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HOMOGENEOUS COMPACT GEOMETRIES

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Abstract. We classify compact homogeneous geometries of irreducible spherical type and rank at least 2 which admit a transitive action of a compact connected group, up to equivariant 2-coverings. We apply our classification to polar actions on compact symmetric spaces.

We classify compact homogeneous geometries which look locally like compact spherical buildings. Geometries which look locally like buildings arise naturally in various recognition problems in group theory. Tits' seminal paper *A local approach to buildings* [Ti2] is devoted to them. Among other things, Tits proved there that a geometry which looks locally like a building can be 2-covered by a building if and only if a local geometric obstruction vanishes. The condition is that the links of all corank 3 simplices of type C_3 and H_3 admit coverings by buildings.

There exists a famous finite geometry of type C_3 which is not covered by any building, the so-called Neumaier Geometry [Neu] (see also 1.18 below). It seems to be an open problem if there exist other (finite) examples of non-building C_m geometries, and if there exist geometries of type H_3 (note that we assume geometries to be thick). Assuming a transitive group action, Aschbacher classified all finite homogeneous geometries of type C_3, see [Asch] and [Yos]. Using this result, Aschbacher classified the finite homogeneous geometries with irreducible spherical diagrams [Asch, Thm. 3]. Our Theorem A below may be viewed as a Lie group analog of his classification. More results and references can be found in Pasini's book [Pas].

We are here concerned with the classification of geometries on which compact Lie groups act transitively. Such geometries arise in the classification of polar actions. For example, Thorbergsson's classification of isoparametric submanifolds in spheres [Th] relied heavily on the Burns–Spatzier classification of compact connected spherical buildings admitting a strongly transitive action [BuSp]. In the last section we describe an application of our results to polar actions on compact symmetric spaces.

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Our results are as follows. With Grundhöfer and Knarr, the first author obtained in the mid-1990s a complete classification of irreducible homogeneous compact connected spherical buildings. We recall the result (which is built on earlier work of Salzmann, Löwen, and Burns–Spatzier [CPP], [BuSp]).

Theorem ([GKK1], [GKK2], [GKVW]). *Let Δ be a compact building of irreducible spherical type and rank at least 2, with connected panels. Assume that its topological automorphism group acts transitively on the chambers of Δ. Then Δ is the spherical building associated to a noncompact simple Lie group.*

Using a combination of this and results in Tits' local approach [Ti2], we prove the following main results. The exceptional C_3 geometry that appears in Theorem A was discovered by Podestà–Thorbergsson [PoTh]. We describe this geometry in detail in Section 3B. For the unexplained definitions we refer to our paper below, in particular to Sections 1 and 2. A map between geometries is called a 2-covering if it is bijective on the links of all simplices of corank 2, see 1.10 below.

Theorem A. *Let Δ be a compact geometry of irreducible spherical type and rank at least 2, with connected panels. Assume that a compact group acts continuously and transitively on the chambers of Δ.*

If Δ is not of type C_3, then there exists a simple noncompact Lie group S, a compact chamber-transitive subgroup $K \subseteq S$ and a K-equivariant 2-covering $\widetilde{\Delta} \to \Delta$, where $\widetilde{\Delta}$ is the canonical spherical building associated to S.

If Δ is of type C_3, then either there exists a building $\widetilde{\Delta}$ and a 2-covering $\widetilde{\Delta} \to \Delta$ as in the previous case, or Δ is isomorphic to the unique exceptional homogeneous compact C_3 geometry which cannot be 2-covered by any building.

More general results are proved in 5.4, 5.3, 4.1, 3.18. In this way we obtain a complete classification of homogeneous compact geometries with connected panels whose irreducible factors are of spherical type and rank at least 2, up to equivariant 2-coverings. In certain situations the conclusion of Theorem A may be strengthened. For example, we prove the following in 2.24.

Proposition. *Let Δ be a homogeneous compact geometry as in Theorem A. If Δ is of type A_m or E_6 or if all panels are 2-dimensional, then Δ is the building associated to a noncompact simple Lie group S, and the compact connected group induced on Δ is a maximal compact subgroup of S.*

One application of this classification is the following result, which builds heavily on results by the second author [Lyt]. See 5.5 below for more details, an outline of proof, and how this relates to independent work on the classification of polar actions in positive curvature by Fang–Grove–Thorbergsson [FGTh].

Theorem B. *Suppose that $G \times X \to X$ is a polar action of a compact connected Lie group G on a symmetric space X of compact type. Then, possibly after replacing G by a larger orbit equivalent group, we have splittings $G = G_1 \times \cdots \times G_m$ and $X = X_1 \times \cdots \times X_m$, such that the action of G_i on X_i is either trivial or hyperpolar or the space X_i has rank 1, for $i = 1, \ldots m$.*

The following problems seem to be open.

Problem 1. *Are there geometries of type* H_3, H_4, F_4, *or* C_m, $m \geq 4$, *that are not* 2-*covered by buildings*?

Note that we assume geometries to be thick. Possibly, in some cases affirmative answers can be obtained along the lines of [BeKa].

Problem 2. *Is the topological automorphism group of a compact geometry of spherical type locally compact in the compact-open topology*?

Our approach avoids this question. However, we show that the compact groups that appear are automatically Lie groups, provided that the panels are connected.

Problem 3. *Does the conclusion of Theorem* A *still hold if we just assume that the topological automorphism group acts transitively on the chambers*?

Problem 4. *Are there non-homogeneous compact geometries of irreducible spherical type and rank at least* 3 *that are not* 2-*covered by buildings*?

We remark that non-homogeneous compact geometries which are 2-covered by buildings arise naturally from polar foliations, see for example [DoVa].

Problem 5. *Is the exceptional* C_3 *geometry from Section* 3B *simply connected? Is there an analogy with the Neumaier Geometry? Can this geometry be defined over other fields*?

The paper is organized as follows. In Section 1 we introduce the relevant combinatorial notions and explain Tits' results. In Section 2 we introduce a convenient category of homogeneous compact geometries, and we show the existence of universal objects. In Section 3 we review the known examples of homogeneous compact geometries of type C_3, and in Section 4 we prove that this list is complete. In the final Section 5 we combine our classification results and prove, among other things, Theorem A and explain the main steps for the proof of Theorem B.

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1. Geometries and buildings

In order to make this paper self-contained, we first introduce some elementary combinatorial terminology. For the following facts and definitions we refer to Tits [Ti1], [Ti2]. Additional material can be found in [Buek], [BuPa], [Pas]. The geometries which we consider here are a standard tool in the structure theory of the finite simple groups. We allow ourselves a few small deviations from Tits' terminology. These will be indicated where they appear.

1.1. Chamber complexes. Let V be a (nonempty) set and Δ a collection of finite subsets of V. If V is closed under going down (i.e., $\alpha \subseteq \beta \in \Delta$ implies $\alpha \in \Delta$) and if $V = \bigcup \Delta$, then the poset $(\Delta, \subseteq)$ is called a *simplicial complex*. The elements of V are called *vertices* and the elements of Δ are called *simplices*. The *rank* of a simplex α is the number of its vertices, $\mathrm{rk}(\alpha) = \mathrm{card}(\alpha)$. Two vertices which are contained in a simplex are called *adjacent*. If the simplex α is contained in the simplex β, we call α a *face* of β. A simplicial complex is called a *flag complex* if every finite set of pairwise adjacent vertices is a simplex ('every non-simplex contains a non-edge'). The *link* of a simplex α is the subcomplex

$$\mathrm{lk}_\Delta(\alpha) = \mathrm{lk}(\alpha) = \{\beta \in \Delta \mid \alpha \cap \beta = \varnothing \text{ and } \alpha \cup \beta \in \Delta\}$$

and the *residue* of α is the set $\Delta_{\geq\alpha} = \{\beta \in \Delta \mid \beta \supseteq \alpha\}$ of all simplices having α as a face. The link and the residue of α are poset-isomorphic. We note that Δ is the link of the empty simplex.

A *simplicial map* between simplicial complexes is a map between their vertex sets which maps simplices to simplices. We call a simplicial map *regular* if its restriction to every simplex is bijective; these are Tits' *morphisms* [Ti1, 1.1]. The *geometric realization* $|\Delta|$ of Δ consists of all functions $\xi : V \to [0,1]$ whose support $\mathrm{supp}(\xi) = \{v \in V \mid \xi(v) > 0\}$ is a simplex, and with $\sum_{v \in V} \xi(v) = 1$. We also write

$$\xi = \sum_{v \in V} v \cdot \xi(v).$$

The weak topology turns $|\Delta|$ into a CW complex which we denote by $|\Delta|_w$. A simplicial map $f : \Delta \to \Delta'$ induces (by piecewise linear continuation) a continuous map $|\Delta|_w \to |\Delta'|_w$ which we denote by the same symbol f.

A simplicial complex is called *pure* if every simplex is contained in a maximal simplex and if all maximal simplices have the same rank n. In this case we say that Δ has rank n and we call the simplices of rank n *chambers*. The set of all chambers is denoted $\mathrm{Cham}(\Delta)$. The *corank* of a simplex α is then defined as

$$\mathrm{cor}(\alpha) = n - \mathrm{rk}(\alpha)$$

(the corank coincides with the codimension of the simplex in the geometric realization). The residue of a corank 1 simplex is called a *panel*. Abusing notation slightly, we call the link of such a simplex also a panel. Given a simplex α and $k \geq \mathrm{cor}(\alpha)$, we denote by $\mathcal{E}_k(\Delta, \alpha)$ the union of the links of the corank k faces of α,

$$\mathcal{E}_k(\Delta, \alpha) = \bigcup \{\mathrm{lk}_\Delta(\beta) \mid \beta \subseteq \alpha \text{ and } \mathrm{cor}(\beta) = k\}.$$

A *gallery* in a pure simplicial complex is a sequence of chambers $(\gamma_0, \ldots, \gamma_r)$, where $\gamma_{i-1} \cap \gamma_i$ has corank at most 1. A gallery *stammers* if $\gamma_{i-1} = \gamma_i$ holds for some i. A pure simplicial complex where any two chambers can be connected by some gallery is called a *chamber complex*. A gallery $(\gamma_0, \ldots, \gamma_r)$ is called *minimal* if there is no gallery from γ_0 to γ_r with less than $r + 1$ chambers. If every panel contains at least 3 different chambers, the chamber complex is called *thick*.

1.2. Geometries. Suppose that Δ is a thick chamber complex of rank n with vertex set V and that I is a finite set of n elements. A *type function* is a map $t : V \to I$ whose restriction to every simplex is injective. We view the type function also as a regular simplicial map $t : \Delta \to 2^I$ and extend it to the geometric realizations, $t : |\Delta| \to |2^I|$. The latter map is, for obvious reasons, sometimes called the *accordion map*. We call (Δ, t) a *geometry* if Δ has the following two properties.

(1) Δ is a flag complex.
(2) The link of every nonmaximal simplex is a chamber complex.

We remark that what we call here a geometry is called a *thick residually connected geometry* in [Ti2]. The *type* (resp. *cotype*) of a simplex α is $t(\alpha)$ (resp. $I - t(\alpha)$). If α is a simplex of cotype J, then $\mathrm{lk}(\alpha)$ is a geometry over J. The simplicial join of two geometries is again a geometry. The *type of a nonstammering gallery* $(\gamma_0, \ldots, \gamma_r)$ is the sequence $(j_1, \ldots, j_r) \in I^r$, where j_k is the cotype of $\gamma_{k-1} \cap \gamma_k$. Automorphisms and homomorphisms of geometries are defined in the obvious way; they are regular simplicial maps which preserve types.

The idea behind this is that the vertices in a geometry are points, lines, planes and so on. The type function says what kind of geometric object a given vertex is and the simplices are the flags. The set of all simplices of a given type $J \subseteq I$ is the *flag variety* $V_J(\Delta)$. The chambers are thus the maximal flags, $V_I = \mathrm{Cham}(\Delta)$. A gallery shows how one maximal flag can be altered into another maximal flag by exchanging one vertex at a time. The type of the gallery records what types of exchanges occur.

1.3. Generalized *n*-gons. Let $n \geq 2$ be an integer. A geometry of rank 2 is a bipartite simplicial graph. It is called a *generalized n-gon* if it has girth $2n$ and diameter n, i.e., if it contains no circles of length less than $2n$ and if the combinatorial distance between two vertices is at most n.

A generalized digon is the same as a complete bipartite graph, i.e., the simplicial join of two vertex sets (of cardinalities at least 3, because of the thickness assumption). A generalized triangle is the same as an abstract projective plane; one type gives the points and the other the lines. The axioms above then say that any two distinct lines intersect in a unique point, and that any two distinct points lie on a unique line.

Lemma 1.4. *Let Δ be a simplicial flag complex with a type function $t \colon \Delta \to 2^I$. Suppose that the link of every vertex v is a thick chamber complex of rank* $\mathrm{card}(I) - 1$. *If $|\Delta|_w$ is connected as a topological space, then Δ is a geometry.*

Proof. The simplicial complex Δ is pure (since this a local condition). We have to show that it is gallery-connected. Since the 1-skeleton $\Delta^{(1)}$ is connected, it suffices to show that any two chambers that have a vertex v in common can be joined by a gallery. But this is true since $\mathrm{lk}(v)$ is a chamber complex. □

1.5. Geometries of type *M*. Suppose that $M : I \times I \to \mathbb{N}$ is a Coxeter matrix, i.e., $M_{i,j} = M_{j,i} \geq 2$ for all $i \neq j$, and $M_{i,i} = 1$ for all i. A geometry (Δ, t) is *of type M* if the link of every simplex α of corank 2 and cotype $\{i, j\}$ is a generalized $M_{i,j}$-gon.

We put $M_\alpha = M_{i,j}$ for short. The link of a simplex α of cotype J is a geometry of type M', where M' is the restriction of M to $J \times J$. The Coxeter group associated to M is

$$W = \langle I \mid (ij)^{M_{i,j}} = 1 \rangle,$$

see [Hum, 5.1]. The Coxeter group and diagram for M' will be called the Coxeter group and diagram of the simplex α. A gallery is called *reduced* if the word which is represented by its type in W is reduced in the sense of Coxeter groups, see

[Hum, 5.2]. Recall that a Coxeter group is called *spherical* if it is finite. We will be mainly concerned with geometries of spherical type.

For the irreducible spherical Coxeter groups we use the standard names A_k, C_k, D_k and so on as in [Hum]. By C_3 and H_3 we mean in particular the octahedral and the icosahedral group. The dihedral group of order $2n$ is denoted

$$\mathsf{I}_2(n) = \langle a, b \mid a^2 = b^2 = (ab)^n = 1 \rangle.$$

If Δ is a geometry of type M whose Coxeter diagram is not connected, then Δ is in a natural way a join of two geometries, see [Ti2, 6.1.3]. It therefore suffices in many cases to consider geometries with connected Coxeter diagrams. A geometry of type A_1 is a set without further structure. Therefore a geometry whose Coxeter diagram has an isolated node is a join of a set with a geometry. For this reason we will often exclude geometries whose Coxeter diagrams have isolated nodes.

Lemma 1.6. *Suppose that Δ is a geometry of type M. Then every minimal gallery is reduced. In particular there is a uniform upper bound on the length of minimal galleries if M is of spherical type.*

Proof. This is an easy consequence of the reduction process of words in Coxeter groups, see 3.4.1–3.4.4 in [Ti2]. $\square$

1.7. Homogeneous geometries. If a group G acts (by type preserving automorphisms) transitively on the chambers of a geometry Δ, we call the pair (G, Δ) a *homogeneous geometry.* We denote the stabilizer of a simplex α by G_α. If (G, Δ) is homogeneous, then $(G_\alpha, \mathrm{lk}_\Delta(\alpha))$ is also homogeneous. The following fact about the bounded generation of stabilizers will be important on several occasions.

Lemma 1.8. *Let (G, Δ) be a homogeneous geometry of type M. Let γ be a chamber and suppose that $\beta \subseteq \gamma$ is a face of corank at least 1 whose Coxeter group is of spherical type. Let $\alpha_1, \ldots, \alpha_t \subseteq \gamma$ be the faces of corank 1 which contain β. Let s be the length of the longest word in the Coxeter group of β. Then the st-fold multiplication map*

$$\left(G_{\alpha_1} \times \cdots \times G_{\alpha_t}\right)^s \to G$$

which sends a sequence of st group elements to their product has G_β as its image.

Proof. It is clear that the image of the multiplication map is contained in G_β, since each of the groups G_{α_k} is contained in G_β. Suppose that g is in G_β. Then there is a gallery $\gamma = \gamma_0, \gamma_1, \ldots, \gamma_{r-1}, \gamma_r = g(\gamma)$ in $\Delta_{\geq \beta}$, with $r \leq s$. We show by induction on r that g is in the image of the multiplication map. For $r = 0, 1$ we have $g \in G_{\alpha_1} \cup \cdots \cup G_{\alpha_t}$. For $r > 1$ we find $h \in G_{\alpha_1} \cup \cdots \cup G_{\alpha_t}$ with $h(\gamma_0) = \gamma_1$. Then $h^{-1}g(\gamma)$ can be connected to γ by a gallery of length $r - 1$ in $\Delta_{\geq \beta}$. By the induction hypothesis $h^{-1}g$ can be written as a product of $r - 1$ elements from $G_{\alpha_1} \cup \cdots \cup G_{\alpha_t}$ and the claim follows. $\square$

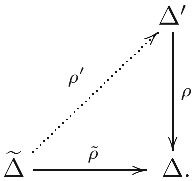
1.9. Simple complexes of groups. A *simple complex of groups* $\mathcal{G}$ is a cofunctor from a poset to the category of group monomorphisms, see [BrHa, II.12.11]. In other words, it assigns in a functorial way to every poset element α a group G_α and to every inequality $\beta \leq \alpha$ a group monomorphism $G_\beta \leftarrow G_\alpha$, such that all resulting triangles of maps commute. If H is a group, then a simple homomorphism $\varphi : \mathcal{G} \to H$ consists of a collection of homomorphisms $\varphi_\alpha : G_\alpha \to H$ such that all resulting triangles commute.

Let γ be a chamber in a homogeneous geometry (G, Δ). The stabilizers G_α of the nonempty simplices $\varnothing \neq \alpha \subseteq \gamma$ form in a natural way a simple complex of groups $\mathcal{G}$, with a simple homomorphism $\mathcal{G} \to G$. The pair (G, Δ) is completely determined by this datum $\mathcal{G} \to G$. We will see that in certain situations (G, Δ) is already determined by $\mathcal{G}$. This is for example true if $|\Delta|_w$ is simply connected as a topological space, since then $G = \varinjlim \mathcal{G}$, see [Ti3] or [BrHa, II.12.18]. However, this condition is not so easy to check in our setting of compact Lie groups and we will replace the 'abstract' colimit $\varinjlim \mathcal{G}$ in 2.27 by a compact group $\widehat{G}$ which serves essentially the same purpose in the category of compact groups. We remark that an analogous construction works for finite geometries and groups.

Finally, we need Tits' notion of a k-covering of geometries [Ti2].

1.10. *k*-Coverings. Let Δ and Δ' be chamber complexes of rank n and let $\rho : \Delta \to \Delta'$ be a surjective regular simplicial map. We call ρ a *k-covering* if for every simplex $\alpha \in \Delta$ of corank at most k, the induced map $\mathrm{lk}_\Delta(\alpha) \to \mathrm{lk}_{\Delta'}(\rho(\alpha))$ is an isomorphism. If ρ is an $(n-1)$-covering, then $|\Delta|_w \to |\Delta'|_w$ is a covering in the topological sense. We call an $(n-1)$-covering a *covering* for short. As Tits remarks, one should view k-coverings as 'branched coverings'. We note the following: if $\rho : \Delta \to \Delta'$ is a covering and if Δ' is a flag complex, then Δ is also a flag complex (since this is a local condition). For k-coverings between geometries we always assume that they preserve types.

1.11. Universal *k*-coverings. A k-covering $\tilde\rho : \widetilde\Delta \to \Delta$ is called *universal* if it has the following property: for every k-covering $\rho : \Delta' \to \Delta$ and every pair of chambers $\gamma' \in \Delta'$ and $\tilde\gamma \in \widetilde\Delta$ with $\tilde\rho(\tilde\gamma) = \rho(\gamma')$, there is a unique k-covering $\rho' : \widetilde\Delta \to \Delta'$ with $\rho'(\tilde\gamma) = \gamma'$ and $\tilde\rho = \rho \circ \rho'$.



Applying this universal property twice, we have the following.

Lemma 1.12. *Let $\rho : \widetilde\Delta \to \Delta$ be a universal k-covering of geometries of type M, for $k \geq 2$. Suppose g is an automorphism of Δ. Given any two chambers $\gamma_1, \gamma_2 \in \widetilde\Delta$ with $g(\rho(\gamma_1)) = \rho(\gamma_2)$, there exists a unique automorphism $\tilde g$ of $\widetilde\Delta$ with $\rho \circ \tilde g = g \circ \rho$ and $\tilde g(\gamma_1) = \gamma_2$.* $\square$

We call the lifts of the identity *deck transformations.* The following is an immediate consequence of the previous lemma.

Proposition 1.13. *Let $\rho : \widetilde\Delta \to \Delta$ be a universal k-covering of geometries of type M, for $k \geq 2$. Suppose that $H \subseteq \mathrm{Aut}(\Delta)$ acts transitively on the chambers of*

Δ. Let $\widetilde{H} \subseteq \mathrm{Aut}(\widetilde{\Delta})$ denote the collection of all lifts of the elements of H and let $F \subseteq \widetilde{H}$ denote the collection of all deck transformations. Then we have the following.

1. *$\widetilde{H}$ is a group acting transitively on the chambers of $\widetilde{\Delta}$ and $F \subseteq \widetilde{H}$ is a normal subgroup.*
2. *The map ρ is equivariant with respect to the map $\widetilde{H} \to H \cong \widetilde{H}/F$.*
3. *If $\alpha \in \widetilde{\Delta}$ is a simplex of corank at most k, then $\widetilde{H}_\alpha \cap F = \{\mathrm{id}\}$ and $\widetilde{H}_\alpha$ maps isomorphically onto $H_{\rho(\alpha)}$.*
4. *For a simplex α of corank at most k in $\tilde{\Delta}$, the $\widetilde{H}$-stabilizer of $\rho(\alpha)$ splits as a semidirect product $\widetilde{H}_{\rho(\alpha)} = \widetilde{H}_\alpha F$.*

Proof. From 1.12 we see that products and inverses of lifts are again lifts. Thus $\widetilde{H}$ is a group. The natural map $\widetilde{H} \to H$ which assigns to a lift $\tilde{g}$ the automorphism g which was lifted is an epimorphism with kernel F. Therefore we have (1) and (2).

Suppose that $\alpha \in \widetilde{\Delta}$ has corank at most k and that $\gamma \supseteq \alpha$ is a chamber. A deck transformation which sends γ to a chamber in $lk(\alpha)$ must fix γ and is therefore the identity. Thus $F \cap \widetilde{H}_\alpha = \{\mathrm{id}\}$. If $g \in H$ fixes $\rho(\alpha)$, then we find a unique chamber $\gamma' \in \widetilde{\Delta}_{\geq\alpha}$ with $\rho(\gamma') = g(\rho(\gamma))$ and hence a lift $\tilde{g} \in \widetilde{H}_\alpha$ of g. This shows that $\widetilde{H}_\alpha \to H_{\rho(\alpha)}$ is surjective, and therefore an isomorphism. Thus we have (3).

For (4) we note that $\widetilde{H}_\alpha F$ fixes $\rho(\alpha)$. Conversely, suppose that $\tilde{g}$ in $\widetilde{H}$ fixes $\rho(\alpha)$ and that γ is a chamber containing α. Then $\rho(\tilde{g}(\gamma))$ contains $\rho(\alpha)$. There exists an element $f \in F$ such that $f(\tilde{g}(\gamma)) \in \widetilde{\Delta}_\alpha$, because F acts transitively on the preimage of $g(\rho(\gamma))$. This proves (4). □

Remark 1.14. The existence of a universal 2-covering of a geometry seems in general to be an open problem. The existence of a universal $(n-1)$-covering is not an issue; see also Pasini [Pas, Chap. 12]. For $n \geq 2$, the topological universal covering $\widetilde{|\Delta|_w} \to |\Delta|_w$ is the universal $(n-1)$-covering, as one sees from 1.4. We remark also that an analog of the construction that we give in 2.27 below gives universal homogeneous geometries in the class of finite homogeneous geometries of spherical type. In any case, we have the following important fact.

Theorem 1.15 (Tits). *Suppose that $\rho : \widetilde{\Delta} \to \Delta$ is a 2-covering of geometries. If $\widetilde{\Delta}$ is a building, then ρ is universal.*

Proof. This follows from Theorem 3 and 2.2 in [Ti2]. □

We close this section with the following deep result due to Tits. It says that the only obstruction to the existence of a 2-covering by a building lies in the rank 3 links. We remark that (thick) buildings of type H_3 and H_4 do not exist, see [Ti1, Addenda]. (Tits' result applies also to non-thick geometries.)

Theorem 1.16 (Tits). *Let Δ be a geometry of type M. Then the following are equivalent.*

1. *There exists a building $\widetilde{\Delta}$ and a 2-covering $\widetilde{\Delta} \to \Delta$.*
2. *For every simplex $\alpha \in \Delta$ of corank 3 whose Coxeter diagram is of type C_3 or H_3, there exists a building Γ and a 2-covering $\Gamma \to \mathrm{lk}_\Delta(\alpha)$.*

Proof. Our assumptions allow us to go back and forth between chamber systems and geometries. The result follows thus from 5.3 in [Ti2]. □

1.17. The following facts illustrate two interesting cases:

(a) Every geometry Δ of type A_n is a projective geometry and in particular a building, see [Ti2, 6.1.5]. Therefore $\mathrm{id} : \Delta \to \Delta$ is the universal 2-covering (and Δ admits no quotients).

(b) Suppose that Δ is a geometry of type C_3 and that we call the three types of vertices points, lines and hyperlines as in [Ti2, p. 542]. Then Δ is a building if and only if any two lines which have at least two distinct points in common are equal, i.e., if there are no digons, see [Ti2, 6.2.3].

