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Jacques Tits: "Œuvres. Collected Works, Vols. I–IV". Edited by Francis Buekenhout, Bernhard Matthias Mühlherr, Jean-Pierre Tignol, Hendrik Van Maldeghem

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Jacques Tits is one of the outstanding mathematicians of our time. His published (and unpublished) works, most of which are collected in these impressive four volumes, encompass a wide area of modern mathematics. The central and recurring topics in his œuvre are certainly group theory and geometry. However, group theory itself is a vast area. During the last six decades, Jacques Tits has contributed important results to finite group theory, to Lie theory, to algebraic group theory, to combinatorial and geometric group theory, to Kac-Moody theory, and to arithmetic geometry. Probably he is one of the few present-day mathematicians who may claim to have an overview of group theory in its entirety. In some of these areas, he was among those who laid the foundations, while in others, he pushed existing results much further. The task to describe what is in these four volumes is thus somewhat intimidating. I will try to highlight and describe some of their contents, led partially by my own mathematical interests and knowledge. The four volumes group his work, of course, in a chronological order. In what follows, I will try to group some of Tits' contributions by their topics.

Multiply Transitive Groups and Lie Groups A group G acting on a set X is called k-transitive if given any two k-tuples of distinct elements of X, there is a group element g that maps one k-tuple to the other. We assume here that X contains at least k elements. The action is called sharply k-transitive if the group element in question is uniquely determined. The reader should be warned here that the terminology

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has changed over the years, and also varies in Tits' articles—what we call sharply k-transitive here was called earlier k-transitive.

In any case, the symmetric group on k letters is an example of a sharply k-transitive group. At the same time, its action is ℓ-transitive for all $\ell \leq k$. If F is a field, then the projective linear group $\mathrm{PGL}_2(F)$ acts sharply 3-transitively on the projective line $P = F \cup \{\infty\}$. Surprisingly, there exist no infinite sharply k-transitive groups for $k \geq 4$. Two of the sporadic simple groups, the Mathieu groups M_{11} and M_{12}, are among the finite sharply 4- and 5-transitive groups. Tits' PhD Thesis [6] discusses these transitivity properties, first in the case of the projective linear groups, and then in general. It contains, in particular, the classification of all sharply k-transitive groups with $k \geq 4$, a remarkable result.

In the long article [7] (roughly equivalent to a *Habilitationsschrift*) Tits obtains a large number of interesting results about homogeneous spaces of Lie groups and locally compact groups, based on the transitivity properties of the actions. My favorite is the classification of 2-transitive Lie groups. Suppose that a Lie group G acts faithfully (i.e., with trivial kernel) and 2-transitively on a connected manifold X. In present-day terminology, he proves that there are precisely three cases.

(1) The identity component G° of G is the projective special linear group over the real numbers, the complex numbers, the quaternions or the Cayley division algebra, acting on the corresponding projective space.
(2) G° is the connected isometry group of a hyperbolic space over the real numbers, the complex numbers, the quaternions or the Cayley division algebra, acting on its visual boundary.
(3) $G = H \ltimes \mathbb{R}^m$, where $H \subseteq \mathrm{GL}_m\mathbb{R}$ is a closed linear Lie group acting transitively on the nonzero vectors of $\mathbb{R}^m$.

In particular, the topological space X is either a projective space or a sphere of a real vector space. Related results on the topology of X had been obtained at the same time, and independently, by A. Borel. However, Borel did not classify the corresponding Lie groups.

In the same article, Tits obtains many more equally interesting classification results. For example, he classifies transitive faithful G-actions on manifolds where every point stabilizer G_x acts transitively on the tangent directions at x. The results in [7] seem to be less known than they deserve. Some particular cases were rediscovered and reproved several times by others. This is probably also due to the fact that prior to the publication of Tits' collected works, it was difficult to get hold of a copy of [7].

