}\|$ as a vector subspace of $T(\|\mathfrak{g}\|)$. Now we elevate this quotient to the level of the weakly complete unital algebras. From Theorem 1.3 we know that $\mathfrak{g} \subseteq \mathbf{U}(\mathfrak{g})$. This give us an embedding of weakly complete vector spaces $|\mathfrak{g}| \to |\mathbf{U}(\mathfrak{g})|$ where $\mathbf{U}(\mathfrak{g})$ is a weakly complete unital algebra. Then Theorem 8.1 provides us with a unique morphism $q_{\mathfrak{g}}\colon \mathbf{T}(|\mathfrak{g}|) \to \mathbf{U}(\mathfrak{g})$ extending the identity function $|\mathfrak{g}| \to \mathfrak{g}$. As a morphism of weakly complete algebras, $q_{\mathfrak{g}}$ has a closed image (see e.g. [11], Theorem A7.12), and by Theorem 8.1(i) has a dense image. Thus $q_{\mathfrak{g}}$ is surjective and thus a quotient map (again by [11], A7.12). We summarize this in the following Theorem whose proof is clear from what we know:

Theorem 8.7. *There is a canonical quotient morphism of weakly complete algebras* $q_{\mathfrak{g}}\colon \mathbf{T}(|\mathfrak{g}|) \to \mathbf{U}(\mathfrak{g})$ *such that the following diagram is commutative:*

$$\begin{array}{ccccccc}
\|\mathfrak{g}\| & \xrightarrow{\text{incl}} & T(\|\mathfrak{g}\|) & = & \langle\|\mathfrak{g}\|\rangle & \xrightarrow{j_{\|\mathfrak{g}\|}} & \mathbf{T}(|\mathfrak{g}|)\\
{\scriptstyle\text{id}}\big\downarrow & & {\scriptstyle p_{\mathfrak{g}}}\big\downarrow & & & & \big\downarrow{\scriptstyle q_{\mathfrak{g}}}\\
\mathfrak{g} & \xrightarrow[\text{incl}]{} & U(\mathfrak{g}) & = & \langle\mathfrak{g}\rangle & \xrightarrow[\text{incl}]{} & \mathbf{U}(\mathfrak{g}).
\end{array}$$

Remark 8.8. The quotient morphism $q_{\mathfrak{g}}$ respects augmentations in the sense that $\alpha_{\mathfrak{g}} \circ q_{\mathfrak{g}} = \alpha_{|\mathfrak{g}|}$.

9. Appendix: Some facts on weakly complete symmetric Hopf algebras

Definition 9.1. Let A be a weakly complete symmetric Hopf algebra, i.e. a group object in the monoidal category $(\mathcal{W}, \otimes_W)$ of weakly complete vector spaces (see [11], Appendix 7 and Definition A3.62), with comultiplication $c\colon A \to A \otimes A$ and coidentity $k\colon A \to \mathbb{K}$.

(i) An element $a \in A$ is called *grouplike* if $c(a) = a \otimes a$ and $k(a) = 1$. The subgroup of grouplike elements in the group of units $A^\times$ will be denoted $\mathbb{G}(A)$.

(ii) An element $a \in A$ is called *primitive*, if $c(a) = a \otimes 1 + 1 \otimes a$. The Lie algebra of primitive elements of A_{Lie} will be denoted $\mathbb{P}(A)$. ∎

Any weakly complete unital algebra A has an everywhere defined exponential function $\exp\colon A_{\text{Lie}} \to A^\times$ into the pro-Lie group $A^\times$ of invertible elements defined as $\exp x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots$. As a function $\exp\colon A_{\text{Lie}} \to A^\times$ it is the exponential function of the pro-Lie group $A^\times$ in the sense of pro-Lie groups.

Theorem 9.2. (Weakly Complete Symmetric Hopf Algebras) *If A is a weakly complete symmetric Hopf algebra, then the set $\mathbb{G}(A)$ of grouplike elements is a closed pro-Lie subgroup of the pro-Lie group $A^\times$, and the set $\mathbb{P}(A)$ of primitive elements is a closed Lie subalgebra of the profinite-dimensional Lie algebra A_{Lie} and* $\exp(\mathbb{P}(A)) \subseteq \mathbb{G}(A)$ *in such a fashion that the restriction and corestriction of* $\exp$ *is the exponential function* $\exp_{\mathbb{G}(A)}\colon \mathbb{P}(A) \to \mathbb{G}(A)$ *of the pro-Lie group* $\mathbb{G}(A)$.

