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Abstract homomorphisms from locally compact groups to discrete groups ☆

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ABSTRACT

We show that every abstract homomorphism φ from a locally compact group L to a graph product G_Γ, endowed with the discrete topology, is either continuous or $\varphi(L)$ lies in a 'small' parabolic subgroup. In particular, every locally compact group topology on a graph product whose graph is not 'small' is discrete. This extends earlier work by Morris-Nickolas.
We also show the following. If L is a locally compact group and if G is a discrete group which contains no infinite torsion group and no infinitely generated abelian group, then every abstract homomorphism $\varphi : L \to G$ is either continuous, or $\varphi(L)$ is contained in the normalizer of a finite nontrivial subgroup of G. As an application we obtain results concerning the continuity of homomorphisms from locally compact groups to Artin and Coxeter groups.
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1. Introduction

We investigate the following type of question.

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Let L be a locally compact group and let G be a discrete group. Under what conditions on the group G is an abstract (i.e. not necessarily continuous) homomorphism $\varphi : L \to G$ automatically continuous?

There are many results in this direction in the literature, see [11], [17], [21], [32] or [35]. In particular, Dudley [21] proved that every abstract homomorphism from a locally compact group to a free group is automatically continuous. This was generalized by Morris and Nickolas [32]. They proved that every abstract homomorphism from a locally compact group to a free product of groups is either continuous, or the image of the homomorphism is conjugate to a subgroup of one of the factors of the free product.

Our first aim is to prove similar results for the case where the codomain G of an abstract homomorphism $L \to G$ is a graph product of arbitrary groups. Given a simplicial graph $\Gamma = (V, E)$ and a collection of groups $\mathcal{G} = \{G_u \mid u \in V\}$, the *graph product* G_Γ is defined as the quotient

$$G_\Gamma = \left(\mathop{*}_{u \in V} G_u \right) / \langle\langle [G_v, G_w] \text{ for } \{v, w\} \in E \rangle\rangle.$$

We call G_Γ *finite dimensional* if there exists a uniform bound on the sizes of cliques in Γ.

Throughout, L denotes a Hausdorff locally compact group with identity component L°, and G denotes a discrete group. We call L *almost connected* if the totally disconnected group L/L° is compact. By an *abstract homomorphism* we mean a group homomorphism between topological groups which is not assumed to be continuous. We remark that every abstract homomorphism whose codomain is discrete is open.

Proposition A. *Let $\varphi : L \to G_\Gamma$ be an abstract homomorphism from an almost connected locally compact group L to a finite dimensional graph product G_Γ. Then $\varphi(L)$ lies in a complete parabolic subgroup of G_Γ.*

Using Proposition A, we show the following more general result.

Theorem B. *Let φ be an abstract homomorphism from a locally compact group L to a finite dimensional graph product G_Γ. Then either φ is continuous, or $\varphi(L)$ lies in a conjugate of a parabolic subgroup $G_{S \cup \mathrm{lk}(S)}$, where $S \neq \varnothing$ is a clique. If every composite $L \xrightarrow{\varphi} G_\Gamma \xrightarrow{r_v} G_v$ is continuous, then φ is continuous.*

In particular, every locally compact group topology on a finite dimensional graph product G_Γ is discrete, unless Γ is contained in the link of a clique. In the latter case, G_Γ is a direct product of vertex groups and a smaller graph product, and then a locally compact topology on G_Γ may indeed be nondiscrete.

Our remaining results deal with a certain class $\mathcal{G}$ of discrete groups. Let $\mathcal{G}$ denote the class of all groups G with the following two properties:

(i) Every torsion subgroup $T \subseteq G$ is finite, and
(ii) Every abelian subgroup $A \subseteq G$ is a (possibly infinite) direct sum of cyclic groups.

The abelian subgroups A in such a group G are thus of the form $A = F \times \mathbb{Z}^{(J)}$, where F is a finite abelian group and $\mathbb{Z}^{(J)}$ is free abelian of (possibly infinite) rank card(J). We remark that subgroups of free abelian groups are again free abelian [25, A1.9].

We study abstract homomorphisms from locally compact groups to groups in this class. We show in Section 7 that the class $\mathcal{G}$ is huge. It is closed under finite products, under coproducts, and under passage to subgroups, see Proposition 7.1. For example, every finitely generated hyperbolic group, every right-angled Artin group, every Artin group of finite type and every Coxeter group is in this class, see Propositions 7.2, 7.3 and 7.4. Furthermore, the groups $\mathrm{GL}_n(\mathbb{Z})$, $\mathrm{Out}(F_n)$ and the mapping class groups $\mathrm{Mod}(S_g)$ of compact orientable surfaces of genus g are in this class, see Proposition 7.5. Further examples of groups which are in the class $\mathcal{G}$ are diagram groups, see [24, Theorem 16]. In particular, the Thompson's group F is a diagram group and this group contains a free abelian group which is not finitely generated.

