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On Weakly Complete Group Algebras of Compact Groups

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Abstract. A topological vector space over the real or complex field $\mathbb{K}$ is *weakly complete* if it is isomorphic to a power $\mathbb{K}^J$. For each topological group G there is a *weakly complete topological group Hopf algebra* $\mathbb{K}[G]$ over $\mathbb{K}=\mathbb{R}$ or $\mathbb{C}$, for which three insights are contributed:

Firstly, *there is a comprehensive structure theorem saying that the topological algebra* $\mathbb{K}[G]$ *is the cartesian product of its finite dimensional minimal ideals whose structure is clarified.*

Secondly, *for a compact* abelian *group* G *and its character group* $\widehat{G}$, *the weakly complete* complex *Hopf algebra* $\mathbb{C}[G]$ *is the product algebra* $\mathbb{C}^{\widehat{G}}$ *with the comultiplication* $c\colon \mathbb{C}^{\widehat{G}} \to \mathbb{C}^{\widehat{G}\times\widehat{G}} \cong \mathbb{C}^{\widehat{G}}\otimes\mathbb{C}^{\widehat{G}}$, $c(F)(\chi_1,\chi_2)=F(\chi_1+\chi_2)$ *for* $F\colon \widehat{G}\to\mathbb{C}$ *in* $\mathbb{C}^{\widehat{G}}$. *The subgroup* $\Gamma(\mathbb{C}^{\widehat{G}})$ *of grouplike elements of the group of units of the algebra* $\mathbb{C}^{\widehat{G}}$ *is* $\mathrm{Hom}(\widehat{G},(\mathbb{C}\setminus\{0\},.))$ *while the vector subspace of primitive elements is* $\mathrm{Hom}(\widehat{G},(\mathbb{C},+))$. This forces the group $\Gamma(\mathbb{R}[G])\subseteq\Gamma(\mathbb{C}[G])$ to be $\mathrm{Hom}(\widehat{G},\mathbb{S}^1)\cong\widehat{\widehat{G}}\cong G$ with the complex circle group $\mathbb{S}^1$. While the relation $\Gamma(\mathbb{R}[G])\cong G$ remains true for *any* compact group, $\Gamma(\mathbb{C}[G])\cong G$ holds for a compact abelian group G if and only if it is profinite.

Thirdly, for each pro-Lie algebra L a weakly complete universal enveloping Hopf algebra $\mathbf{U}_{\mathbb{K}}(L)$ over $\mathbb{K}$ exists such that *for each* connected *compact group* G *the weakly complete real group Hopf algebra* $\mathbb{R}[G]$ *is a quotient Hopf algebra of* $\mathbf{U}_{\mathbb{R}}(\mathfrak{L}(G))$ *with the (pro-)Lie algebra* $\mathfrak{L}(G)$ *of* G. *The group* $\Gamma(\mathbf{U}_{\mathbb{R}}(\mathfrak{L}(G)))$ *of grouplike elements of the weakly complete enveloping algebra of* $\mathfrak{L}(G)$ *maps onto* $\Gamma(\mathbb{R}[G])\cong G$ *and is therefore nontrivial* in contrast to the case of the discrete classical enveloping Hopf algebra of an abstract Lie algebra.

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1. Introduction

For much of the material surrounding a theory of group Hopf algebras in the category of weakly complete real or complex vector spaces we refer to [3] and the forthcoming fourth edition of [6]. Some additional facts are presented here. A topological vector space over a locally compact field $\mathbb{K}$ is called *weakly complete* if it is isomorphic to $\mathbb{K}^J$ for some set J. This text considers $\mathbb{K}=\mathbb{R}$ or $\mathbb{K}=\mathbb{C}$ only and deals

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with the categories $\mathcal{G}$ of topological groups, respectively, $\mathcal{WA}$ of weakly complete topological algebras. Inside each $\mathcal{WA}$-object A we have the subset A^{-1} of all units (i.e., invertible elements) which turns out to be a $\mathcal{G}$-object in the subspace topology. In [3] it was shown that the functor $A \to A^{-1} : \mathcal{WA} \to \mathbb{G}$ has a left adjoint functor $G \mapsto \mathbb{K}[G] : \mathcal{G} \to \mathcal{WA}$. Automatically, $\mathbb{K}[G]$ is a topological Hopf algebra. Then for each topological group G there is a $\mathcal{G}$-morphism $\eta_G \colon G \to \mathbb{K}[G]^{-1}$ such that for each $\mathcal{WA}$-object A and $\mathcal{G}$-morphism $f \colon G \to A^{-1}$ there is a unique $\mathcal{WA}$-morphism $f' \colon \mathbb{K}[G] \to A$ such that $f(g) = f'(\eta_G(g))$. The weakly complete algebra $A = \mathbb{K}[G]$ is seen to have a comultiplication $c \colon A \to A \otimes A$ making it into a symmetric Hopf algebra. The subset

$$\mathbb{G}(A) = \{a \in A^{-1} : c(a) = a \otimes a\}$$

is a subgroup of A^{-1} and its elements are called *grouplike*. If G is a discrete group, then $\mathbb{K}[G]$ is the traditional *group algebra* over $\mathbb{K}$ and η_G is an embedding and its image is $\mathbb{G}(\mathbb{K}[G])$. In [3] it was shown that for a compact group G the map η_G is an embedding and for $\mathbb{K} = \mathbb{R}$ induces an isomorphism onto $\mathbb{G}(\mathbb{R}[G])$. We shall see in this text that *this fails over the complex ground field even for all compact abelian groups which are not profinite.* The assignment $G \mapsto \mathbb{G}(\mathbb{C}[G])$ may be viewed as as 'complexification' of the compact group G. This viewpoint becomes important if one considers the module category of $\mathbb{C}[G]$ in $\mathcal{W}$, i.e. the representations of G on weakly complete complex vector spaces, but we will not pursue this in the present note.

Since we shall deal with compact groups throughout this paper, we shall always consider G as a subgroup of the group $\mathbb{K}[G]^{-1}$ of units of $\mathbb{K}[G]$. In particular, this means $G \subseteq \mathbb{R}[G] \subseteq \mathbb{C}[G]$.

The article [3] and the 4th Edition of the book [6] identify a category $\mathcal{H}$ of weakly complete topological Hopf algebras for which the functor $G \to \mathbb{K}[G]$ implements an equivalence of the category of compact groups and the category $\mathcal{H}$. The vector space continuous dual of $\mathbb{K}[G]$ turns out to be the traditional representation algebra $R(G, \mathbb{R})$. This approach yields a new access to the Tannaka-Hochschild duality of the categories of compact groups and "reduced" real Hopf algebras.

In all of this, the precise nature of the weakly complete Hopf algebras $\mathbb{K}[G]$ even for compact groups remained somewhat obscure in the nondiscrete case. The present paper will present a precise piece of information on *the topological algebra structure of* $\mathbb{K}[G]$ in terms of a direct product of its finite dimensional minimal ideals whose precise structure links this presentation with the classical information of finite dimensional G-modules (cf. e.g. [6], Chapters 3 and 4). Certain complications have to be overcome on that level if one insists on an explicit identification of the algebra and ideal structure of $\mathbb{R}[G]$ as the case $\mathbb{C}[G]$ is easier.

For *abelian* compact groups G we shall present a very direct access to *the structure of the weakly complete Hopf algebra of* $\mathbb{C}[G]$ by identifying an isomorphism between $\mathbb{C}[G]$ and $\mathbb{C}^{\widehat{G}}$ and by further identifying the group of grouplike elements to be isomorphic to $(\mathfrak{L}(G), +) \oplus G$ with the pro-Lie algebra $\mathfrak{L}(G)$ of G (see [6]). The subgroup G of $\mathbb{C}[G]^{-1}$ therefore agrees with the group of grouplike elements of $\mathbb{C}[G]$ if and only if $\mathfrak{L}(G) = \{0\}$ if and only if G is totally disconnected.

The presence of weakly complete Lie algebras over $\mathbb{K}$ in the group $\mathbb{K}$-Hopf algebra of a compact group motivates a proof of the existence of a weakly complete universal enveloping algebra over $\mathbb{K}$ for $\mathcal{WA}$ Lie algebras over $\mathbb{K}$. In contrast to the case of enveloping algebras on the purely algebraic side, the $\mathcal{WA}$ enveloping Hopf algebras will sometimes have grouplike elements. The universal property of the $\mathcal{WA}$ enveloping Hopf algebras will show that each $\mathcal{WA}$ Hopf-group algebra $\mathbb{K}[G]$ of a compact *connected* group G is a quotient algebra of the $\mathcal{WA}$ enveloping algebra of $\mathfrak{L}(G)$. So $\mathcal{WA}$ enveloping algebras have a tendency of being larger than $\mathcal{WA}$ group algebras.

For a special class of profinite dimensional Lie algebras a similar but different approach to appropriate enveloping algebras is considered in [4].

A preprint of the present material appeared in the Series of Preprints of the Mathematical Research Institute of Oberwolfach [5].

2. Weakly Complete Hopf Algebras

For the basic theory of weakly complete Hopf algebras we may safely refer to [3] and [6], 4th Edition. For the present discussion we need a reminder of some basic concepts.