1.18. The Neumaier Geometry. We briefly explain the one known finite geometry of type C_3 which is not covered by a building. Let V_1 be a set consisting of seven *points* and let $V_2 = \binom{V_1}{3}$ denote the set of all 3-element subsets of V_1. These are the *lines* of the geometry. There are 30 ways of making V_1 into a projective plane by choosing 7 appropriate lines in V_2; let $X \subseteq \binom{V_2}{7}$ be this set. Finally, let $G = \mathrm{Alt}(V_1) = \mathrm{Alt}(7)$ and let $V_3 \subseteq X$ be one of the two 15-element G-orbits in X. The elements of this orbit are the *planes* of the geometry. Put $V = V_1 \cup V_2 \cup V_3$ and define two vertices $v, w \in V$ to be adjacent if $v \in w$ or $w \in v$. Let Δ denote the corresponding flag complex. Then Δ is a geometry of type C_3. We note that points (vertices of type 1) and planes (vertices of type 3) are always incident. See Neumaier [Neu] and Pasini [Pas, 6.4.2] for more details.

2. Compact geometries

Now we consider actions of compact Lie groups on geometries. This leads to a different topology on $|\Delta|$. The next definition is very much in the spirit of Burns–Spatzier [BuSp]; see also [GKVW, 6.1]. Suppose that Δ is a geometry over I. Given a simplex α of type $J \subseteq I$, let $\alpha(j)$ denote its unique vertex of type $j \in J$. In this way we can view α as a map $J \to V$ or as a J-tuple of vertices, $\alpha \in V^J$.

Definition 2.1. Let Δ be a geometry of type M over I. Suppose that the vertex set V of Δ carries a compact Hausdorff topology and that for every $J \subseteq I$, the flag variety V_J (viewed as a subset of the compact space V^J) is closed. Then we call Δ a *compact geometry.* The proof of [GKVW, 6.6] applies verbatim and shows that for every simplex $\alpha \in \Delta$, the link $\mathrm{lk}(\alpha)$ is again a compact geometry. We say that Δ has *connected panels* if the panels are connected in this topology.

Examples of compact geometries arise as follows from groups. Suppose that (G, Δ) is a homogeneous geometry of type M and that G is a locally compact group. If every simplex stabilizer G_α is closed and cocompact (i.e., G/G_α is compact), then V carries a compact topology and the flag varieties are also compact, hence closed. We then call (G, Δ) a *homogeneous compact geometry.* The spherical buildings associated to semisimple or reductive isotropic algebraic groups over local fields are particular examples of homogeneous compact geometries.

The topology on V can be used to define a new topology on $|\Delta|$ as follows. Consider the map

$$p : \mathrm{Cham}(\Delta) \times |2^I| \to |\Delta|$$
$$\Big(\gamma, \textstyle\sum_{i\in I} i \cdot \xi(i)\Big) \mapsto \textstyle\sum_{i\in I} \gamma(i) \cdot \xi(i)$$

Both $\mathrm{Cham}(\Delta)$ and $|2^I|_w$ are compact and we endow $|\Delta|$ with the quotient topology with respect to the map p. The resulting compact space is denoted $|\Delta|_K$. The identity map $|\Delta|_w \to |\Delta|_K$ is clearly continuous, and we call the topology of $|\Delta|_K$ the *coarse topology* on $|\Delta|$.

Lemma 2.2. *The space $|\Delta|_K$ is compact Hausdorff. If Δ has rank at least 2, then $|\Delta|_K$ is path-connected.*

Proof. From the continuity of the natural maps

$$\mathrm{Cham}(\Delta) \times |2^I|_w \to |\Delta|_K \xrightarrow{t} |2^I|_w$$

we see that we can separate points which have different t-images.

Suppose now that $x, y \in |\Delta|_K$ have the same type $\xi = t(x) = t(y) \in |2^I|$. We let $J = \mathrm{supp}(\xi) = \{j \in I \mid \xi(j) > 0\}$ denote the support of ξ and we put

$$u(\xi) = \{\zeta \in |2^I| \mid \mathrm{supp}(\zeta) \supseteq \mathrm{supp}(\xi)\}.$$

Then $u(\xi)$ is an open neighborhood of ξ. Let $U \subseteq V_J$ be open and let $U_C \subseteq \mathrm{Cham}(\Delta)$ denote the open set of all chambers whose face of type J is in U. We claim that $U_C \times u(\xi)$ is p-saturated. Indeed, if $(\gamma, \zeta) \in U_C \times u(\xi)$ and if

$$p(\gamma, \zeta) = \textstyle\sum \gamma(i) \cdot \zeta(i) = \textstyle\sum \gamma'(i) \cdot \zeta'(i) = p(\gamma', \zeta'),$$

then $\zeta = \zeta'$ and $t(\gamma \cap \gamma') \supseteq J$. It follows that the p-image of $U_C \times u(\xi)$ is open.

Now for x, y as above, we choose disjoint open neighborhoods $X, Y \subseteq V_J$ of the type J simplices containing them. Then the p-images of $X_C \times u(\xi)$ and $Y_C \times u(\xi)$ are disjoint open neighborhoods.

Finally, we note that $|\Delta|_w$ is path-connected if Δ has rank at least 2, so the same is true for $|\Delta|_K$. □

Lemma 2.3. *Let Δ be a compact geometry with connected panels. Then all flag varieties V_J are connected (in the coarse topology).*

Proof. We show first that $\mathrm{Cham}(\Delta)$ is connected. If $(\gamma_0, \ldots, \gamma_r)$ is a gallery, then γ_{k-1}, γ_k are in a common panel and hence in a connected subset. Since Δ is gallery-connected, $\mathrm{Cham}(\Delta)$ is connected. For $J \subseteq I$ we have a continuous surjective map $\mathrm{Cham}(\Delta) \to V_J$, hence V_J is also connected. □

Lemma 2.4. *Let Δ be a geometry of type M over I. Suppose that $\varnothing \subsetneq J \subsetneq I$ and that $M_{j,k} = 2$ holds for all $j \in J$ and $k \in K = I - J$. Then Δ is a join of two geometries Δ_1, Δ_2 of types $M|_{J\times J}$ and $M|_{K\times K}$. If Δ is a compact geometry, then this decomposition is compatible with the topology.*

Proof. The proof in [GKVW, 6.7] applies verbatim. □

A compact homogeneous geometry of type A_1 is just a compact space with a transitive group action. Therefore we will often assume that the Coxeter diagram of the geometry has no isolated nodes. A compact homogeneous geometry (K, Δ) of type $\mathsf{A}_1 \times \mathsf{A}_1$ consists of two compact spaces X, Y and a transitive K-action on $X \times Y$ which is equivariant with respect to the maps $X \xleftarrow{\mathrm{pr}_1} X \times Y \xrightarrow{\mathrm{pr}_2} Y$. Suppose that $X = \mathbb{S}^m$, that $Y = \mathbb{S}^n$ and that K is a compact connected Lie group acting faithfully and transitively on $\mathbb{S}^m \leftarrow \mathbb{S}^m \times \mathbb{S}^n \rightarrow \mathbb{S}^n$. We note that such a group K embeds into $\mathrm{SO}(m+1) \times \mathrm{SO}(n)$, see [CPP, 96.20]. In this case, a classification is possible. The result that we need is as follows.

Lemma 2.5. *Let $K \subseteq \mathrm{SO}(m+1) \times \mathrm{SO}(n+1)$ be a compact connected group acting transitively on $\mathbb{S}^m \times \mathbb{S}^n$. Let K_1 and K_2 denote the projections of K to $\mathrm{SO}(m+1)$ and $\mathrm{SO}(n+1)$, respectively. Assume that $m = 1, 2, 4, 8$ and that $K_1 = \mathrm{SO}(m+1)$. If $m = 1$ assume in addition that $K_2 = \mathrm{SO}(n+1)$ or that K_2 is a compact simple Lie group. Then $K = K_1 \times K_2$, unless $m = 2$, $n = 4k - 1$ and $K = \mathrm{Sp}(1) \cdot \mathrm{Sp}(k)$ acting on $\mathrm{Pu}(\mathbb{H}) \oplus \mathbb{H}^k$ via $(a, g) \cdot (u, v) = (au\bar{a}, gv\bar{a})$.*

Proof. We decompose the Lie algebra $\mathrm{Lie}(K)$ into the ideals $\mathfrak{h}_1 = \mathrm{Lie}(K) \cap (\mathfrak{so}(m+1) \oplus 0)$, $\mathfrak{h}_2 = \mathrm{Lie}(K) \cap (0 \oplus \mathfrak{so}(n+1))$ and a supplement $\mathfrak{h}_0$, such that $\mathrm{Lie}(K) = \mathfrak{h}_1 \oplus \mathfrak{h}_2 \oplus \mathfrak{h}_0$ and $K = (H_1 \times H_2) \cdot H_0$, where H_i is the closed connected normal subgroup with Lie algebra $\mathfrak{h}_i$. Since $\mathrm{Lie}(K)/\mathfrak{h}_2 \cong \mathfrak{h}_1 \oplus \mathfrak{h}_0 \cong \mathfrak{so}(m+1)$ is either 1-dimensional or simple, we have necessarily $\mathfrak{h}_1 = 0$ or $\mathfrak{h}_0 = 0$. We consider these two cases separately.

(a) Assume that $\mathfrak{h}_0 = 0$. Then we have a product decomposition of the Lie algebra and therefore $K = H_1 \times H_2 = K_1 \times K_2$.

(b) Assume that $\mathfrak{h}_1 = 0$. Then we have $\mathfrak{h}_0 \cong \mathfrak{so}(m+1)$ and thus the Lie algebra of the group induced on the second factor $\mathbb{S}^n$ is $\mathfrak{so}(m+1) \oplus \mathfrak{h}_2$. We compare this with the classification of transitive actions of compact connected Lie groups on spheres, see [Oni, p. 227] or [CPP, 96.20–23] or [Kr5, 6.1]. We note also that the K-stabilizer of a nonzero vector $v \oplus 0 \in \mathbb{R}^{m+1} \oplus \mathbb{R}^{n+1}$ acts transitively on $\mathbb{S}^n$.

If $m = 8$ then $n = 8, 15$ and $\mathfrak{h}_2 = 0$. However, a group with Lie algebra $\mathfrak{so}(8)$ cannot act transitively on $\mathbb{S}^8$ or $\mathbb{S}^{15}$, so this case cannot occur. If $m = 4$, then $n = 4, 7$ and $\mathfrak{h}_2 \subseteq \mathfrak{so}(3)$. Again, a group with Lie algebra $\mathfrak{so}(4) \oplus \mathfrak{h}_2$ cannot act transitively on $\mathbb{S}^4$ or $\mathbb{S}^7$. If $m = 2$, then H_0 cannot act transitively on $\mathbb{S}^m \times \mathbb{S}^n$ by a similar argument as in the case $m = 8$. Hence H_0 is not transitive on $\mathbb{S}^n$ and the only remaining possibility is that $n = 4k - 1$ and $\mathfrak{h}_2 = \mathfrak{sp}(k)$. The case $m = 1$, with $\mathrm{Lie}(K) = \mathbb{R} \oplus \mathfrak{h}_2$ is excluded by our assumptions. □

A general classification of transitive action on products of spheres can be found in Onishchik [Oni, p. 274], Straume [Str, Table II]. The transitive $\mathrm{SO}(4)$-action on $\mathbb{S}^2 \times \mathbb{S}^3 \subseteq \mathrm{Pu}(\mathbb{H}) \oplus \mathbb{H}$ will play a role for the exceptional C_3 geometry.

The following result is a main ingredient in our classification of homogeneous compact geometries. A spherical building is *Moufang* if it has a 'large' automorphism group; see [Ti1, p. 274]. The buildings associated to reductive isotropic algebraic groups have this property and conversely, the spherical Moufang buildings can be classified in terms of certain algebraic data [TW]. A deep result due to Tits says that all irreducible spherical buildings of rank at least 3 are Moufang, see [Ti1, 4.1.2] [We]. Spherical buildings of rank 2 need not be Moufang.

Theorem 2.6. *Let (G, Δ) be a homogeneous compact geometry of type M with connected panels. Suppose that α is a simplex of corank 2 and cotype $\{i, j\}$, and with $M_{i,j} = M_\alpha \geq 3$. Then $M_\alpha \in \{3, 4, 6\}$ and $\mathrm{lk}(\alpha)$ is a compact connected Moufang M_α-gon (and explicitly known).*

The panels of cotype i and j are homeomorphic to spheres (in the coarse topology), of dimensions $m_i, m_j \geq 1$. If G is compact, then the panel stabilizers act linearly (i.e., as subgroups of orthogonal groups) on these panels. The space $|\mathrm{lk}(\alpha)|_K$ is homeomorphic to a sphere of dimension $M_{i,j}(m_i + m_j) + 1$.

If $M_{i,j} = 3$, then $m_i = m_j = 1, 2, 4, 8$, if $M_{i,j} = 6$, then $m_i = m_j = 1, 2$ and if $M_{i,j} = 4$, then either $1 \in \{m_i, m_j\}$ or $m_i = m_j = 2$ or $m_i + m_j$ is odd (with further number-theoretic restrictions).

In particular, there are no homogeneous compact geometries of type H_3 with connected panels.

Proof. The link $\mathrm{lk}(\alpha)$ is a homogeneous compact generalized M_α-gon with connected panels. By the main results in [Szm, Kn, GKK1, GKK2] we have $M_\alpha \in \{3, 4, 6\}$ and $\mathrm{lk}(\alpha)$ is the compact connected Moufang M_α-gon associated to a simple real Lie group S of $\mathbb{R}$-rank 2. Moreover, $|\mathrm{lk}(\alpha)|_K$ is G-equivariantly homeomorphic to unit sphere in the tangent space of the symmetric space S/G, where $G \subseteq S$ is a maximal compact subgroup. The principal G-orbits have dimension $M_\alpha(m_1 + m_2)$ and codimension 1 in this sphere. A different, purely topological proof that $|\mathrm{lk}(\alpha)|_K$ is a sphere of this dimension is given in [Kn]. A complete classification of these compact geometries and their chamber-transitive closed connected subgroups is given in [GKK2]. □

Corollary 2.7. *Let (G, Δ) be a homogeneous compact geometry of type M with connected panels. Suppose also that G is compact. If the Coxeter diagram of M has no isolated nodes and if all panels have in (the coarse topology) dimension at least 2, then the commutator group $[G, G]$ of G acts transitively on the chambers of Δ.*

Proof. Let $\alpha \in \Delta$ be a simplex of corank 1 and let m denote the dimension of the sphere $|\mathrm{lk}(\alpha)|_K \cong \mathbb{S}^m$. The stabilizer G_α induces a transitive subgroup of $\mathrm{O}(m+1)$ on this panel. Since $m \geq 2$, the commutator group of G_α still acts transitively on $\mathrm{lk}(\alpha)$, see [Oni, p. 94]. In particular, $([G, G])_\alpha$ acts transitively on $\mathrm{lk}(\alpha)$. Since any two chambers can be connected by some gallery, $[G, G]$ acts transitively on the chambers. □

Lemma 2.8. *Let (G, Δ) be a homogeneous compact geometry of type M with connected panels. If G is compact and acts faithfully on Δ, then G_γ acts faithfully on $\mathcal{E}_1(\Delta, \gamma)$, for every chamber γ.*

Proof. Suppose that $g \in G_\gamma$ fixes $\mathcal{E}_1(\Delta, \gamma)$ pointwise. Let $\alpha \subseteq \gamma$ be a face of corank 2. If $M_\alpha \geq 3$, then g fixes $\mathrm{lk}(\alpha)$ pointwise by [GKK2, 2.2]. If $M_\alpha = 2$, then $\mathrm{lk}(\alpha)$ is a join of two panels and therefore g fixes $\mathrm{lk}(\alpha)$ pointwise. Thus g fixes $\mathcal{E}_2(\Delta, \gamma)$ pointwise. If (γ, γ') is a gallery, then $\mathcal{E}_1(\Delta, \gamma') \subseteq \mathcal{E}_2(\Delta, \gamma)$, hence g fixes $\mathcal{E}_1(\Delta, \gamma')$

pointwise. Since Δ is gallery-connected, we conclude that g acts trivially on Δ. $\square$

In order to show that compact groups acting transitively on compact geometries are automatically Lie groups, we use the following fact.

Lemma 2.9. *Let K be a compact group and $H \trianglelefteq K$ a closed normal subgroup. If H and K/H are Lie groups, then K is a Lie group as well.*

Proof. We show that K has no small subgroups, see [HoMo, 2.40]. Let $W \subseteq K/H$ be a neighborhood of the identity which contains no nontrivial subgroup and let V be its preimage in K. Let $U \subseteq K$ be a neighborhood of the identity such that $U \cap H$ does not contain a nontrivial subgroup of H. Then $U \cap V$ contains no nontrivial subgroup of K. $\square$

Theorem 2.10. *Let (G, Δ) be a homogeneous compact geometry of spherical type M, with connected panels. Assume that G is compact and acts effectively, and that the Coxeter diagram of Δ has no isolated nodes. Then G is a compact Lie group.*

Proof. We first show that certain simplex stabilizers are compact Lie groups. Let γ be a chamber. By 2.8, G_γ acts faithfully on $\mathcal{E}_1(\Delta, \gamma)$. From 2.6 we see that G_γ injects into a finite product of orthogonal groups. Thus G_γ is a Lie group. Now let $\alpha \subseteq \gamma$ be a face of corank 1. Let $N \trianglelefteq G_\alpha$ denote the kernel of the action of G_α on the panel $\mathrm{lk}(\alpha)$. Then N is a closed subgroup of G_γ and hence a Lie group. The quotient G_α/N is by 2.6 a closed subgroup of an orthogonal group and therefore also a Lie group. By 2.9, G_α is a Lie group.

Let now $\alpha_1, \dots, \alpha_t$ denote the corank 1 faces of γ. Let s be the length of the longest word in the Coxeter group of Δ. Recall from 1.8 that we have a surjective multiplication map $\big(G_{\alpha_1} \times \cdots \times G_{\alpha_t}\big)^s \to G$. If we compose it with the projection $G \to G/\overline{[G,G]}$, it becomes a surjective continuous homomorphism, since the target group is abelian. Thus $G/\overline{[G,G]}$ is a compact abelian Lie group. From the multiplication map we see also that G has only finitely many path components, and that the path components of G are closed, and therefore open. In particular, G° is an open and path-connected subgroup. Since $\mathrm{Cham}(\Delta)$ is connected by 2.3, the identity component G° acts transitively and (G°, Δ) is a homogeneous compact geometry. It now suffices to show that G° is a compact Lie group, and for this we may as well assume that $G = G^\circ$ is connected.

Then G is a central quotient $\big(Z \times \prod_{\nu \in N} S_\nu\big)/D$, where $(S_\nu)_{\nu \in N}$ is a (possibly infinite) family of simply connected compact almost simple Lie groups, Z is a compact connected abelian group, and D is a compact totally disconnected central subgroup of the product $Z \times \prod_{\nu \in N} S_\nu$, see [HoMo, 9.24]. We claim that the index set N of the product is finite. Otherwise, G admits a homomorphism onto a semisimple Lie group H of dimension strictly bigger than $r = s\dim(G_{\alpha_1} \times \cdots \times G_{\alpha_t})$. The composite $\big(G_{\alpha_1} \times \cdots \times G_{\alpha_t}\big)^s \to G \to H$ is a smooth map between Lie groups. Therefore its image has (by Sard's Theorem, see, e.g., [Mil, Chap. 3]) dimension at most r, a contradiction. So the index set N is finite, and $[G,G] = \overline{[G,G]}$ is a compact semisimple Lie group. By 2.9, the group G is a compact Lie group. $\square$

The following byproduct of the proof will be useful later.

Corollary 2.11. *Under the assumptions of 2.10, the identity component G° acts transitively on* $\mathrm{Cham}(\Delta)$ *and* (G°, Δ) *is a homogeneous compact geometry.* □

We do not know the answer to the following problem (see Problem 2 and Problem 3 in the introduction). For compact connected buildings, it is in both cases affirmative.

Problem 2.12. *Suppose that M has no isolated nodes. Is the automorphism group of a compact geometry of type M locally compact in the compact-open topology? If the geometry is homogeneous, does there necessarily exist a compact chamber-transitive group?*

A first application of 2.10 is that there is an upper bound for the topological dimension of the chamber set.

Corollary 2.13. *Let (G, Δ) be a homogeneous compact geometry of spherical type M, with connected panels. Assume that G is compact and acts effectively, and that the Coxeter diagram of Δ has no isolated nodes. Each panel of cotype i is by* 2.6 *a sphere, of dimension m_i. Let $(i_1, \ldots, i_r)$ be a representation of the longest word in the Coxeter group W of M and let γ be a chamber. Then*

$$\dim(G) - \dim(G_\gamma) \leq m_{i_1} + m_{i_2} + \cdots + m_{i_r}.$$

Proof. Let $\alpha \subseteq \gamma$ be a face of corank 1 and cotype i. The canonical map

$$\mathrm{Cham}(\Delta) \cong G/G_\gamma \to G/G_\alpha \cong V_{I-\{i\}}$$

is a locally trivial $G_\alpha/G_\gamma = \mathbb{S}^{m_i}$-bundle. We fix a chamber $\gamma = \gamma_0$ and a sequence $(i_1, \ldots, i_r) \subseteq I^r$. Pulling these sphere bundles back several times, we see that the space of stammering galleries (the 'Bott–Samelson cycles')

$$\{(\gamma_0, \ldots, \gamma_r) \in \mathrm{Cham}(\Delta)^{r+1} \mid I - \{i_k\} \subseteq t(\gamma_{k-1} \cap \gamma_k),\ k = 1, \ldots r\}$$

is a smooth manifold of dimension $m_{i_1} + \cdots + m_{i_r}$, see also [Kr4, 7.9]. The map sending such a stammering gallery $(\gamma_0, \ldots \gamma_r)$ to γ_r is smooth, hence its image has (by Sard's Theorem) dimension at most $m_{i_1} + \cdots + m_{i_r}$. By 1.6, every chamber can be reached from γ_0 by a gallery whose type is reduced. Since there are only finitely many reduced words in a spherical Coxeter group we obtain an upper bound for the dimension. From the Bruhat order on the Coxeter group we see that this upper bound is of the form that we claim, (and does not depend on the chosen representation of the longest word), see [Hum, 5.10] and [Kr4, 7.9]. □

If Δ is a homogeneous compact building with connected panels and if the Coxeter diagram is spherical and has no isolated nodes, then $|\Delta|_K$ is homeomorphic to a sphere. This is the 'Topological Solomon–Tits-Theorem', which has been proved in various degrees of generality, see [Mit], [Kn], [Kr1]. For a homogeneous compact geometry, $|\Delta|_K$ need not be a manifold. However, we have the following result for geometries of rank 3.

Proposition 2.14. *Suppose that (G, Δ) is a homogeneous compact geometry of irreducible spherical type M with connected panels and that G is compact. If Δ has rank* 3*, then $|\Delta|_K$ is a closed connected topological manifold of dimension* $\dim(G/G_\gamma) + 2$.

Proof. Replacing G by G/N, where N is the kernel of the action, we may by 2.10 assume that G is a compact Lie group acting faithfully and transitively on the chambers. We put $I = \{1, 2, 3\}$ and we fix a chamber $\gamma \in \Delta$. Recall from 2.1 the closed quotient maps

$$\begin{gathered}\mathrm{Cham}(\Delta) \times |2^I|_w \xrightarrow{p} |\Delta|_K \xrightarrow{t} |2^I|_w,\\ (\gamma', \zeta) \mapsto \sum \gamma'(i) \cdot \zeta(i) \mapsto \zeta.\end{gathered}$$

We use the same notation as in the proof of 2.2. Suppose that $x \in |\Delta|_K$ has type $t(x) = \xi \in |2^I|$. Let $J = \mathrm{supp}(\xi) = \{j \in I \mid \xi(j) \neq 0\}$ and $u(\xi) = \{\zeta \in |2^I| \mid \mathrm{supp}(\zeta) \supseteq \mathrm{supp}(\xi)\}$. Let $W = \mathrm{Cham}(\Delta) \times u(\xi)$. This set is p-saturated, hence its p-image is open (compare the proof of 2.2) and a neighborhood of x. We claim that $p(W)$ is a tube around the orbit $G(x) \subseteq |\Delta|_K$, see Bredon [Bred, II.4]. Let $\alpha \subseteq \gamma$ denote the face of type J. We put

$$r\big(\sum \gamma'(i) \cdot \zeta(i)\big) = \sum \gamma'(i) \cdot \xi(i).$$

From the commutative diagram

$$\begin{array}{ccc} W & \longrightarrow & G/G_\gamma \\ {\scriptstyle p}\downarrow & & \downarrow \\ p(W) & \xrightarrow{r} & G(x) \end{array}$$

we see that r is continuous, since the preimage in W of an open set in $G(x) \cong G/G_\alpha$ is p-saturated (by the same arguments as in the proof of 2.2). Thus r is an equivariant retraction. From the chamber-transitivity of G_α on $\mathrm{lk}(\alpha)$ we have that $G_\alpha(p(\{\gamma\} \times u(\xi)) = r^{-1}(x)$. By [Bred, II.4.2], the set $p(W)$ is a tube with slice $S = r^{-1}(x)$ and

$$p(W) \cong G \times_{G_\alpha} S.$$

By construction, the slice S is G_α-equivariantly homeomorphic to a product of $\mathbb{R}^{\mathrm{card}(J)-1}$ and the open cone over $|\mathrm{lk}(\alpha)|_K$. If $I = J$, then the slice S is thus an open 2-disk. If $J = \{1, 2\}$, then S is an open $m + 2$-disk, since $|\mathrm{lk}(\alpha)|_K$ is an m-sphere. If $J = \{1\}$, then $|\mathrm{lk}(\alpha)|_K$ is G_α-equivariantly homeomorphic to unit sphere in a polar representation of G_α of cohomogeneity 2 (here we use the classification 2.6). Thus $p(W)$ is in each case equivariantly homeomorphic to an open disk bundle over G/G_α and therefore a manifold. □

The previous proof works also for irreducible spherical types of higher rank if we assume that all proper links arise from polar representations.

We collect a few more elementary facts about homogeneous compact geometries.