We obtain the following results.

Proposition C. *Let φ be an abstract homomorphism from a locally compact group L to a group G in the class $\mathcal{G}$. Then φ factors through the canonical projection $\pi : L \to L/L^\circ$. If L is almost connected, then $\varphi(L)$ is finite.*

Theorem D. *Let φ be an abstract homomorphism from a locally compact group L to a group G in the class $\mathcal{G}$. Then either φ is continuous, or $\varphi(L)$ lies in the normalizer of a finite non-trivial subgroup of G.*

The following is an immediate consequence of Theorem D.

Corollary E. *Every abstract homomorphism from a locally compact group L to a torsion free group G in the class $\mathcal{G}$ is continuous. In particular, every abstract homomorphism from a locally compact group to a right-angled Artin group or to an Artin group of finite type is continuous.*

Related results on abstract homomorphisms into right-angled Artin groups and into Artin groups of non-exceptional finite type were recently proved in [18].

Our proofs depend heavily on a theorem of Iwasawa on the structure of connected locally compact groups and on a theorem of van Dantzig on the existence of compact open subgroups in totally disconnected groups. For the proof of Proposition A we use the structure of the right-angled building X_Γ associated to a graph product G_Γ.

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2. Graph products

In this section we briefly present the main definitions and properties concerning graph products. These groups are defined by presentations of a special form.

A *simplicial graph* $\Gamma = (V, E)$ consists of a set V of *vertices* and a set E of 2-element subsets of V which are called *edges*. We allow infinite graphs. Given a subset $S \subseteq V$, the *graph generated by* S is the graph with vertex set S and edge set $E|_S = \{\{v, w\} \in E \mid v, w \in S\}$. We call S a *clique* if $E|_S = \{\{v, w\} \mid v, w \in S \text{ with } v \neq w\} = \binom{S}{2}$. We count the empty set as a clique. We say that Γ has *finite dimension* if there is a uniform upper bound on the cardinality of cliques in Γ. For a subset $S \subseteq V$ we define its *link* as

$$\mathrm{lk}(S) = \{w \in V \mid \{v, w\} \in E \text{ for all } v \in S\}.$$

Definition 2.1. Let Γ be a simplicial graph, as defined above. Suppose that for every vertex $v \in V$ we are given a nontrivial[1] abstract group G_v. The *graph product* G_Γ is the group obtained from the free product of the G_v, for $v \in V$, by adding the commutator relations $gh = hg$ for all $g \in G_v$, $h \in G_w$ with $\{v, w\} \in E$, i.e.

$$G_\Gamma = \left(\mathop{*}_{u \in V} G_u \right) / \langle\langle [G_v, G_w] \text{ for } \{v, w\} \in E \rangle\rangle.$$

Graph products are special instances of graphs of groups, and in particular colimits in the category of groups [20, §5]. We call the graph product *finite dimensional* if Γ has finite dimension as defined above, i.e. if there is an upper bound on the size of cliques in Γ.

The first examples to consider are the extremes. If $E = \varnothing$, then G_Γ is the free product of the groups G_v, for $v \in V$. On the other hand, if $E = \binom{V}{2}$ is the set of all 2-element subsets of V, then G_Γ is the direct sum of the G_v, for $v \in V$. So graph products interpolate between free products and direct sums of groups.

2.2. Parabolic subgroups

Let $\Gamma = (V, E)$ be a simplicial graph, let G_Γ denote the graph product of a family of groups $\{G_v \mid v \in V\}$ and let S be a subset of V. The subgroup G_S of G_Γ generated by the G_v, for $v \in S$, is again a graph product, corresponding to the subgraph $\Gamma' = (S, E|_S)$.

[1] We need nontrivial vertex groups in order to obtain a building. Alternatively, one may remove all vertices v from Γ whose vertex group G_v is trivial, without changing the resulting graph product.

This follows from the Normal Form Theorem [23, Thm. 3.9]. There is also a retraction homomorphism

$$r_S : G_\Gamma \to G_S$$

which is obtained by substituting the trivial group for G_v for all $v \in V - S$ [3, Section 3].

If $S \subseteq V$ is a subset (resp. a clique), then G_S is called a *special parabolic subgroup* (resp. a *special complete parabolic subgroup*). The conjugates in G_Γ of the special (complete) parabolic subgroups are called (*complete*) *parabolic subgroups*. We note that parabolic subgroups behave well. For $R, S \subseteq V$ and $a, b \in G_\Gamma$ we have

$$aG_R a^{-1} \subseteq bG_S b^{-1} \Rightarrow R \subseteq S \tag{1}$$

see [3, Corollary 3.8]. If $gG_S g^{-1} \subseteq G_S$, then by [3, Lemma 3.9]

$$gG_S g^{-1} = G_S. \tag{2}$$

Let X be a subset of G_Γ. If the set of all parabolic subgroups containing X has a minimal element, then this minimal parabolic subgroup containing X is unique by the remarks above. In this case, it is called the *parabolic closure* of X and denoted by $\mathrm{Pc}(X)$. The parabolic closure always exists if Γ is finite or if X is finite [3, Proposition 3.10].