Definition 2.1. Let A be a weakly complete symmetric Hopf algebra, i.e. a group object in the monoidal category $(\mathcal{W}, \otimes_W)$ of weakly complete vector spaces (see [6], Appendix 7 and Definition A3.62), with comultiplication $c\colon A \to A \otimes A$ and coidentity $k\colon A \to \mathbb{K}$.
An element $a \in A$ is called *grouplike* if $c(a) = a \otimes a$ and $k(a) = 1$. The subgroup of grouplike elements in the group of units A^{-1} will be denoted $\mathbb{G}(A)$.
An element $a \in A$ is called *primitive*, if $c(a) = a \otimes 1 + 1 \otimes a$. The Lie algebra of primitive elements of A_{Lie}, i.e. the weakly complete Lie algebra obtained by endowing the weakly complete vector space underlying A with the Lie bracket obtained by $[a, b] = ab - ba$, will be denoted $\Pi(A)$.

We recall that any weakly complete symmetric Hopf algebra A has an exponential function $\exp_A\colon A \to A^{-1}$ as explained in [3], Theorem 3.12 or in [6], 4th edition, A7.41.

Theorem 2.2. *Let A be a weakly complete symmetric Hopf algebra. Then the following statements hold:*

(i) *The set $\Gamma(A)$ of grouplike elements of a weakly complete symmetric Hopf algebra A is a closed subgroup of $(A, \cdot)$ and therefore is a pro-Lie group.*

(ii) *The set $\Pi(A)$ of primitive elements of A is a closed Lie subalgebra of A_{Lie} and therefore is a pro-Lie algebra.*

(iii) $\Pi(A) \cong \mathfrak{L}(\mathbb{G}(A))$ *and the exponential function* $\exp_A$ *of A induces the exponential function* $\exp_{\Gamma(A)}\colon \Pi(A) \to \Gamma(A)$ *of the pro-Lie group $\Gamma(A)$.*

For a proof see e.g. [3], Theorem 6.15.

Definition 2.3. For an arbitrary topological group G we define $R(G, \mathbb{K}) \subseteq C(G, \mathbb{K})$ to be that set of continuous functions $f\colon G \to \mathbb{K}$ for which the linear span of the set of translations ${}_gf$, ${}_gf(h) = f(hg)$, is a finite dimensional vector subspace of $C(G, \mathbb{K})$. The functions in $R(G, \mathbb{K})$ are called *representative functions.*

Clearly $R(G,\mathbb{K})$ is a subalgebra of $C(G,\mathbb{K})$ also known as the *representation algebra* of G. In [3], Theorem 7.7(a) the following duality result was shown.

Theorem 2.4. (The Dual of a Weakly Complete Group Algebra $\mathbb{K}[G]$)
For an arbitrary topological group G, the function

$$F_G\colon \mathbb{K}[G]' \to \mathcal{R}(G,\mathbb{K}), \quad F_G(\omega) = \omega \circ \eta_G$$

is a natural isomorphism of Hopf algebras.

This applies, of course, to compact groups, in which case the Hopf algebra $R(G,\mathbb{K})$ is a well-known object.
For easy reference we record the following facts in the case of a compact group G for which we recall $G \subseteq \mathbb{K}[G]$:

Theorem 2.5. *For any* compact *topological group G, the following statements hold:*
(i) *We have $G \subseteq \Gamma(\mathbb{K}(G)) \subseteq \mathbb{K}[G]^{-1}$*
(ii) *In the case of $\mathbb{K} = \mathbb{R}$ the equality $G = \Gamma(\mathbb{R}[G])$ holds.*

For (1) see [3], 5.4, and for (ii) see [3], 8.7.
For the complex case, we shall see later in this paper that in many cases, a compact group G is a *proper* subgroup of $\Gamma(\mathbb{C}[G])$.

3. Some preservation properties of $\mathbb{K}[-]$

Let us explicitly formulate and prove some preservation properties of our functor $\mathbb{K}[-]$. Left adjoint functors preserve epics. A morphism of compact groups is an epimorphism if and only if it is surjective (see [6], RA3.17). Therefore the following lemma is to be expected.

Lemma 3.1. *For every surjective morphism $f\colon G \to H$ of compact groups the morphism $\mathbb{K}[f]\colon \mathbb{K}[G] \to \mathbb{K}[H]$ of weakly complete $\mathbb{K}$- Hopf algebras is surjective.*

Proof. From the surjectivity of $f\colon G \to H$ we conclude that

$$f(\mathrm{span}(G)) = \mathrm{span}(f(G)) = \mathrm{span}(H)$$

is dense in $\mathbb{K}[H]$ by Proposition 5.3 of [3], and likewise $\mathbb{K}(f)(\mathbb{K}[G])$ is dense in $\mathbb{K}[H]$. But $\mathbb{K}[f]$ is, in particular, a $\mathcal{W}$-morphism, that is, a morphism of weakly complete vector spaces. Every such has a closed image by [6], Theorem 7.30(iv). Hence $\mathbb{K}(f)(\mathbb{K}[G]) = \mathbb{K}[H]$. ∎

This particular left adjoint functor $\mathbb{K}[-]$, however, also preserves the injectivity of morphisms:

Theorem 3.2. *If G is a closed subgroup of the* compact *group H, then $\mathbb{K}[G] \subseteq \mathbb{K}[H]$ (up to natural isomorphism).*

Proof. From the injectivity of a morphism of compact groups $j\colon G \to H$ we derive the surjectivity of $C(j,\mathbb{K})\colon C(H,\mathbb{K}) \to C(G,\mathbb{K})$ by the Tietze Extension Theorem. Now we set $M := C(f,\mathbb{K})\big(R(H,\mathbb{K})\big) \subseteq R(G,\mathbb{K})$. Since $\mathbb{R}(H,\mathbb{K})$ is dense in $C(H,\mathbb{K})$ in the norm topology, M is dense in $R(G,\mathbb{K})$ in the norm topology.

Then it is dense in $L^2(G,\mathbb{K})$ in the L^2-topology, and M is a G-module. In the case of $\mathbb{K}=\mathbb{R}$ we can now apply Lemma 8.11 of [3] and conclude that $M=R(G,\mathbb{R})$. Thus $R(G,j)\colon R(H,\mathbb{R})\to R(G,\mathbb{R})$ is surjective. By Theorem 7.7 of [3] this implies that $\mathbb{R}[f]'\colon \mathbb{R}[H]'\to\mathbb{R}[G]'$ is surjective. The duality between $\mathbb{K}$-vector spaces and weakly complete $\mathbb{K}$-vector spaces shows that $\mathbb{R}[f]\colon\mathbb{R}[G]\to\mathbb{R}[H]$ is injective. This proves the theorem for $\mathbb{K}=\mathbb{R}$. But then the commuting diagram

$$\begin{array}{ccc}
\mathbb{C}\otimes\mathbb{R}[G] & \xrightarrow{\mathbb{C}\otimes\mathbb{R}[f]} & \mathbb{C}\otimes\mathbb{R}[H]\\
\cong\big\downarrow & & \big\downarrow\cong\\
\mathbb{C}[G] & \xrightarrow[\mathbb{C}[f]]{} & \mathbb{C}[H]
\end{array}$$

shows that $\mathbb{C}[f]$ is also injective. In the category of weakly complete vector spaces every injective morphism is an embedding by duality since every surjective morphism of vector spaces is a coretraction. ∎

Corollary 3.3. *Let G_0 denote the identity component of the compact group G. Then*

(i) *The Hopf algebra $\mathbb{K}[G_0]$ is a Hopf subalgebra of $\mathbb{K}[G]$.*

(ii) *$\mathbb{R}[G_0]$ is algebraically and topologically generated by $\Pi(\mathbb{R}[G])\cong\mathfrak{L}(G)$.*

Proof. (i) is a consequence of Theorem 3.2.

(ii) Note that the compact group G_0 is algebraically and topologically generated by $\exp_G(\mathfrak{L}(G))$ (cf. [7], Corollary 4.22, p. 191), and that $\overline{\mathrm{span}(G_0)}=\mathbb{R}[G_0]$ by [3], Corollary 5.3. ∎

4. A principal structure theorem of $\mathbb{K}[G]$

Let G be a compact group and let E be a finite dimensional vector space over $\mathbb{K}\in\{\mathbb{R},\mathbb{C}\}$. We recall that the *character of a representation* $\rho\colon G\to\mathrm{End}_{\mathbb{K}}(E)$ is the continuous map $g\mapsto \mathrm{tr}_{\mathbb{K}}(\rho(g))$. We also say that χ is the character of the G-module E. A representation, respectively, a G module, is determined by its character up to isomorphism. A character is called *irreducible* if the corresponding representation is irreducible (over the ground field $\mathbb{K}$), equivalently, the corresponding G-module is simple. We denote the set of all irreducible characters of G over $\mathbb{K}$ by $\widehat{G}_{\mathbb{K}}$. For each character χ of G, we select a finite dimensional G-module $E_{\chi,\mathbb{K}}$ having χ as its character. If ε is an irreducible character, then the ring $\mathbb{L}_{\varepsilon,\mathbb{K}}=\mathrm{End}_G(E_{\varepsilon,\mathbb{K}})$ of all $\mathbb{K}$-linear endomorphisms of $E_{\varepsilon,\mathbb{K}}$ which commute with the G-action is, by Schur's Lemma, a finite dimensional division ring over $\mathbb{K}$. Hence

$$\begin{array}{lcll}
\mathbb{L}_{\varepsilon,\mathbb{K}} & = & \mathbb{C} & \text{if}\quad \mathbb{K}=\mathbb{C},\\
\mathbb{L}_{\varepsilon,\mathbb{K}} & \in & \{\mathbb{R},\mathbb{C},\mathbb{H}\} & \text{if}\quad \mathbb{K}=\mathbb{R},
\end{array}$$

where $\mathbb{H}$, as is usual, denotes the skew-field of quaternions. We view $E_{\varepsilon,\mathbb{K}}$ as a right module over $\mathbb{L}_{\varepsilon,\mathbb{K}}$. We denote the corresponding representation by

$$\rho_{\varepsilon,\mathbb{K}}: G\to \mathrm{End}_{\mathbb{L}_{\varepsilon,\mathbb{K}}}(E_{\varepsilon,\mathbb{K}})\subseteq \mathrm{End}_{\mathbb{K}}(E_{\varepsilon,\mathbb{K}}).$$

Before we enter the presentation of the principal theorem on the weakly complete group algebra $\mathbb{K}[G]$ of a compact group we elaborate on some basic ideas of finite dimensional representation theory, indeed extending some of the presentation such as it can be found e.g. in Chapter 3 of [6]. The first lemma extends the details of Proposition 3.21 of [6] and the comments which precede it.