The Structure of Lie Groups and Algebraic Groups The article [7] depends very much on the structure theory of semisimple Lie groups. As Tits himself pointed out, the work on transitivity properties of Lie groups led him to the combinatorial structure of parabolic subgroups, and finally to the definition of buildings. Also, Dynkin diagrams (called "figure de Schläfli") of simple Lie groups appear in [7], with a reference to Dynkin's work. In the early 60's, Tits wrote a whole series of papers on real forms of the exceptional Lie groups E_6, E_7 and E_8. In these articles, his focus shifts already from Lie groups to semisimple algebraic groups over arbitrary fields.

At the same time the associated combinatorial structures emerge, which led him subsequently to the discovery of buildings.

I also want to mention the short Lecture Notes volume [11] on the representation theory of Lie groups. In combination with [13], where rational representations of reductive groups over arbitrary fields (and in particular of real Lie groups) are classified, these tables are indispensable for any Lie theorist.

His joint paper [1] with A. Borel on the structure of reductive algebraic groups over arbitrary fields still stands as a landmark paper and as a standard reference. The classification and the structure theory of semisimple algebraic groups over arbitrary fields is a recurring topic in Tits' work. His Boulder article [9] summarizes his results in a very nice and reader-friendly way. History and terminology are not always fair: what is nowadays often called the "Satake diagram" of a semisimple algebraic group should properly be called the "Tits diagram". These decorated Dynkin diagrams, together with the datum of the so-called anisotropic kernel, classify algebraic groups up to isogeny over arbitrary fields. This result is certainly one of the major mathematical achievements of Jacques Tits.

Another recurring theme in Tits' work is the underlying "abstract" group structure of a simple algebraic group or Lie group. In [8] he shows that if G is an absolutely simple algebraic group, defined over a sufficiently large field F, and if $G^\dagger \subseteq G(F)$ is the subgroup generated by the F-rational unipotent elements, then $G^\dagger/\mathrm{Cen}(G^\dagger)$ is simple as an abstract group. His proof uses spherical buildings and (B, N)-pairs in a systematic way. In another joint paper with A. Borel [15] he proves (essentially) that if the groups of rational points of two absolutely simple isotropic algebraic groups G and G', possibly defined over different fields F and F', are isomorphic as abstract groups, then G and G' are isomorphic as algebraic groups. (The precise statement is somewhat more elaborate, but this is basically the result.) Tits needed this for his work on the automorphism groups of spherical buildings associated to semisimple algebraic groups in [16]. It would be interesting to see if nowadays, with the advanced structure theory of buildings at hand, buildings could be used to prove this deep result in a completely geometric way.

Buildings The discovery of the various kinds of buildings is probably the mathematical topic which is associated most often with Jacques Tits. It had been known at least since the 19th century that the projective geometry, that is, the collection of all nontrivial subspaces of a given vector space V, is a very useful tool for studying the general linear group $\mathrm{GL}(V)$. It was also known that this projective geometry can be described by a relatively short list of geometric incidence axioms. The fundamental theorem of projective geometry asserts that

(1) every projective geometry of rank at least 3 (in the synthetic sense) arises from some vector space over some field, and that
(2) every automorphism of this geometry comes from a semi-linear automorphism of the vector space.

These are extremely useful facts, as they guarantee, in the presence of very few combinatorial axioms, the existence of a field, a vector space, and a linear group. Similar constructions were known for the other classical groups, like symplectic groups or orthogonal groups. However, it was not so clear which combinatorial structures should correspond to the exceptional groups like E_6, E_7 and E_8. Geometers such as H. Freudenthal devoted a large number of papers to this question. However, the combinatorial constructions looked complicated and a unifying theme was missing.