Lemma 2.15. *Suppose that $\rho : (G', \Delta') \to (G, \Delta)$ is a continuous equivariant k-covering between homogeneous compact geometries of type M with connected panels, for $k \geq 2$, and that G' is compact and acts faithfully on Δ'. If $\alpha \in \Delta'$ is a simplex of corank at most k, then $G'_\alpha \to G_{\rho(\alpha)}$ is injective.*

Proof. Let γ be a chamber containing α. Then $\mathcal{E}_1(\Delta', \gamma) \xrightarrow{\rho} \mathcal{E}_1(\Delta, \rho(\gamma))$ is a G'_γ-equivariant bijection. By 2.8, the group G'_γ acts faithfully on $\mathcal{E}_1(\Delta', \gamma)$, hence $G'_\gamma \to G_{\rho(\gamma)}$ is injective. By assumption, ρ maps $\mathrm{lk}_{\Delta'}(\alpha)$ bijectively onto $\mathrm{lk}_\Delta(\rho(\alpha))$. So if $g \in G'_\alpha$ is in the kernel of $G'_\alpha \to G_{\rho(\alpha)}$, then $g \in G'_\gamma$, and therefore $g = 1$. □

Definition 2.16. We call a homogeneous compact geometry (G, Δ) *minimal* if G has no closed normal chamber-transitive subgroup $N \subsetneq G$.

Such minimal actions are called *irreducible* in Onishchik [Oni], but this terminology would conflict with our terminology for geometries. Since compact Lie groups satisfy the descending chain condition, we have the following fact.

Lemma 2.17. *Suppose that the Coxeter diagram of M is spherical and has no isolated nodes and that (G, Δ) is a homogeneous compact geometry of type M with connected panels. If G is compact and acts faithfully, then there exists a closed connected normal subgroup $K \trianglelefteq G^\circ$ such that (K, Δ) is minimal.*

Proof. The group G° acts transitively on the chambers by 2.11. Among all closed normal connected chamber transitive subgroups of G°, let $K \subseteq G^\circ$ be a smallest one. Every closed connected normal subgroup of K is also normal in G°, hence (K, Δ) is minimal. □

Under the assumptions of 2.17, the group K is necessarily connected (by 2.11) and if all panels have dimension at least 2, then K is semisimple (by 2.7).

2.18. In the setting of 2.17, the group G° can be recovered from K as follows. Let α be a simplex and put $N = \mathrm{Nor}_K(K_\alpha)$. Then $H = N/K_\alpha$ acts from the right on K/K_α. It is not difficult to see that in this action, H is isomorphic to the centralizer of K in the symmetric group of the set K/K_α. Now let $L \trianglelefteq G^\circ$ be a connected normal complement of K, i.e., $G = K \cdot L$ is a central product with $K \cap L$ finite. The group L is therefore a closed connected subgroup of H. See [Kr3, 3.5 and 3.6] or Onishchik [Oni, p. 75] for more details. Note that this applies to every nonempty simplex α. In particular, we have $K = G^\circ$ if one of the K-stabilizers is self-normalizing in K.

2.19. The category $\mathbf{HCG}(M)$. Our aim is the classification of compact homogeneous geometries of a given spherical type M. To this end, we consider the following category $\mathbf{HCG}(M)$. Its objects are homogeneous compact geometries (G, Δ) of spherical type M with connected panels, where G is a compact group acting transitively and faithfully on $\mathrm{Cham}(\Delta)$. The morphisms are equivariant 2-coverings which are continuous with respect to the coarse topologies on the respective geometries. We note that the continuity condition can be also phrased as follows: the homomorphisms between the groups are continuous.

In what follows, we have sometimes to compare 'abstract' homomorphisms in the sense of 1.2 with homomorphisms which are in addition continuous in the coarse topology. The group of all continuous automorphisms of a compact geometry Δ will be denoted $\mathrm{AutTop}(\Delta)$, in contrast to the group $\mathrm{Aut}(\Delta)$ of all abstract automorphisms of the underlying combinatorial structure.

There is a Moufang spherical building Δ associated to every noncompact simple Lie group S, which can be defined in various ways. For example, there is a Riemannian symmetric space $X = S/K$ of noncompact type whose connected isometry group is S, and whose Tits boundary $\partial_\infty X$ is the (metric) realization $|\Delta|$. The cone topology on $\partial_\infty X$ coincides with the coarse topology on $|\Delta|$. See Eberlein [Eber] and Bridson–Haefliger [BrHa] for more details. The building Δ can also be defined in group-theoretic terms: the Lie group S has a canonical Tits system (or BN-pair) whose building is Δ, see Warner [War]. The latter approach is used in the next result.

Theorem 2.20. *Let Δ denote the Moufang building associated to a centerless simple real Lie group S of real rank $k \geq 2$ and let $K \subseteq S$ be a maximal compact subgroup. Then (K, Δ) is in $\mathbf{HCG}(M)$, where M is the relative diagram of S, see* [Hel, Chap. X Table VI].

If S is absolutely simple (i.e., if $\mathrm{Lie}(S) \otimes_{\mathbb{R}} \mathbb{C}$ is simple), then every automorphism of Δ is continuous in the coarse topology, $\mathrm{Aut}(\Delta) = \mathrm{AutTop}(\Delta)$. *Moreover,* $\mathrm{Aut}(\Delta) \subseteq \mathrm{Aut}_{\mathbb{R}}(\mathrm{Lie}(S))$ *is a second countable Lie group.*

If S is a complex Lie group, then $\mathrm{Aut}(\Delta)$ *is a semidirect product of the group* $\mathrm{Aut}_{\mathbb{C}}(\Delta)$ *of $\mathbb{C}$-algebraic automorphisms of Δ and the (uncountable) automorphism group of the field $\mathbb{C}$. The group* $\mathrm{AutTop}(\Delta)$ *is a semidirect product of* $\mathrm{Aut}_{\mathbb{C}}(\Delta)$ *and* $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ *(this is again a second countable Lie group). An automorphism of Δ which is (in the coarse topology) continuous on at least one panel is continuous everywhere.*

Proof. The simple Lie group S is simple as an abstract group, see, e.g., [CPP, 94.21]. Therefore it coincides with the group $S^\dagger$ generated by the roots groups of Δ. Thus we have $\mathrm{Aut}(\Delta) \subseteq \mathrm{Aut}(S)$. From the Iwasawa decomposition $S = KAU$ and the fact that the S-stabilizer of a chamber is the Borel subgroup $B = MAU$, with $M = \mathrm{Cen}_K(A)$, we see that K acts transitively on the chambers.

If S is absolutely simple, then its abstract automorphism group $\mathrm{Aut}(S)$ coincides with $\mathrm{Aut}(\mathrm{Lie}(S))$ and is itself a Lie group by Freudenthal's Continuity Theorem [Freu1], see also [BoTi] or [Kr6]. If S is a complex Lie group, then its abstract automorphism group $\mathrm{Aut}(S)$ is a semidirect product of $\mathrm{Aut}_{\mathbb{C}}(\mathrm{Lie}(S))$ and $\mathrm{Aut}(\mathbb{C})$, see [BoTi] or [Kr6]. From the description of the building through the flag varieties of S it is clear that the group $\mathrm{Aut}(\mathbb{C})$ acts indeed on Δ, and that the action is continuous if and only if the field automorphism is continuous. See also Chap. 5 in [Ti1] for more details about the automorphism group of a spherical building over an arbitrary field.

For the last claim, suppose that the abstract automorphism g is continuous on some panel. Since S acts transitively on the chambers, we may assume that g fixes a chamber γ, and that g is continuous on a panel of cotype i containing γ. Let $i \neq j$ be another cotype, and let $\alpha \subseteq \gamma$ be of cotype $\{i, j\}$. If $M_\alpha = 3, 4, 6$, then there is a continuous bijection between the two panels which commutes with g. This is a special property of the complex algebraic generalized polygons: there exit so-called *projective points*, see [Kr2, 2.10]. Therefore g is also continuous on the panel of cotype j. Since the Coxeter diagram of M is connected, we see that g is continuous on $\mathcal{E}_1(\Delta,\gamma)$, and hence everywhere, see [GKVW, 6.16] (or the arguments in [BuSp, 5.1]). □

The previous theorem says in particular that we have a good class of objects in our category $\mathbf{HCG}(M)$. The next result shows that the continuity of 2-coverings is almost automatic if the covering geometry is a building. In the proof we require the following lemma.

Lemma 2.21. *Let S be a noncompact simple centerless Lie group. Then S is absolutely simple if and only if* $\mathrm{Lie}(S)$ *has a simple rank* 1 *Levi factor which is not of type* $\mathfrak{sl}_2\mathbb{C}$.

Proof. This is Lemma 10 in [Kr6]. It follows also from the classification of the real simple Lie algebras and a case-by-case inspection of their root groups, see Chap. X, Table VI in [Hel]. □

We note that $\mathrm{PSL}_2\mathbb{C}$ is the only connected Lie group acting 2-transitively on $\mathbb{S}^2$, see [Kr5]. Therefore a simple centerless Lie group is complex if and only if all its root groups are of real dimension 2.

Theorem 2.22. *Suppose that (G,Δ) is a homogeneous compact geometry in the category $\mathbf{HCG}(M)$ and that the diagram M is spherical and without isolated nodes. Assume that $\widetilde{\Delta}$ is a building and that $\rho:\widetilde{\Delta}\to\Delta$ is an abstract* 2*-covering. Then $\widetilde{\Delta}$ is the Moufang building associated to a semisimple Lie group S of noncompact type. Moreover, there exists a compact chamber-transitive subgroup $K\subseteq S$ and an abstract automorphism φ of $\widetilde{\Delta}$ such that $\rho\circ\varphi:(K,\widetilde{\Delta})\to(G,\Delta)$ is a morphism in $\mathbf{HCG}(M)$, i.e., an equivariant continuous* 2*-covering.*

Proof. Suppose that $\beta\in\widetilde{\Delta}$ is a simplex of corank 2, with $M_\beta>2$. Then $\mathrm{lk}_{\widetilde{\Delta}}(\beta)\cong\mathrm{lk}_\Delta(\rho(\beta))$ is by 2.6 a Moufang generalized M_β-gon associated to a simple noncompact Lie group. Since we excluded factors of type A_1, the irreducible factors of the building $\widetilde{\Delta}$ are Moufang buildings associated to real simple Lie groups. This holds because the panels encode the defining field(s) of an irreducible spherical Moufang building, see Tits–Weiss [TW, 40.22].

We now fix a chamber $\gamma\in\widetilde{\Delta}$, with corank 1 faces $\alpha_1,\dots,\alpha_t$. For $i\neq j$ we put $\alpha_{i,j}=\alpha_i\cap\alpha_j$.

Claim. *There exists an automorphism φ of $\widetilde{\Delta}$ fixing γ such that*

$$\mathcal{E}_2(\widetilde{\Delta},\gamma)\xrightarrow{\rho\circ\varphi}\mathcal{E}_2(\Delta,\rho(\varphi(\gamma)))$$

is a homeomorphism in the coarse topology.

Suppose first that S is absolutely simple. If $M_{\alpha_{i,j}}>2$, then $\mathrm{lk}_{\widetilde{\Delta}}(\alpha_{i,j})\xrightarrow{\rho}\mathrm{lk}_\Delta(\rho(\alpha_{i,j}))$ is a homeomorphism by 2.20. Since the Coxeter diagram is irreducible and of rank at least 2, $\mathcal{E}_2(\widetilde{\Delta},\gamma)\xrightarrow{\rho}\mathcal{E}_2(\Delta,\rho(\gamma))$ is a homeomorphism.

Suppose next that S is a complex simple Lie group and that $M_{\alpha_{i,j}}>2$. By 2.20 we find a field automorphism φ of $\mathbb{C}$ such that φ acts on $\widetilde{\Delta}$, fixes γ, and such that $\mathrm{lk}_{\widetilde{\Delta}}(\alpha_{i,j}) \xrightarrow{\rho\circ\varphi} \mathrm{lk}_\Delta(\rho(\alpha_{i,j}))$ is a homeomorphism. It follows from 2.20 that $\mathrm{lk}_{\widetilde{\Delta}}(\alpha_{i,k}) \xrightarrow{\rho\circ\varphi} \mathrm{lk}_\Delta(\rho(\alpha_{i,k}))$ is a homeomorphism whenever $M_{\alpha_{i,k}} > 2$. An easy induction shows now that $\mathcal{E}_2(\widetilde{\Delta},\gamma) \xrightarrow{\rho} \mathcal{E}_2(\Delta,\rho(\gamma))$ is a homeomorphism.

Finally, suppose that S has several simple factors. Then the Coxeter diagram of M has several components and both $\widetilde{\Delta}$ and Δ factor as joins. This factorization is compatible with ρ and we may apply the previous arguments to the irreducible factors. This finishes the proof of the claim.

Replacing ρ by $\rho\circ\varphi$, we assume from now on that $\mathcal{E}_2(\widetilde{\Delta},\gamma) \xrightarrow{\rho} \mathcal{E}_2(\Delta,\rho(\gamma))$ is a homeomorphism. We let $K \subseteq \mathrm{Aut}(\widetilde{\Delta})$ denote the collection all lifts of the elements of G, see 1.13. It remains to prove that K acts continuously and is compact. To this end we now consider an arbitrary corank 1 face $\alpha = \alpha_i \subseteq \gamma$.

Claim. *The stabilizer K_α acts faithfully and continuously on $\mathcal{E}_2(\widetilde{\Delta},\alpha)$.*

We have

$$\mathcal{E}_1(\widetilde{\Delta},\gamma) \subseteq \mathcal{E}_2(\widetilde{\Delta},\alpha) \subseteq \mathcal{E}_2(\widetilde{\Delta},\gamma).$$

Suppose that $g \in K_\alpha$ acts trivially on $\mathcal{E}_2(\widetilde{\Delta},\alpha)$. Then g fixes γ and acts trivially on $\mathcal{E}_1(\Delta,\rho(\gamma))$, hence g is a lift of the identity fixing a chamber. By 1.12, the deck transformation g is the identity. From the ρ-equivariance we see that K_α acts continuously on $\mathcal{E}_2(\widetilde{\Delta},\alpha) \cong \mathcal{E}_2(\Delta,\rho(\alpha))$.

Claim. *The stabilizer K_α fixes a simplex α' opposite α.*

Let $\beta \subseteq \alpha$ be a corank 2 simplex. Then K_α acts on the generalized polygon $\Gamma = \mathrm{lk}_{\widetilde{\Delta}}(\beta)$. In this action, it centralizes a Cartan involution of $\mathrm{Aut}(\Gamma)$, because it acts in the same way as the compact group $G_{\rho(\alpha)}$ on Γ. Therefore it fixes a vertex opposite $\alpha-\beta$ in Γ. Thus K_α fixes a corank 1 face in $\widetilde{\Delta}$ having a corank 2 face in common with α. Continuing in this way, we obtain a geodesic gallery-like sequence of corank 1 faces fixed by K_α. Eventually, this sequence reaches a corank 1 face opposite α.

Claim. *The group K_α acts continuously on $\widetilde{\Delta}$ and is compact.*

We noticed already that K_α acts continuously on $\mathcal{E}_2(\widetilde{\Delta},\alpha)$. Since $\mathcal{E}_1(\widetilde{\Delta},\gamma) \subseteq \mathcal{E}_2(\widetilde{\Delta},\alpha)$, this implies by [GKVW, 6.16] that K_α acts continuously on $\widetilde{\Delta}$. We noted above that K_α fixes a simplex α' opposite α. Let $L = \mathrm{AutTop}(\widetilde{\Delta})_{\alpha,\alpha'}$ denote the stabilizer of α,α'. The group L acts faithfully on the set $B = \mathcal{E}_2(\widetilde{\Delta},\alpha)$, and B is compact in the coarse topology. The identity map from L with the Lie topology to L with the compact-open topology with respect to the L-action on B is continuous, and $K_\alpha \subseteq L$ has a compact image in the latter. Thus $K_\alpha \subseteq L$ is closed in the Lie topology and therefore a second countable Lie group. It follows from the open mapping theorem that K_α is compact in the Lie topology.

The claim of the theorem follows now. From 1.8 we see that K and all stabilizers in K are compact. Let s denote the length of the longest word in the Coxeter group of M. We have by 1.8 a commutative diagram

$$\begin{array}{ccc}
\big(K_{\alpha_1} \times \cdots \times K_{\alpha_t}\big)^s & \xrightarrow{\text{closed}} & K \\
\text{homeomorphism} \big\downarrow & & \vdots \\
\big(G_{\rho(\alpha_1)} \times \cdots \times G_{\rho(\alpha_t)}\big)^s & \xrightarrow{\text{continuous}} & G.
\end{array}$$

Therefore the dotted homomorphism is continuous. □

Remark 2.23. The proof of 2.22 above relies on properties of Moufang buildings and Lie groups. There is a completely different proof which constructs the topology on the abstract building $\widetilde{\Delta}$ from the topology of Δ, without using the group, see Lytchak [Lyt] and Fang–Grove–Thorbergsson [FGTh].

Under the assumptions of the previous Theorem 2.22, $G = K/F$ where $F \subseteq K$ is, by 1.13, a closed normal subgroup which intersects the stabilizers of corank k simplices trivially (where $k \geq 2$ is the largest integer such that ρ is a k-covering). Since we know the possibilities for the compact group K (at least for the irreducible case) from [EH2], a great deal can be said about the possibilities for F. We indicate for a few examples how such a classification works.

Proposition 2.24. *Assume that (G, Δ) is a homogeneous compact geometry in* $\mathbf{HCG}(M)$ *and that the Coxeter diagram of M is irreducible. In the following three situations, Δ is necessarily the building associated to a simple Lie group S, and G° is a maximal compact subgroup of S.*

(1) *The diagram M is of type A_n.*
(2) *All panels have dimension* 2.
(3) *The diagram M is of type E_6.*

Proof. A geometry of type A_n is always a building by [Ti2, 6.1.5]. By the previous theorem, Δ is the building associated to a simple Lie group S (for $n \geq 3$ this is due to Kolmogoroff [Kolm]). From [EH2] we see that G° is a maximal compact subgroup of S. Thus we have the result (1).

Assume now that all panels have dimension 2. By 4.1 below, a C_3 geometry with 2-dimensional panels is 2-covered by a building. From 1.16 we see that Δ is 2-covered by a building $\widetilde{\Delta}$. By 2.22, the building corresponds to a simple centerless Lie group S. By the remark following 2.21, the Lie group S is complex. Thus a maximal compact subgroup $K \subseteq S$ is centerless simple. By [EH2], K has no chamber-transitive proper closed subgroups, hence $K = G^\circ$.

For (3) we note that all panels are either 1- or 2-dimensional, and the 2-dimensional case is covered by (2). In the 1-dimensional case, we have $G^\circ = \mathrm{PSp}(4)$ by [EH2], and this group is simple. □

For the buildings of type E_7 and E_8 with 1-dimensional panels, the Lie algebra of G° is simple, but G° has nontrivial finite center.

In order to proceed with the classification of homogeneous compact geometries, we need a substitute for the building, i.e., a good universal object in the class of homogeneous compact geometries. The remainder of this section will be devoted to the construction of this compact universal geometry.

2.25. Simple complexes of groups in $\mathbf{HCG}(M)$. Let M be a Coxeter matrix of spherical type over the index set I. Let $\mathcal{G}$ be a simple complex of compact groups and continuous homomorphisms, indexed by the poset of nonempty subsets of I, i.e., $\mathcal{G} = \{G_J \mid \varnothing \neq J \subseteq I\}$.

We consider the following category $\mathbf{HCG}_{\mathcal{G}}(M)$. Its objects are quadruples $(G, \Delta, \gamma, \psi)$, where (G, Δ) is a geometry in $\mathbf{HCG}(M)$ and γ is a chamber of Δ, and ψ is an isomorphism between $\mathcal{G}$ and the simple complex of groups $\{G_\alpha \mid \varnothing \neq \alpha \subseteq \gamma\}$. We assume that for each group $G_J \in \mathcal{G}$ we have $\psi(G_J) = G_\alpha$, where α is the unique face of type J of γ.

A morphism in $\mathbf{HCG}_{\mathcal{G}}(M)$ is an equivariant morphism between the geometries in $\mathbf{HCG}(M)$ which preserves the preferred chambers and which commutes with the isomorphisms between $\mathcal{G}$ and the stabilizer complex. We remark that such a morphism is unique.

Our aim is to show that there is a universal object in this category. The main ingredient is the following construction.

2.26. The basic coset construction. Let (G, Δ) be a homogeneous compact geometry in $\mathbf{HCG}(M)$. Let $\gamma \in \Delta$ be a chamber and let $\mathcal{G}$ denote the simple complex of groups formed by the stabilizers G_α, for $\varnothing \neq \alpha \subseteq \gamma$, see 1.9. Suppose that H is a topological group and that $\psi : \mathcal{G} \to H$ is a continuous simple homomorphism (i.e., that each homomorphism $\psi : G_\alpha \to H$ is continuous and that all triangles commute). In this situation we construct a new homogeneous compact geometry (G', Δ') in $\mathbf{HCG}(M)$ and a covering

$$\rho : (G', \Delta') \to (G, \Delta)$$

as follows.

For $g \in G_\alpha$ put $\psi'(g) = (\psi(g), g) \in H \times G$. This defines a continuous and injective simple homomorphism $\psi' : \mathcal{G} \to H \times G$. We put $G'_{\alpha'} = \psi'(G_\alpha) \subseteq H \times G$ and we let $G' \subseteq H \times G$ denote the group which is algebraically generated by the $G'_{\alpha'}$. In order to construct Δ', we use the following standard method, see Tits [Ti1, 1.4] and Bridson–Haefliger [BrHa, II.12.18–22].

Let $v_1, \ldots, v_t$ denote the vertices of the chamber γ. The set of cosets $G'/G'_{v'_1} \cup \cdots \cup G'/G'_{v'_t}$ covers G'. We let Δ' denote the nerve of this cover. It is easy to see that the simplices of Δ' correspond bijectively to the cosets $gG'_{\alpha'}$, for $\varnothing \neq \alpha \subseteq \gamma$ and $g \in G'$. The inclusion of simplices corresponds to the reversed inclusion of cosets. In particular we see that Δ' is a pure simplicial complex. The residue $\Delta'_{\geq G'_{\alpha'}}$ of the simplex $G'_{\alpha'}$ consists of all cosets $gG'_{\beta'}$ with $g \in G'_{\alpha'}$ and $\beta \supseteq \alpha$. Moreover, there is a well-defined type function on Δ' which maps $gG'_{v'_i}$ to the type $t(v_i)$. We note also that the projection $\mathrm{pr}_2 : H \times G \to G$ induces a continuous surjective homomorphism $p : G' \to G$, and a regular simplicial map $p : \Delta' \to \Delta$ which maps $gG'_{\alpha'}$ to $\mathrm{pr}_2(g)(\alpha)$.

Claim. *Δ' is a thick chamber complex.*

Every element $g \in G'$ can be written as a product $g = g_1 \cdots g_r$, where g_k is in the stabilizer of a corank 1 face of γ. This gives a gallery from $G'_{\gamma'}$ to $gG'_{\gamma'}$. The panels of Δ' have the same cardinalities as the panels of Δ, hence Δ' is thick.

Claim. *Δ' is a geometry over I, of the same type M, and $p : \Delta' \to \Delta$ is a covering.*

From the description of the residues above we see that the link of a nonempty simplex in Δ' maps isomorphically onto a link in Δ. Thus p is a covering. In particular, Δ' is a flag complex.

Claim. *(G', Δ') is a homogeneous compact geometry. The group G' is compact and acts faithfully.*

Obviously, Δ' is a homogeneous geometry. The groups $G'_{\alpha'}$ are by construction compact. From the bounded generation 1.8 we see that G' is also compact. Suppose that $(h, g) \in G'_\gamma$ acts trivially on Δ'. Then $g = \mathrm{id}_\Delta$. Since $(h, g) \in G'_{\gamma'}$, we have $h = 1$.

We record a few more useful facts about (G', Δ').

Fact. *The subgroup of H generated by the $\psi(G_\alpha)$ is compact.*

This group is the image of the compact group G' under $\mathrm{pr}_1 : H \times G \to H$.

Fact. *Let $F \subseteq G'$ denote the kernel of $G' \xrightarrow{p} G$. Then F intersects every simplex stabilizer trivially, i.e., F acts freely on Δ'. The G'-stabilizer of $\alpha \in \Delta$ is a semidirect product $G'_\alpha = G'_{\alpha'} F$.*

Consider an element $(h, id_\Delta) \in F \cap G'_{\alpha'}$. Since $G_\alpha \xrightarrow{\psi'} G'_{\alpha'}$ is bijective, we have $h = 1$. Suppose now that the group element $(h, g) \in G'$ fixes the simplex α. Then we have $g \in G_\alpha$. Let $h_1 = \psi(g)$. Then we have $\psi'(g^{-1}) = (h_1^{-1}, g^{-1}) \in G'_{\alpha'}$ and $(h, g)(h_1^{-1}, g^{-1}) = (hh_1^{-1}, id_\Delta) \in F$.

We now use the Basic Coset Construction 2.26 in order to construct a universal object in $\mathbf{HCG}_\mathcal{G}(M)$.

Theorem 2.27. *Suppose that M is spherical without isolated nodes over the index set I, that $\mathcal{G}$ is a simple complex of compact groups and continuous homomorphisms over the collection of the nonempty subsets $J \subseteq I$ and that $\mathbf{HCG}_\mathcal{G}(M)$ is not empty. Then there exists a homogeneous compact geometry $(\widehat{G}, \widehat{\Delta}, \hat{\gamma}, \hat{\psi})$ in $\mathbf{HCG}_\mathcal{G}(M)$ which has a unique morphism ρ to every $(G, \Delta, \gamma, \psi)$ in $\mathbf{HCG}_\mathcal{G}(M)$.*

Proof. We choose a 'transversal' in $\mathbf{HCG}_\mathcal{G}(M)$, i.e., a family $(G_\nu, \Delta_\nu, \gamma_\nu, \psi_\nu)_{\nu \in N}$ of objects in $\mathbf{HCG}_\mathcal{G}(M)$ which contains one member of each isomorphism class. Such a family exists since there are only countably many isomorphism classes of compact Lie groups (every compact Lie group can be realized as an algebraic matrix group). The cardinality of the index set N is not important here; we need only the fact that such a set exists. The ψ_ν fit together to a continuous simple homomorphism $\psi : \mathcal{G} \to \prod_{\nu \in N} G_\nu$. Let $\widehat{G} \subseteq \prod_{\nu \in N} G_\nu$ denote the group generated algebraically by the groups $\psi(G_J)$. The basic coset construction 2.26 gives us a homogeneous compact geometry $(\widehat{G}, \widehat{\Delta})$ and for each ν a continuous equivariant covering $\rho_\nu : (\widehat{G}, \widehat{\Delta}) \to (G_\nu, \Delta_\nu)$. This morphism is unique, as we remarked above. $\square$

Definition 2.28. We call the pair $(\widehat{G}, \widehat{\Delta})$ constructed in 2.27 a *universal homogeneous compact geometry* for the pair $(\mathcal{G}, M)$ (obviously this homogeneous compact geometry is unique up to isomorphism). If an element of the class $\mathbf{HCG}_{\mathcal{G}}(M)$ can be covered by a building $\widetilde{\Delta}$, then this building is the universal homogeneous compact geometry by Tits' result 1.15 and by 2.22.