Lemma 4.1. *Let E be a finite dimensional vector space over $\mathbb{K}$ and $\rho\colon G \longrightarrow \mathrm{End}_{\mathbb{K}}(E)$ an irreducible representation of a group G. Let A denote the $\mathbb{K}$-span of the set $\{\rho(g) \mid g \in G\}$. Then $A = \mathrm{End}_{\mathbb{L}}(E)$, where $\mathbb{L} = \mathrm{End}_A(E) = \mathrm{End}_G(E)$, $\mathbb{L} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$.*

Proof. First of all we note that A is a $\mathbb{K}$-algebra containing id_E. Hence every A-submodule of the additive group E is a G-invariant linear subspace, and vice versa. Therefore E is a simple A-module and so Jacobson's Density Theorem applies, which, for the sake of completeness, we cite here in its entirety (see e.g. [2]).

Theorem 4.2. (Jacobson's Density Theorem) *Let $M \neq 0$ be an (additive) abelian group, let $A \subseteq \mathrm{End}(M)$ be a subring and suppose that M is simple as a left A-module. Put $\mathbb{L} = \mathrm{End}_A(M)$. Then $\mathbb{L}$ is a division ring and M is a right $\mathbb{L}$-module in a natural way. For every $2k$-tuple $(x_1, \ldots, x_k, y_1, \ldots, y_k) \in M^{2k}$, such that the elements $x_1, \ldots, x_k$ are linearly independent, there exists $a \in A$ such that $a(x_i) = y_i$ holds for all $i = 1, \ldots k$.*

Now the division ring $\mathbb{L}$ is a finite dimensional $\mathbb{K}$-algebra over $\mathbb{K}$, and hence is isomorphic to $\mathbb{R}, \mathbb{C}$, or $\mathbb{H}$. Moreover, $A \subseteq \mathrm{End}_{\mathbb{L}}(E)$. Let $x_1, \ldots, x_m$ be a $\mathbb{L}$-basis for E, and let $\phi \in \mathrm{End}_{\mathbb{L}}(E)$ be arbitrary. Then there exists an element $a \in A$ such that $a(x_i) = \phi(x_i)$ holds for all $i = 1, \ldots, m$. Therefore $\phi = a$ and thus $\mathrm{End}_{\mathbb{L}}(E) = A$. ∎

The following result now extends [6], Lemma 3.14.

Lemma 4.3. *Let E and F be finite dimensional vector spaces over $\mathbb{K}$ and suppose that $\rho\colon G \to \mathrm{End}_{\mathbb{K}}(E)$ and $\sigma\colon H \to \mathrm{End}_{\mathbb{K}}(F)$ are irreducible representations of groups G, H. Suppose also that $\mathrm{End}_G(E) = \mathbb{L} = \mathrm{End}_H(F)$. Then $\mathrm{Hom}_{\mathbb{L}}(F, E)$ is an irreducible $G \times H$-module over $\mathbb{K}$, where $(g, h)(f) = \rho(g) \circ f \circ \sigma(h^{-1})$.*

Proof. We define $A \subseteq \mathrm{End}_{\mathbb{K}}(E)$ and $B \subseteq \mathrm{End}_{\mathbb{K}}(F)$ as in Corollary 4.1. Then $A = \mathrm{End}_{\mathbb{L}}(E)$ and $B = \mathrm{End}_{\mathbb{L}}(F)$. The $\mathbb{K}$-vector space $\mathrm{Hom}(F, E)$ is in a natural way a right A-module and a left B-module. For every nonzero $f \in \mathrm{Hom}_{\mathbb{L}}(F, E)$ we have $AfB = \mathrm{Hom}_{\mathbb{L}}(F, E)$. Therefore $\mathrm{Hom}_{\mathbb{L}}(F, E)$ is simple as $G \times H$-module over $\mathbb{K}$. ∎

We are now ready to prove a principal structure theorem for the weakly complete group algebra $\mathbb{K}[G]$ of a compact group G for either $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. For each $\varepsilon \in \widehat{G}_{\mathbb{K}}$ we have the G-module $E_{\varepsilon,\mathbb{K}}$ and the corresponding irreducible representation $\rho_{\varepsilon,\mathbb{K}}\colon G \to \mathrm{End}_{\varepsilon,\mathbb{K}}(E_{\varepsilon,\mathbb{K}})$ into the group of units of the concrete *matrix ring* $M_\varepsilon := \mathrm{End}_{\varepsilon,\mathbb{K}}(E_{\varepsilon,\mathbb{K}}))$ over $\mathbb{L} = \mathbb{L}_{\varepsilon,\mathbb{K}}$ of $\mathbb{L}$-dimension $(\dim_{\mathbb{L}} E_{\varepsilon,\mathbb{K}})^2$. Accordingly there is a unique function $\rho_G\colon G \to \prod_{\varepsilon\in\widehat{G}_{\mathbb{K}}} M_\varepsilon$ which is an injective group morphism into the multiplicative group of units of the product defined by the universal property of the product such that the following diagram commutes for all $\chi \in \widehat{G}_{\mathbb{K}}$:

$$\begin{array}{ccc}
G & \xrightarrow{\rho_G} & \prod_{\varepsilon\in\widehat{G}_{\mathbb{K}}} M_\varepsilon \\
{\scriptstyle =}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{pr}_\chi} \\
G & \xrightarrow[\rho_{\chi,\mathbb{K}}]{} & M_\chi
\end{array}$$

Theorem 4.4. *For any compact group G the weakly complete symmetric Hopf algebra $\mathbb{K}[G]$ is a direct product*

$$\mathbb{K}[G] = \prod_{\epsilon \in \widehat{G}_{\mathbb{K}}} \mathbb{K}_\epsilon[G]$$

of finite dimensional minimal two-sided ideals $\mathbb{K}_\epsilon[G]$ such that for each $\varepsilon \in \widehat{G}_{\mathbb{K}}$ there is a $\mathbb{K}$-algebra isomorphism

$$\mathbb{K}_\varepsilon[G] \cong M_\varepsilon = \mathrm{End}_{\mathbb{L}_{\varepsilon,\mathbb{K}}}(E_{\varepsilon,\mathbb{K}}).$$

In particular, each of these two-sided ideals $\mathbb{K}_\varepsilon[G]$ is a two sided simple $G \times G$-module and as an algebra is isomorphic to a full matrix ring over $\mathbb{L}$.

Remark 4.5. The diagram

$$\begin{tikzcd} G \arrow[r, "\eta_G"] \arrow[d, "="'] & \mathbb{K}[G] \arrow[d, "\cong"] \\ G \arrow[r, "\rho_G"'] \arrow[d, "="'] & \prod_{\varepsilon \in \widehat{G}_{\mathbb{K}}} M_\varepsilon \arrow[d, "\mathrm{pr}_\chi"] \\ G \arrow[r, "\rho_{\chi,\mathbb{K}}"'] & M_\chi \end{tikzcd}$$

commutes for all $\chi \in \widehat{G}_{\mathbb{K}}$.

Proof. By Theorem 2.4 and [6] Theorem 3.28, the topological dual $\mathbb{K}[G]' \cong R(G,\mathbb{K})$ is the direct sum of the finite dimensional two-sided G-submodules $R_\epsilon(G,\mathbb{K})$ as ϵ ranges through the set of irreducible characters in $\widehat{G}_{\mathbb{K}}$. The $G \times G$-module $R_\epsilon(G,\mathbb{K})$ is defined in [6] as the image of the linear map

$$\phi : E'_{\varepsilon,\mathbb{K}} \otimes_{\mathbb{K}} E_{\varepsilon,\mathbb{K}} \longrightarrow R(G,\mathbb{K}),$$

where

$$\phi(u \otimes v)(g) = \langle u, \rho_{\varepsilon,\mathbb{K}}(g)v\rangle.$$

If we put $\psi(f)(g) = \mathrm{tr}_{\mathbb{K}}(\rho_{\varepsilon,\mathbb{K}}(g)f)$, for $f \in \mathrm{End}_{\mathbb{K}}(E_{\varepsilon,\mathbb{K}})$ and $g \in G$, then the diagram

$$\begin{tikzcd} E'_{\varepsilon,\mathbb{K}} \otimes_{\mathbb{K}} E_{\varepsilon,\mathbb{K}} \arrow[r, "\phi"] \arrow[d, "s"'] & R_\varepsilon(G,\mathbb{K}) \arrow[d, "="'] \\ \mathrm{End}_{\mathbb{K}}(E_{\varepsilon,\mathbb{K}}) \arrow[r, "\psi"] & R_\varepsilon(G,\mathbb{K}) \end{tikzcd}$$

commutes, where $s(u \otimes v) = [w \mapsto v\langle u, w\rangle]$. We recall that group $G \times G$ acts on $R_\varepsilon(G,\mathbb{K})$ via $(a,b)(\lambda) = [g \mapsto \lambda(a^{-1}gb)]$. If we put

$$(a,b)(u \otimes v) = (u \circ \rho_{\varepsilon,\mathbb{K}}(a^{-1})) \otimes \rho_{\varepsilon,\mathbb{K}}(b)v \quad \text{and} \quad (a,b)(f) = \rho_{\varepsilon,\mathbb{K}}(b) \circ f \circ \rho_{\varepsilon,\mathbb{K}}(a^{-1}),$$

then all maps in this diagram are $G \times G$-equivariant.