Let $H \subseteq G_\Gamma$ be a subgroup. We denote by $\mathrm{Nor}_{G_\Gamma}(H)$ the normalizer of H in G_Γ. For a parabolic subgroup of G_Γ there is a good description of the normalizer.

Lemma 2.3. *[3, Lemma 3.12 and Proposition 3.13]*

(i) *Let $H \subseteq G_\Gamma$ be a subgroup. Suppose that the parabolic closure of H in G_Γ exists. Then* $\mathrm{Nor}_{G_\Gamma}(H) \subseteq \mathrm{Nor}_{G_\Gamma}(\mathrm{Pc}(H))$.

(ii) *Let G_S be a non-trivial special parabolic subgroup of G_Γ. Then* $\mathrm{Nor}_{G_\Gamma}(G_S) = G_{S\cup \mathrm{lk}(S)}$.

3. Actions on cube complexes

A detailed description of CAT(0) cube complexes can be found in [12] and in [36]. Let $\mathcal{C}$ denote the class of finite dimensional CAT(0) cube complexes and let $\mathcal{A}$ denote the subclass of $\mathcal{C}$ consisting of simplicial trees. Inspired by Serre's fixed point property F$\mathcal{A}$, Bass introduced the property F$\mathcal{A}'$ in [5]. A group G has property F$\mathcal{A}'$ if every simplicial action of G on every member T of $\mathcal{A}$ is locally elliptic, i.e. if each $g \in G$ fixes some point on the tree T. A generalization of property F$\mathcal{A}'$ was defined in [28]. A group G has property F$\mathcal{C}'$ if every simplicial action of G on every member X of $\mathcal{C}$ is locally elliptic, i.e. if each $g \in G$ fixes some point on X. Bass proved in [5] that every profinite group has property F$\mathcal{A}'$. His result was generalized by Alperin to compact groups in [1] and to almost connected locally compact groups in [2].

The next result was proved by Caprace in [14, Theorem 2.5].

Proposition 3.1. *Let X be a locally Euclidean* CAT(0) *cell complex with finitely many isometry types of cells, and L be a compact group acting as an abstract group on X by cellular isometries. Then every element of L is elliptic. In particular, every compact group has property* $\mathrm{F}\mathcal{C}'$.

We recall that a group G is called *divisible* if $\{g^n \mid g \in G\} = G$ holds for all integers $n \geq 1$. Another result which we will need in order to prove Proposition A is the following.

Lemma 3.2. *[14, Theorem 2.5 Claim 7] Every divisible group has property* $\mathrm{F}\mathcal{C}'$.

The following result is due to Sageev and follows from the proof of Theorem 5.1 in [36], see also [28, Theorem A].

Proposition 3.3. *Let G be a finitely generated group acting by simplicial isometries on a finite dimensional* CAT(0) *cube complex. If the G-action is locally elliptic, then G has a global fixed point.*

The last fact we need for the proof of Proposition A concerning global fixed points is the following easy consequence of the Bruhat-Tits Fixed Point Theorem [30, Lemma 2.1].

Lemma 3.4. *Suppose that a group H acts isometrically on a complete* CAT(0) *space. If $H = H_1 H_2 \cdots H_r$ is a product of finitely many subgroups H_j each fixing some point in X, then H has a global fixed point.*

3.5. Graph products, cube complexes and the building

Associated to finite dimensional graph products are certain finite dimensional CAT(0) cube complexes. We briefly describe the construction of these spaces. For a graph product G_Γ we consider the poset

$$P = \{gG_T \mid g \in G_\Gamma \text{ and } T \text{ is a clique}\},$$

ordered by inclusion (we recall that we allow empty cliques). The group G_Γ acts by left multiplication on this poset and hence simplicially on the flag complex X_Γ associated to this coset poset. This flag complex has a canonical cubical structure. With respect to this structure X_Γ is the Davis realization of a right-angled building, [20, Theorem 5.1]. By [20, Theorem 11.1] the Davis realization of every building is a complete CAT(0) space. Hence X_Γ is a finite dimensional CAT(0) cube complex, and G_Γ acts isometrically on X_Γ. The *chambers* of X_Γ correspond to the cosets of the trivial subgroup, i.e. to the elements of G_Γ. The G_Γ-stabilizer of a chamber (a maximal cube) is therefore trivial. The *vertices* of X_Γ correspond to the cosets of the G_S, where $S \subseteq V$ is an inclusion-maximal clique. The action of G_Γ on X_Γ preserves the canonical cubical structure.