(See e.g. [2], [11], [10].) A simple observation tells us something about the geometry of the set $\exp A$. Indeed, for $t \in \mathbb{K}$ we have $\exp(t1 + x) = e^t \exp x$. Thus we have

Remark 9.3. In any weakly complete unital algebra A, we have

$$\exp A = \begin{cases} \mathbb{R}_< \exp A, & \text{if } \mathbb{K} = \mathbb{R}, \\ (\mathbb{C}^\times) \exp A, & \text{if } \mathbb{K} = \mathbb{C}. \end{cases} \tag{51}$$

Now we assume that A *has a coidentity* $\alpha\colon A \to \mathbb{K}$ *which is a morphism of unital algebras,* and we call A an *augmented algebra.* We set

$$\mathbb{I} = \ker \alpha = \{a \in A : \alpha(a) = 0\}.$$

Then $\mathbb{I}$ is a maximal ideal of A and $A/\mathbb{I} \cong \mathbb{K}$ and since $\mathbb{K}1$ is central we have

$$A = \mathbb{K}1 \oplus \mathbb{I} \tag{52}$$

as a direct sum of closed subalgebras.

Lemma 9.4. *Let A be an augmented weakly complete unital algebra. Then we have $1 + \mathbb{I} \subseteq \exp A$.*

Proof. Let $x \in \mathbb{I}$ and set $a = 1 - x \in 1 + \mathbb{I}$. By Lemma 5.4 (ii) we find a morphism $\phi\colon \mathbb{K}\langle X\rangle \to A$ of weakly complete unital algebras such that $\phi(X) = x$. By Theorem 5.10 (ii), in $\mathbb{K}\langle X\rangle$ the element $Y = \log(1 - X) = (\log(1 - X_f))_{f\in\mathfrak{P}_\mathbb{K}}$ is well defined. Then $\exp_{\mathbb{K}\langle X\rangle}(Y) = 1 - X$ and so

$$a = 1 - \phi(X) = \phi(1 - X) = \phi(\exp_{\mathbb{K}\langle X\rangle}(Y)) = \exp_A(\phi(Y)) \in \exp_A(A). \qquad \blacksquare$$

Lemma 9.5.

If $\mathbb{K} = \mathbb{R}$ *then* $\quad \mathbb{R}_<(1 + \mathbb{I}) = (\mathbb{R}_< 1) \oplus \mathbb{I} \subseteq \mathbb{R}1 \oplus \mathbb{I} = A,$

and if $\mathbb{K} = \mathbb{C}$, *then* $\quad (\mathbb{C}^\times)(1 + \mathbb{I}) = (\mathbb{C}^\times)(1 \oplus \mathbb{I}) = A \setminus \mathbb{I}.$

Proof. In view of (52) above, the proof is elementary. ■

Theorem 9.6. *In any weakly complete algebra A with augmentation $\alpha\colon A \to \mathbb{K}$ let $\mathbb{I} := \alpha^{-1}(0)$.*

(i) $\mathbb{K} = \mathbb{R}$*: Then* $\exp(A) = \alpha^{-1}(\mathbb{R}_<) = (\mathbb{R}_< 1) \oplus \mathbb{I}$.

(ii) $\mathbb{K} = \mathbb{C}$*: Then* $\exp A = \alpha^{-1}(\mathbb{C}^\times) = A \setminus \mathbb{I} = A^\times$

Proof. By Lemma 9.4 we have $1 + \mathbb{I} \subseteq \exp(A)$. By Lemma 9.3 $e^\mathbb{K}(1 + \mathbb{I}) \subseteq \exp A$. Since $e^\mathbb{K} = \mathbb{R}_<$ for $\mathbb{K} = \mathbb{R}$ and $e^\mathbb{K} = \mathbb{C}^\times$ for $\mathbb{K} = \mathbb{C}$, Lemma 9.5 completes the proof. ■

These simple facts complement Theorem 9.2.

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