Suppose that $\mathbb{K} = \mathbb{L}$. Then $\mathrm{End}_{\mathbb{K}}(E_{\varepsilon,\mathbb{K}}) = \mathrm{End}_L(E_{\varepsilon,\mathbb{K}})$ is simple as a $G \times G$-module by Lemma 4.3 and thus ψ is an isomorphism.

Suppose next that $\mathbb{K} \subsetneq \mathbb{L}$. Then $\mathbb{K} = \mathbb{R}$ and $\mathbb{L} = \mathbb{C}$ or $\mathbb{L} = \mathbb{H}$. By the averaging process in [6] Lemma 2.15 there exists a G-invariant positive definite $\mathbb{L}$-hermitian form $(\cdot|\cdot)$ on E, semilinear in the first argument and linear in the second argument.

This allows us to rewrite $R_\varepsilon(G,\mathbb{K})$ as the span of the maps $g \mapsto \mathrm{Re}(u|gv)$, for $u, v \in E$. The G-invariance of $(\cdot|\cdot)$ yields that $\mathrm{Re}(au|gbv) = \mathrm{Re}(u|a^{-1}gbv)$. If we consider the algebra inclusion

$$j\colon \mathrm{End}_{\mathbb{L}}(E_{\varepsilon,\mathbb{K}}) \to \mathrm{End}_{\mathbb{K}}(E_{\varepsilon,\mathbb{K}}),$$

then $\mathrm{Re}(u|v) = \mathrm{tr}_{\mathbb{K}}[w \mapsto v(u|w)]$ holds for the trace map of $\mathrm{End}_{\mathbb{K}}(E_{\varepsilon,\mathbb{K}})$. It follows that the map $\psi \circ j$ in the diagram

$$\begin{tikzcd}
E'_{\varepsilon,\mathbb{K}} \otimes_{\mathbb{K}} E_{\varepsilon,\mathbb{K}} \arrow[r, "\phi"] \arrow[d, "s"'] & R_\varepsilon(G,\mathbb{K}) \arrow[d, "="'] \\
\mathrm{End}_{\varepsilon,\mathbb{K}}(E_{\varepsilon,\mathbb{K}}) \arrow[r, "\psi"] & R_\varepsilon(G,\mathbb{K}) \\
\mathrm{End}_{\mathbb{L}}(E_{\varepsilon,\mathbb{K}}) \arrow[u, "j"]
\end{tikzcd}$$

is surjective and $G \times G$-equivariant. Since $\mathrm{End}_{\mathbb{L}}(E_{\varepsilon,\mathbb{K}})$ is a simple $G \times G$-module over $\mathbb{K}$ by Lemma 4.3, the map $\psi \circ j$ is an isomorphism.

For the remaining part of the proof we apply standard duality theory. We put

$$R^\chi = \bigoplus_{\chi \neq \varepsilon \in \widehat{G}_{\mathbb{K}}} R_\varepsilon(G,\mathbb{K})$$

and define $\mathbb{K}_\chi[G]$ as the annihilator of R^χ. The annihilator mechanism supplies us with the diagram

$$\begin{tikzcd}
\mathbb{K}[G] \arrow[r, leftrightarrow, "\perp"] \arrow[d, dash] & \{0\} \arrow[d, dash] \\
\mathbb{K}_\chi[G] \arrow[r, leftrightarrow, "\perp"] \arrow[d, dash] & R^\chi \arrow[d, dash] \\
\{0\} \arrow[r, leftrightarrow, "\perp"] & R(G,\mathbb{K}).
\end{tikzcd} \quad \Big\} \cong R_\chi(G,\mathbb{K})$$

By the duality of $\mathcal{V}_{\mathbb{K}}$ and $\mathcal{W}_{\mathbb{K}}$ it follows that $\mathbb{K}[G] \cong \prod_{\epsilon \in \widehat{G}_K} \mathbb{K}_\epsilon[G]$ with

$$\mathbb{K}_\epsilon[G] \cong R_\epsilon(G,\mathbb{K})'.$$

Now, if any closed vector subspace J of $\mathbb{K}[G]$ satisfies $G \cdot J \subseteq J$ and $J \cdot G \subseteq J$, then we also have $\mathrm{span}(G) \cdot J \subseteq J$ and $J \cdot \mathrm{span}(G) \subseteq J$ (where we view G as a subset of $\mathbb{K}[G]$). Then Proposition 5.3 of [3] says that $\overline{\mathrm{span}(G)} = \mathbb{K}[G]$, and so $\mathbb{K}[G] \cdot J \subseteq J$ and $J \cdot \mathbb{K}[G] \subseteq J$. That is, J is a closed two-sided ideal of $\mathbb{K}[G]$. Therefore each $\mathbb{K}_\epsilon[G]$ is a two-sided ideal in $\mathbb{K}[G]$.

It remains to clarify the multiplicative structure of the ideals $\mathbb{K}_\varepsilon[G]$. If we consider $\varepsilon \in \widehat{G}_{\mathbb{K}}$ and the representation $\rho_{\varepsilon,\mathbb{K}}$, then the map

$$G \xrightarrow{\rho_{\varepsilon,\mathbb{K}}} \mathrm{Gl}_{\mathbb{L}_{\varepsilon,\mathbb{K}}}(E_{\varepsilon,\mathbb{K}}) = \mathrm{End}_{\mathbb{L}_{\varepsilon,\mathbb{K}}}(E_{\varepsilon,\mathbb{K}})^{-1} \xrightarrow{\mathrm{inc}} \mathrm{End}_{\mathbb{L}_{\varepsilon,\mathbb{K}}}(E_{\varepsilon,\mathbb{K}})$$

and the universal property of $\mathbb{K}[G]$ described in the Weakly Complete Group Algebra Theorem 5.1 of [3] provides a morphism of weakly complete algebras

$$\pi_\varepsilon\colon \mathbb{K}[G] \to \mathrm{End}_{\mathbb{L}_{\varepsilon,\mathbb{K}}}(E_{\varepsilon,\mathbb{K}})$$

extending $\rho_{\varepsilon,\mathbb{K}}$. We also have the product projection of weakly complete algebras $\mathrm{pr}_\epsilon\colon \mathbb{K}[G] \to \mathbb{K}_\epsilon[G]$. Both maps π_ε and pr_ϵ have the same kernel $\prod_{\epsilon\neq\epsilon'\in\widehat{G}_{\mathbb{K}}} A_{\epsilon'}$. So there is an injective morphism

$$\alpha\colon \mathbb{K}_\epsilon[G] \to \mathrm{End}_{\mathbb{L}_{\varepsilon,\mathbb{K}}}(E_{\varepsilon,\mathbb{K}})$$

such that $\pi_\varepsilon = \alpha\circ \mathrm{pr}_\epsilon$. Since both algebras have the same dimension, α is an isomorphism of $\mathbb{K}$-algebras. ■

Corollary 4.6. *There is an isomorphism of $G\times G$-modules*

$$R(G,\mathbb{K}) = \bigoplus_{\varepsilon\in\widehat{G}_{\mathbb{K}}} \mathrm{End}_{\mathbb{L}_{\varepsilon,\mathbb{K}}}(E_{\varepsilon,\mathbb{K}}).$$

Thus the multiplicity m of $E_{\varepsilon,\mathbb{K}}$ as a G-module in $R(G,\mathbb{K})$ is

$$m = \dim_{\mathbb{L}_{\varepsilon,\mathbb{K}}}(E_{\varepsilon,\mathbb{K}}) = \frac{\dim_{\mathbb{K}} E_{\varepsilon,\mathbb{K}}}{\dim_{\mathbb{K}}\mathbb{L}}.$$

This conclusion is well-known for $\mathbb{K} = \mathbb{C}$ (see e.g Theorems 3.22 and 3.28 in [6]) but we could not readily find a reference for $\mathbb{K} = \mathbb{R}$. While the algebra structure of the weakly complete symmetric Hopf algebra $\mathbb{K}[G]$ is satisfactorily elucidated in Theorem 4.4, the comultiplication seems to be not easily accessible due to complications of the way how the representation theory of $G\times G$ reduces to that of G in general. In the case of commutative compact groups G and the complex ground field $\mathbb{C}$ these complications go away, and so we shall clarify the situation in these circumstances in the subsequent section.

5. The weakly complete group algebras of compact Abelian groups: an alternative view

We have seen the usefulness of the concept of a weakly complete group algebra $\mathbb{K}[G]$ over the real or complex numbers. We obtained its existence from the Adjoint Functor Existence Theorem. This is rather remote from a concrete construction. It may therefore be helpful to see the whole apparatus in a much more concrete way at least for a substantial subcategory of the category of compact groups, namely, the category of compact abelian groups for which we already have a familiar duality theory due to Pontryagin and Van Kampen (see e.g. [6], Chapter 7).