The group $\widehat{G}$ has the following universal property.

Proposition 2.29. *Suppose that $(\widehat{G}, \widehat{\Delta})$ is a universal homogeneous compact geometry for the pair $(\mathcal{G}, M)$. Suppose that H is a topological group and that $\varphi : \mathcal{G} \to H$ is a simple continuous homomorphism. Then there is a unique continuous homomorphism $\psi : \widehat{G} \to H$ such that the diagram*

$$\begin{array}{ccc} \mathcal{G} & \longrightarrow & \widehat{G} \\ & {\scriptstyle\varphi}\searrow & \downarrow{\scriptstyle\psi} \\ & & H \end{array}$$

commutes.

Proof. The Basic Coset Construction 2.26 applied to $\mathcal{G} \to H \times \widehat{G}$ gives us a geometry (G', Δ') and a map $G' \to H$. From the universal property of $(\widehat{G}, \widehat{\Delta})$ we have a homomorphism $\widehat{G} \to G' \to H$. The uniqueness is clear. $\square$

Finally, we note that we can pass to a minimal universal homogeneous compact geometry.

Proposition 2.30. *Suppose that M is of spherical type and without isolated nodes. Suppose that $\mathcal{G}$ is a simple complex of compact groups and that $\mathbf{HCG}_{\mathcal{G}}(M)$ is nonempty. Then there exists a simple complex of compact groups $\mathcal{K}$ formed by subgroups $K_J \subseteq G_J$ such that $\mathbf{HCG}_{\mathcal{K}}(M)$ is nonempty, with the following properties.*

1. *The universal homogeneous compact geometry $(\widehat{K}, \widehat{\Delta})$ in $\mathbf{HCG}_{\mathcal{K}}(M)$ is minimal.*
2. *For every geometry (G, Δ) in $\mathbf{HCG}_{\mathcal{G}}(M)$ there is an equivariant covering*

$$(\widehat{K}, \widehat{\Delta}) \to (G, \Delta).$$

Proof. We construct a sequence of equivariant coverings

$$\cdots \to (G_{k+1}, \Delta_{k+1}) \to (G_k, \Delta_k) \to \cdots \to (G_0, \Delta_0)$$

and simple complexes of compact groups $\mathcal{G}_k$ as follows. Let $\mathcal{G}_0 = \mathcal{G}$ and let (G_0, Δ_0) denote the corresponding universal homogeneous compact geometry in $\mathbf{HCG}_{\mathcal{G}_0}(M)$. Given $\mathcal{G}_k$ and (G_k, Δ_k), we choose a closed chamber transitive subgroup $H \subseteq G_k$ such that (H, Δ_k) is minimal. Let $\mathcal{G}_{k+1}$ denote the simple complex of groups formed by the simplex stabilizers of H, and let (G_{k+1}, Δ_{k+1}) denote the corresponding universal homogeneous compact geometry in $\mathbf{HCG}_{\mathcal{G}_{k+1}}(M)$. If $\mathcal{G}_{k+1} \neq \mathcal{G}_k$, then the stabilizers have become strictly smaller. Since there are no infinite descending sequences of closed compact Lie groups, this process becomes stationary in finite time k, and we may put $\mathcal{K} = \mathcal{G}_k$. $\square$

We remark that a completely analogous construction works for the class of finite homogeneous geometries.

3. Homogeneous compact geometries of type C_3

In this section we review the known examples of universal homogeneous compact geometries (G, Δ) of type C_3. In Section 4 we will show that this list of examples is complete: such a geometry is either a building (a polar space of rank 3), or the exceptional geometry discovered by Podestà–Thorbergsson [PoTh], which we describe in Section 3B. Some of the results in the present section will be used in the classification. We begin with the classical geometries, the buildings of type C_3.

3A. Projective and polar spaces and their Veronese representations

Almost all buildings of type C_3 arise from hermitian forms. We review the relevant linear algebra, since we will use it in our classification in Section 4. Buildings of type C_n are also called *polar spaces* of rank n. Buildings of type C_2 are called *generalized quadrangles.*

3.1. Polar spaces. Let $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ and let σ be an involution on $\mathbb{F}$, i.e., an additive map with $a^{\sigma^2} = a$ and $(ab)^\sigma = b^\sigma a^\sigma$, for all $a, b \in \mathbb{F}$. The involution σ extends in a natural way to matrices, acting by matrix transposition combined with entry-wise application of σ. Let V be a finite dimensional right $\mathbb{F}$-module and let $\varepsilon = \pm 1$. A nondegenerate (ε, σ)*-hermitian form* is a biadditive map $h : V \times V \to \mathbb{F}$ with the properties

$$h(v, w) = \varepsilon h(w, v)^\sigma \qquad h(va, wb) = a^\sigma h(v, w) b \qquad h(V, w) = 0 \;\Rightarrow\; w = 0.$$

The relevant examples are

symmetric bilinear forms with $(\varepsilon, \sigma) = (1, \mathrm{id})$ and $\mathbb{F} = \mathbb{R}, \mathbb{C}$,

symplectic forms with $(\varepsilon, \sigma) = (-1, \mathrm{id})$ and $\mathbb{F} = \mathbb{R}, \mathbb{C}$, and

(ε, σ)*-hermitian forms* with $a^\sigma = \bar{a}$ and $\mathbb{F} = \mathbb{C}, \mathbb{H}$.

A nonzero subspace $W \subseteq V$ is called *totally isotropic* if $W \subseteq W^{\perp_h}$. The form h is called *isotropic* if there exist totally isotropic subspaces. The maximal dimension k of a totally isotropic subspace is the *Witt index* of h. The corresponding geometry Δ has as its vertices the collection of all isotropic subspaces. The simplices in Δ are the ascending chains of isotropic subspaces. This simplicial complex is a building of type C_k, unless $\dim(V) = 2k$ and $(\varepsilon, \sigma) = (1, \mathrm{id})$. In the latter case, a slightly modified simplicial complex is a building of type D_k, see [Ti1, 7.12]. We refer to [Ti1, Chap. 7, 8] for more details.

The automorphism group of this building is (an extension of) the projective unitary group of the form h. Its identity component S is a noncompact simple Lie group of classical type and (S, Δ) is a homogeneous compact geometry. If $G \subseteq S$ is a maximal compact subgroup, then also (G, Δ) is a compact homogeneous geometry.

For each of these polar spaces mentioned above, it is possible to describe the associated polar representation in terms of certain tensors and geometric algebra. We first recall the definition of a polar representation.

3.2. Polar representations. An orthogonal representation of a compact Lie group is called *polar* if there exists a linear subspace that meets every orbit orthogonally. Polar representations were classified up to orbit equivalence by Hsiang–Lawson [HsLa] and Dadok [Dad]. See Eschenburg–Heintze [EH1, EH2] for a modern account. The result is that every polar representation is orbit equivalent to an *s-representation.* An s-representation is defined as follows. Let S be a semisimple centerless Lie group of noncompact type, let $G \subseteq S$ be a maximal compact subgroup and let $\mathrm{Lie}(S) = \mathrm{Lie}(G) \oplus \mathfrak{P}$ be the corresponding Cartan decomposition. The adjoint representation of G on $\mathfrak{P}$ is the associated s-representation. It is polar, and if Δ is the associated building, then $|\Delta|_K$ is G-equivariantly homeomorphic to the unit sphere $\mathbb{S}(\mathfrak{P}) \subseteq \mathfrak{P}$.

3.3. Polar representations for certain polar spaces. Suppose that $\varepsilon = 1$ and $a^\sigma = \bar{a}$, for $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$. Let f_k denote the standard positive definite hermitian form on $\mathbb{F}^k$, i.e.,

$$f_k(v, w) = \sum_{j=1}^{k} \bar{v}_j w_j.$$

Let $\mathrm{U}_k\mathbb{F}$ denote the corresponding unitary group,

$$\mathrm{U}_k\mathbb{F} = \{g \in \mathbb{F}^{k\times k} \mid g^\sigma g = 1\}$$

(recall that $(x_{i,j})^\sigma = (\bar{x}_{j,i})$). Consider the hermitian form

$$h = (-f_k) \oplus f_\ell$$

on $V = \mathbb{F}^{k+\ell}$, with $k \leq \ell$ (resp. $k < \ell$ for $\mathbb{F} = \mathbb{R}$). The Witt index of h is k and $\mathrm{U}(k) \times \mathrm{U}(\ell)$ is a maximal compact subgroup of the unitary group

$$\mathrm{U}(h) = \mathrm{U}_{k,\ell}\mathbb{F} = \{g \in \mathrm{GL}(V) \mid h(-,-) = h(g(-), g(-))\}.$$

We identify the tensor product $\mathbb{F}^k \otimes_{\mathbb{F}} (\mathbb{F}^\ell)^\sigma$ with the $\mathbb{R}$-module $\mathbb{F}^{k\times\ell}$ and we note that $\mathrm{U}_k\mathbb{F} \times \mathrm{U}_\ell\mathbb{F}$ acts in a natural way on $\mathbb{F}^{k\times\ell}$, via

$$(g_1, g_2, X) \longmapsto g_1 X g_2^\sigma.$$

This is the polar representation we are interested in.

There are natural projections $\mathbb{F}^k \xleftarrow{\mathrm{pr}_1} \mathbb{F}^{k+\ell} \xrightarrow{\mathrm{pr}_2} \mathbb{F}^\ell$. For every t-dimensional totally isotropic subspace $W \subseteq \mathbb{F}^{k+\ell}$ there exists a basis $w_1, \ldots, w_t$ such that $\{u_1 = \mathrm{pr}_1(w_1), \ldots, u_t = \mathrm{pr}_1(w_t)\} \subseteq \mathbb{F}^k$ and $\{v_1 = \mathrm{pr}_2(w_1), \ldots, v_t = \mathrm{pr}_2(w_t)\} \subseteq \mathbb{F}^\ell$ are orthonormal. The map which sends the subspace W to $(1/\sqrt{t})\big(u_1 \otimes v_1^\sigma + \cdots + u_t \otimes v_t^\sigma\big) \in \mathbb{F}^k \otimes_{\mathbb{F}} (\mathbb{F}^\ell)^\sigma$ is well-defined and $\mathrm{U}_k\mathbb{F} \times \mathrm{U}_\ell\mathbb{F}$-equivariant. This map extends to a mapping

$$|\Delta| \to \mathbb{S}(\mathbb{F}^k \otimes_{\mathbb{F}} (\mathbb{F}^\ell)^\sigma) = \mathbb{S}^{k\cdot\ell\cdot\dim_{\mathbb{R}}\mathbb{F}-1}$$

which is a homeomorphism in the coarse topology. This map is called the *Veronese representation* of Δ. The Veronese representation of Δ lends itself to computations of vertex stabilizers in $\mathrm{U}_k\mathbb{F} \times \mathrm{U}_\ell\mathbb{F}$.

Finally, we note that for $\mathbb{F} = \mathbb{R}, \mathbb{C}$ and $k < \ell$ the groups $\mathrm{SO}(k) \times \mathrm{SO}(\ell)$ and $\mathrm{SU}(k) \times \mathrm{SU}(\ell)$ act transitively on the chambers. According to Eschenburg–Heintze [EH2] and [GKK2] these are the smallest compact chamber-transitive groups K, unless we are in one of the following exceptional cases:

$(k, \ell) = (2, 7)$ and $K = \mathrm{SO}(2) \cdot \mathrm{G}_2$,
$(k, \ell) = (2, 8)$ and $K = \mathrm{SO}(2) \cdot \mathrm{Spin}(7)$,
$(k, \ell) = (3, 8)$ and $K = \mathrm{SO}(3) \cdot \mathrm{Spin}(7)$.

We remark that for the other types of (ε, σ)-hermitian forms, similar models for Veronese representations can be worked out in terms of tensor products and exterior products. These will not be needed here. However, we need also polar representations for the classical projective geometries over $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ and the Cayley algebra $\mathbb{O}$.

3.4. Polar representations for projective geometries. Let $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ and $V = \mathbb{F}^{n+1}$, endowed with the standard hermitian form f_{n+1}. The projective geometry over $\mathbb{F}$ (of type A_n) is the simplicial complex Δ whose vertices are the proper nonzero subspaces of V. The simplices are the partial flags. The noncompact Lie group $\mathrm{GL}(V) = \mathrm{GL}_{n+1}\mathbb{F}$ acts transitively on the chambers of Δ. A maximal compact subgroup is the unitary group $\mathrm{U}_{n+1}\mathbb{F}$. Suppose that $W \subseteq V$ is a t-dimensional subspace, with an orthonormal basis $w_1, \ldots, w_t$. The map which sends W to the traceless hermitian matrix $\big(w_1 \otimes w_1^\sigma + \cdots + w_t \otimes w_t^\sigma\big) - t\mathbf{1}$ is well-defined and $\mathrm{U}_{n+1}\mathbb{F}$-equivariant. It extends to a map

$$|\Delta| \to \mathbb{S}^\ell,$$

where $\ell = n(n+1)\dim_{\mathbb{R}}(\mathbb{F})/2 + n - 1$. One can check that this map induces a homeomorphism $|\Delta|_K \to \mathbb{S}^\ell$. This is the *Veronese representation* of Δ. Again, the computation of vertex stabilizers in $\mathrm{U}_{n+1}\mathbb{F}$ is easily done in this representation. We note that for $n = 2$, the minimal transitive faithful compact groups are $\mathrm{SO}(3)$, $\mathrm{PSU}(3)$ and $\mathrm{PSp}(3)$.

The projective Cayley plane has no simple description in terms of $\mathbb{O}^3$, since this is not a module over the Cayley algebra $\mathbb{O}$. Nevertheless, the right-hand side of the Veronese representation in terms of traceless hermitian 3×3-matrices over $\mathbb{O}$ makes sense and leads to a Veronese representation of this geometry, The compact group in question is the centerless simple compact Lie group F_4, with vertex stabilizers $\mathrm{Spin}(9)$ and chamber stabilizer $\mathrm{Spin}(8)$. We refer to Freudenthal [Freu2], Salzmann et al. [CPP, Chap. 1], and to Section 3B below.

3.5. The nonembeddable polar space. Over the reals, there is one polar space Δ of type C_3 which is not associated to a hermitian form, see [Ti1, Chap. 9]. Instead, it is related to the Cayley algebra. The corresponding simple noncompact Lie group is of type $\mathrm{E}_{7(-25)}$ and its maximal compact subgroup is $G = \mathrm{E}_6 \cdot \mathrm{SO}(2)$. (In Cartan's classification, this is the noncompact case E VII, see Helgason [Hel, Chap. X, Table V].) Its Veronese representation $|\Delta| \to \mathbb{S}^{53}$ arises from the corresponding s-representation as in 3.2. The panels have dimensions 8 and 1 and the links of the vertices are projective Cayley planes and generalized quadrangles belonging to the symmetric bilinear form $h = (-f_2) \oplus f_{10}$ on $\mathbb{R}^{2+10}$. Apparently, no 'simple model' for Δ and its Veronese representation is known; the abstract construction is purely Lie-theoretic.

3B. The exceptional C_3 geometry

We now construct the exceptional geometry of type C_3 that was found by Podestà–Thorbergsson [PoTh, 2B.3]. We use the Veronese representation of the Cayley plane as a focal manifold of an isoparametric foliation in $\mathbb{S}^{25}$, corresponding to the s-representation of the symmetric space for $(\mathfrak{e}_{6(-26)}, \mathfrak{f}_4)$. For the description of the Cayley plane which we use, see also Cartan [Cart], Console–Olmos [CoOl], Freudenthal [Freu2], Karcher [Kar], Knarr–Kramer [KnKr] and Salzmann et al. [CPP, Chap. 1].

3.6. The action of SU(3) on $\mathbb{O}$. We first recall some algebraic facts. The real Cayley division algebra $\mathbb{O}$ is bi-associative (any two elements generate an associative subalgebra) and therefore in a natural way a (right) complex vector space, see [CPP, 11.13]. The norm of $\mathbb{O}$ is a quadratic form which induces a positive definite complex hermitian form on $\mathbb{O}$. As a unitary $\mathbb{C}$-basis of $\mathbb{O}$ we fix the elements $1, \boldsymbol{j}, \boldsymbol{\ell}, \boldsymbol{j\ell} \in \mathbb{O}$, see [CPP, 11.34]. The $\mathrm{Aut}(\mathbb{O})$-stabilizer of $\boldsymbol{i} \in \mathbb{C}$ acts $\mathbb{C}$-linearly and can be identified with the matrix group

$$\mathrm{Aut}_{\mathbb{C}}(\mathbb{O}) \cong \mathrm{SU}(3).$$

In order to specify such an isomorphism with the matrix group, we use the ordered $\mathbb{C}$-basis $(\boldsymbol{j}, \boldsymbol{\ell}, \boldsymbol{j\ell})$ of $\mathbb{C}^\perp \subseteq \mathbb{O}$. Every element of $u \in \mathbb{O}$ has a unique expression as

$$u = u_0 + \boldsymbol{j}u_1 + \boldsymbol{\ell}u_2 + (\boldsymbol{j\ell})u_3 \quad \text{with } u_0, u_1, u_2, u_3 \in \mathbb{C}.$$

We define an isomorphism $\mathbb{C}^\perp \xrightarrow{\cong} \mathbb{C}^3$ via

$$\boldsymbol{j}u_1 + \boldsymbol{\ell}u_2 + (\boldsymbol{j\ell})u_3 \mapsto (u_1, u_2, u_3).$$

For $z \in \mathbb{C}$ and $u \in \mathbb{C}^\perp$ we have $zu = u\bar{z}$. In what follows, we will use this identity frequently. The $\mathbb{C}$-component of a product

$$uv = (u_0 + \boldsymbol{j}u_1 + \boldsymbol{\ell}u_2 + (\boldsymbol{j\ell})u_3)(v_0 + \boldsymbol{j}v_1 + \boldsymbol{\ell}v_2 + (\boldsymbol{j\ell})v_3)$$

is therefore $u_0v_0 - (\bar{u}_1v_1 + \bar{u}_2v_2 + \bar{u}_3v_3)$. The group $\mathrm{Aut}_{\mathbb{C}}(\mathbb{O})$ preserves this quantity and fixes u_0v_0. Hence $\mathrm{Aut}_{\mathbb{C}}(\mathbb{O})$ preserves the hermitian form $\bar{u}_1v_1 + \bar{u}_2v_2 + \bar{u}_3v_3$ on $\mathbb{C}^\perp \cong \mathbb{C}^3$. It follows that $\mathrm{Aut}_{\mathbb{C}}(\mathbb{O}) \cong \mathrm{SU}(3)$ acts via standard matrix multiplication from the left on $\mathbb{C}^3 \cong \mathbb{C}^\perp$.

3.7. The model of $\mathbb{O}\mathbf{P}^2$. We view the Cayley plane $\mathbb{O}\mathrm{P}^2$ as the set of all idempotent hermitian 3×3-matrices over $\mathbb{O}$ with trace 1 (the *rank* 1 *projectors*). This is slightly different from 3.4 above, where we considered traceless hermitian matrices. The change of the trace simplifies matrices without changing the stabilizers. The euclidean inner product of two $\mathbb{O}$-hermitian 3×3-matrices is defined as

$$\langle X, Y \rangle = \mathrm{trace}(XY) = \sum_{i=1}^{3} X_{i,i}Y_{i,i} + 2 \sum_{1 \le i < j \le 3} \mathrm{Re}(X_{i,j}\bar{Y}_{i,j}).$$

The euclidean distance between two elements $\xi, \xi' \in \mathbb{O}\mathrm{P}^2$ is given by

$$\|\xi - \xi'\|^2 = 2 - 2\langle \xi, \xi' \rangle,$$

because $\|\xi\|^2 = \|\xi'\|^2 = 1$. The Cayley plane $\mathbb{O}\mathrm{P}^2$ is in particular a Riemannian submanifold of the 26-dimensional euclidean space of $\mathbb{O}$-hermitian 3×3-matrices with trace 1, and the compact group F_4 acts transitively and isometrically on $\mathbb{O}\mathrm{P}^2$. A point with affine coordinates $(x, y) \in \mathbb{O} \times \mathbb{O}$ is identified with the projector

$$\frac{1}{x\bar{x} + y\bar{y} + 1} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} (\bar{x} \ \bar{y} \ 1) = \frac{1}{x\bar{x} + y\bar{y} + 1} \begin{pmatrix} x\bar{x} & x\bar{y} & x \\ y\bar{x} & y\bar{y} & y \\ \bar{x} & \bar{y} & 1 \end{pmatrix},$$

see [CPP, p. 84]. By means of this coordinate chart we view the *affine Cayley plane* $\mathbb{O} \times \mathbb{O}$ as an open dense subset of $\mathbb{O}\mathrm{P}^2$. We note that under this chart the image of a real line in $\mathbb{O} \times \mathbb{O}$ passing through the origin is a geodesic in $\mathbb{O}\mathrm{P}^2$. Also, the chart is conformal at the origin $(x, y) = (0, 0)$, as is easily seen by differentiating. The complement of the range of the chart is the cut locus L of the point $(0, 0)$ in $\mathbb{O}\mathrm{P}^2$, or, in terms of projective geometry, the projective line at infinity of the affine Cayley plane $\mathbb{O} \times \mathbb{O}$, an 8-sphere.

3.8. The action of $\mathbf{SU(3) \times SU(3)}$ on $\mathbb{O}\mathbf{P^2}$. The group SU(3) acts in the standard way isometrically on the set of all hermitian 3×3-matrices over $\mathbb{O}$ with trace 1, preserving $\mathbb{O}\mathrm{P}^2$, and with $\mathbb{C}\mathrm{P}^2$ as one orbit, via

$$g(X) = gXg^{-1}.$$

Due to the bi-associativity of $\mathbb{O}$, this matrix product is well-defined. The action is faithful, since $\mathbb{O}$ is not commutative. On the other hand, $\mathrm{SU}(3) = \mathrm{Aut}_{\mathbb{C}}(\mathbb{O})$ acts as in 3.6 entry-wise on the $\mathbb{O}$-hermitian matrices. In this way, the compact group

$$K = \mathrm{Aut}_{\mathbb{C}}(\mathbb{O}) \times \mathrm{SU}(3) = \mathrm{SU}(3) \times \mathrm{SU}(3)$$

acts isometrically on $\mathbb{O}\mathrm{P}^2$. Our aim is to understand the orbit structure of this action. We begin with the point

$$q = (0, 0) \in \mathbb{O} \times \mathbb{O}.$$

3.9. The $\mathbb{C}\mathbf{P^2}$-orbit and its normal isotropy representation. The affine coordinates $(0, 0)$ are complex and therefore the K-orbit of q is $K(q) = \mathbb{C}\mathrm{P}^2 \subseteq \mathbb{O}\mathrm{P}^2$. Since $\mathrm{Aut}_{\mathbb{C}}(\mathbb{O})$ acts trivially on $\mathbb{C}\mathrm{P}^2$, the K-stabilizer of q is isomorphic to $\mathrm{Aut}_{\mathbb{C}}(\mathbb{O}) \times \mathrm{U}(2)$. The projector corresponding to $q = (0, 0)$ is $\left(\begin{smallmatrix} 0&0&0 \\ 0&0&0 \\ 0&0&1 \end{smallmatrix}\right)$ and thus K_q consists of the block matrices of the form

$$Y_1 \times \begin{pmatrix} Y_2 & 0 \\ 0 & y \end{pmatrix} \in \mathrm{SU}(3) \times \mathrm{SU}(3),$$

with $Y_1 \in \mathrm{SU}(3)$ and $Y_2 \in \mathrm{U}(2)$, and $y = \overline{\det(Y_2)}$. The group K_q stabilizes the polar line (the cut locus) L of q in $\mathbb{O}\mathrm{P}^2$ and acts on the affine Cayley plane $\mathbb{O} \times \mathbb{O}$.

In this way we are reduced to a linear action. The representation of K_q on $\mathbb{O} \times \mathbb{O}$ splits off a representation of (real) dimension 4 on $\mathbb{C} \times \mathbb{C}$, with $\mathrm{Aut}_{\mathbb{C}}(\mathbb{O})$ acting trivially and $\mathrm{U}(2)$ acting via matrix multiplication from the left by $\det(Y_2)Y_2$ on $\mathbb{C}^2$.