This changed completely with Tits' discoveries. Firstly, he showed in a uniform way that to every group G with a Tits system or (B, N)-pair, that is, with two subgroups $B, N \subseteq G$ satisfying a list of four axioms, one can associate a combinatorial structure, the building $\Delta(G)$. The building is a simplicial complex, and G acts simplicially on it. Secondly, the building itself can be characterized by a very short set of combinatorial axioms. There is an associated Weyl group $W = N/B \cap N$, which describes the so-called type of the building. If W is finite, then the building is called spherical. Tits' discovery was that one can find such a (B, N)-pair in every simple isotropic F-algebraic group G. If the F-rank of G is at least 2, then $G(F) \rtimes \mathrm{Aut}(F)$ is (essentially) the group of all combinatorial automorphisms of the building Δ. The $G(F)$-action on the building can then be used in order to study G and its subgroups. This result is proved in Tits' Lecture Note [16]. The other important result proved in [16] is the classification of spherical buildings of rank at least 3. These spherical buildings arise from algebraic data (such as fields, division algebras, Cayley algebras, quadratic forms). In fact almost all of them arise from simple F-algebraic groups. The fundamental theorem of projective geometry is a special case of this result.

This classification result from [16], which one might call the fundamental theorem of spherical buildings, is the reason why buildings turned out to be so useful in group theory (including the classification of the finite simple groups) and geometry. Again, the presence of a combinatorial structure Δ satisfying a short list of simple axioms guarantees the presence of a field F and (in most cases) a simple F-algebraic group acting on Δ. This had, for example, striking applications in Riemannian geometry in connection with manifolds of nonpositive curvature and with isoparametric foliations.

At the very end of [16], the Moufang condition appears already. The Moufang condition, a transitivity property of the automorphism group of the building, turned out to be an essential property of spherical buildings. In [16] Tits shows that every spherical building of rank at least 3 satisfies the Moufang condition. The proof, albeit long, is strictly combinatorial. Once this is shown, the algebra of root groups and commutator relations may be used to translate the classification into an algebraic problem. To this end, Tits studies and classifies certain Moufang polygons, that is, spherical buildings of rank 2 satisfying the Moufang condition, in several of his papers. The complete classification was carried out by Tits and R.M. Weiss in the monograph [29] which is an almost mandatory supplement to Tits' collected works.

There is one other paper on buildings [18] that I want to mention here, where the "local approach" to buildings is carried out. The paper is important for several reasons. Firstly, it introduces a completely different approach to buildings as chamber systems, that is, as generalized Cayley graphs. This approach to buildings (which is shown to be functorially equivalent to the older one) is particularly well-suited for group-theoretic questions, while the older simplicial approach fits better into the world of metric geometry and nonpositive curvature. Secondly, this article gives a criterion when a simplicial complex that looks locally like a building is actually a building or a quotient of a building. This result has become increasingly important during the last years, for example in Riemannian geometry. It has also led to the construction of exotic buildings and lattices, starting with a finite complex of finite groups. This article goes well with the fundamental papers by M. Gromov on geometric group theory that appeared at about the same time.

Reductive Algebraic Groups over Local Fields If G is a reductive isotropic group over a local field F, then there is the Tits system (B, N) and its spherical building Δ, as described in the previous section. However the valuation of F gives rise to a richer subgroup structure of $G(F)$ and eventually to a second Tits system (I, N), with the same subgroup N, but a different group $I \subseteq G(F)$, the Iwahori subgroup. The second building, X, associated to (I, N) is not of spherical type. Its Weyl group $\overline{W} = N/N \cap I$ is an infinite euclidean reflection group, and X is a euclidean building. The simplicial complex X is unbounded and carries a metric of nonpositive curvature. There is a close relation between the two buildings: the spherical building Δ appears as the "boundary at infinity" of the euclidean building X. In a series of joint papers with F. Bruhat [3–5], Tits developed the structure theory both of euclidean buildings and of reductive groups over local fields. In [23] he outlined how the classification of spherical buildings can be used to achieve a classification of euclidean buildings of dimension at least 3. In this article, he considers in fact a much wider class of metric spaces of nonpositive curvature, which need not be buildings in the combinatorial sense. Nowadays it is common to call these spaces nondiscrete euclidean buildings. Similarly to $\mathbb{R}$-trees, these spaces may branch everywhere.