In this section let G be a compact abelian group and $\widehat{G} = \mathcal{CAB}(G,\mathbb{T})$ (with the category $\mathcal{CAB}$ of compact abelian groups and $\mathbb{T} = \mathbb{R}/\mathbb{Z}$) its discrete character group. These groups are written additively. For $\mathbb{K} = \mathbb{C}$ there is a natural bijection $\widehat{G} \to \widehat{G}_{\mathbb{C}}$ from the character group to the set of equivalence classes of complex simple G-modules (cf. [6], Lemma 2.30 (p.43), Exercise E3.10 (p.66)), and also Proposition 3.56 (p.87) for some information on $\widehat{G}_{\mathbb{R}}$). This bijection associates with a character $\chi\in\widehat{G} = \mathrm{Hom}(G,\mathbb{T})$ the class of the module $E_\chi = \mathbb{C}$, $\chi\cdot c = e^{2\pi i\chi}c$. Accordingly, [6] Theorem 3.28 (12) reads $R(G,\mathbb{C}) = \sum_{\chi\in\widehat{G}} \mathbb{C}\cdot f_\chi$, for a suitable basis f_χ, $\chi\in\widehat{G}$, $f_\chi(g) = e^{2\pi i\chi(g)}$. In other words, as a G-module, $R(G,\mathbb{C}) \cong \mathbb{C}^{(\widehat{G})}$. Accordingly, we expect $\mathbb{C}[G]$ to be uncomplicated. Our Theorem 4.4 makes this clear:

The complex algebra $\mathbb{C}[G]$ *may be naturally identified with the componentwise algebra* $\mathbb{C}^{\widehat{G}}$.

In the abelian case, our understanding of the comultiplication of $\mathbb{C}[G] = \mathbb{C}^{\widehat{G}}$ is much more explicit than in the general situation of Theorem 4.4. Each character $\chi\colon G \to \mathbb{T}$ determines a morphism $f_\chi\colon G \to \mathbb{C}^{-1} = \mathbb{C}^\times$, $f_\chi(g) = e^{2\pi i\chi(g)}$, $g \in G \subseteq \mathbb{C}^{\widehat{G}}$. By the universal property of $\mathbb{C}[G] = \mathbb{C}^{\widehat{G}}$, this value agrees with the χ-th projection of $g \in G \subseteq \mathbb{C}^{\widehat{G}}$. Hence

$$(\forall g \in G, \chi \in \widehat{G})\, \eta_G(g)(\chi) = e^{2\pi i\langle \chi, g\rangle}.$$

Accordingly, if we write $\mathbb{S}^1 = \{z \in \mathbb{C}; |z| = 1\}$, then $g \in \operatorname{Hom}(\widehat{G}, \mathbb{S}^1) \cong \widehat{\widehat{G}} \cong G$. Then in view of $G \subseteq \mathbb{R}[G] \subseteq \mathbb{C}[G]$ we have

$$\operatorname{Hom}(\widehat{G}, \mathbb{S}^1) \subseteq \overline{\operatorname{span}_{\mathbb{R}}(\operatorname{Hom}(\widehat{G}, \mathbb{S}^1))} = \mathbb{R}[G] \subseteq \mathbb{C}[G] = \mathbb{C}^{\widehat{G}}.$$

Recall from [3], Theorem 5.5 that we have an isomorphism

$$\alpha_G\colon \mathbb{C}[G \times G] \to \mathbb{C}[G] \otimes_{\mathcal{W}} \mathbb{C}[G],$$

and from [3] Lemma 5.12 we recall the comultiplication $\gamma_G\colon \mathbb{C}[G] \to \mathbb{C}[G] \otimes_{\mathcal{W}} \mathbb{C}[G]$ to be the composition

$$\mathbb{C}[G] \xrightarrow{\ \delta_G\ } \mathbb{C}[G \times G] \xrightarrow{\ \alpha_G\ } \mathbb{C}[G] \otimes_{\mathcal{W}} \mathbb{C}[G].$$

Now for a compact abelian group G, the diagonal morphism $\delta_G\colon G \to G \times G$ has the group operation of $\widehat{G}$ as its dual, namely:

$$\widehat{\delta_G}\colon \widehat{G} \times \widehat{G} \to \widehat{G}. \quad \widehat{\delta_G}(\chi_1, \chi_2) = \chi_1 + \chi_2,$$

as we write abelian group operations additively in general. If now we also write $\mathbb{C}[G] \otimes_{\mathcal{W}} \mathbb{C}[G] = \mathbb{C}^{\widehat{G} \times \widehat{G}}$ (identifying $\phi \otimes \psi$ with $(\chi_1, \chi_2) \mapsto \phi(\chi_1)\psi(\chi_2)$), then we have

$$\gamma_G = \mathbb{C}^{\widehat{\delta_G}}\colon \mathbb{C}^{\widehat{G}} \to \mathbb{C}^{\widehat{G} \times \widehat{G}}, \text{i.e., } (\forall \phi \in \mathbb{C}^{\widehat{G}}), \gamma_G(\phi)(\chi_1, \chi_2) = \phi(\chi_1 + \chi_2).$$

This allows us to determine explicitly the elements of the group $\mathbb{G}(\mathbb{C}^{\widehat{G}})$ of all grouplike elements: Indeed a nonzero element $\phi \in \mathbb{C}^{\widehat{G}}$ is in $\mathbb{G}(\mathbb{C}^{\widehat{G}})$ if and only if

$$\gamma_G(\phi) = \phi \otimes \phi \quad \text{in} \quad \mathbb{C}^{\widehat{G}} \otimes_{\mathcal{W}} \mathbb{C}^{\widehat{G}} = \mathbb{C}^{\widehat{G} \times \widehat{G}},$$

where $(\phi \otimes \phi)(\chi_1, \chi_2) = \phi(\chi_1)\phi(\chi_2)$. This is the case if and only if

$$(\forall \phi_1, \phi_2 \in \widehat{G})\, \phi(\chi_1 + \chi_2) = \gamma_G(\phi)(\chi_1, \chi_2) = (\phi \otimes \phi)(\chi_1, \chi_2) = \phi(\chi_1)\phi(\chi_2),$$

that is, if and only if ϕ is a morphism of groups from $\widehat{G}$ to $\mathbb{C}^\times = (\mathbb{C} \setminus \{0\}, \cdot)$.
Similarly, an element $\phi \in \mathbb{C}^{\widehat{G}}$ is primitive if and only if

$$\phi(\chi_1 + \chi_2) = \gamma_G(\phi)(\chi_1, \chi_2) = \big((\phi \otimes 1) + 1 \otimes \phi)\big)(\chi_1, \chi_2) = \phi(\chi_1) + \phi(\chi_2)$$

if and only if $\phi\colon \widehat{G} \to (\mathbb{C}, +)$ is a morphism of topological groups.
Let us summarize this discourse:

Theorem 5.1. (The Group Hopf Algebra $\mathbb{C}[G]$ for Compact Abelian G) *Let G be a compact abelian group and A its weakly complete commutative symmetric group Hopf algebra $\mathbb{C}[G]$ and let $\widehat{G} = \mathrm{Hom}(G, \mathbb{T})$ be its character.*

(i) *Then A may be identified with $\mathbb{C}^{\widehat{G}}$ such that $g\colon \widehat{G} \to A^{-1}$ is defined by*

$$(\forall \chi \in \widehat{G})\, g(\chi) = e^{2\pi i \langle \chi, g\rangle} \in \mathbb{S}^1,$$

where $\mathbb{S}^1 = \{z \in \mathbb{C} : |z| = 1\} \subseteq \mathbb{C}^\times$. The natural image of G in A^{-1} is

$$G = \mathrm{Hom}(\widehat{G}, \mathbb{S}^1) \cong \widehat{\widehat{G}}, \quad \textit{and} \quad G = \mathrm{Hom}(\widehat{G}, \mathbb{S}^1) \subseteq \mathbb{R}[G] \subseteq \mathbb{C}[G] = \mathbb{C}^{\widehat{G}}.$$

(ii) *If, as is possible in the category of weakly complete vector spaces, the weakly complete vector spaces $A \otimes_{\mathcal{W}} A$ and $\mathbb{C}^{\widehat{G} \times \widehat{G}}$ are identified, then the comultiplication $\gamma_G\colon A \to A \otimes_{\mathcal{W}} A$ of A is given by*

$$(\forall \phi\colon \widehat{G} \to \mathbb{C},\ \chi_1, \chi_2 \in \widehat{G}) \quad \gamma_G(\phi)(\chi_1, \chi_2) = \phi(\chi_1 + \chi_2) \in \mathbb{C}.$$

(iii) *The group of grouplike elements of A is*

$$\mathbb{G}(A) = \mathrm{Hom}(\widehat{G}, \mathbb{C}^\times) \subseteq \mathbb{C}^{\widehat{G}}.$$

(iv) *The weakly complete Lie algebra of primitive elements of A is*

$$\mathbb{P}(A) = \mathrm{Hom}(\widehat{G}, \mathbb{C}) \subseteq \mathbb{C}^{\widehat{G}}. \qquad \blacksquare$$

We write $\mathbb{R}_+^\times$ for the multiplicative subgroup $\{z \in \mathbb{C} : 0 < z \in \mathbb{R} \subseteq \mathbb{C}\}$ of $\mathbb{C}^\times$.