On the complement $\mathbb{C}^\perp \times \mathbb{C}^\perp \cong \mathbb{C}^{2\times 3}$ we have the following representation. We represent a point with affine coordinates

$$(u, v) = (\boldsymbol{j}u_1 + \boldsymbol{\ell}u_2 + (\boldsymbol{j\ell})u_3, \boldsymbol{j}v_1 + \boldsymbol{\ell}v_2 + (\boldsymbol{j\ell})v_3) \in \mathbb{C}^\perp \times \mathbb{C}^\perp$$

as

$$\begin{pmatrix} u_1 & v_1 \\ u_2 & v_2 \\ u_3 & v_3 \end{pmatrix} \in \mathbb{C}^{2\times 3}$$

Then $\mathrm{Aut}_{\mathbb{C}}(\mathbb{O}) \cong \mathrm{SU}(3)$ acts in the standard way from the left on $\mathbb{C}^{2\times 3}$. We put

$$Y_2 = \begin{pmatrix} c & -a\bar{s} \\ s & a\bar{c} \end{pmatrix},$$

where c, s, a are complex numbers with $c\bar{c} + s\bar{s} = a\bar{a} = 1$, and $y = \bar{a}$. Then $\left(\begin{smallmatrix} Y_2 & 0 \\ 0 & y \end{smallmatrix}\right)$ maps the point $(u, v) \in \mathbb{C}^\perp \times \mathbb{C}^\perp$ to

$$\begin{aligned}(cua - a\bar{s}va, sua + a\bar{c}va) &= (u\bar{c}a - vs, u\bar{s}a + vc) \\ &= (u, v)\begin{pmatrix} \bar{c}a & \bar{s}a \\ -s & c \end{pmatrix} = (u, v)\begin{pmatrix} \bar{c} & \bar{s} \\ -s\bar{a} & c\bar{a} \end{pmatrix} a.\end{aligned}$$

Hence Y_2 acts on $\mathbb{C}^{2\times 3}$ through matrix multiplication from the right by $Y_2^*\bar{y}$, where $Y_2^* = \bar{Y}_2^T$ and $\bar{y} = \det(Y_2)$. Summing this up, we have the K_q-action

$$Y_1 \times \begin{pmatrix} Y_2 & 0 \\ 0 & y \end{pmatrix} : \begin{pmatrix} u_1 & v_1 \\ u_2 & v_2 \\ u_3 & v_3 \end{pmatrix} \mapsto Y_1 \begin{pmatrix} u_1 & v_1 \\ u_2 & v_2 \\ u_3 & v_3 \end{pmatrix} Y_2^*\bar{y}.$$

In particular, the action of K_q on $\mathbb{C}^{2\times 3}$ is orbit equivalent with the polar action described in 3.3. Since the tangent space $T_q\mathbb{O}\mathrm{P}^2$ splits also K_q-equivariantly as $T_q\mathbb{O}\mathrm{P}^2 = T_q\mathbb{C}\mathrm{P}^2 \oplus \perp_q\mathbb{C}\mathrm{P}^2$, this gives us at the same time the normal isotropy representation of K_q. In particular, the normal isotropy representation of K_q on $\perp_q\mathbb{C}\mathrm{P}^2 \subseteq T_q\mathbb{O}\mathrm{P}^2$ is polar.

We put

$$d = (0, \boldsymbol{j}) \qquad \text{and} \qquad p = (-\boldsymbol{\ell}, \boldsymbol{j}).$$

Suppose that λ, μ are nonnegative reals and consider the point

$$o = o_{\lambda,\mu} = p\lambda + d\mu = (-\lambda\boldsymbol{\ell}, (\lambda + \mu)\boldsymbol{j}) \in \mathbb{C}^\perp \times \mathbb{C}^\perp.$$

The corresponding point in $\mathbb{C}^{2\times 3}$ is $\left(\begin{smallmatrix} 0 & \lambda+\mu \\ -\lambda & 0 \\ 0 & 0 \end{smallmatrix}\right)$. Since the action of K_q on $\perp_q\mathbb{C}\mathrm{P}^2$ is polar, every K_q-orbit in $\perp_q\mathbb{C}\mathrm{P}^2$ contains exactly one such point $o_{\lambda,\mu}$. The K_q-stabilizer of $o_{\lambda,\mu}$ can easily be computed. For $\lambda, \mu > 0$, it consists of the matrices

$$\begin{pmatrix} \bar{z} & & \\ & zy & \\ & & \bar{y} \end{pmatrix} \times \begin{pmatrix} z & & \\ & \overline{zy} & \\ & & y \end{pmatrix} \in \mathrm{SU}(3) \times \mathrm{SU}(3),$$

with $y, z \in \mathrm{U}(1)$. For $\lambda = 0 < \mu$, it consists of the matrices

$$\begin{pmatrix} \bar{z} & \\ & Z_2 \end{pmatrix} \times \begin{pmatrix} z & & \\ & \overline{yz} & \\ & & y \end{pmatrix} \in \mathrm{SU}(3) \times \mathrm{SU}(3),$$

with $Z_2 \in \mathrm{U}(2)$. For $\mu = 0 < \lambda$ it consists of matrices of the form

$$\begin{pmatrix} \bar{Y}_2 & \\ & \bar{y} \end{pmatrix} \times \begin{pmatrix} Y_2 & \\ & y \end{pmatrix} \in \mathrm{SU}(3) \times \mathrm{SU}(3),$$

with $Y_2 \in \mathrm{U}(2)$.

3.10. Euclidean distances between orbits. The projector $\xi_{\lambda,\mu} \in \mathbb{O}\mathrm{P}^2$ corresponding to $o_{\lambda,\mu}$ is

$$\xi = \xi_{\lambda,\mu} = \frac{1}{\lambda^2 + (\lambda+\mu)^2 + 1} \begin{pmatrix} \lambda^2 & -\lambda(\lambda+\mu)\boldsymbol{j}\boldsymbol{\ell} & -\lambda\boldsymbol{\ell} \\ \lambda(\lambda+\mu)\boldsymbol{j}\boldsymbol{\ell} & (\lambda+\mu)^2 & (\lambda+\mu)\boldsymbol{j} \\ \lambda\boldsymbol{\ell} & -(\lambda+\mu)\boldsymbol{j} & 1. \end{pmatrix}$$

We note that the off-diagonal entries of this matrix are all Cayley numbers which are perpendicular to $\mathbb{C}$. We denote the euclidean distance between ξ and $\mathbb{C}\mathrm{P}^2$ by

$$\delta(\xi) = \min\{\|\zeta - \xi\| \mid \zeta \in \mathbb{C}\mathrm{P}^2\}.$$

In order to compute this distance, we note that every point in $\mathbb{C}\mathrm{P}^2 \subseteq \mathbb{O}\mathrm{P}^2$ is of the form

$$\zeta = \begin{pmatrix} |u|^2 & u\bar{v} & u\bar{w} \\ v\bar{u} & |v|^2 & v\bar{w} \\ w\bar{u} & w\bar{v} & |w|^2 \end{pmatrix},$$

where u, v, w are complex numbers with $|u|^2 + |v|^2 + |w|^2 = 1$. The point q corresponds to $(u, v, w) = (0, 0, 1)$. The euclidean inner product between $\xi = \xi_{\lambda,\mu}$ and ζ is given by

$$\langle \xi, \zeta \rangle = \frac{\lambda^2|u|^2 + (\lambda+\mu)^2|v|^2 + |w|^2}{\lambda^2 + (\lambda+\mu)^2 + 1},$$

because the off-diagonal entries of ξ are perpendicular to $\mathbb{C}$. From this formula we see the following. We have

$$\delta(\xi) = \|q - \xi\| \quad \text{if and only if} \quad \lambda \le 1 \text{ and } \lambda + \mu \le 1.$$

This condition defines a linear simplex (recall that $\lambda, \mu \ge 0$). From the formula for $\langle \xi, \zeta \rangle$, the following is immediate:

(1) If $\lambda + \mu < 1$, then q is the unique point in $\mathbb{C}\mathrm{P}^2$ at distance $\delta(\xi_{\lambda,\mu})$ from $\xi_{\lambda,\mu}$. In particular, $K_{\xi_{\lambda,\mu}} \subseteq K_q$.

(2) If $\lambda + \mu = 1 \neq \lambda$, then every point with complex coordinates $(u, v, w) \in \mathbb{S}^5$ and $u = 0$ realizes the distance δ. This condition defines a complex projective line in $\mathbb{C}\mathrm{P}^2$. Also, the point $\tilde{q}$ with complex coordinates $(u, v, w) = (1, 0, 0)$ is in this case the unique point in $\mathbb{C}\mathrm{P}^2$ at maximal distance from $\xi_{\lambda,\mu}$, hence $K_{\xi_{\lambda,\mu}} \subseteq K_{\tilde{q}}$.

(3) If $\lambda + \mu = 1 = \lambda$, then every point ζ in $\mathbb{C}\mathrm{P}^2$ has distance $\delta(\xi_{\lambda,\mu})$ from $\xi_{\lambda,\mu}$.

Lemma 3.11. *Every K-orbit contains a unique point $\xi_{\lambda,\mu}$ with $0 \le \lambda, \mu$ and $\lambda + \mu \le 1$.*

Proof. Let $\eta \in \mathbb{O}\mathrm{P}^2$ and let $\zeta \in \mathbb{C}\mathrm{P}^2$ be a point that has minimal euclidean distance from η. There exists $g \in K$ with $g(\zeta) = q$. Then $g(\eta)$ is not in the cut locus L of q, since $L \cap \mathbb{C}\mathrm{P}^2$ contains points which are strictly closer to any given point in L than q (we omit this short calculation). Hence $g(\eta) \notin L$. If one of the off-diagonal entries of the projector $g(\eta)$ is not perpendicular to $\mathbb{C}$, then the inner product shows that q is not the closest point to $g(\eta)$ on $\mathbb{C}\mathrm{P}^2$. Thus $g(\eta)$ is, as a point in $\mathbb{O} \times \mathbb{O}$, perpendicular to $\mathbb{C} \times \mathbb{C}$. Since we have a polar action on the normal space of q, there exists $h \in K_q$ such that $hg(\eta) = \xi_{\lambda,\mu}$, for some $\lambda, \mu \ge 0$. By the observations above, we have $\lambda + \mu \le 1$.

It remains to show the uniqueness. Let $\xi_{\lambda,\mu}$ be a point in the simplex. If $\lambda + \mu < 1$ and if $g(\xi_{\lambda,\mu}) = \xi_{\lambda',\mu'}$ is in the simplex, then $g(q) = q$, because q is the unique nearest point to $\xi_{\lambda,\mu}$. Therefore $(\lambda, \mu) = (\lambda', \mu')$, because the action of K_q on the normal space is polar. If $\lambda + \mu = 1 \ne \lambda$ and if $g(\xi_{\lambda,\mu}) = \xi_{\lambda',\mu'}$, then we see from the geometric description above that $\lambda' + \mu' = 1 \ne \lambda'$. The number λ is determined by the distance of $\xi_{\lambda,\mu}$ from $\mathbb{C}\mathrm{P}^2$, hence $(\lambda, \mu) = (\lambda', \mu')$. Finally, p is the unique point in the simplex that has constant distance from $\mathbb{C}\mathrm{P}^2$. □

The uniqueness statement of the previous lemma follows also from the fact that the K-action is polar, which we prove below. Also, we have worked with the euclidean distance, rather than with the inner metric of the Riemannian manifold $\mathbb{O}\mathrm{P}^2$. We will come back to this. But first we determine the stabilizers of the $\xi_{\lambda,\mu}$, where $\lambda + \mu = 1$.

3.12. The remaining orbit types. Suppose that $\lambda + \mu = 1 \ne \lambda$. Then the point $\tilde{q}$ with complex coordinates $(u, v, w) = (1, 0, 0)$ uniquely maximizes the euclidean distance from $\xi_{\lambda,\mu}$, as we noticed above. Thus $K_{\xi_{\lambda,\mu}} \subseteq K_{\tilde{q}}$. The involution $h = \left(\begin{smallmatrix} & & 1 \\ & -1 & \\ 1 & & \end{smallmatrix}\right) \times \left(\begin{smallmatrix} & & 1 \\ & -1 & \\ 1 & & \end{smallmatrix}\right) \in \mathrm{SU}(3) \times \mathrm{SU}(3)$ interchanges q and $\tilde{q}$.

For $\lambda \ne 0$, it maps $p\lambda + d\mu = (-\lambda \boldsymbol{\ell}, \boldsymbol{j})$ to $\big(-\frac{1}{\lambda}\boldsymbol{\ell}, -\frac{1}{\lambda}\boldsymbol{j}\big)$. The K-stabilizer of $p\lambda + d\mu$ consists then of the matrices

$$\begin{pmatrix} \bar{z} & \\ & \bar{Z}_1 \end{pmatrix} \times \begin{pmatrix} z & \\ & Z_1 \end{pmatrix} \in \mathrm{SU}(3) \times \mathrm{SU}(3),$$

with $Z_1 \in \mathrm{U}(2)$. By continuity, these matrices also fix p and d.

Suppose that $\lambda = 0$. The involution maps the projector coresponding to $d = (0, \boldsymbol{j})$ to

$$\theta = \tfrac{1}{2} \begin{pmatrix} 1 \\ -\boldsymbol{j\ell} \\ 0 \end{pmatrix} (1\ \boldsymbol{j\ell}\ 0) = \tfrac{1}{2} \begin{pmatrix} 1 & \boldsymbol{j\ell} & 0 \\ \overline{\boldsymbol{j\ell}} & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The element $\mathrm{id} \times \left(\begin{smallmatrix} c & -a\bar{s} & \\ s & a\bar{c} & \\ & & \bar{a} \end{smallmatrix}\right) \in \mathrm{Aut}_{\mathbb{C}}(\mathbb{O}) \times \mathrm{SU}(3)$ maps θ to $\frac{1}{2} \left(\begin{smallmatrix} 1 & \boldsymbol{j\ell}a & 0 \\ \overline{\boldsymbol{j\ell}a} & 1 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right)$ and therefore K_d consists of the matrices of the form

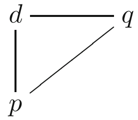
$$\begin{pmatrix} \bar{z} & \\ & Z_2 \end{pmatrix} \times \begin{pmatrix} z & \\ & Z_1 \end{pmatrix} \in \mathrm{SU}(3) \times \mathrm{SU}(3),$$

with $Z_1, Z_2 \in \mathrm{U}(2)$.

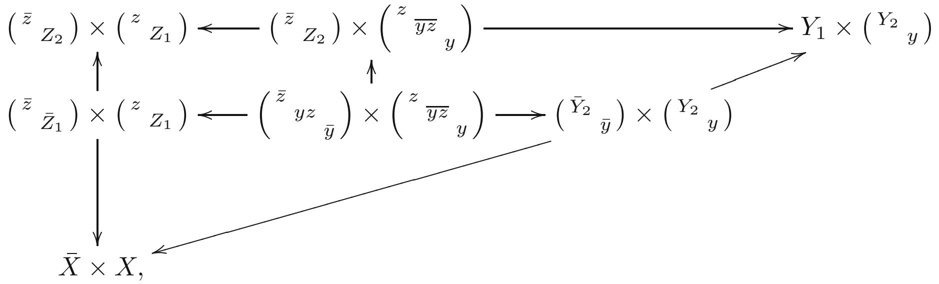
Lemma 3.13. *We have* $K_p = \{\bar X \times X \,|\, X \in \mathrm{SU}(3)\}$.

Proof. Let $H = \{\bar X \times X \,|\, X \in \mathrm{SU}(3)\}$. The block matrices $\begin{pmatrix}\bar Y & \\ & 1\end{pmatrix} \times \begin{pmatrix} Y & \\ & 1\end{pmatrix}$ and $\begin{pmatrix}1 & \\ & \bar Y\end{pmatrix} \times \begin{pmatrix} 1 & \\ & Y\end{pmatrix}$ with $Y \in \mathrm{SU}(2)$ fix p and generate H, hence H fixes p. The Lie algebra of H is maximal in $\mathfrak{su}(3) \oplus \mathfrak{su}(3)$, because $\mathfrak{su}(3)$ is simple. Thus $H = (K_p)^\circ$. If $1 \times A$ is in the kernel of $\mathrm{pr}_1 : K_p \to \mathrm{SU}(3)$, then $A \in \mathrm{Cen}(\mathrm{SU}(3))$. Such an element fixes p only if $A = 1$. Similarly, if $A \times 1$ fixes p, then $A = 1$. It follows that $H = K_p$. □

3.14. The corresponding complex of groups. The kernel of the K-action is the group $Z = K_{p,d,q} \cap \mathrm{Cen}(\mathrm{SU}(3) \times \mathrm{SU}(3)) \cong \mathbb{Z}/3$ and we put $G = K/Z$. The simple complex of groups in K formed by the seven types of stabilizers, corresponding to the faces of the simplex



looks as follows:



with $Y_2, Z_1, Z_2 \in \mathrm{U}(2)$ and $Y_1, X \in \mathrm{SU}(3)$. The corresponding simple complex of groups in G is obtained by taking matrices mod Z.

An isometric action of a Lie group G on a complete Riemannian manifold M is called *polar* if there exists a complete submanifold $\Sigma \subseteq M$ which meets every orbit orthogonally, i.e.,

$$G(\Sigma) = M \quad \text{and} \quad T_\sigma\Sigma \perp T_\sigma G(\sigma) \text{ holds for every } \sigma \in \Sigma.$$

This is the case for our action. We define an immersion $\sigma : \mathbb{S}^2 \to \mathbb{O}\mathrm{P}^2$ by putting

$$\sigma(x,y,z) = \begin{pmatrix} x^2 & -\boldsymbol{j\ell} xy & -\boldsymbol{\ell} zx \\ \boldsymbol{j\ell} xy & y^2 & \boldsymbol{j} yz \\ \boldsymbol{\ell} zx & -\boldsymbol{j} yz & z^2 \end{pmatrix} \in \mathbb{O}\mathrm{P}^2$$

and we put $\Sigma = \sigma(\mathbb{S}^2)$. The surface Σ is isometric to $\mathbb{R}\mathrm{P}^2$. The following is proved in [PoTh].

Theorem 3.15 (Podestà–Thorbergsson)**.** *The action of* $G = (\mathrm{SU}(3) \times \mathrm{SU}(3))/Z$ *on* $\mathbb{O}\mathrm{P}^2$ *is polar and* Σ *is a section.*

Proof. The simplex which we considered above is contained in Σ, hence $G(\Sigma) = \mathbb{O}\mathrm{P}^2$ by 3.11. Let $\xi = \sigma(x,y,z) \in \Sigma$. We claim that $T_\xi\Sigma \perp T_\xi G(\xi)$.

Let $\dot g \in \mathfrak{su}(3)$. We view $\dot g$ as an element of $\mathfrak{su}(3)\oplus 0 \subseteq \mathrm{Lie}(\mathrm{SU}(3)\times\mathrm{SU}(3))$. Then $\dot g$ acts via ordinary 3×3-matrix multiplication as $\xi \longmapsto \dot g\xi - \xi\dot g \in T_\xi G(\xi)$. Now let $\dot\xi \in T_\xi\Sigma$. A short and elementary calculation shows that the matrix product $\xi\dot\xi$ is a matrix whose off-diagonal entries are Cayley numbers perpendicular to $\mathbb{C}$, while the entries on the diagonal are real. The same holds for $\dot\xi\xi$. We denote the real part of a Cayley number a by $\mathrm{Re}(a)$ and extend this entry-wise to matrices. Then we have

$$\begin{aligned}\langle \dot g\xi - \xi\dot g, \dot\xi\rangle &= \mathrm{trace}((\dot g\xi - \xi\dot g)\dot\xi)\\ &= \mathrm{Re}(\mathrm{trace}(\dot g\xi\dot\xi) - \mathrm{trace}(\xi\dot g\dot\xi))\\ &= \mathrm{Re}(\mathrm{trace}(\dot g\xi\dot\xi)) - \mathrm{Re}(\mathrm{trace}(\xi\dot g\dot\xi))\\ &= \mathrm{trace}(\mathrm{Re}(\dot g\xi\dot\xi)) - \mathrm{trace}(\mathrm{Re}(\dot g\dot\xi\xi))\\ &= 0.\end{aligned}$$

Now let $\dot h$ be an element of $\mathrm{Lie}(\mathrm{Aut}_{\mathbb{C}}(\mathbb{O})) = \mathfrak{su}(3)$. Because $\dot h$ has imaginary entries on its diagonal, we have $\langle \dot h(\boldsymbol{j}), \boldsymbol{j}\rangle = \langle \dot h(\boldsymbol{\ell}), \boldsymbol{\ell}\rangle = \langle \dot h(\boldsymbol{j\ell}), \boldsymbol{j\ell}\rangle = 0$. On the three real diagonal entries of ξ, the infinitesimal automorphism $\dot h$ acts as multiplication by 0. Therefore $\langle \dot\xi, \dot h(\xi)\rangle = 0$.

This shows that $T_\xi\Sigma \perp T_\xi G(\xi)$. □

3.16. The Riemannian metric. The linear simplex which we considered in $\mathbb{O}\times\mathbb{O}$ is contained in Σ and has geodesic edges (and constant curvature) in $\mathbb{O}\mathrm{P}^2$. The quotient $G\backslash\mathbb{O}\mathrm{P}^2$ is isometric to a spherical simplex of shape C_3.

The previous results give us a geometric description of the orbits $G(d)$ and $G(p)$. The orbit $G(p)$ consists of all points in $\mathbb{O}\mathrm{P}^2$ having maximal (inner or euclidean) distance from $\mathbb{C}\mathrm{P}^2$. The orbit $G(d)$ consists of all points which have the property that a (euclidean or inner-metric) ball around them touches $\mathbb{C}\mathrm{P}^2$ in a 2-sphere, and which have maximal distance from $\mathbb{C}\mathrm{P}^2$ with respect to this property. We remark that the embedding of $\mathbb{C}\mathrm{P}^2$ is *tight*: every euclidean ball that touches $\mathbb{C}\mathrm{P}^2$ does this either in a unique point, along a 2-sphere, or everywhere.

Proposition 3.17. *Let Δ denote the simplicial complex whose nerve is the covering of G by the cosets of G_p, G_d and G_q, as defined in* 3.14. *Then (G,Δ) is a homogeneous compact geometry of type C_3 which is not a building. We have $|\Delta|_K = \mathbb{O}\mathrm{P}^2$.*

Proof. We can identify the nonempty simplices with the cosets of the various G_α, for $\varnothing \neq \alpha \subseteq \{p,d,q\}$. From the diagram in 3.14 above it is clear that the link of G_p is isomorphic to the 2-dimensional complex projective geometry. From 3.3 we see that the link of G_q is isomorphic to the generalized quadrangle corresponding to the hermitian form $h = (-f_2)\oplus f_3$ on $\mathbb{C}^{2+3}$. The link of G_d is isomorphic to the generalized digon $\mathbb{S}^2 \leftarrow \mathbb{S}^2\times\mathbb{S}^3 \rightarrow \mathbb{S}^3$. In particular, $\mathrm{lk}(d)$ is a complete bipartite graph. It follows that every triangle in the 1-skeleton $\Delta^{(1)}$ which contains d is filled by a 2-simplex. From the transitive action of G we conclude that Δ is a flag complex. From the diagram 3.14 we see that $G = G_pG_q$. Thus Δ is (gallery-) connected and by 1.4 a geometry of type C_3.

Since $G = G_pG_q$, the plane stabilizer G_p acts transitively on the set of points G/G_q. In other words, a point and a plane in Δ are always incident (such geometries are called *flat* in [Pas]). This cannot hold in a polar space.

Finally we note that we have a G-equivariant bijective map $|\Delta|_K \to \mathbb{O}\mathrm{P}^2$ which sends $g(G_p \cdot \lambda + G_d \cdot \mu + G_q \cdot \nu) \in |\Delta|$ to $g(\xi_{\lambda,\mu,\nu}) \in \mathbb{O}\mathrm{P}^2$. □

Theorem 3.18. *Let $\mathcal{G}$ denote the simple complex of groups from* 3.14. *Up to isomorphism, there is exactly one homogeneous compact geometry (G, Δ) of type C_3 belonging to this complex of groups.*

Proof. Let $(\widehat{G}, \widehat{\Delta})$ denote the universal homogeneous compact geometry for (G, Δ), as in 2.27. We denote the vertex stabilizers corresponding to $\mathcal{G} \to \widehat{G}$ by $\widehat{G}_\alpha$, for $\alpha \subseteq \{p, d, q\}$. We have by 2.26 a surjective equivariant map $(\widehat{G}, \widehat{\Delta}) \to (G, \Delta)$. Let $F \subseteq \widehat{G}$ denote its kernel. Thus $\mathrm{Lie}(\widehat{G}) \cong \mathrm{Lie}(G) \oplus \mathrm{Lie}(F)$. Let $\mathrm{pr}_2 : \mathrm{Lie}(\widehat{G}) \to \mathrm{Lie}(F)$ denote the projection onto the second summand and suppose that $\mathrm{Lie}(F) \neq 0$. Since $\widehat{G}$ is generated by $\widehat{G}_p \cup \widehat{G}_q$, see 1.8, either $\mathrm{pr}_2(\mathrm{Lie}(\widehat{G}_p)) \neq 0$ or $\mathrm{pr}_2(\mathrm{Lie}(\widehat{G}_q)) \neq 0$. Moreover, we have $\dim(F) \leq 7$ by 2.13. Since $\mathfrak{su}(3)$ is simple and 8-dimensional, we have $\mathrm{pr}_2(\mathrm{Lie}(\widehat{G}_p)) = 0$ and $\mathrm{pr}_2 : \mathrm{Lie}(\widehat{G}_q) \to \mathrm{Lie}(F)$ annihilates the $\mathfrak{su}(3)$-summand. From the diagram above and the fact that the pr_2-image of $\mathrm{Lie}(\widehat{G}_q)$ is nontrivial, we see that pr_2 is not trivial on $\mathrm{Lie}(\widehat{G}_{p,q})$. This is a contradiction, since $\mathrm{Lie}(\widehat{G}_{p,q}) \subseteq \mathrm{Lie}(\widehat{G}_p)$. Thus $\mathrm{Lie}(F) = 0$ and F is finite. Since F acts freely by 2.26, $F \subseteq \pi_1(|\Delta|_K) = \pi_1(\mathbb{O}\mathrm{P}^2) = 1$. This shows that (G, Δ) is universal.

Finally, we note that the $\mathbb{Z}/2$-Lefschetz number of every self-homeomorphism φ of $\mathbb{O}\mathrm{P}^2$ is 1, hence φ has a fixed point. Therefore $|\Delta|_K$ admits no continuous free action and in particular no quotients, see, e.g., Brown [Brow, p. 42] or [CPP, 55.19]. □

The following result is a consequence of our classification in Section 4.