Euclidean buildings are the counterparts in arithmetic geometry of the Riemannian symmetric spaces of noncompact type. They found numerous applications in various fields, and notably provide important tools for studying or constructing representations of reductive groups, for the Langlands program, for Shimura varieties, and for Deligne-Lusztig theory. Yet, they are also important in Riemannian geometry and geometric group theory, where the (wide open) rank rigidity problem predicts that euclidean buildings and symmetric spaces are basically the only irreducible locally compact higher rank spaces of nonpositive curvature admitting a cocompact group action.

Kac-Moody Groups and Twin Buildings Kac-Moody algebras, which are infinite-dimensional generalizations of semisimple Lie algebras, were discovered independently by V. Kac and R. Moody. These algebras attracted immediately the attention of theoretical physicists, and they play nowadays a role in various physical models. The construction of Kac-Moody groups, however, turned out to be more complicated. For complex affine Kac-Moody algebras one can construct corresponding groups as loop groups of compact Lie groups. However, this method fails for other algebras. In [22, 24, 26, 27] Tits shows that there is a group functor that associates to every Kac-Moody datum and every field F a split Kac-Moody group $G(F)$. The construction resembles the construction of complex simple Lie groups in terms of the Steinberg relations. Associated to G, there are now two Tits systems $(B_\pm, N)$ and two buildings $\Delta_\pm$, sharing the same Weyl group W. The group $G(F)$ acts on both buildings, preserving a map $\delta^* : \Delta^+ \times \Delta^- \to W$, the codistance function. This structure is called a twin building. Twin buildings over finite fields have been used, for example, by B. Rémy to construct new simple locally compact groups and lattices acting on spaces of nonpositive curvature.

Coxeter Groups, the Tits Alternative, Trees, and the Monster The geometry and structure of Coxeter groups is essential for understanding buildings. Tits has devoted several papers to Coxeter groups, to their structure and to the solution of the word problem. Bourbaki's volume [2] was heavily influenced by Tits. These articles are still the standard references for Coxeter groups and root systems.

Such questions belong to the wider area of geometric group theory. Perhaps the most famous contribution of Tits here is what is nowadays called the Tits alternative [14]: If Γ is a finitely generated linear group, then Γ either is virtually solvable, or it contains the free group on two generators.

Trees are special cases of euclidean buildings. Tits studies group actions on trees and related automorphism groups in [12, 17]. He proves a simplicity criterion for groups acting on trees in [12]. In [17] he introduces what is nowadays called an $\mathbb{R}$-tree, and he proves that a solvable group acting without fixed point on such a tree fixes one end or two ends. $\mathbb{R}$-trees have since then become quite important in geometric group theory and in Riemannian geometry, for example in connection with degenerations of hyperbolic structures on closed manifolds.

Tits' work had also a significant impact on finite group theory. The classification of the finite spherical buildings of rank at least 3, carried out in [16], plays an important role in the classification of the finite simple groups. Tits investigated also the sporadic finite simple groups, and the Tits group of order $2^{11}3^3 5^2 13$ was discovered by him [8]. He wrote two papers on the Monster, simplifying Griess' construction [19, 21] and one on the Moonshine module [25].

The Volumes Jacques Tits' works appeared in four volumes, published by the European Mathematics Society and edited by F. Buekenhout, B. Mühlherr, J.-P. Tignol and H. Van Maldeghem. The editors have done a great job collecting and reproducing Tits' works. This was apparently not always an easy task, as there are several papers that were difficult to obtain (and to reproduce). Also, they included some manuscripts which were previously unpublished. In particular, the editors have included (and re-typed) two important sets of lecture notes from Yale [10, 20]. Each volume closes with an interesting list of comments and notes by the editors, pointing towards further developments, corrections and complements. Everyone interested in group theory should have access to these four volumes. What is not included are the *Résumés de Cours* articles with Tits' contributions from 1973 to 2000. But these are available from the Société Mathématique de France in a nice separate volume [28]. And finally, the monograph "Moufang Polygons" by Tits and Weiss [29] should be mentioned here. The combination of these six books encompasses some of the finest group theory and geometry of our time.

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