Corollary 5.2. *For a compact abelian group G and the weakly complete commutative unital algebra $A := \mathbb{C}[G]$ we have a commutative diagram*

$$\begin{array}{ccccc}
\mathbb{P}(A) & = & \mathrm{Hom}(\widehat{G}, \mathbb{R}) + \mathrm{Hom}(\widehat{G}, i\mathbb{R}) & \xrightarrow{\cong} & \mathfrak{L}(G) \times \mathfrak{L}(G) \\
{\scriptstyle \exp_A}\big\downarrow & & & & \big\downarrow {\scriptstyle \mathrm{id}_{\mathfrak{L}(G)} \times \exp_G} \\
\mathbb{G}(A) & = & \mathrm{Hom}(\widehat{G}, \mathbb{R}_+^\times) \cdot \mathrm{Hom}(\widehat{G}, \mathbb{S}^1) & \xrightarrow[\cong]{} & \mathfrak{L}(G) \times G.
\end{array}$$

The unique maximal compact subgroup of $\mathbb{G}(A)$ is $G = \mathrm{Hom}(\widehat{G}, \mathbb{S}^1)$.

Proof. There is an elementary isomorphism of topological groups

$$(r, t + \mathbb{Z}) \mapsto e^r e^{2\pi i t} = e^{r + 2\pi i t} : \mathbb{R} \times \mathbb{T} \to \mathbb{C}^\times.$$

Accordingly, $\mathbb{G}(A) = \mathrm{Hom}(\widehat{G}, \mathbb{C}^\times) \cong \mathrm{Hom}(\widehat{G}, \mathbb{R}) \oplus \mathrm{Hom}(\widehat{G}, \mathbb{T})$.

Now $\mathrm{Hom}(\widehat{G}, \mathbb{R}) \cong \mathrm{Hom}(\mathbb{R}, G)$ (cf. [6], Proposition 7.11(iii)), $\mathrm{Hom}(\mathbb{R}, G) = \mathfrak{L}(G)$ by [6], Definition 5.7 (cf. Proposition 7.36ff., Theorem 7.66) and $\mathrm{Hom}(\widehat{G}, \mathbb{T}) = \widehat{\widehat{G}} \cong G$ by [6], Theorem 2.32. For the exponential function $\exp_A$ of a weakly complete unital symmetric Hopf algebra is treated in Theorem 2.2 above. $\blacksquare$

A compact abelian group is totally disconnected (i.e. profinite) if and only if $\mathfrak{L}(G) = \{0\}$ (cf. [6], Corollary 7.72).

Remark 5.3. For a compact abelian group G the equality $G = \mathbb{G}(\mathbb{C}[G])$ holds if and only if G is totally disconnected (i.e. profinite).

Proof. By Theorem 5.1, the equality holds if and only if $L(G) = \{0\}$ if and only if $\mathrm{Hom}(\widehat{G}, \mathbb{R}) = \{0\}$ if and only if $\widehat{G}$ is a torsion group (cf. e.g. [6], Propositions A1.33, A1.39) if and only if G is totally disconnected (see Corollary 8.5). ∎

In particular, e.g., $\mathbb{T} \neq \mathbb{G}(\mathbb{C}[\mathbb{T}])$.

Now we understand $\mathbb{C}[G] = \mathbb{C}^{\widehat{G}}$ rather explicitly, but $\mathbb{R}[G]$ only rather implicitly. However, Theorem 4.4 applies with $\mathbb{K} = \mathbb{R}$ in order to shed some light on its intrinsic structure.

We define the function $\sigma_G \colon \mathbb{C}[G] \to \mathbb{C}[G]$ as follows: For $\chi \in \widehat{G}$ we set $\check{\chi}(g) = \chi(-g) = -\chi(g)$. Then we define

$$(\forall \phi \in \mathbb{C}^{\widehat{G}})\, \sigma(\phi)(\chi) = \overline{\phi}(\check{\chi}).$$

Exercise 5.4. For a compact abelian group G, the function σ_G is an involution of weakly complete real algebras of $\mathbb{C}[G]$ whose precise fixed point algebra is $\mathbb{R}[G]$. Accordingly, $\mathbb{C}[G] = \mathbb{R}[G] \oplus i\mathbb{R}[G]$.

Remark 5.5. If $\aleph$ is any cardinal and $\widehat{G}$ is any abelian group with torsion free rank $\aleph$, then $\mathrm{Hom}(\widehat{G}, \mathbb{R}) \cong \mathbb{R}^{\aleph}$.

Proof. In [6], Theorem 8.20, pp.387ff. it is discussed that G contains totally disconnected compact subgroups Δ such that the annihilator in the character group of G, say, $\Delta^\perp \subseteq \widehat{G}$ is free, and $\widehat{G}/\Delta^\perp$ is a torsion group. This means that G/Δ is a torus. We note that the inclusion $\Delta^\perp \to \widehat{G}$ induces an isomorphism $\mathbb{K} \otimes_{\mathbb{Z}} \Delta^\perp \to \mathbb{K} \otimes_{\mathbb{Z}} \widehat{G}$ and the (torsion free) rank of $\widehat{G}$ is $\operatorname{rank} \Delta^\perp$. If $\Delta^\perp \cong \mathbb{Z}^{(X)}$ for a set X of cardinality $\operatorname{rank} \Delta^\perp$, then $\mathrm{Hom}(\widehat{G}, \mathbb{K}) \cong \mathbb{K}^X$. ∎

5.1. The exponential function of $\mathbb{C}[G] = \mathbb{C}^{\widehat{G}}$

We recall from Theorem 2.2 that every weakly complete associative unital algebra W has an exponential function, which is immediate in the case of $W = \mathbb{C}^{\widehat{G}}$ as it is calculated componentwise. If the weakly complete algebra W is even a Hopf algebra, such as $\mathbb{C}^{\widehat{G}}$, then the group $\Gamma(W)$ of grouplike elements is a pro-Lie group and $\Pi(W)$ is the (real!) Lie algebra of the pro-Lie group $\Gamma(W)$ ([3], Theorem 6.15). If indeed $W = \mathbb{C}^{\widehat{G}} = \mathbb{C}[G]$ for a compact abelian group G, then the exponential function $\exp_{\Gamma(W)} \colon \mathfrak{L}(\Gamma(W)) \to \Gamma(W))$ of $\Gamma(W)$ is the restriction of the (componentwise!) exponential function $\exp \colon \mathbb{C}^{\widehat{G}} \to (\mathbb{C}^{\widehat{G}})^{-1} = ((\mathbb{C}^\times, .))^{\widehat{G}}$ to $\mathfrak{L}(\Gamma(W)) = \mathrm{Hom}(\widehat{G}, \mathbb{C})$.

6. The weakly complete enveloping algebra of a weakly compact Lie algebra

We have observed that for compact groups G the weakly complete real group algebra $\mathbb{R}[G]$ contains a substantial volume of different materials: the pro-Lie group G itself, its Lie algebra $\mathfrak{L}(G)$, the exponential function between them and, as was discussed in detail in [3], a substantial portion of the Radon measure theory of G. The topological Hopf algebra $\mathbb{K}[G]$ is, in a sense, universally generated by G. So it seems natural to ask the question whether $\mathfrak{L}(G)$ generates $\mathbb{K}[G]$ in a universal way – perhaps in some fashion that would resemble the universal enveloping algebra of a Lie algebra such as it is presented in the famous POINCARÉ-BIRKHOFF-WITT-Theorem (see e.g. [1], §2, n° 7, Théorème 1., p. 30). This is not exactly the case, but a few aspects should be discussed.

So we let $\mathbb{K}$ again denote one of the topological fields $\mathbb{R}$ or $\mathbb{C}$. Let $\mathcal{WA}$ denote the category of weakly complete associative unital algebras over $\mathbb{K}$ and and $\mathcal{WL}$ the category of weakly complete Lie algebras over $\mathbb{K}$. The functor $A \mapsto A_{\mathrm{Lie}}$ which associates with a weakly complete associative algebra A the weakly complete Lie algebra obtained by considering on the weakly complete vector space A the Lie algebra obtained with respect to the Lie bracket $[x, y] = xy - yx$ is called the *underlying Lie algebra functor*.

Since A is a strict projective limit of finite dimensional K-algebras by [3], Theorem 3.2, then A_{Lie} is a strict projective limit of finite dimensional $\mathbb{K}$-Lie algebras, briefly called *pro-Lie algebras*. Every pro-Lie algebra is weakly complete. (Caution: A comment following Theorem 3.12 of [3] exhibits an example of a weakly complete $\mathbb{K}$-Lie algebra which is not a pro-Lie algebra!)

Lemma 6.1. *The* underlying Lie algebra *functor $A \mapsto A_{\mathrm{Lie}}$ from $\mathcal{WA}$ to $\mathcal{WL}$ has a left adjoint $\mathbf{U}\colon \mathcal{WL} \to \mathcal{WA}$.*

Proof. The category $\mathcal{WL}$ is complete. (Exercise. Cf. Theorem A3.48 of [6], p. 781.) The "Solution Set Condition" (of Definition A3.59 in [6], p. 786) holds. (Exercise: Cf. the proof of [3], Section 5.1 "The solution set condition".) Hence $\mathbf{U}$ exists by the Adjoint Functor Existence Theorem (i.e., Theorem A3.60 of [6], p. 786). ∎

In other words, for each weakly complete Lie algebra L there is a natural morphism $\lambda_L\colon L \to \mathbf{U}(L)$ such that for each continuous Lie algebra morphism $f\colon L \to A_{\mathrm{Lie}}$ for a weakly complete associative unital algebra A there is a unique $\mathcal{WA}$-morphism $f'\colon \mathbf{U}(L) \to A$ such that $f = f'_{\mathrm{Lie}} \circ \lambda_L$.

$$
\begin{array}{ccccc}
 & \mathcal{WL} & & & \mathcal{WA} \\
\hline
L & \xrightarrow{\lambda_L} & \mathbf{U}(L)_{\mathrm{Lie}} & & \mathbf{U}(L) \\
{\scriptstyle \forall f}\big\downarrow & & \big\downarrow{\scriptstyle f'_{\mathrm{Lie}}} & & \big\downarrow{\scriptstyle \exists ! f'} \\
A_{\mathrm{Lie}} & \xrightarrow[\mathrm{id}]{} & A_{\mathrm{Lie}} & & A.
\end{array}
$$

If necessary we shall write $\mathbf{U}_{\mathbb{K}}$ instead of $\mathbf{U}$ whenever the ground field should be emphasized. We shall call $\mathbf{U}_{\mathbb{K}}(L)$ *the weakly complete enveloping algebra* of L (over $\mathbb{K}$).