One nice property of this action is the following: if a subgroup $H \subseteq G_\Gamma$ has a global fixed point in X_Γ, then there exists a vertex in X_Γ which is fixed by H. This follows from the fact that the action is type preserving. Furthermore, the stabilizer of a vertex gG_T is equal to gG_Tg^{-1}.

Lemma 3.6. *Let G_Γ be a finite dimensional graph product and let H be a subgroup. If the action of H on the building X_Γ is locally elliptic, then H has a global fixed point.*

Proof. For each finite subset $X \subseteq H$, the finitely generated group $\langle X\rangle$ acts locally elliptically on X_Γ. Thus $\langle X\rangle$ has by Proposition 3.3 a fixed vertex gG_S, for some $g \in G_\Gamma$ and some maximal clique S. It follows that the parabolic closure of X is of the form $\mathrm{Pc}(X) = gG_{S_X}g^{-1}$, where S_X is a clique depending uniquely on X. Since there is an upper bound on the size of cliques in Γ, there exists a finite set $Z \subseteq H$ such that S_Z is maximal among all cliques S_X, for $X \subseteq H$ finite. We claim that $H \subseteq \mathrm{Pc}(Z)$.

Let $h \in H$ and put $X = Z \cup \{h\}$. If we put $\mathrm{Pc}(X) = aG_{S_X}a^{-1}$ and $\mathrm{Pc}(Z) = bG_{S_Z}b^{-1}$, then

$$aG_{S_X}a^{-1} \supseteq bG_{S_Z}b^{-1}$$

because $X \supseteq Z$. Then $S_X \supseteq S_Z$ holds by 2.2(1). From the maximality of S_Z we conclude that $S_Z = S_X$. Then $aG_{S_X}a^{-1} = bG_{S_Z}b^{-1}$ by 2.2(2). It follows that $H \subseteq \mathrm{Pc}(Z) = bG_{S_Z}b^{-1}$, and thus H has a global fixed point. □

4. The proofs of Proposition A and Theorem B

Proposition A. *Let $\varphi : L \to G_\Gamma$ be an abstract homomorphism from an almost connected locally compact group L to a finite dimensional graph product G_Γ. Then $\varphi(L)$ lies in a complete parabolic subgroup of G_Γ.*

Proof. The group L acts via

$$L \to G_\Gamma \to \mathrm{Isom}(X_\Gamma)$$

isometrically and simplicially on the right-angled building X_Γ.

Suppose first that L is compact. Then the L-action is by Proposition 3.1 locally elliptic. Hence there is global fixed point by Lemma 3.6.

Suppose next that L is connected. By Iwasawa's decomposition [27, Theorem 13] we have

$$L = H_1H_2\cdots H_rK,$$

where K is a connected compact group and $H_i \cong \mathbb{R}$ for $i = 1, \ldots, r$. Each group H_j has a fixed point by Lemma 3.2, and K has a fixed point by the result in the previous paragraph. Hence L has a fixed point by Lemma 3.4.

Now we consider the general case. If L is almost connected, then the identity component L° has a global fixed point by the previous paragraph. The fixed point set $Z \subseteq X_\Gamma$ of L° is a convex CAT(0) cube complex, because the L-action is simplicial and type-preserving. By Proposition 3.1, the action of L/L° on Z is locally elliptic. Hence the action of L on X is locally elliptic as well. Another application of Lemma 3.6 shows that L has a global fixed point. □

Now we may prove Theorem B.

Theorem B. *Let φ be an abstract homomorphism from a locally compact group L to a finite dimensional graph product G_Γ. Then either φ is continuous, or $\varphi(L)$ lies in a conjugate of a parabolic subgroup $G_{S\cup \mathrm{lk}(S)}$, where $S \neq \varnothing$ is a clique. If every composite $L \xrightarrow{\varphi} G_\Gamma \xrightarrow{r_v} G_v$ is continuous, then φ is continuous.*

Proof. Let L° be the connected component of the identity in L. We distinguish two cases.

Case 1: $\varphi(L^\circ)$ is not trivial.

By Proposition A we know that $\varphi(L^\circ) \subseteq gG_Tg^{-1}$ where $T \subseteq V$ is a clique and $g \in G_\Gamma$. Hence $\mathrm{Pc}(\varphi(L^\circ)) = hG_Sh^{-1}$, where $\varnothing \neq S \subseteq T$ and $h \in G_\Gamma$. Since $\varphi(L)$ normalizes $\varphi(L^\circ)$, we have by Lemma 2.3 that $\varphi(L) \subseteq \mathrm{Nor}_{G_\Gamma}(\mathrm{Pc}(\varphi(L^\circ)))$. This normalizer is of the form $hG_{S\cup \mathrm{lk}(S)}h^{-1}$, for some $h \in G_\Gamma$. We note that in Case 1, the homomorphism φ is not continuous, since the image of a connected group under continuous map is always connected and a connected subgroup of a discrete group is trivial.