Proposition 3.19. *Suppose that (G, Δ) is the exceptional compact homogeneous C_3 geometry from* 3.17 *and suppose that a compact connected group H acts continuously, faithfully and transitively on the chambers. Then H is conjugate to the group G in the group of topological automorphisms of Δ.*

Proof. The group H is a compact connected Lie group by 2.10. We consider the chamber $\gamma = \{p, d, q\}$. The fundamental group of the set of chambers G/G_γ is finite. Therefore the semisimple commutator group $K = [H, H]$ acts transitively on the chambers, see [Oni, p. 94]. From the long exact homotopy sequence for the transitive action of K on $K/K_p \cong \mathrm{SU}(3)$ we see that K_p is connected and semisimple. Similarly, we see from the transitive action of K on $K/K_q \cong \mathbb{C}\mathrm{P}^2$ that K_q has a 1-dimensional center. This is all that is needed in 4.24 in order to determine the possibilities for the simple complex of groups $\mathcal{K}$ formed by the stabilizers. Thus there are at most two possibilities for $\mathcal{K}$, and the corresponding universal homogeneous compact geometries are by 4.1 either a polar space or the exceptional geometry. Since Δ is not covered by a building, the compact universal covering is (G, Δ). Therefore we have a continuous isomorphism $(G, \Delta) \to (K, \Delta)$. Finally, we have $H = K$ because the connected K-normalizer of K_p is K_p, see 2.18. □

Remark 3.20. There is another way to approach the exceptional geometry. Starting from the fact that the G-action on $\mathbb{O}\mathrm{P}^2$ is polar and has a spherical simplex as its metric orbit space, one may consider the simple complex of groups formed by the stabilizers corresponding to the faces of the simplex. The horizontal simplicial complex corresponding to the action can be shown to be a compact geometry of type C_3, whose coarse realization is homeomorphic to $\mathbb{O}\mathrm{P}^2$, see Lytchak [Lyt] and Fang–Grove–Thorbergsson [FGTh]. The proof of the main theorem in [Lyt] shows that this geometry cannot be covered by a building. The classification in the following sections shows that there is at most one candidate for such a simple complex of groups. Therefore, this candidate must describe our geometry. This very implicit way can be used to obtain the stabilizer of our action without explicit computations. The proof that this action is polar with a nice quotient, however, seems to require some calculations, as in [PoTh, p. 151–154].

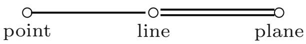
4. The classification of the universal homogeneous compact geometries of type C_3

Our aim in this section is the classification of the universal homogeneous compact geometries of type C_3. The main result of this section is as follows.

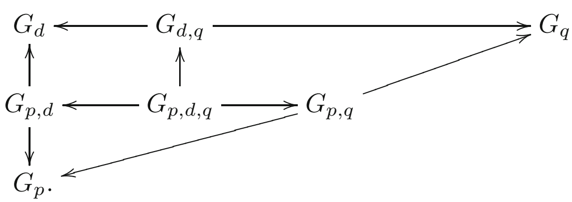
Theorem 4.1. *Let (G, Δ) be a homogeneous compact geometry of type C_3 with connected panels. Assume that G is compact and acts faithfully, and let $(\widehat{G}, \widehat{\Delta})$ denote the corresponding universal compact homogeneous geometry, as in* 2.27. *Then $\widehat{\Delta}$ is either a building or the exceptional geometry described in Section* 3B.

In order to prove this theorem, we classify the possibilities for the simple complex of groups $\mathcal{G}$. In view of 2.30 we assume that the homogeneous geometry (G, Δ) is both minimal and universal. The proof will be given at the end of Section 4.

4.2. Notation. We fix some notation that will be used throughout Section 4. We assume that (G, Δ) is a homogeneous compact geometry of type C_3 with connected panels. The Lie group G is compact and connected and acts faithfully. In the geometry Δ we have three types of vertices, called *points*, *lines* and *planes*:



We fix a chamber $\gamma = \{p, d, q\}$, where p is a plane, d is a line and q is a point (so $\mathrm{lk}(p)$ is a projective **p**lane, $\mathrm{lk}(d)$ is a generalized **d**igon and $\mathrm{lk}(q)$ is a generalized **q**uadrangle). The simple complex of groups $\mathcal{G}$ looks like this:



We note also that

$$G = \langle G_p \cup G_q \rangle = \langle G_p \cup G_d \rangle = \langle G_q \cup G_d \rangle,$$

because $G = \langle G_{p,d} \cup G_{d,q} \cup G_{q,p} \rangle$. The link $\mathrm{lk}(p)$ is one of the four compact Moufang planes over $\mathbb{R}, \mathbb{C}, \mathbb{H}$ or $\mathbb{O}$. Accordingly, the panels $\mathrm{lk}(\{p,d\})$ and $\mathrm{lk}(\{p,q\})$ are, in the coarse topology, spheres of dimension $m = 1, 2, 4, 8$. The link $\mathrm{lk}(q)$ is a compact connected Moufang quadrangle, the panel $\mathrm{lk}(\{d,q\})$ is an n-sphere, and the link $\mathrm{lk}(d)$ is a generalized digon. It is given by the two G_d-equivariant maps

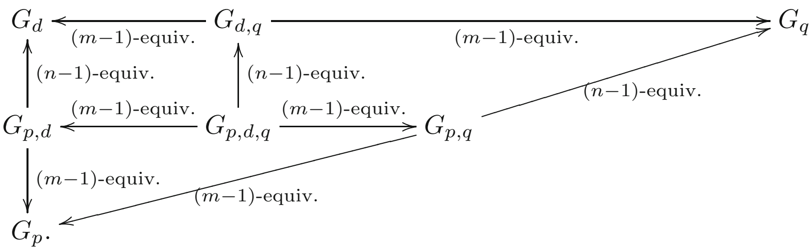
$$\mathbb{S}^m \leftarrow \mathbb{S}^m \times \mathbb{S}^n \rightarrow \mathbb{S}^n$$

as in 2.5. We have

$$\dim(G/G_\gamma) \leq 6m + 3n$$

by 2.13. If $m, n \geq 2$, then G is semisimple by 2.7.

4.3. Homotopy properties of $\mathcal{G}$. Recall that a continuous map is called a *k-equivalence* if it induces an isomorphism on the homotopy groups in degrees less than k and an epimorphism in degree k. The following diagram shows the low-dimensional homotopy properties of the maps in $\mathcal{G}$:



They follow from the fact that the quotients of the various isotropy groups are spheres, products of spheres or compact generalized polygons. For example, $G_q/G_{d,q}$ is the point space of a compact generalized quadrangle with topological parameters (m, n) and admits therefore a CW decomposition $G_q/G_{d,q} = e^0 \cup e^m \cup e^{m+n} \cup e^{m+n+m}$, see [Kr1, 3.4]. The long exact homotopy sequence of the fibration $G_{d,q} \to G_q \to G_q/G_{d,q}$ yields the $(m-1)$-connectivity of the homomorphism $G_{d,q} \to G_q$. The reasoning for the other homomorphisms is similar, using the results in *loc.cit.*

We now consider the homotopy groups π_0 and π_1. A compact connected Lie group is divisible (because tori are divisible and every element is contained in some torus, see, e.g., [HoMo, 6.30 or 9.35]). This implies the following. If H is a compact Lie group and if $\varphi : H \to F$ is a homomorphism to a finite group F, then φ factors through $\pi_0(H) = H/H^\circ$. In particular, φ is automatically continuous.

Lemma 4.4. *If $m, n > 1$, then all seven isotropy groups appearing in $\mathcal{G}$ are connected.*

If $m > n = 1$, then $\pi_0(G_d) = \pi_0(G_{d,q}) = \pi_0(G_q) = 1$.
If $n > m = 1$, then $\pi_0(G_p) = 1$.

Proof. If $m, n > 1$, then all maps in $\mathcal{G}$ are 1-equivalences and induce therefore isomorphisms on π_0. By the universal property 2.29 of G, there is a homomorphism $G \to \pi_0(G_{p,d,q})$ which is surjective, because the natural map $G_{p,q,z} \to \pi_0(G_{p,d,q})$ is surjective. The group G is connected and therefore $\pi_0(G_{p,d,q}) = 1$. If $m > n = 1$, then 4.3 shows similarly that $\pi_0(G_d) = \pi_0(G_{d,q}) = \pi_0(G_q) = 1$. The case $n > m = 1$ is analogous. $\square$

We need a similar result for the fundamental groups in order to control the torus factors of the stabilizers. This requires some low-dimensional cohomology.

Lemma 4.5. *Let K be a compact connected Lie group. There are natural isomorphisms*

$$\begin{array}{ccccc} \mathrm{Hom}(K,\mathbb{S}^1) & \xrightarrow{\cong} & H^1(K) & \xrightarrow{\cong} & \mathrm{Hom}(\pi_1(K),\mathbb{Z}) \\ \cong\uparrow & & \cong\uparrow & & \cong\uparrow \\ \mathrm{Hom}(K/[K,K],\mathbb{S}^1) & \xrightarrow{\cong} & H^1(K/[K,K]) & \xrightarrow{\cong} & \mathrm{Hom}(\pi_1(K/[K,K]),\mathbb{Z}), \end{array}$$

where H^1 denotes 1-dimensional singular cohomology and $\mathrm{Hom}(K,\mathbb{S}^1)$ denotes the group of continuous homomorphisms $K \to \mathbb{S}^1$.

Proof. For every path-connected space X we have by the Universal Coefficient Theorem

$$H^1(X) \cong \mathrm{Hom}(H_1(X),\mathbb{Z}) \cong \mathrm{Hom}(\pi_1(X),\mathbb{Z}),$$

see, e.g., [Mas, XII.4.6 and VIII.7.1]. We note that $K/[K,K]$ is a torus, hence $\pi_1(K/[K,K])$ is free abelian. The fundamental group of the semisimple group $[K,K]$ is finite, see [HoMo, 5.76]. From the split short exact sequence

$$1 \to \pi_1([K,K]) \to \pi_1(K) \to \pi_1(K/[K,K]) \to 1$$

we have therefore an isomorphism

$$\mathrm{Hom}(\pi_1(K),\mathbb{Z}) \xleftarrow{\cong} \mathrm{Hom}(\pi_1(K/[K,K]),\mathbb{Z}).$$

Since $\mathbb{S}^1$ is abelian, we also have a natural isomorphism

$$\mathrm{Hom}(K,\mathbb{S}^1) \xleftarrow{\cong} \mathrm{Hom}(K/[K,K],\mathbb{S}^1).$$

Since $\mathbb{S}^1 \simeq K(\mathbb{Z},1)$ is an Eilenberg–MacLane space representing 1-dimensional cohomology with integral coefficients, see, e.g., [Whi, V.7.5 and 7.14], we have for every connected Lie group H a natural map

$$\mathrm{Hom}(H,\mathbb{S}^1) \to [H,\mathbb{S}^1]_0 \cong H^1(H).$$

For the torus $H = K/[K,K]$ this map is an isomorphism, see [HoMo, 8.57(ii)]. $\square$

Corollary 4.6. *Let $\varphi : H \to K$ be a continuous homomorphism between compact connected Lie groups. If $\varphi_* : \pi_1(H) \to \pi_1(K)$ is bijective (surjective), then the abelianization $H/[H,H] \to K/[K,K]$ is bijective (surjective).*

Proof. If φ_* is bijective/surjective on the fundamental groups, then the map in 1-dimensional cohomology is bijective/injective by duality and 4.5. Thus

$$\mathrm{Hom}(H/[H,H],\mathbb{S}^1) \leftarrow \mathrm{Hom}(K/[K,K],\mathbb{S}^1)$$

is bijective/injective. Dualizing again, we obtain the claim by (Pontrjagin) duality. □

We now apply this result to $\mathcal{G}$ in order to control the torus factors. Recall that G is semisimple if $m, n \geq 2$.

Proposition 4.7. *If $m = 2 < n$, then G_p is semisimple and G_q and G_d have 1-dimensional centers, and $G_{p,d,q}$ has a 2-dimensional center. If $m, n \geq 3$, then all groups appearing in $\mathcal{G}$ are semisimple.*

Proof. A compact connected Lie group is perfect if and only if it is semisimple, see [HoMo, 6.16]. By 4.4 all groups G_α are connected. We consider the abelianizations $H_\alpha = G_\alpha/[G_\alpha, G_\alpha]$. Let $\mathcal{H}$ denote the diagram formed by these seven abelian groups H_α. Suppose that this diagram has a continuous homomorphism to some abelian topological group H. By the universal property of G, there is a unique homomorphism $G \to H$ commuting with the maps $G_\alpha \to H_\alpha \to H$. Since G is perfect, each composite map $G_\alpha \to G \to H$ is constant. It follows that the seven maps $H_\alpha \to H$ are also constant.

If $m, n \geq 3$, then all maps in $\mathcal{H}$ are isomorphisms by 4.6. From the previous paragraph we conclude that all groups in $\mathcal{H}$ are trivial. If $m = 2 < n$, then all groups in $\mathcal{H}$ surject naturally onto H_p. Again by the previous paragraph, $H_p = 1$. For $\alpha = \{p, d\}, \{p, q\}$ we have $G_p/G_\alpha \cong \mathbb{C}\mathrm{P}^2$ and $G_\alpha/G_{p,d,q} \cong \mathbb{S}^2$ and therefore short exact sequences

$$0 \to \mathbb{Z} \to \pi_1(G_\alpha) \to \pi_1(G_p) \to 0,$$
$$0 \to \mathbb{Z} \to \pi_1(G_{p,d,q}) \to \pi_1(G_\alpha) \to 0.$$

Thus $\dim H_\alpha = 1$ and $\dim H_{p,d,q} = 2$ by 4.5. □

4.8. The Lie algebra diagram Lie($\mathcal{G}$). Passing to the Lie algebras of the groups in $\mathcal{G}$, we obtain a commutative diagram of Lie algebra inclusions which we denote by $\mathrm{Lie}(\mathcal{G})$. The next proposition reduces in many cases the classification of the possible complexes $\mathcal{G}$ to the much simpler classification of the complexes of Lie algebras $\mathrm{Lie}(\mathcal{G})$. For $\varnothing \neq \alpha \subseteq \gamma$, we denote by $\widetilde{G_\alpha}$ the simply connected group with Lie algebra $\mathrm{Lie}(G_\alpha)$. In this way we obtain from $\mathrm{Lie}(\mathcal{G})$ a commutative diagram of simply connected Lie groups which we denote by $\widetilde{\mathcal{G}}$. We note that $\mathrm{Lie}(\mathcal{G})$ and $\widetilde{\mathcal{G}}$ encode exactly the same information, see, e.g., [HoMo, 5.42 and A2.26].

The group $\widetilde{G_\alpha}$ is the universal covering of $(G_\alpha)^\circ$ and we have a central extension

$$1 \to \pi_1(G_\alpha) \to \widetilde{G_\alpha} \to (G_\alpha)^\circ \to 1.$$

The identification of $\pi_1(G_\alpha)$ with the kernel of this map is compatible with the maps on the fundamental groups in $\mathcal{G}$.

Proposition 4.9. *If $m, n \geq 2$, then $\mathcal{G}$ is uniquely determined by* $\mathrm{Lie}(\mathcal{G})$.

Proof. We begin with a small observation. Let $z \in G_{p,d,q}$. If z is central in G_p and in G_q, then $z \in \mathrm{Cen}(G)$, because $G = \langle G_p \cup G_q \rangle$. It follows that $z = 1$, since G acts faithfully.

By Lemma 4.4, all groups G_α in $\mathcal{G}$ are connected. Therefore $\widetilde{\mathcal{G}}$ consists of the universal coverings of the G_α. We let $\pi_1 \cong \pi_1(G_{p,d,q})$ denote the kernel of the map $\widetilde{G_{p,d,q}} \to G_{p,d,q}$. From 4.3 we see that for each $\varnothing \neq \alpha \subseteq \gamma$, the group π_1 maps onto the kernel of $\widetilde{G_\alpha} \to G_\alpha$.

The group π_1 can now be characterized as follows. It consists of all elements $z \in \widetilde{G_{p,d,q}}$ whose images are central in each $\widetilde{G_\alpha}$. Indeed, every $z \in \pi_1$ has this property. Conversely, if $z \in \widetilde{G_{p,d,q}}$ has this property, then its image in every G_α is central and thus its image in $G_{p,d,q}$ is trivial by the small observation above. Thus π_1 is determined by $\widetilde{\mathcal{G}}$. It follows that $\mathcal{G}$ is determined by $\mathrm{Lie}(\mathcal{G})$. □

4.10. Kernels. We introduce some more notation. We denote by A, B and C the kernels of the actions of G_p, G_q and G_d on $\mathrm{lk}(p)$, $\mathrm{lk}(q)$ and $\mathrm{lk}(d)$, respectively. Their respective Lie algebras are denoted by $\mathfrak{a}$, $\mathfrak{b}$ and $\mathfrak{c}$. We choose supplements $\mathfrak{p}$, $\mathfrak{d}$ and $\mathfrak{q}$, such that

$$\begin{array}{lll} \mathrm{Lie}(G_p) = \mathfrak{g}_p = \mathfrak{p} \oplus \mathfrak{a}, & \mathrm{Lie}(G_q) = \mathfrak{g}_q = \mathfrak{q} \oplus \mathfrak{b}, & \mathrm{Lie}(G_d) = \mathfrak{g}_d = \mathfrak{d} \oplus \mathfrak{c}, \\ \mathrm{Lie}(G_p/A) \cong \mathfrak{p}, & \mathrm{Lie}(G_q/B) \cong \mathfrak{q}, & \mathrm{Lie}(G_d/C) \cong \mathfrak{d}, \end{array}$$

see, e.g., [HoMo, 5.78]. By 2.8 we have

$$A \cap B = B \cap C = C \cap A = 1 \qquad \text{and} \qquad \mathfrak{a} \cap \mathfrak{b} = \mathfrak{b} \cap \mathfrak{c} = \mathfrak{c} \cap \mathfrak{a} = 0.$$

Moreover, A is contained in $G_{p,d,q}$ and acts by 2.8 faithfully and as a subgroup of $\mathrm{O}(n)$ on $|\mathrm{lk}(\{d,q\})|_K \cong \mathbb{S}^n$. Similarly, B acts faithfully as a subgroup of $\mathrm{O}(m)$ on $|\mathrm{lk}(\{p,d\})|_K \cong \mathbb{S}^m$, and C acts faithfully as a subgroup of $\mathrm{O}(m)$ on $|\mathrm{lk}(\{p,q\})|_K \cong \mathbb{S}^m$.

Lemma 4.11. *If $m > n = 1$, then $A = 1$.*

Proof. By 4.4, the group $G_{d,q}$ is connected. Therefore it induces the group $\mathrm{SO}(2)$ on the 1-sphere $\mathrm{lk}(\{d,q\})$. The group $G_{p,d,q}$ acts thus trivially on $\mathrm{lk}(\{d,q\})$. In particular, A acts trivially on $\mathcal{E}_1(\{p,d,q\})$, hence $A = 1$ by 2.8. □

Lemma 4.12. *Suppose that $m > n = 1$ and that G_p is connected. Then $\mathcal{G}$ is determined by* $\mathrm{Lie}(\mathcal{G})$ *and the subdiagram*

$$\begin{array}{ccccc} G_{p,d} & \longleftarrow & G_{p,d,q} & \longrightarrow & G_{p,q} \\ \downarrow & & & \swarrow & \\ G_p. & & & & \end{array}$$

Proof. The proof is similar to the proof of 4.9 above. From our assumptions and 4.4 we see that all seven groups in $\mathcal{G}$ are connected. We define the diagram $\widetilde{\mathcal{G}}$ as in 4.8. The groups $\widetilde{G_\alpha}$ are thus the universal covering groups of the G_α. In $\widetilde{\mathcal{G}}$ we consider the two maps $\widetilde{G_d} \xleftarrow{\varphi} \widetilde{G_{d,q}} \xrightarrow{\psi} \widetilde{G_q}$. Since both $G_q/G_{d,q}$ and $G_d/G_{d,q}$ are 1-connected, we have

$$\widetilde{G_d}/\varphi(\widetilde{G_{d,q}}) = G_d/G_{d,q} \qquad \text{and} \qquad \widetilde{G_q}/\psi(\widetilde{G_{d,q}}) = G_q/G_{d,q}.$$

An element $z \in \widetilde{G_{d,q}}$ which acts trivially on $G_q/G_{d,q}$ acts trivially on $\mathrm{lk}(q)$. If it acts in addition trivially on $G_d/G_{d,q}$, then it acts trivially on $G_{p,d}/G_{p,d,q}$ and hence on $\mathcal{E}_1(\{p,d,q\})$. By 2.8, it acts then trivially on Δ. Let $\pi_1 \subseteq \widetilde{G_{d,q}}$ be the subgroup consisting of these elements. Then π_1 is precisely the kernel of the map $\widetilde{G_{d,q}} \to G_{d,q}$, i.e., $\pi_1 = \pi_1(G_{d,q})$. By 4.3, the group π_1 maps onto $\pi_1(G_d)$ and onto $\pi_1(G_q)$. Therefore $\widetilde{\mathcal{G}}$ determines the diagram $G_d \leftarrow G_{d,q} \to G_q$ completely. Since all groups in $\mathcal{G}$ are connected, the maps in $\widetilde{\mathcal{G}}$ determine also the maps in $\mathcal{G}$. □

For $m = 1$ we have to deal with stabilizers that are not connected. The identity components of the stabilizers form a subdiagram of $\mathcal{G}$ which we denote by $\mathcal{G}^\circ$.

Lemma 4.13. *Suppose that $n > m = 1$. Then $\mathcal{G}^\circ$ is determined by* $\mathrm{Lie}(\mathcal{G})$ *and the subdiagram*

$$\begin{array}{ccccc} G_{p,d} & \longleftarrow & G_{p,d,q} & \longrightarrow & G_{p,q} \\ \downarrow & & & \swarrow & \\ G_p. & & & & \end{array}$$

Proof. We argue similarly as in the proof of 4.12. For the four simplices α with $p \in \alpha \subseteq \{p,d,q\}$, we know already the kernels $\pi_1(G_\alpha)$ of the central extensions

$$\pi_1(G_\alpha) \to \widetilde{G_\alpha} \to (G_\alpha)^\circ,$$

since we know the groups G_α. For $\varnothing \neq \beta = \alpha - \{p\}$ the homomorphism $\pi_1(G_\alpha) \to \pi_1(G_\beta)$ is onto by 4.3, and this homomorphism, which is the restriction of $\widetilde{G_\alpha} \to \widetilde{G_\beta}$, is in turn determined by the homomorphism $\mathrm{Lie}(G_\alpha) \to \mathrm{Lie}(G_\beta)$. Therefore we know also the groups $(G_\beta)^\circ$, and, since they are connected, the maps between them. □

The problem is then to pass from $\mathcal{G}^\circ$ to $\mathcal{G}$. This requires the following homological fact, which allows us to determine G_α once we know $(G_\alpha)^\circ$ and G_α/N, where N is the kernel of G_α on $\mathrm{lk}(\alpha)$. See also Hilgert–Neeb [HiNe, 18.2] for a slightly more special result.

Lemma 4.14. *Suppose F and H are Lie groups, that F is connected and that $F \xrightarrow{p} H$ is an open and continuous homomorphism with discrete kernel D. Consider the category $\mathcal{C}$ of all Lie group homomorphisms $F \xrightarrow{i} E \xrightarrow{q} H$, where q is a surjective covering map and i is an open inclusion, such that $q \circ i = p$:*

$$\begin{array}{ccc} F & \overset{i}{\dashrightarrow} & E \\ {\scriptstyle p}\downarrow & & \vdots\,{\scriptstyle q} \\ H^\circ & \hookrightarrow & H. \end{array}$$

If the category $\mathcal{C}$ is nonempty, then its isomorphism classes are parametrized by the cohomology group $H^2(\pi_0(H), D)$. The group $\pi_0(H)$ acts on D by conjugation, and the cohomology is taken with respect to this action.

Proof. Suppose that $F \xrightarrow{i} E \xrightarrow{q} H$ is in $\mathcal{C}$. We view $\pi_0(H)$ as the group of path components of H. If X is a path component of H, then $E_X \to X$ is, as a bundle map, isomorphic to the bundle map $F \to H^\circ$. Hence every other Lie group solution $F \to E' \to H$ is isomorphic to one living on the same covering space E, but possibly with a different group multiplication.

We denote the given multiplication on E by a dot $\cdot$ and we assume that $* : E \times E \to E$ is another Lie group multiplication on E, compatible with q. Suppose that $X, Y, Z \in \pi_0(H)$ are path components with $XY = Z$. Then $E_X * E_Y = E_Z = E_X \cdot E_Y$, and for all $x \in E_X$, $y \in E_Y$ we have $q(x * y) = q(x \cdot y)$. The map

$$c : E \times E \to D, \qquad (x, y) \longmapsto (x * y) \cdot (x \cdot y)^{-1}$$

is locally constant and factors to a map

$$c : \pi_0(H) \times \pi_0(H) \to D.$$

The associativity of $*$ implies the cocycle condition. More precisely, we have the identities

$$c(U, VW) \cdot Uc(V, W)U^{-1} = c(UV, W) \cdot c(U, V) \quad \text{and} \quad c(H^\circ, U) = c(U, H^\circ) = 1$$

which say that c is a normalized 2-cocycle. Conversely, if c is a locally constant map that satisfies these two conditions, then $x * y = c(x, y) \cdot x \cdot y$ defines a new Lie group multiplication on E, as is easily checked. Finally, we have the group of deck transformations acting on these multiplications. These maps yield precisely the coboundaries, and the claim follows. $\square$

Note that the proof gives a method to construct all other multiplications from a given one. The case which is interesting for us is when $\pi_0(H) \cong \mathbb{Z}/2 \cong D$. Then the action of $\pi_0(H)$ on D is trivial and we have

$$H^2(\pi_0(H); D) = H^2(\mathbb{R}\mathrm{P}^\infty; \mathbb{Z}/2) \cong \mathbb{Z}/2.$$

Hence there are two multiplications in this case.