Example 6.2. Let $L = \mathbb{K}$, the smallest possible nonzero Lie algebra over $\mathbb{K}$. Then $\mathbf{U}(L) = \mathbb{K}\langle X \rangle$ (see [3], Definition following Corollary 3.3), and define $\lambda_L\colon L \to \mathbf{U}(L)_{\mathrm{Lie}}$ by $\lambda_L(t) = t{\cdot}X$. Indeed the universal property is satisfied by [3], Corollary 3.4. Namely, let $f\colon \mathbb{K} \to A_{\mathrm{Lie}}$ be a morphism of weakly complete Lie algebras. Then there is a unique morphism $f'\colon \mathbf{U}(L) \to A$ such that $f'(X) = f(1)$ by [3], Corollary 3.4. Then $f'(t{\cdot}X) = t{\cdot}F'(X) = t{\cdot}f(1) = f(t)$.

Thus by Lemma 3.5 of [3] and the subsequent remarks we have:

> *The weakly complete enveloping algebra* $\mathbf{U}_{\mathbb{C}}(\mathbb{C})$ *over* $\mathbb{C}$ *of the smallest nonzero complex Lie algebra is isomorphic to the weakly complete commutative algebra* $\mathbb{C}[[X]]^{\mathbb{C}}$ *with the complex power series algebra* $\mathbb{C}[[X]] \cong \mathbb{C}^{\mathbb{N}_0}$, $\mathbb{N}_0 = \{0, 1, 2, \dots\}$.

The size of the weakly complete enveloping algebras therefore is considerable.

Proposition 6.3. *The universal enveloping functor* $\mathbf{U}$ *is multiplicative, that is, there is a natural isomorphism* $\alpha_{L_1L_2}\colon \mathbf{U}(L_1 \times L_2) \to \mathbf{U}(L_1) \otimes_{\mathcal{W}} \mathbf{U}(L_2)$.

Proof. We have a natural bilinear inclusion map of weakly complete vector spaces $j\colon \mathbf{U}(L_1) \times \mathbf{U}(L_2) \to \mathbf{U}(L_1) \otimes_{\mathcal{W}} \mathbf{U}(L_2)$ yielding

$$L_1 \times L_2 \xrightarrow{\lambda_1 \times \lambda_2} \mathbf{U}(L_1)_{\mathrm{Lie}} \times \mathbf{U}(L_2)_{\mathrm{Lie}} \xrightarrow{j} \mathbf{U}(L_1)_{\mathrm{Lie}} \otimes_{\mathcal{W}} \mathbf{U}(L_2)_{\mathrm{Lie}}$$

and
$$\mathbf{U}(L_1)_{\mathrm{Lie}} \otimes_{\mathcal{W}} \mathbf{U}(L_2)_{\mathrm{Lie}} = (\mathbf{U}(L_1) \otimes_{\mathcal{W}} \mathbf{U}(L_2))_{\mathrm{Lie}},$$

the composition α_0 of which is a morphism of weakly complete Lie algebras. Hence the universal property yields a morphism of weakly complete associative algebras

$$\alpha\colon \mathbf{U}(L_1 \times L_2) \to \mathbf{U}(L_1) \otimes_{\mathcal{W}} \mathbf{U}(L_2) \tag{1}$$

such that $\alpha_0 = \alpha_{\mathrm{Lie}} \circ \lambda_{L_1} \otimes \lambda_{L_2}$.

The functorial property of $\mathbf{U}$ allows us to argue that each of $\mathbf{U}(L_m)$, $m = 1, 2$ is a retract of $\mathbf{U}(L_1 \times L_2)$ so that we may assume $\mathbf{U}(L_m) \subseteq \mathbf{U}(L_1 \times L_2)$, $m = 1, 2$. Now the multiplication in $\mathbf{U}(L_1 \times L_2)$ gives rise to a continuous bilinear map $\mathbf{U}(L_1) \times \mathbf{U}(L_2) \to \mathbf{U}(L_1 \times L_2)$, and then the universal property of the tensor product of weakly complete vector spaces yields the morphism

$$\beta\colon \mathbf{U}(L_1) \otimes_{\mathcal{W}} \mathbf{U}(L_2) \to \mathbf{U}(L_1 \times L_2). \tag{2}$$

Similarly to the proof of [3], Theorem 5.5 (preceding the statement of the theorem) we argue that α and β are inverses of each other, and so α of (1) is the desired isomorphism $\alpha_{L_1L_2}$. ∎

Lemma 6.4. *For any weakly complete unital algebra* A, *the vector space morphism* $\Delta_A\colon A \to A \otimes_{\mathcal{W}} A$, $\Delta_A(a) = a \otimes 1 + 1 \otimes a$ *is a morphism of weakly complete Lie algebras* $A_{\mathrm{Lie}} \to (A \otimes_W A)_{\mathrm{Lie}}$.

Proof. Since the functions $a \mapsto a \otimes 1, \quad 1 \otimes a$ are morphisms of topological vector spaces, so is Δ_A For $y_1, y_2 \in A$, write $z_j := \Delta_A(y_j) = y_j \otimes 1 + 1 \otimes y_j$ Then just as in the classical case, we calculate

$$\begin{aligned}
[\Delta_A(y_1), \Delta_A(y_2)] &= [z_1, z_2] = z_1 z_2 - z_2 z_1 \\
&= (y_1 \otimes 1 + 1 \otimes y_1)(y_2 \otimes 1 + 1 \otimes y_2) \\
&\qquad\qquad - (y_2 \otimes 1 + 1 \otimes y_2)(y_1 \otimes 1 + 1 \otimes y_1) \\
&= (y_1 y_2 \otimes 1 + y_1 \otimes y_2 + y_2 \otimes y_1 + 1 \otimes y_1 y_2) \\
&\qquad\qquad - (y_2 y_1 \otimes 1 + y_2 \otimes y_1 + y_1 \otimes y_2 + 1 \otimes y_2 y_1) \\
&= [y_1, y_2] \otimes 1 + 1 \otimes [y_1, y_2] = \Delta_A [y_1, y_2].
\end{aligned}$$

Thus Δ_A is a morphism of Lie algebras as asserted. ∎

Now we consider a weakly complete Lie algebra L and recall that $\lambda_L\colon L \to U(L)_{\rm Lie}$ is a morphism of weakly complete Lie algebras. Thus by Lemma 6.4,

$$p_L = \Delta_{U(L)} \circ \lambda_L\colon L \to (\mathbf{U}(L) \otimes \mathbf{U}(L))_{\rm Lie}$$

is a morphism of weakly complete Lie algebras. Now by the universal property of $\mathbf{U}$, p_L induces a unique natural morphism of weakly complete associative unital algebras $\gamma_L\colon\ \mathbf{U}(L) \to \mathbf{U}(L) \otimes_{\mathcal{W}} \mathbf{U}(L)$ such that $p_L = (\gamma_L)_{\rm Lie} \circ \lambda_L$. This yields the following insight, where $k_L\colon L \to \{0\}$ denotes the constant morphism.

Corollary 6.5. *Each weakly complete enveloping algebra* $\mathbf{U}(L)$ *is a weakly complete Hopf algebra with the comultiplication* γ_L *and the coidentity* $\mathbf{U}(k_L)\colon\ \mathbf{U}(L) \to \mathbb{K}$.

Proof. Observe that γ_L is a morphism of weakly complete unital algebras satisfying $\gamma_L(y) = y \otimes 1 + 1 \otimes y$ for $y = \lambda(x)$, $x \in L$. The associativity of this comultiplication is readily checked as in the case of abstract enveloping algebras. The constant morphism of weakly complete Lie algebras $L \to \{0\}$ yields a morphism of weakly complete unital algebras $\mathbf{U}(L) \to \mathbf{U}(\{0\}) = \mathbb{K}$ which is the coidentity of the Hopf algebra. ∎

Our results from [3] regarding weakly complete associative unital algebras and Hopf algebras over $\mathbb{K}$ apply to the present situation.

Theorem 6.6. (The Weakly Complete Enveloping Algebra) *Let L be a weakly complete Lie algebra. Then the following statements hold:*

(i) $\mathbf{U}(L)$ *is a strict projective limit of finite-dimensional associative unital algebras and the group of units* $\mathbf{U}(L)^{-1}$ *is dense in* $\mathbf{U}(L)$. *It is an almost connected pro-Lie group (which is connected in the case of* $\mathbb{K} = \mathbb{C}$*). The algebra* $\mathbf{U}(L)$ *has an exponential function* $\exp\colon \mathbf{U}(L)_{\rm Lie} \to \mathbf{U}(L)^{-1}$,

(ii) *The pro-Lie algebra* $\Pi(\mathbf{U}(L))$ *of primitive elements of* $\mathbf{U}(L)$ *contains* $\lambda_L(L)$,

(iii) *The subalgebra generated by* $\lambda_L(L)$ *in* $\mathbf{U}(L)$ *is dense in* $\mathbf{U}(L)$.