Case 2: $\varphi(L^\circ)$ is trivial.

Then φ factors through an abstract homomorphism $\psi : L/L^\circ \to G_\Gamma$, and L/L° is a totally disconnected locally compact group. By van Dantzig's Theorem [10, III§4, No. 6] there exists a compact open subgroup K in L/L°.

Subcase 2a: There is a compact open subgroup $K \subseteq L/L^\circ$ such that $\psi(K)$ is trivial.

Then the kernel of ψ is open in L/L° and hence ψ and φ are continuous.

Subcase 2b: There is no compact open subgroup $K \subseteq L/L^\circ$ such that $\psi(K)$ is trivial.

Let $\mathcal{K}$ denote the collection of all compact open subgroups of L/L°. We note that L acts on $\mathcal{K}$ by conjugation. For $K \in \mathcal{K}$ we put $\mathrm{Pc}(\psi(K)) = gG_{S_K}g^{-1}$. Thus S_K is a clique in Γ which depends uniquely on K. We choose $M \in \mathcal{K}$ in such a way that S_M is minimal and we note that $S_M \neq \varnothing$. Given $a \in L/L^\circ$ we have $M \cap aMa^{-1} \in \mathcal{K}$ and

$$\mathrm{Pc}(\psi(aMa^{-1})) = \psi(a)\,\mathrm{Pc}(\psi(M))\psi(a)^{-1}.$$

From 2.2(1) and

$$\mathrm{Pc}(\psi(M)) \supseteq \mathrm{Pc}(\psi(M) \cap \psi(aMa^{-1})) \subseteq \mathrm{Pc}(\psi(aMa^{-1}))$$

we obtain that

$$S_M \supseteq S_{M\cap aMa^{-1}} \subseteq S_{aMa^{-1}}.$$

Since both $S_{aMa^{-1}}$ and S_M are minimal we conclude that

$$S_M = S_{M\cap aMa^{-1}} = S_{aMa^{-1}}$$

and that

$$\mathrm{Pc}(\psi(M)) = \mathrm{Pc}(\psi(aMa^{-1})) = \psi(a)\,\mathrm{Pc}(\psi(M))\psi(a)^{-1}.$$

Therefore $\psi(a)$ normalizes $\mathrm{Pc}(\psi(M))$, whence

$$\varphi(L) = \psi(L/L^\circ) \subseteq hG_{S_M\cup \mathrm{lk}(S_M)}h^{-1},$$

for some $h \in G_\Gamma$ by Lemma 2.3.

Suppose now towards a contradiction that φ is not continuous, but that each composite $L \xrightarrow{\varphi} G_\Gamma \xrightarrow{r_v} G_v$ is continuous. Then $\varphi(L) \subseteq gG_{S\cup \mathrm{lk}(S)}g^{-1}$ for some nonempty clique S. There is a direct product decomposition

$$G_{S\cup\mathrm{lk}(S)} = G_S \times G_{\mathrm{lk}(S)} = \prod_{v\in S} G_v \times G_{\mathrm{lk}(S)}$$

and therefore φ factors as a product of commuting homomorphisms

$$\varphi(a) = g\prod_{v\in S}\varphi_v(a)\varphi_{\mathrm{lk}(S)}(a)g^{-1},$$

with $\varphi_v = \varphi\circ r_v$ and $\varphi_{\mathrm{lk}(S)} = \varphi\circ r_{\mathrm{lk}(S)}$. Here we use the retractions r introduced in Section 2. Since the φ_v are all continuous, $\varphi_{\mathrm{lk}(S)}$ is not continuous. Hence we find a clique $T \subseteq \mathrm{lk}(S)$ such that $\varphi_{\mathrm{lk}(S)}(L) \subseteq hG_{T\cup(\mathrm{lk}(T)\cap\mathrm{lk}(S))}h^{-1}$. But then $S\cup T$ is a clique (because $T \subseteq \mathrm{lk}(S)$) which is strictly bigger than S. If we continue in this fashion, we end up after finitely many steps with an empty link, because Γ has finite dimension. Thus φ is a finite product of commuting continuous homomorphisms, and therefore itself continuous. This is a contradiction. □

The referee has pointed out that if every composite $L \xrightarrow{\varphi} G_\Gamma \xrightarrow{r_v} G_v$ is continuous, then φ is continuous, whether or not the graph product if finite dimensional (see proof of Theorem 3.3 in [17]).

5. The proof of Proposition C

We consider abstract homomorphisms from locally compact groups L into groups G which are in the class $\mathcal{G}$. We recall from the introduction that for such a group G, every torsion subgroup $T \subseteq G$ is finite, and every abelian subgroup $A \subseteq G$ is a (possibly infinite) direct sum of cyclic groups. In particular, such a group G has no nontrivial divisible abelian subgroups.