Example 4.15. Let $F = \mathrm{SO}(2)$, $D = \{\pm 1\}$ and $H = \mathrm{O}(2)/D$. There are two Lie groups E which fit into the diagram

$$\begin{array}{ccc} \mathrm{SO}(2) & \longrightarrow & E \\ \downarrow & & \downarrow \\ \mathrm{SO}(2)/D & \longrightarrow & \mathrm{O}(2)/D. \end{array}$$

One group is $E = \mathrm{O}(2)$, the other group is the 'fake O(2)', which is $E' = \mathrm{U}(1) \cup \boldsymbol{j}\mathrm{U}(1) \subseteq \mathrm{Sp}(1)$. For all $g \in \mathrm{O}(2) - \mathrm{SO}(2)$ we have $g^2 = 1$, whereas $g^2 = -1$ holds for all $g \in E' - \mathrm{U}(1)$. The group E' is formally obtained from O(2) by putting

$$u * v = \begin{cases} uv & \text{if } (\det(u), \det(v)) \neq (-1,-1), \\ -uv & \text{if } (\det(u), \det(v)) = (-1,-1). \end{cases}$$

Now we start the actual classification. As we noted above, we can identify $\mathfrak{q}$ with $\mathrm{Lie}(G_q/B)$, etc. In this way we obtain three diagrams for the Lie algebras of the groups acting faithfully on the links:

$$\begin{array}{ccc} \mathfrak{d} & \longleftarrow & \mathfrak{d}_q \\ \uparrow & & \uparrow \\ \mathfrak{d}_p & \longleftarrow & \mathfrak{d}_{p,q} \end{array} \qquad \begin{array}{ccc} \mathfrak{q}_d & \longrightarrow & \mathfrak{q} \\ \uparrow & & \nearrow \\ \mathfrak{q}_{p,d} & \longrightarrow & \mathfrak{q}_p \end{array}$$

$$\begin{array}{ccccc} \mathfrak{p}_d & \longleftarrow & \mathfrak{p}_{d,q} & \longrightarrow & \mathfrak{p}_q \\ \downarrow & & & \swarrow & \\ \mathfrak{p}. & & & & \end{array}$$

The groups belonging to these Lie algebras are known by [GKK1], [GKK2]. From this, we determine $\mathrm{Lie}(\mathcal{G})$ in the following way. We first determine the possible isomorphisms

$$\begin{array}{ccc} \mathfrak{q}_{p,d} \oplus \mathfrak{b} & \longrightarrow & \mathfrak{q}_p \oplus \mathfrak{b} \\ & \cong \Big\uparrow \iota & \\ \mathfrak{p}_{d,q} \oplus \mathfrak{a} & \longrightarrow & \mathfrak{p}_q \oplus \mathfrak{a}. \end{array}$$

Once this is done, it turns out in each case that there is just one possibility for the structure of $\mathfrak{d}$, and one possibility to fill in $\mathfrak{g}_d$. These data determine $\mathrm{Lie}(\mathcal{G})$. If $m, n \geq 2$, this determines $\mathcal{G}$. In the cases where $1 \in \{m, n\}$, further work is needed.

We now consider the cases $m = 1, 2, 4, 8$ separately. Accordingly, $(G_p/A)^\circ$ is one of the groups

$$\mathrm{SO}(3), \quad \mathrm{PSU}(3), \quad \mathrm{Sp}(3), \quad \text{or} \quad \mathrm{F}_4,$$

see [CPP, 63.8] and [GKK2]. Also, G_p/A is necessarily connected, unless we are in the case $m = 2$, where G_p/A may have two components. We begin with the case $m = 8$.

4A. The classification of $\mathcal{G}$ for $m = 8$

By 2.6, $\mathrm{lk}(p)$ is the projective plane over the Cayley algebra. The subalgebras $\mathfrak{p}_d, \mathfrak{p}_{d,q}, \mathfrak{p}_q \subseteq \mathfrak{p}$ form the following diagram, with the standard inclusions corresponding to the Cayley plane:

$$\begin{array}{ccccc} \mathfrak{so}(9) & \longleftarrow & \mathfrak{so}(8) & \longrightarrow & \mathfrak{so}(9) \\ \downarrow & & & \swarrow & \\ \mathfrak{f}_4. & & & & \end{array}$$

According to the Main Theorem in [GKK2] there is just one possibility for $\mathfrak{q}$, with $n = 1$. The corresponding part of the diagram for the subalgebras $\mathfrak{q}_d, \mathfrak{q}_p, \mathfrak{q}_{p,d} \subseteq \mathfrak{q}$ is as follows:

$$\begin{array}{ccccc} \mathfrak{so}(8)\oplus\mathbb{R} & & \longrightarrow & & \mathfrak{so}(10)\oplus\mathbb{R} \\ \uparrow & & & \nearrow & \\ \mathfrak{so}(8) & \longrightarrow & \mathfrak{so}(9). & & \end{array}$$

From 4.10 we see that $\mathfrak{a} = \mathfrak{b} = 0$. Up to inner automorphism, there is a unique possibility for the isomorphism ι. By 4.4, the group G_d is connected, and by 2.5 it induces $\mathrm{SO}(9) \times \mathrm{SO}(2)$ on $\mathrm{lk}(d)$, in its standard action on $\mathbb{S}^8 \times \mathbb{S}^1$. Up to inner automorphism, the isomorphisms which identify $\mathfrak{q}_{p,d} \to \mathfrak{q}_d$ and $\mathfrak{d}_{p,q} \to \mathfrak{d}_q$ are parametrized by a nonzero real number r (acting on the $\mathbb{R}$-summand). This determines the diagram $\mathrm{Lie}(\mathcal{G})$. The diagrams for different values of r are isomorphic. Thus, there is a unique possibility for $\mathrm{Lie}(\mathcal{G})$.

We have $A = 1$ by 4.11. All automorphisms of $\mathfrak{f}_4$ are inner and the corresponding compact Lie group F_4 is both centerless and simply connected, see, e.g., [CPP, 94.33]. In particular, $G_p \cong \mathrm{F}_4$ is connected and we have determined the subdiagram

$$\begin{array}{ccccc} G_{p,d} & \longleftarrow & G_{p,d,q} & \longrightarrow & G_{d,q} \\ \downarrow & & \swarrow & & \\ G_p. & & & & \end{array}$$

By 4.12, this diagram together with $\mathrm{Lie}(\mathcal{G})$ determines $\mathcal{G}$ uniquely.

Proposition 4.16. *If $m = 8$, then there is up to isomorphism a unique possibility for the diagram $\mathcal{G}$.* $\square$

This unique possibility for $\mathcal{G}$ is realized by the nonembeddable polar space, see 3.5. The minimal universal group is $\widehat{G} = \mathrm{E}_6 \cdot \mathbb{S}^1$, see [Hel, Chap. X, Table V] and [EH2].

4B. The classification of $\mathcal{G}$ for $m = 4$

By 2.6, $\mathrm{lk}(p)$ is the projective plane over the quaternions. The subalgebras $\mathfrak{p}_q$, $\mathfrak{p}_d$, $\mathfrak{p}_{d,q}$ form the following diagram, with the 'obvious' inclusions. We decorate the arrows by the kernels of actions on the respective spheres:

$$\begin{array}{ccccc} \mathfrak{sp}(1)\oplus\mathfrak{sp}(2) & \xleftarrow{\mathfrak{sp}(1)\oplus 0\oplus 0} & \mathfrak{sp}(1)\oplus\mathfrak{sp}(1)\oplus\mathfrak{sp}(1) & \xrightarrow{0\oplus 0\oplus\mathfrak{sp}(1)} & \mathfrak{sp}(2)\oplus\mathfrak{sp}(1) \\ \downarrow & & \swarrow & & \\ \mathfrak{sp}(3). & & & & \end{array}$$

According to the Main Theorem in [GKK2] there are the following possibilities for $\mathfrak{q}$, with $n \in \{1, 5\} \cup (3 + 4\mathbb{N})$.

4.17. ($n = 4\ell + 3$, for $\ell \geq 2$). There is one possibility for $\mathfrak{q}$, which is as follows:

$$\begin{array}{ccc} \mathfrak{sp}(1) \oplus \mathfrak{sp}(1) \oplus \mathfrak{sp}(\ell+1) & \longrightarrow & \mathfrak{sp}(2) \oplus \mathfrak{sp}(\ell+2) \\ \uparrow{\scriptstyle \mathfrak{sp}(1)\oplus 0 \oplus 0} & & \nearrow \\ \mathfrak{sp}(1) \oplus \mathfrak{sp}(1) \oplus \mathfrak{sp}(\ell) & \xrightarrow[0\oplus 0 \oplus \mathfrak{sp}(\ell)]{} & \mathfrak{sp}(2) \oplus \mathfrak{sp}(\ell). \end{array}$$

Thus $\mathfrak{a} = \mathfrak{sp}(\ell)$ and $\mathfrak{b} = \mathfrak{sp}(1)$. Up to inner automorphisms, there is a unique identification between $\mathfrak{p}_{d,q} \oplus \mathfrak{a} \to \mathfrak{p}_q \oplus \mathfrak{a}$ and $\mathfrak{q}_{p,d} \oplus \mathfrak{b} \to \mathfrak{q}_p \oplus \mathfrak{b}$ which is compatible with the action on $\mathbb{S}^4$. From 2.5 we see that G_d induces the group $\mathrm{SO}(5) \times (\mathrm{Sp}(\ell+1) \cdot \mathrm{Sp}(1))$ on $\mathbb{S}^4 \times \mathbb{S}^{4\ell+3}$. This determines $\mathfrak{g}_d \cong \mathfrak{sp}(2) \oplus \mathfrak{sp}(\ell+1) \oplus \mathfrak{sp}(1)$ and the remaining homomorphisms in $\mathrm{Lie}(\mathcal{G})$. By 4.9, $\mathcal{G}$ is determined by $\mathrm{Lie}(\mathcal{G})$. $\square$

4.18. ($n = 7$). There is one possibility for $\mathfrak{q}$, which is as follows:

$$\begin{array}{ccc} \mathfrak{sp}(1) \oplus \mathfrak{sp}(1) \oplus \mathfrak{sp}(2) & \longrightarrow & \mathfrak{sp}(2) \oplus \mathfrak{sp}(3) \\ \uparrow{\scriptstyle \mathfrak{sp}(1)\oplus 0 \oplus 0} & & \nearrow \\ \mathfrak{sp}(1) \oplus \mathfrak{sp}(1) \oplus \mathfrak{sp}(1) & \xrightarrow[0\oplus 0 \oplus \mathfrak{sp}(1)]{} & \mathfrak{sp}(2) \oplus \mathfrak{sp}(1). \end{array}$$

From this and 2.5 we see that G_d induces the group $\mathrm{SO}(5) \times (\mathrm{Sp}(2) \cdot \mathrm{Sp}(1))$ in its standard action on $\mathbb{S}^4 \times \mathbb{S}^7$. In particular, $\mathfrak{g}_{p,d,q}$ contains $\mathfrak{sp}(1) \oplus \mathfrak{sp}(1) \oplus \mathfrak{sp}(1) \oplus \mathfrak{sp}(1)$. From this we see that $\mathfrak{a} = \mathfrak{sp}(1)$ and $\mathfrak{b} = \mathfrak{sp}(1)$. The isomorphism ι is unique up to inner automorphisms. We end up with a unique diagram $\mathrm{Lie}(\mathcal{G})$ as in 4.17, with $\ell = 1$. This diagram determines $\mathcal{G}$ by 4.9. $\square$

4.19. ($n = 3$). There is one possibility for $\mathfrak{q}$, which is as follows:

$$\begin{array}{ccc} \mathfrak{sp}(1) \oplus \mathfrak{sp}(1) \oplus \mathfrak{sp}(1) & \longrightarrow & \mathfrak{sp}(2) \oplus \mathfrak{sp}(2) \\ \uparrow{\scriptstyle \mathfrak{sp}(1)\oplus 0} & & \nearrow \\ \mathfrak{sp}(1) \oplus \mathfrak{sp}(1) & \xrightarrow[0\oplus 0]{} & \mathfrak{sp}(2). \end{array}$$

From this and 2.5 we see that G_d induces the group $\mathrm{SO}(5) \times \mathrm{SO}(4)$ on $\mathbb{S}^4 \times \mathbb{S}^3$. Thus we have $\mathfrak{b} = \mathfrak{sp}(1)$ and $\mathfrak{a} = 0$. The isomorphism ι is unique up to inner automorphisms and $\mathrm{Lie}(\mathcal{G})$ is uniquely determined. This determines $\mathcal{G}$ by 4.9. $\square$

4.20. ($n = 5$). There is one generalized quadrangle, but two transitive connected groups, of type $\mathfrak{su}(5)$ and $\mathfrak{su}(5) \oplus \mathbb{R}$, respectively. By 2.7 and 4.7, the group G_q is semisimple, hence the ideal $\mathfrak{q}$ is also semisimple. The possibility for $\mathfrak{q}$ is thus as follows:

$$\begin{array}{ccc} \mathfrak{su}(3) \oplus \mathfrak{su}(2) & \longrightarrow & \mathfrak{su}(5) \\ \uparrow{\scriptstyle 0\oplus \mathfrak{su}(2)} & & \nearrow \\ \mathfrak{su}(2) \oplus \mathfrak{su}(2) & \xrightarrow[0\oplus 0]{} & \mathfrak{so}(5). \end{array}$$

From this and 2.5 we see that the group induced by G_d on $\mathbb{S}^4 \times \mathbb{S}^5$ is $\mathrm{SO}(5) \times \mathrm{SU}(3)$. It follows that $\mathfrak{b} = \mathfrak{sp}(1)$ and $\mathfrak{a} = 0$. The isomorphism ι is unique up to inner automorphisms and the diagram $\mathrm{Lie}(\mathcal{G})$ is uniquely determined. This determines $\mathcal{G}$ by 4.9. $\square$

4.21. ($n = 1$). There is a unique possibility for $\mathfrak{q}$, which is as follows:

$$\begin{array}{ccccc} \mathbb{R} \oplus \mathfrak{so}(4) & & \longrightarrow & & \mathbb{R} \oplus \mathfrak{so}(6) \\ \uparrow{\scriptstyle 0} & & & \nearrow & \\ \mathfrak{so}(4) & \xrightarrow[0]{} & \mathfrak{so}(5). & & \end{array}$$

By 4.4, the group G_d is connected. From this and 2.5 we see that the group induced by G_d on $\mathbb{S}^4 \times \mathbb{S}^1$ is $\mathrm{SO}(5) \times \mathrm{SO}(2)$. Also, we have $\mathfrak{b} = \mathfrak{sp}(1)$ and $\mathfrak{a} = 0$. The isomorphism ι is unique up to inner automorphisms and the diagram $\mathrm{Lie}(\mathcal{G})$ is uniquely determined by this. We have $A = 1$ by 4.11, hence $G_p = \mathrm{PSp}(3)$. From 4.12 we see that $\mathcal{G}$ is uniquely determined. $\square$

These are all possibilities for $m = 4$. In each case, there exists a building Δ corresponding to $\mathcal{G}$. The possibilities for G are given by [Hel, Chap. X, Table V] and [EH2]. They are as follows.

4.22. If $n = 4\ell + 3$, with $\ell \geq 0$, then Δ is the polar space associated to the quaternionic $(1, [a \mapsto \bar{a}])$-hermitian form

$$h = (-f_3) \oplus f_{3+\ell}$$

on $\mathbb{H}^{3+(3+\ell)}$. In this case $G = \big(\mathrm{Sp}(3) \times \mathrm{Sp}(3+\ell)\big)/\langle(-1,-1)\rangle$.

If $n = 1, 5$, then Δ is the polar space associated to the (unique) quaternionic $(-1, [a \mapsto \bar{a}])$-hermitian form on $\mathbb{H}^6$ or $\mathbb{H}^7$. Either $G = \mathrm{U}(6)/\langle -1 \rangle$, with $n = 1$, or $G = \mathrm{SU}(7), \mathrm{U}(7)/\langle -1 \rangle$, with $n = 5$.

4C. The classification of $\mathcal{G}$ for $m = 2$

By 2.6, $\mathrm{lk}(p)$ is the projective plane over $\mathbb{C}$. The subalgebras $\mathfrak{p}_q, \mathfrak{p}_d, \mathfrak{p}_{d,q}$ form the following diagram, with the 'obvious' inclusions. We decorate the arrows by the kernels of actions on the respective spheres:

$$\begin{array}{ccccc} \mathbb{R} \oplus \mathfrak{su}(2) & \xleftarrow[\mathbb{R}\oplus 0]{} & \mathbb{R} \oplus \mathbb{R} & \xrightarrow[0\oplus\mathbb{R}]{} & \mathfrak{su}(2) \oplus \mathbb{R} \\ \downarrow & & & \swarrow & \\ \mathfrak{su}(3). & & & & \end{array}$$

This case $m = 2$ is more complicated since we have to deal with reductive Lie algebras, where the complement of an ideal is not necessarily unique. We fix some more notation. We identify the Lie algebra $\mathfrak{su}(3)$ with the algebra of complex traceless skew-hermitian 3×3 matrices, and the upper line in the diagram above with the following inclusions in $\mathfrak{su}(3)$:

$$\begin{array}{ccccc} \mathfrak{p}_d & \longleftarrow & \mathfrak{p}_{d,q} & \longrightarrow & \mathfrak{p}_q \\ \begin{pmatrix} * & & \\ & * & * \\ & * & * \end{pmatrix} & \longleftarrow & \begin{pmatrix} * & & \\ & * & \\ & & * \end{pmatrix} & \longrightarrow & \begin{pmatrix} * & * & \\ * & * & \\ & & * \end{pmatrix}. \end{array}$$

We have the following four 1-dimensional subalgebras of $\mathfrak{p}_{d,q} \cong \mathbb{R}^2$. Each pair of them spans $\mathfrak{p}_{d,q}$, and we use them below.

$$\mathfrak{z}_d = \mathrm{Cen}(\mathfrak{p}_d) = \left\{ \begin{pmatrix} -2si & & \\ & si & \\ & & si \end{pmatrix} \middle| \, s \in \mathbb{R} \right\}, \quad \mathfrak{t}_d = \mathfrak{p}_{d,q} \cap [\mathfrak{p}_d, \mathfrak{p}_d] = \left\{ \begin{pmatrix} 0 & & \\ & si & \\ & & -si \end{pmatrix} \middle| \, s \in \mathbb{R} \right\},$$
$$\mathfrak{z}_q = \mathrm{Cen}(\mathfrak{p}_q) = \left\{ \begin{pmatrix} si & & \\ & si & \\ & & -2si \end{pmatrix} \middle| \, s \in \mathbb{R} \right\}, \quad \mathfrak{t}_q = \mathfrak{p}_{d,q} \cap [\mathfrak{p}_q, \mathfrak{p}_q] = \left\{ \begin{pmatrix} si & & \\ & -si & \\ & & 0 \end{pmatrix} \middle| \, s \in \mathbb{R} \right\}.$$

According to [GKK1, GKK2], we have $n \in \{2\} \cup (2\mathbb{N}+1)$, and there are the following possibilities for $\mathfrak{q}$.

4.23. ($n = 2\ell + 1$ and $\ell \geq 2$). By 4.7, the group G_p is semisimple and G_d and G_q have 1-dimensional centers. The Lie algebra $\mathfrak{a}$ is then also semisimple. We let $L = G_q/B$ denote the group induced by G_q on $\mathrm{lk}(q)$, with $\mathfrak{q} \cong \mathrm{Lie}(L)$. From the Main Theorem in [GKK2] we see that there are two possibilities for L, both acting on the same generalized quadrangle. These actions can be understood from the two orbit equivalent polar representations of $\mathrm{SU}(2)\times\mathrm{SU}(\ell+2)$ and $\mathrm{U}(2)\times\mathrm{SU}(\ell+2)$ on $\mathbb{C}^{2\times(\ell+2)}$, as described in 3.3. The semisimple commutator group $[L, L]$ acts transitively on $\mathrm{lk}(q)$. We denote its Lie algebra by $\mathfrak{q}' = [\mathfrak{q}, \mathfrak{q}]$. The diagram for $\mathfrak{q}'$ looks as follows:

$$\begin{array}{ccccc} \mathbb{R} \oplus \mathfrak{su}(\ell+1) & & \longrightarrow & & \mathfrak{su}(2) \oplus \mathfrak{su}(\ell+2) \\ \uparrow{\scriptstyle 0\oplus 0} & & & \nearrow & \\ \mathbb{R} \oplus \mathfrak{su}(\ell) & \xrightarrow[0\oplus\mathfrak{su}(\ell)]{} & \mathfrak{su}(2) \oplus \mathfrak{su}(\ell). & & \end{array}$$

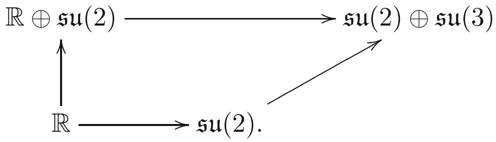
If $\mathfrak{q}$ is not semisimple (and we will see shortly that this is indeed the case), then $\mathfrak{q}_\alpha = \mathfrak{q}'_\alpha \oplus \mathbb{R}$ in the diagram above. We note also that $\mathfrak{su}(\ell) \subseteq \mathfrak{a}$. From the polar representation we see that the group induced by $[L, L]_d$ on $\mathrm{lk}(\{d, q\}) \cong \mathbb{S}^{2\ell+1}$ is $\mathrm{U}(\ell+1)$. From this and 2.5 we see that the group induced by G_d on $\mathrm{lk}(d)$ is $\mathrm{SO}(3) \times \mathrm{U}(\ell+1)$, in its natural action on $\mathbb{S}^2 \times \mathbb{S}^{2\ell+1}$. Now we determine the isomorphism ι

$$\begin{array}{ccc} \mathfrak{q}_{p,d} \oplus \mathfrak{b} & \longrightarrow & \mathfrak{q}_p \oplus \mathfrak{b} \\ & \cong \big\uparrow \iota & \\ \mathfrak{t}_d \oplus \mathfrak{t}_q \oplus \mathfrak{a} & \longrightarrow & \mathfrak{z}_q \oplus [\mathfrak{p}_q, \mathfrak{p}_q] \oplus \mathfrak{a}. \end{array}$$

Since $\mathfrak{g}_p$ is semisimple, we have $\mathfrak{a} = \mathfrak{su}(\ell)$. The pair $\mathfrak{t}_q \subseteq [\mathfrak{p}_q, \mathfrak{p}_q] \cong \mathfrak{su}(2)$ is identified with the pair $\mathbb{R} \subseteq \mathfrak{su}(2) \subseteq \mathfrak{q}'_p$, and ι is unique up to inner automorphisms on this part. The group corresponding to the algebra $\mathfrak{t}_d$ acts trivially on $|\mathrm{lk}(\{d, q\})|_K \cong \mathbb{S}^n$, because we have a product action on $\mathrm{lk}(d)$. It acts, however, nontrivially on $|\mathrm{lk}(\{p, q\})|_K \cong \mathbb{S}^m$. There is a unique homomorphism $\mathfrak{t}_d \to \mathfrak{q}$ corresponding to such an action. It follows that $\mathfrak{q} = \mathfrak{q}' \oplus \mathbb{R}$ is not semisimple, that $\mathfrak{b} = 0$, and now we have determined the isomorphism $\iota : \mathfrak{p}_d \oplus \mathfrak{a} \to \mathfrak{q}_p$. The structure of $\mathrm{lk}(d)$ was already determined above. Thus $\mathrm{Lie}(\mathcal{G})$ is uniquely determined, and so is $\mathcal{G}$ by 4.9. □

Now we get to the interesting case where the exceptional geometry occurs.

4.24. $(n = 3)$**.** By 4.7, the group G_p is semisimple, and so is $\mathfrak{a}$. From 4.3 and 4.6 we see again that $\mathfrak{g}_q$ has a 1-dimensional center. We use the same notation as in 4.23. The diagram for $\mathfrak{q}'$ is as follows:



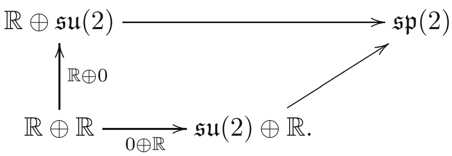
If $\mathfrak{q}$ is not semisimple, then we have again $\mathfrak{q}_\alpha = \mathfrak{q}'_\alpha \oplus \mathbb{R}$ in the diagram above. In either case, $\mathfrak{a} = 0$ (because it is semisimple), hence $\mathfrak{g}_{p,d,q} \cong \mathbb{R}^2$. Let $L = G_q/B$. The group induced by $[L, L]_d$ on $\mathrm{lk}(\{d, q\}) \cong \mathbb{S}^3$ is $\mathrm{U}(2)$. The isomorphism ι identifies the pair $\mathfrak{t}_q \to [\mathfrak{p}_q, \mathfrak{p}_q]$ with the pair $\mathbb{R} \to \mathfrak{su}(2)$ in the diagram above, uniquely up to inner automorphisms. So far, everything is completely analogous to 4.23.

By 2.5, we have two possibilities for the group induced on $\mathrm{lk}(d)$ by G_d. It is either the product action of $\mathrm{SO}(3) \times \mathrm{U}(2)$ on $\mathbb{S}^2 \times \mathbb{S}^3$ or the exceptional action of $\mathrm{SO}(4)$ on $\mathbb{S}^2 \times \mathbb{S}^3$ described after 2.5.

(1) Assume that we are in the case of the product action. Then $\mathfrak{t}_d$ acts trivially on $\mathrm{lk}(\{d, q\})$, but nontrivially on $\mathrm{lk}(\{p, q\})$. As in the case $\ell \geq 1$ before, there is a unique homomorphism $\mathfrak{t}_d \to \mathfrak{q}$ corresponding to this action, and we find that $\mathfrak{q} = \mathfrak{q}' \oplus \mathbb{R}$ is not semisimple. Thus ι is uniquely determined on $\mathfrak{p}_d = [\mathfrak{p}_q, \mathfrak{p}_q] + \mathfrak{t}_d$, and $\mathfrak{b} = 0$. This determines also $\mathfrak{g}_d$ and thus $\mathrm{Lie}(\mathcal{G})$. By 4.9, the complex $\mathcal{G}$ is uniquely determined. This case corresponds to the building.