(iv) *The pro-Lie algebra* $\Pi(\mathbf{U}(L))$ *is the Lie algebra of the pro-Lie group* $\Gamma(\mathbf{U}(L))$ *of grouplike elements of* $\mathbf{U}(L)$. *We use the abbreviation* $G := \Gamma(\mathbf{U}(L))$ *and note that the exponential function* $\exp_G\colon \mathfrak{L}(G) \to G$ *is the restriction and corestriction of the exponential function* $\exp$ *of* $\mathbf{U}(L)$ *to* $\Pi(\mathbf{U}(L))$, *respectively,* G. *The image* $\exp(\mathfrak{L}(G))$ *generates algebraically and topologically the identity component* G_0 *of* G.

Proof. (i) See [3], Theorems 3.2, 3.11, 3.12, 4.1.

(ii) The very definition of the comultiplication γ_L for Corollary 6.5 shows that for any $y \in \lambda_L(L)$, the image under the comultiplication γ_L is $y \otimes 1 + 1 \otimes y$, which means that y is primitive.

(iii) An argument analogous to that in the proof of Proposition 5.3 of [3] showing that, for the case of any topological group T, the subset $\eta_G(T)$ of the weakly complete group algebra $\mathbb{K}[T]$ spans a dense subalgebra, shows here that the closed subalgebra S generated in $\mathbf{U}(L)$ by $\lambda_L(L)$ has the universal property of $\mathbf{U}(L)$ and therefore agrees with $\mathbf{U}(L)$.

(iv) See Theorem 2.2 and [3], Theorem 6.15. ∎

Remark 6.7. We note right away that for any weakly complete Lie algebra L which has at least one nonzero finite dimensional $\mathbb{K}$-linear representation, the morphism $\lambda_L\colon L \to \mathbf{U}(L)_{\mathrm{Lie}}$ is nonzero. By Ado's Theorem, this applies, in particular, to any Lie algebra which has a nontrivial finite dimensional quotient and therefore is true for the Lie algebra $\mathfrak{L}(P)$ of any pro-Lie group P.

Corollary 6.8. (i) *The weakly complete enveloping algebra $\mathbf{U}(L)$ of a weakly complete Lie algebra L with a nontrivial finite dimensional quotient has nontrivial grouplike elements.*

(ii) *If L is a pro-Lie algebra, then $\lambda_L\colon L \to \mathbf{U}(L)_{\mathrm{Lie}}$ maps L isomorphically onto a closed Lie subalgebra of the pro-Lie algebra $\Pi(\mathbf{U}(L))$ of primitive elements.*

Proof. (i) If $\Pi(\mathbf{U}(L))$ is nonzero, then $\mathbf{U}(L)$ has nontrivial grouplike elements by Theorem 6.6(iii), and by (ii) of 6.6, this is the case if λ_L is nonzero which is the case for all L satisfying the hypothesis of the Corollary by the remark preceding it.

(ii) Since each finite dimensional quotient of L has a faithful representation by the Theorem of Ado, and since the finite dimensional quotients separate the points of L, the morphism λ_L is injective. However, injective morphisms of weakly complete vector spaces are open onto their images. ∎

It follows that *for pro-Lie algebras L we may assume that L is in fact a closed Lie subalgebra of primitive elements of $\mathbf{U}(L)$ which generates $\mathbf{U}(L)$ algebraically and topologically as a weakly complete algebra.*

It remains an open question under which circumstances we then have in fact $L = \mathbb{P}(\mathbf{U}(L))$. In the classical setting of the discrete enveloping Hopf algebra in characteristic 0 this is the case: see e.g. [8], Theorem 5.4.

One application of the functor $\mathbf{U}$ is of present interest to us. Recall that for a compact group we naturally identify G with the group of grouplike elements of $\mathbb{R}[G]$ (cf. [3], Theorems 8.7, 8.9 and 8.12), and that $\mathfrak{L}(G)$ may be identified with the pro-Lie algebra $\Pi(\mathbb{R}[G])$ of primitive elements. (Cf. also Theorem 2.2 above.) We may also assume that $\mathfrak{L}(G)$ is contained the set $\Pi(\mathbf{U}(\mathfrak{L}(G)))$ of primitive elements of $\mathbf{U}_{\mathbb{R}}(\mathfrak{L}(G))$.

Theorem 6.9. (i) *Let G be a compact group. Then there is a natural morphism of weakly complete algebras $\omega_G\colon \mathbf{U}_{\mathbb{R}}(\mathfrak{L}(G)) \to \mathbb{R}[G]$ fixing the elements of $\mathfrak{L}(G)$ elementwise.*

(ii) *The image of ω_G is the closed subalgebra $\mathbb{R}[G_0]$ of $\mathbb{R}[G]$.*

(iii) *The pro-Lie group $\Gamma(\mathbf{U}_{\mathbb{R}}(\mathfrak{L}(G)))$ is mapped onto $G_0 = \Gamma(\mathbb{R}[G_0]) \subseteq \mathbb{R}[G]$. The connected pro-Lie group $\Gamma(\mathbf{U}_R(\mathfrak{L}(G)))_0$ maps surjectively onto G_0 and $\mathbb{P}(\mathbf{U}_R(\mathfrak{L}(G)))$ onto $\mathbb{P}(\mathbb{R}[G])$.*

Proof. (i) follows at once from the universal property of $\mathbf{U}$.

(ii) As a morphism of weakly complete Hopf algebras, ω_G has a closed image which is generated as a weakly complete subalgebra by $\mathfrak{L}(G)$ which is $\mathbb{R}[G_0]$ by Corollary 3.3 (ii).

(iii) The morphism ω_G of weakly complete Hopf algebras maps grouplike elements to grouplike elements, whence we have the commutative diagram

$$\begin{tikzcd}
\mathfrak{L}(G)\subseteq \Pi(\mathbf{U}_{\mathbb{R}}(\mathfrak{L}(G))) \arrow[r, "\Pi(\omega_G)"] \arrow[d, "\exp_{\Gamma(\mathbf{U}_{\mathbb{R}}(\mathfrak{L}(G)))}"'] & \Pi(\mathbb{R}(G))=\mathfrak{L}(G) \arrow[d, "\exp_G"] \\
\Gamma(\mathbf{U}_{\mathbb{R}}(\mathfrak{L}(G))) \arrow[r, "\Gamma(\omega_G)"'] & \Gamma(\mathbb{R}[G])=G.
\end{tikzcd}$$

Since $\mathbb{P}(\omega)$ is a retraction and the image of $\exp_G$ topologically generates G_0, the image of $\mathbb{G}(\omega_G)\circ\exp_{\mathbf{U}_{\mathbb{R}}(\mathfrak{L}(G))}$ topologically generates G_0. Since the image of the exponential function of the pro-Lie group $\mathbb{G}(\mathbf{U}_{\mathbb{R}}(\mathfrak{L}(G)))$ generates topologically its identity component, $\mathbb{G}(\omega_G)$ maps this identity component onto G_0.

Since $\mathfrak{L}(G)\subseteq\mathbb{P}(\mathbf{U}_{\mathbb{R}}(\mathfrak{L}(G))$, and since any morphism of Hopf algebras maps a primitive element onto a primitive element we know $\omega_G(\mathbb{P}(\mathbf{U}_{\mathbb{R}}(\mathfrak{L}(G))))=\mathbb{P}(\mathbb{R}[G])$. ■

It remains an open question whether $\mathbb{G}(\mathbf{U}_{\mathbb{R}}(\mathfrak{L}(G)))$ is in fact connected.

An overview of the situation may be helpful:

$$\begin{tikzcd}
 & & & \mathbb{R}[G] \arrow[dd, dash] \\
\mathbf{U}_{\mathbb{R}}(\mathfrak{L}(G)) \arrow[r, "{\omega_G, onto}"] \arrow[d, dash] & \mathbb{R}[G_0] \arrow[dd, dash] & & \\
\Gamma(\mathbf{U}_{\mathbb{R}}(G)) \arrow[rrr] \arrow[d, dash] & & & \Gamma(\mathbb{R}[G])=G \arrow[d, dash] \\
\Gamma(\mathbf{U}_{\mathbb{R}}(\mathfrak{L}(G)))_0 \arrow[r, "onto"] & \Gamma(\mathbb{R}[G_0])=G_0 & = & G_0 \\
\Pi(\mathbf{U}_{\mathbb{R}}(\mathfrak{L}(G))) \arrow[u, "\exp_{\mathbf{U}_{\mathbb{R}}(\mathfrak{L}(G))}"'] \arrow[r, phantom, "="] & \Pi(\mathbb{R}[G_0])=\Pi(\mathbb{R}[G]) \arrow[u, "\exp_{\mathbb{R}[G]}"'] & = \arrow[u, phantom, "="] & \mathfrak{L}(G). \arrow[u, "\exp_G"']
\end{tikzcd}$$

A noteworthy consequence of the preceding results is the insight that

> *for any nonzero weakly complete real Lie algebra* $L=\mathbb{R}^X\times\prod_{j\in J}L_j$ *for any set* X *and any family of compact finite dimensional simple Lie algebras* L_j*, the weakly complete enveloping algebra* $\mathbf{U}(L)$ *has grouplike elements.*

In the discrete situation, the enveloping algebra $U(L)$ of a Lie algebra L for characteristic zero has no grouplike elements.

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