Proposition C. *Let φ be an abstract homomorphism from a locally compact group L to a group G in the class $\mathcal{G}$. Then φ factors through the canonical projection $\pi : L \to L/L^\circ$. If L is almost connected, then $\varphi(L)$ is finite.*

Proof. We first show that every homomorphism $\rho : K \to G$ has finite image if K is compact. Suppose that $g \in K$. We claim that $\rho(g)$ has finite order. The subgroup $H = \overline{\langle g \rangle}$ is compact abelian, whence $\rho(H) = F \times \mathbb{Z}^{(J)}$, where F is a finite abelian group. By Dudley's result [21], a compact group has no nontrivial free abelian quotients. Therefore $\rho(H)$ is finite and in particular, $\rho(g)$ has finite order. Since G contains no infinite torsion groups, $\rho(K)$ is finite.

Now we show that $\varphi(L^\circ)$ is trivial. By Iwasawa's Theorem [27, Theorem 13] there is a decomposition $L^\circ = H_1 \cdots H_r K$, where $H_j \cong \mathbb{R}$ for $j = 1, \ldots, r$ and where K is a compact connected group. The groups $H_1, \ldots, H_r$ are abelian and divisible. From our assumptions on the class $\mathcal{G}$ we see that the abelian groups $\varphi(H_j)$ are trivial, for $j = 1, \ldots, r$. The compact group K is connected and therefore divisible [25, Theorem 9.35]. A finite divisible group is trivial, and therefore $\varphi(K)$ is trivial as well. This shows that $\varphi(L^\circ)$ is trivial.

The first paragraph of the present proof shows then that $\varphi(L)$ is finite if L/L° is compact. □

6. The proof of Theorem D

We are now ready to prove Theorem D.

Theorem D. *Let φ be an abstract homomorphism from a locally compact group L to a group G in the class $\mathcal{G}$. Then either φ is continuous, or $\varphi(L)$ lies in the normalizer of a finite non-trivial subgroup of G.*

Proof. Let L° be the connected component of the identity in L. By Proposition C the homomorphism φ factors through a homomorphism $\psi : L/L^\circ \to G$. The totally disconnected locally compact group L/L° contains by van Dantzig's Theorem [10, III§4, No. 6] compact open subgroups. We distinguish two cases.

Case 1: $\psi(K)$ is trivial for some compact open subgroup $K \subseteq L/L^\circ$.

Then the kernel of ψ is open and therefore ψ and φ are continuous.

Case 2: $\psi(K)$ is nontrivial for every compact open subgroup $K \subseteq L/L^\circ$.

By Proposition C, the image $\psi(K)$ of such a group K is finite. Among the compact open subgroups of L/L° we choose M such that $\psi(M)$ is minimal. Given $g \in L/L^\circ$, we have then that $\psi(gMg^{-1}) = \psi(M \cap gMg^{-1}) = \psi(M)$. It follows that $\psi(g)$ normalizes $\psi(M)$. □

7. Some remarks on the class $\mathcal{G}$

In this last section we show that the class $\mathcal{G}$ contains many groups.

Proposition 7.1. *The class $\mathcal{G}$ is closed under passage to subgroups, under passage to finite products, and under passage to arbitrary coproducts.*

Proof. If $H \subseteq G \in \mathcal{G}$, then clearly $H \in \mathcal{G}$. If $G_1, \ldots, G_r \in \mathcal{G}$ and if $T \subseteq \prod_{j=1}^r G_j$ is a torsion group, then the projection $\pi_j(T) = T_j \subseteq G_j$ is also a torsion group. Hence $T \subseteq \prod_{j=1}^r T_j$ is finite. Similarly, if $A \subseteq \prod_{j=1}^r G_j$ is abelian, then A is contained in the product $\prod_{j=1}^r \pi_j(A)$, which is a direct sum of a finite abelian group and a free abelian group. Hence A itself is a direct sum of a finite abelian group and a free abelian group. Finally suppose that $(G_j)_{j\in J}$ is a family of groups in $\mathcal{G}$. By Kurosh's Subgroup Theorem [4], every subgroup of the coproduct $\coprod_{j\in J} G_j$ is itself a coproduct $F * \coprod_{j\in J} g_j U_j g_j^{-1}$, were F is a free group, $U_j \subseteq G_j$ is a subgroup and the g_j are elements of $\coprod_{i\in J} G_i$. If such a group is abelian, then it is either cyclic or conjugate to a subgroup of one of the free factors. □

Proposition 7.2. *Every hyperbolic group G is in the class $\mathcal{G}$.*

Proof. By a theorem of Gromov [22, Chap. 8, Cor. 36], every torsion subgroup of a hyperbolic group is finite. Furthermore, every abelian subgroup of a hyperbolic group is finitely generated. □

Proposition 7.3. *Let A be an Artin group. If A is a right-angled Artin group or an Artin group of finite type, then A is torsion free and every abelian subgroup of A is finitely generated.*

Proof. Every right-angled Artin group is torsion free by [23, Corollary 3.28]. Moreover A is a CAT(0) group, see [16]. Hence every abelian subgroup of A is finitely generated, see [12, II Corollary 7.6]. If A is an Artin group of finite type, then A is torsion free by [13]. By [8, Corollary 4.2], every abelian subgroup of A is finitely generated. □

We note that it is an open question if every Artin group is torsion free [15, Conjecture 12].

Proposition 7.4. *Let W be a Coxeter group. Then every torsion subgroup of W is finite and every abelian subgroup of W is finitely generated.*

Proof. It was proved in [33, Theorem 14.1] that Coxeter groups are CAT(0) groups. Hence every abelian subgroup of W is finitely generated [12, II Corollary 7.6] and the order of finite subgroups of W is bounded [12, II Corollary 2.8(b)]. Let $T \subseteq W$ be a torsion group. Since W is a linear group [19, Corollary 6.12.11] and since every finitely generated linear torsion group is finite [37, I], it follows that every finitely generated subgroup of T is finite. Since the order of finite subgroups of T is bounded, T is finite. □

Proposition 7.5. *The groups $\mathrm{GL}_n(\mathbb{Z})$, the groups $\mathrm{Out}(F_n)$ of outer automorphisms of free groups and the mapping class groups $\mathrm{Mod}(S_g)$ of orientable surfaces of genus g are in the class $\mathcal{G}$.*

Proof. Since $\mathrm{GL}_n(\mathbb{Z})$ is a linear group, it follows that every finitely generated torsion subgroup is finite [37, I]. Since the order of finite subgroups in $\mathrm{GL}_n(\mathbb{Z})$ is bounded [31], we obtain that every torsion subgroup of $\mathrm{GL}_n(\mathbb{Z})$ is finite. It was proved in [29] that every abelian subgroup of $\mathrm{GL}_n(\mathbb{Z})$ is finitely generated. Hence $\mathrm{GL}_n(\mathbb{Z})$ is in the class $\mathcal{G}$.

The kernel of the map $\mathrm{Out}(F_n) \to \mathrm{GL}_n(\mathbb{Z})$ which is induced by the abelianization of F_n is torsion free [7]. Since every torsion subgroup of $\mathrm{GL}_n(\mathbb{Z})$ is finite, it follows that every torsion subgroup of $\mathrm{Out}(F_n)$ is finite. Every abelian subgroup of $\mathrm{Out}(F_n)$ is finitely generated, see [6]. Thus $\mathrm{Out}(F_n)$ is in the class $\mathcal{G}$.

It was proved in [9, Theorem A] that every abelian subgroup of $\mathrm{Mod}(S_g)$ is finitely generated. Further, it was proved in [34, Theorem 1] that $\mathrm{Mod}(S_g)$ is a linear group. Hence every finitely generated torsion subgroup is finite [37, I]. We know by [26] that the order of finite subgroups in $\mathrm{Mod}(S_g)$ is bounded. Therefore every torsion subgroup of $\mathrm{Mod}(S_g)$ is finite. Thus $\mathrm{Mod}(S_g)$ is in the class $\mathcal{G}$. □

References

[1] R. Alperin, Compact groups acting on trees, Houston J. Math. 6 (4) (1980) 439–441.
[2] R. Alperin, Locally compact groups acting on trees, Pacific J. Math. 100 (1) (1982) 23–32.
[3] Y. Antolin, A. Minasyan, Tits alternatives for graph products, J. Reine Angew. Math. 704 (2015) 55–83.
[4] R. Baer, F. Levi, Freie Produkte und ihre Untergruppen, Compos. Math. 3 (1936) 391–398.
[5] H. Bass, Some remarks on group actions on trees, Comm. Algebra 4 (12) (1976) 1091–1126.
[6] H. Bass, A. Lubotzky, Linear Central Filtrations on Groups, Contemporary Math., vol. 169, AMS, 1994.
[7] G. Baumslag, T. Taylor, The centre of groups with one defining relator, Math. Ann. 175 (1968) 315–319.
[8] M. Bestvina, Non-positively curved aspects of Artin groups of finite type, Geom. Topol. 3 (1999) 269–302.
[9] J.S. Birman, A. Lubotzky, J. McCarthy, Abelian and solvable subgroups of the mapping class groups, Duke Math. J. 50 (4) (1983) 1107–1120.
[10] N. Bourbaki, Topologie generale (Chapitres 1 a 4), Elements de mathematique, Hermann, Paris, 1971.

[11] O. Braun, K.H. Hofmann, L. Kramer, Automatic continuity of abstract homomorphisms between locally compact and polish groups, Transform. Groups (2019), https://doi.org/10.1007/s00031-019-09537-4, in press, arXiv:1611.05801.
[12] M.R. Bridson, A. Haefliger, Metric Spaces of Non-Positive Curvature, Grundlehren der Mathematischen Wissenschaften (Fundamental Principles of Mathematical Sciences), vol. 319, Springer-Verlag, Berlin, 1999.
[13] E. Brieskorn, K. Saito, Artin-Gruppen und Coxeter-Gruppen, Invent. Math. 17 (1972) 245–271.
[14] P.E. Caprace, T. Marquis, Can an anisotropic reductive group admit a Tits system?, Pure Appl. Math. Q. 7 (3) (2011) 539–557, Special Issue: In honor of Jacques Tits.
[15] R. Charney, Problems related to Artin groups, http://people.brandeis.edu/~charney/papers/Artin_probs.pdf.
[16] R. Charney, M. Davis, Finite $K(\pi, 1)$'s for Artin groups, in: F. Quinn (Ed.), Prospects in Topology, in: Ann. of Math. Stud., vol. 138, Princeton Univ. Press, Princeton, 1995, pp. 110–124.
[17] R.C. Conner, S.M. Corson, A note on automatic continuity, arXiv:1710.04787, 2017.
[18] S. Corson, I. Kazachkov, On preservation of automatic continuity, arXiv:1901.09279, 2019.
[19] M.W. Davis, The Geometry and Topology of Coxeter Groups, London Mathematical Society Monographs Series, vol. 32, Princeton University Press, Princeton, NJ, 2008.
[20] M.W. Davis, Buildings are CAT(0), in: Geometry and Cohomology in Group Theory, Durham, 1994, in: London Math. Soc. Lecture Note Ser., vol. 252, Cambridge Univ. Press, Cambridge, 1998, pp. 108–123.
[21] R.M. Dudley, Continuity of homomorphisms, Duke Math. J. 28 (1961) 587–594.
[22] E. Ghys, P. de la Harpe, Sur les groupes hyperboliques d'après Mikhael Gromov (Hyperbolic groups in the theory of Mikhael Gromov), in: Papers from the Swiss Seminar on Hyperbolic Groups Held in Bern, 1988, in: Progress in Mathematics, vol. 83, Birkhäuser Boston, Inc., Boston, MA, 1990 (in French).
[23] E.R. Green, Graph Products of Groups, PhD thesis, University of Leeds, 1990.
[24] V.S. Guba, M.V. Sapir, On subgroups of the R. Thompson group F and other diagram groups, Mat. Sb. 190 (8) (1999) 3–60 (in Russian); translation in Sb. Math. 190 (7–8) (1999) 1077–1130.
[25] K.H. Hofmann, S.A. Morris, The Structure of Compact Groups, third edition, revised and augmented, De Gruyter Studies in Mathematics, vol. 25, De Gruyter, Berlin, 2013, MR3114697.
[26] A. Hurwitz, Über algebraische Gebilde mit eindeutigen Transformationen in sich, Math. Ann. 41 (1893) 403–442.
[27] K. Iwasawa, On some types of topological groups, Ann. of Math. (2) 50 (1949) 507–558.
[28] N. Leder, O. Varghese, A note on locally elliptic actions on cube complexes, arXiv:1810.06927, 2018.
[29] A.I. Mal'cev, On certain classes of infinite solvable groups, Mat. Sb. 28 (1951) 567–588, Amer. Math. Soc. Transl. Ser. 2 2 (1956) 1–21.
[30] T. Marquis, A fixed point theorem for Lie groups acting on buildings and applications to Kac-Moody theory, Forum Math. 27 (1) (2015) 449–466.
[31] H. Minkowski, Zur Theorie der positiven quadratischen Formen, in: Collected Works, B.G. Teubner, vol. I, 1911, pp. 212–218.
[32] S.A. Morris, P. Nickolas, Locally compact group topologies on an algebraic free product of groups, J. Algebra 38 (2) (1976) 393–397.
[33] G. Moussong, Hyperbolic Coxeter Groups, PhD thesis, The Ohio State University, 1988.
[34] I.V. Nikolaev, Mapping class groups are linear, in: Topological Algebras and Their Applications, in: De Gruyter Proc. Math., 2018, pp. 193–200.
[35] G. Paolini, S. Shelah, Polish topologies for graph products of groups, J. Lond. Math. Soc. (2019), https://doi.org/10.1112/jlms.12219.
[36] M. Sageev, Ends of group pairs and non-positively curved cube complexes, Proc. Lond. Math. Soc. (3) 71 (3) (1995) 585–617.
[37] I. Schur, Über Gruppen periodischer linearer Substitutionen, Sitz.ber. K. Preuss. Akad. Wiss. Berl. (1911) 619–627.