(2) Suppose that G_d induces $\mathrm{SO}(4)$ in the exceptional action on $\mathbb{S}^2 \times \mathbb{S}^3$. Let C denote the kernel of the action of G_d on $\mathrm{lk}(d)$, with Lie algebra $\mathfrak{c}$. We have $\mathfrak{g}_d = \mathfrak{c} \oplus \mathfrak{d}$, and $\mathfrak{d} \cong \mathfrak{so}(4)$. Since $\mathfrak{t}_d \subseteq [\mathfrak{p}_d, \mathfrak{p}_d]$, we see that $\mathfrak{d}_{p,q} = \mathfrak{t}_d$. The group $C \subseteq G_{p,d,q}$, on the other hand, acts trivially on $\mathrm{lk}(\{p, d\})$. There is a unique subalgebra in $\mathfrak{p}_{d,q} = \mathfrak{g}_{p,d,q}$ with this property, namely $\mathfrak{z}_d$. Thus $\mathfrak{z}_d = \mathfrak{c}$ acts trivially on $\mathrm{lk}(\{d, q\})$. This determines a unique homomorphism $\mathfrak{z}_d \to \mathfrak{q}$. Thus $\iota : \mathfrak{p}_q \to \mathfrak{q}_p$ is uniquely determined. Also, $\mathfrak{g}_d$ is now uniquely determined, hence the same is true for $\mathrm{Lie}(\mathcal{G})$ and, by 4.9, for $\mathcal{G}$. This case does not correspond to a building, but to the exceptional polar action of $\mathrm{PSU}(3) \times \mathrm{SU}(3)$ on the Cayley plane, as described in Section 3B. □

4.25. $(n = 2)$**.** By 2.7, the group G is semisimple. By [GKK2], there are two non-isomorphic possibilities for $\mathfrak{q} \cong \mathfrak{sp}(2) \cong \mathfrak{so}(5)$, both of which are given by the following diagram, with different homomorphisms. One arrow corresponds to the natural inclusion $\mathfrak{u}(2) \subseteq \mathfrak{so}(5)$ (or $\mathfrak{sp}(1) \oplus \mathfrak{u}(1) \subseteq \mathfrak{sp}(2)$), the other to the natural inclusion $\mathfrak{so}(2) \oplus \mathfrak{so}(3) \subseteq \mathfrak{so}(5)$ (or $\mathfrak{u}(2) \subseteq \mathfrak{sp}(2)$):



From this diagram we see that either $\mathfrak{a} = 0 = \mathfrak{b}$ or $\mathfrak{a} \cong \mathbb{R} \cong \mathfrak{b}$. In particular, $2 \le \dim(G_{p,d,q}) \le 3$, and thus $\dim(G) \le 21$. Since $\mathfrak{p}$ is simple, there exists a simple factor $\mathfrak{h}$ of $\mathfrak{g}$ such that the canonical projection $\mathrm{pr}_{\mathfrak{h}} : \mathfrak{g} \to \mathfrak{h}$ is injective on $\mathfrak{p}$. Since $\mathfrak{q}$ is also simple and $\mathfrak{p} \cap \mathfrak{q} \neq 0$, the map $\mathrm{pr}_{\mathfrak{h}}$ is also injective on $\mathfrak{q}$. Thus $\mathfrak{h}$ is a simple compact Lie algebra which contains copies of $\mathfrak{su}(3)$ and $\mathfrak{so}(5)$, and with $\dim \mathfrak{h} \le 21$. From the list of low-dimensional simple compact Lie algebras [CPP, 94.33] and the low-dimensional representations of $\mathfrak{su}(3)$ and $\mathfrak{so}(5)$, see, e.g., [CPP, 95.10] and [Kr3, Chap. 4], we see readily that $\mathfrak{h} \in \{\mathfrak{su}(4), \mathfrak{so}(7), \mathfrak{sp}(3)\}$. We consider these three cases separately.

The case $\mathfrak{h} = \mathfrak{su}(4)$ is not possible. Suppose to the contrary that $\mathfrak{h} \cong \mathfrak{su}(4)$. We consider the natural representation $\mathbb{C}^4$ of $\mathfrak{su}(4)$. All copies of $\mathrm{pr}_{\mathfrak{h}}(\mathfrak{p}) \cong \mathfrak{su}(3)$ in $\mathfrak{su}(4)$ are conjugate and fix in this 4-dimensional representation a 1-dimensional complex subspace pointwise. Similarly, all copies of $\mathrm{pr}_{\mathfrak{h}}(\mathfrak{q}) \cong \mathfrak{sp}(2)$ in $\mathfrak{su}(4)$ are conjugate, with trivial $\mathfrak{su}(4)$-centralizers. In particular, $\mathrm{pr}_{\mathfrak{h}}(\mathfrak{b}) = 0$ and therefore $\mathrm{pr}_{\mathfrak{h}}(\mathfrak{g}_{p,q}) \cong \mathfrak{su}(2) \oplus \mathbb{R}$. Since $\mathfrak{p}_q \cong \mathfrak{u}(2)$, we see that $\mathrm{pr}_{\mathfrak{h}}(\mathfrak{g}_{p,q}) = \mathrm{pr}_{\mathfrak{h}}(\mathfrak{p}_q) \subseteq \mathrm{pr}_{\mathfrak{h}}(\mathfrak{p})$. Thus $\mathrm{pr}_{\mathfrak{h}}(\mathfrak{g}_{p,q})$ fixes a 1-dimensional subspace in $\mathbb{C}^4$ pointwise. On the other hand, neither the subalgebra $\mathfrak{u}(2) \subseteq \mathfrak{sp}(2) \subseteq \mathfrak{su}(4)$ nor the subalgebra $\mathfrak{sp}(1) \oplus \mathfrak{u}(1) \subseteq \mathfrak{sp}(2) \subseteq \mathfrak{su}(4)$ fix a 1-dimensional subspace in $\mathbb{C}^4$ pointwise. Therefore this case is not possible.

The case $\mathfrak{h} = \mathfrak{so}(7)$ is possible in a unique way. We consider the standard representation $\mathbb{R}^7$ of $\mathfrak{so}(7)$. Since $\dim(\mathfrak{so}(7)) = 21$, we have $\mathfrak{h} = \mathfrak{g}$ and $\mathfrak{a} \cong \mathbb{R} \cong \mathfrak{b}$. The inclusion $\mathfrak{g}_p \cong \mathfrak{u}(3) \subseteq \mathfrak{so}(7)$ is unique up to conjugation and fixes a unique 1-dimensional real subspace pointwise. This determines also how $\mathfrak{su}(2) \cong [\mathfrak{g}_{p,q}, \mathfrak{g}_{p,q}]$ is embedded in $\mathfrak{so}(7)$, namely as a conjugate of its standard real 4-dimensional representation.

The inclusion $\mathfrak{g}_q \cong \mathfrak{so}(2) \oplus \mathfrak{so}(5) \subseteq \mathfrak{so}(7)$ is also unique up to conjugation. We fix once and for all the standard inclusion of this algebra corresponding to the decomposition $\mathbb{R}^7 = \mathbb{R}^2 \oplus \mathbb{R}^5$, and we identify $[\mathfrak{g}_{p,q}, \mathfrak{g}_{p,q}]$ with $\mathfrak{su}(2) \subseteq \mathfrak{so}(5)$ acting on $\mathbb{C}^2 \oplus \mathbb{R} = \mathbb{R}^5$. Then $\mathfrak{g}_{p,q}$ has a unique real 1-dimensional fixed space in $\mathbb{R}^7$, and this determines the subalgebra $\mathfrak{g}_p \cong \mathfrak{u}(3)$ uniquely. This shows that there is at most one possibility for $\mathrm{Lie}(\mathcal{G})$, and by 4.9 also for $\mathcal{G}$.

The case $\mathfrak{h} = \mathfrak{sp}(3)$ is possible in a unique way. We consider the standard representation $\mathbb{H}^3$ of $\mathfrak{sp}(3)$. Since $\dim(\mathfrak{sp}(3)) = 21$, we have $\mathfrak{h} = \mathfrak{g}$ and $\mathfrak{a} \cong \mathbb{R} \cong \mathfrak{b}$. The inclusion $\mathfrak{g}_p \cong \mathfrak{u}(3) \subseteq \mathfrak{sp}(3)$ is unique up to conjugation. It corresponds to the extension of scalars given by $\mathbb{H}^3 = \mathbb{C}^3 \otimes_{\mathbb{C}} \mathbb{H}$. We identify $\mathfrak{g}_{p,q}$ with $\mathfrak{u}(1) \oplus \mathfrak{u}(2)$ in the standard inclusion coming from the splitting $\mathbb{H} \oplus \mathbb{H}^2 = (\mathbb{C} \oplus \mathbb{C}^2) \otimes_{\mathbb{C}} \mathbb{H}$. The inclusion of $\mathfrak{g}_q \cong \mathfrak{u}(1) \oplus \mathfrak{sp}(2) \subseteq \mathfrak{sp}(3)$ is also unique up to conjugation. From the splitting of $\mathbb{H}^3$ we see that there is a unique conjugate of $\mathfrak{g}_q$ containing $\mathfrak{g}_{p,q}$. Thus, there is at most one possibility for $\mathrm{Lie}(\mathcal{G})$, and by 4.9 also for $\mathcal{G}$.

Thus there are precisely two possibilities for $\mathcal{G}$ with $m = n = 2$. Both are realized by polar spaces over the complex numbers, one corresponding to the 5-dimensional nondegenerate quadratic form over $\mathbb{C}$, and the other to the 6-dimensional symplectic form over $\mathbb{C}$. $\square$

4.26. $(\boldsymbol{n=1})$**.** There is a unique possibility for $\mathfrak{q}$, which is as follows:

$$\begin{array}{ccc} \mathbb{R}\oplus\mathbb{R} & \longrightarrow & \mathbb{R}\oplus\mathfrak{so}(4) \\ \uparrow & \nearrow & \\ \mathbb{R} \longrightarrow \mathfrak{so}(3). & & \end{array}$$

The groups G_q, $G_{d,q}$ and G_d are connected by 4.4, and $A=1$. By 2.5, the group induced by G_d on D is $\mathrm{SO}(3)\times\mathrm{SO}(2)$, in its natural action on $\mathbb{S}^2\times\mathbb{S}^1$. In particular, $G_{p,d}$ induces the group $\mathrm{SO}(3)$ on $\mathrm{lk}(\{p,d\})$, and not the group $\mathrm{O}(3)$. Therefore $G_p=\mathrm{PSU}(3)$, and all groups in $\mathcal{G}$ are connected. We now apply Lemma 4.12 and conclude that $\mathcal{G}$ is uniquely determined. $\square$

These are all possibilities for $m=2$. The corresponding universal compact geometries are as follows.

4.27. If $n=2\ell+1$, with $\ell\geq 0$ and $\ell\neq 1$, then Δ is the polar space associated to the complex $(1,[a\mapsto\bar{a}])$-hermitian form

$$h=(-f_3)\oplus f_{3+\ell}$$

on $\mathbb{C}^{3+(3+\ell)}$.

If $n=3$, then Δ is either the polar space associated to the complex $(1,[a\longmapsto\bar{a}])$-hermitian form

$$h=(-f_3)\oplus f_4$$

on $\mathbb{C}^{3+4}$, or the exceptional geometry from Section 3B.

If $n=2$, then Δ is either the polar space associated to the complex symplectic form on $\mathbb{C}^6$ or the polar space associated to the complex quadratic form on $\mathbb{C}^7$.

The compact connected chamber-transitive groups G on the universal geometry Δ are as follows. In the hermitian case we have $G=\mathrm{SU}(3)\cdot\mathrm{U}(3+\ell)$, or $G=\mathrm{SU}(3)\cdot\mathrm{SU}(3+\ell)$ for $\ell\geq 1$. In the symplectic case, the group is $G=\mathrm{Sp}(3)/\langle -1\rangle$, and in the orthogonal case it is $G=\mathrm{SO}(7)$. This follows from [Hel, Chap. X, Table V] and [EH2]. In the case of the exceptional C_3 geometry, $G=\mathrm{PSU}(3)\times\mathrm{SU}(3)$.

4D. The classification of $\mathcal{G}$ for $m=1$

By 2.6, the link $\mathrm{lk}(p)$ is the projective plane over $\mathbb{R}$. This will again be the starting point for our classification. The Lie algebra $\mathfrak{p}$ is isomorphic to $\mathfrak{so}(3)$, and G_p induces the group

$$K=G_p/A\cong\mathrm{SO}(3)$$

on $\mathrm{lk}(p)$. We have $K_d\cong\mathrm{O}(2)\cong K_q$ and $K_{d,q}\cong\mathbb{Z}/2\oplus\mathbb{Z}/2$. In particular, the groups $G_{p,d}$, $G_{p,d,q}$ and $G_{p,q}$ are not connected. We put

$$M=G_d/C \qquad\text{and}\qquad L=G_q/B.$$

By 2.5 we have a product action of M° on $\mathbb{S}^1\times\mathbb{S}^n$. The group M is not connected, because K_d induces the group $\mathrm{O}(2)$ on the 1-sphere $\mathrm{lk}(\{p,d\})$. We note also that both B and C inject into $\mathrm{O}(1)\cong\mathbb{Z}/2$, whence $\mathfrak{b}=\mathfrak{c}=0$.

4.28. The structure of $\mathrm{lk}(q)$ and of $L = G_q/B$. If $n \geq 2$, then the generalized quadrangle $\mathrm{lk}(q)$ belongs by [GKK2] to the symmetric bilinear form $(-f_2) \oplus f_{n+2}$ on $\mathbb{R}^{2+(n+2)}$. The action of

$$L = G_q/B$$

is given by a polar representation of $L \subseteq \mathrm{O}(2) \cdot \mathrm{O}(n+2)$ on $\mathbb{R}^{2\times(n+2)}$ which is orbit equivalent to the polar representation of $\mathrm{O}(2) \cdot \mathrm{O}(n+2)$ described in 3.3. The dot $\cdot$ indicates that the element $(-1,-1) = (-\mathrm{id}_{\mathbb{R}^2}, -\mathrm{id}_{\mathbb{R}^{n+2}})$ acts trivially. By [GKK2], the identity component of L is either $\mathrm{SO}(2)\cdot\mathrm{SO}(n+2)$, or $\mathrm{SO}(2) \times \mathrm{G}_2$ for $n = 5$, or $\mathrm{SO}(2) \cdot \mathrm{Spin}(7)$ for $n = 6$. These connected groups induce $\mathrm{SO}(2)$ on $\mathrm{lk}(\{p, q\})$, as one can easily check in the polar representation. Since we know that $G_{p,q}$ induces the group $\mathrm{O}(2)$ on $\mathrm{lk}(\{p, q\})$, we see that L is not connected. We have $\mathfrak{q}_{p,d} = \mathfrak{so}(n)$, or, for the two exceptional actions, $\mathfrak{q}_{p,d} = \mathfrak{su}(2)$ or $\mathfrak{q}_{p,d} = \mathfrak{su}(3)$. In any case, the Lie algebra $\mathfrak{a} = \mathfrak{q}_{p,d} = \mathfrak{g}_{p,d,q}$ is semisimple for $n \geq 3$. We will see that the group $\mathrm{SO}(2) \cdot \mathrm{Spin}(7)$ cannot occur in this setting.

If $n = 1$, then there are two possibilities. Either $\mathrm{lk}(q)$ is the generalized quadrangle of the symmetric bilinear form $(-f_2)\oplus f_3$ on $\mathbb{R}^{2+3}$ and $L^\circ = \mathrm{SO}(2)\times\mathrm{SO}(3)$, or $\mathrm{lk}(d)$ is the generalized quadrangle associated to the standard symplectic form on $\mathbb{R}^4$. The group L° is then $\mathrm{U}(2)/\{\pm 1\} \cong \mathrm{SO}(2) \times \mathrm{SO}(3)$. These two generalized quadrangles are dual to each other.

Lemma 4.29. *We have $G_p = \mathrm{SO}(3) \times A$. If $n \geq 2$, then A is connected.*

Proof. Let P be a compact connected supplement of A° in $(G_p)^\circ$, such that $(G_p)^\circ = P \cdot A^\circ$. We claim that $P = \mathrm{SU}(2)$ is not possible. Assume to the contrary that $P = \mathrm{SU}(2)$. Suppose first that $n \geq 3$. We put $Z = (P_d)^\circ$. This circle group contains the unique nontrivial central element z of P. Since $\mathfrak{g}_{p,d,q} = \mathfrak{a}$ is semisimple by 4.28, we have $Z = \mathrm{Cen}((G_{p,d})^\circ)^\circ$. Since we have a product action of M° on $\mathrm{lk}(d)$ and since Z is connected, Z acts trivially on $\mathrm{lk}(d, q)$ under the equivariant projection $\mathbb{S}^1 \times \mathbb{S}^n \to \mathbb{S}^n$. In particular, z acts trivially on $\mathcal{E}_1(\{p, d, q\})$. This is a contradiction to 2.8, hence $P = \mathrm{SO}(3)$ is centerless. Suppose now that $n \leq 2$ and that $P = \mathrm{SU}(2)$. Then $G_{p,d,q}$ contains the quaternion group $Q = \{\pm 1, \pm\boldsymbol{i}, \pm\boldsymbol{j}, \pm\boldsymbol{k}\}$. On the other hand, $G_{p,d,q}$ embeds into $\mathrm{O}(1) \times \mathrm{O}(1) \times \mathrm{O}(n)$. But this is impossible: every 1- or 2-dimensional real representation of Q annihilates the element -1.

Thus $P = \mathrm{SO}(3)$ is centerless simple. It follows that $A\cap P = 1$, hence $G_p = P \ltimes A$ is a semidirect product, and P centralizes A°. If $n \geq 2$, then G_p is connected by 4.4, whence $A^\circ = A$. If $n = 1$, then A is discrete and therefore centralized by P. □

As a consequence of the proof, we note that

$$G_{p,d} \cong G_{p,q} \cong \mathrm{O}(2) \times A$$

and that this product splitting is canonical, for all n.

Corollary 4.30. *If $n \geq 2$, then $C \subseteq (G_d)^\circ$ and $B \subseteq (G_q)^\circ$ and*

$$\pi_0(M) \cong \pi_0(G_d) \cong \pi_0(G_{p,d}) \cong \pi_0(G_{p,q}) \cong \pi_0(G_q) \cong \pi_0(L) \cong \mathbb{Z}/2.$$

Proof. From 4.29 we know that $\pi_0(G_{p,d}) \cong \pi_0(G_{p,q}) \cong \mathbb{Z}/2$. By 4.3, we also have $\pi_0(G_d) \cong \pi_0(G_q) \cong \mathbb{Z}/2$. The groups M and L are not connected, but they cannot have more components than G_d and G_q have, hence $\pi_0(M) \cong \pi_0(L) \cong \mathbb{Z}/2$. We have $C \subseteq G_d$, and if $C \not\subseteq (G_d)^\circ$, then M would be connected. Similarly, B has to be contained in $(G_q)^\circ$. $\square$

Corollary 4.31. *The case $n = 6$ with $\mathfrak{q} = \mathbb{R} \oplus \mathfrak{so}(7)$ cannot occur.*

Proof. Consider the 8-dimensional real irreducible representation of $\mathrm{Spin}(7)$. This representation is of real type, see [CPP, p. 625]. Since the nontrivial center of $\mathrm{Spin}(7)$ acts faithfully on $\mathbb{R}^8$, we have $-\mathrm{id}_{\mathbb{R}^8} \in \mathrm{Spin}(7)$.

Assume now to the contrary that $\mathfrak{q} = \mathbb{R} \oplus \mathfrak{so}(7)$, with $n = 6$. Because $\mathrm{Spin}(7)$ is self-normalizing in $\mathrm{O}(8)$ and L is not connected, we have necessarily

$$L = (\mathrm{O}(2) \times \mathrm{Spin}(7))/\langle(-1,-1)\rangle.$$

Let $\mathrm{SU}^{-}(3)$ denote the group generated by $\mathrm{SU}(3)$ and by complex conjugation on $\mathbb{C}^3$. From the polar representation on $\mathbb{R}^{2\times 8}$ we see that

$$L_p = \mathrm{S}(\mathrm{O}(2) \times \mathrm{SU}^{-}(3))/\langle(-1,-1)\rangle.$$

But $L_p = \mathrm{S}(\mathrm{O}(2) \times \mathrm{SU}^{-}(3))/\langle(-1,-1)\rangle$ cannot be written as a quotient of $G_{p,q} = \mathrm{O}(2) \times \mathrm{SU}(3)$. The reason for this is that the adjoint representation of $\mathrm{S}(\mathrm{O}(2) \times \mathrm{SU}^{-}(3))/\langle(-1,-1)\rangle$ splits off a module $\mathfrak{su}(3)$ with $\mathrm{SU}^{-}(3)$ acting faithfully on it, which is not the case for $\mathrm{O}(2) \times \mathrm{SU}(3)$. This is a contradiction to 4.29. $\square$

Lemma 4.32. *Corresponding to each diagram*

$$\begin{array}{ccc} \mathfrak{q}_d & \longrightarrow & \mathfrak{q} \\ \uparrow & & \uparrow \\ \mathfrak{q}_{p,d} & \longrightarrow & \mathfrak{q}_p \end{array}$$

as in 4.28*, there is, up to isomorphism, at most one possibility for the diagram* $\mathrm{Lie}(\mathcal{G})$*.*

Proof. If $n \neq 2$, then $\mathfrak{a}$ is semisimple and $\mathfrak{g}_{p,d}$ and $\mathfrak{g}_{p,q}$ have 1-dimensional centers, which must correspond to the circle groups acting on the 1-spheres $\mathrm{lk}(\{p,d\})$ and $\mathrm{lk}(\{p,q\})$. If $n = 2$, then $\mathfrak{a} \cong \mathbb{R}$ is not semisimple. But since we have $G_{p,q} = \mathrm{SO}(2) \times \mathrm{O}(2)$ by 4.29, the Lie algebra of the circle group $(K_q)^\circ$ is distinguished in $\mathfrak{g}_{p,q}$ by the fact that $\mathrm{Ad}(G_{p,q})$ acts nontrivially on it. The same applies to $\mathfrak{g}_{p,d}$. Thus, $\mathrm{Lie}(K_d)$ and $\mathrm{Lie}(K_q)$ are in any case distinguished subalgebras, and the possible isomorphisms

$$\begin{array}{ccc} \mathfrak{d}_p & \longleftarrow & \mathfrak{d}_{p,q} \\ & \cong\uparrow & \\ \mathrm{Lie}(K_d) \oplus \mathfrak{a} & \longleftarrow & \mathfrak{a} \end{array} \qquad \begin{array}{ccc} \mathfrak{q}_{p,d} & \longrightarrow & \mathfrak{q}_p \\ & \cong\uparrow & \\ \mathfrak{a} & \longrightarrow & \mathrm{Lie}(K_q) \oplus \mathfrak{a} \end{array}$$

are parametrized by nonzero reals. However, all choices of these parameters lead to isomorphic diagrams for the Lie groups. All other identification maps in $\mathrm{Lie}(\mathcal{G})$ are also unique up to automorphisms. $\square$

Recall from 4.13 that $\mathcal{G}^\circ$ denotes the subdiagram of $\mathcal{G}$ consisting of the identity components of the stabilizers.

Corollary 4.33. *Suppose that $n \geq 2$. Corresponding to each diagram*

$$\begin{array}{ccc} \mathfrak{q}_d & \longrightarrow & \mathfrak{q} \\ \uparrow & & \uparrow \\ \mathfrak{q}_{p,d} & \longrightarrow & \mathfrak{q}_p \end{array}$$

as in 4.28*, there is, up to isomorphism, at most one possibility for the diagram $\mathcal{G}^\circ$.*

Proof. This follows from 4.29, 4.32 and 4.13. $\square$

We identify $K_d \leftarrow K_{d,q} \rightarrow K_q$ with the matrix groups

$$\begin{pmatrix} * & & \\ & * & * \\ & * & * \end{pmatrix} \longleftarrow \begin{pmatrix} * & & \\ & * & \\ & & * \end{pmatrix} \longrightarrow \begin{pmatrix} * & * & \\ * & * & \\ & & * \end{pmatrix}$$

in SO(3) and we put

$$u = \begin{pmatrix} 1 & & \\ & -1 & \\ & & -1 \end{pmatrix} \qquad \text{and} \qquad v = \begin{pmatrix} -1 & & \\ & -1 & \\ & & 1 \end{pmatrix}$$

Proposition 4.34. *Suppose that $n \geq 2$. Corresponding to each diagram*

$$\begin{array}{ccc} \mathfrak{q}_d & \longrightarrow & \mathfrak{q} \\ \uparrow & & \uparrow \\ \mathfrak{q}_{p,d} & \longrightarrow & \mathfrak{q}_p \end{array}$$

as in 4.28*, there is, up to isomorphism, at most one possibility for the diagram $\mathcal{G}$.*

Proof. There is a unique possibility for the diagram $\mathcal{G}^\circ$ by 4.33. The group $H = (G_q)^\circ$ induces the group L° on $\mathrm{lk}(q)$ and acts transitively on the chambers of this link. From 4.3 we see that $H_p = (G_{p,q})^\circ$, whence $H_{p,d} = H_p \cap G_{p,d,q}$ and $H_d = H_{p,d}(G_{d,q})^\circ$. Thus the action of $(G_q)^\circ$ on $\mathrm{lk}(q)$ is uniquely determined by $\mathcal{G}^\circ$, and so is the kernel B. Similarly, the action of $(G_d)^\circ$ on $\mathrm{lk}(d)$ and the kernel C are uniquely determined.

We note that, by 4.3, the group G_q is generated by $H \cup \{u\}$. We know that u acts on the 1-sphere $\mathrm{lk}(\{p, q\})$ as a reflection. On the other hand, u is contained in $(G_d)^\circ$ and therefore we know in particular how it acts on $\mathrm{lk}(\{d, q\})$. Thus, the image of u in L is uniquely determined, and hence L is uniquely determined.

We put $B = \{1, z\}$. If $z = 1$, then $L = G_q$ is uniquely determined, and so is the image of u in L. If $z \neq 1$, then we apply 4.14 to the problem

$$\begin{array}{ccc} (G_q)^\circ & \dashrightarrow & G_q \\ \downarrow & & \vdots \\ L^\circ & \longrightarrow & L. \end{array}$$

By 4.14 and the remarks following it, there are two possibilities for the multiplication on G_q. We know that $u^2 = 1$. One of the two possible multiplications would give us $u * u = (uz) * (uz) = z$, which is wrong. So the correct multiplication on G_q is uniquely determined. There are two possible targets for u in G_q, differing by z, which act in the same way on $\mathrm{lk}(q)$. The element u acts trivially on $\mathrm{lk}(\{p, d\})$, while the product uz acts as a reflection on $\mathrm{lk}(\{p, d\})$, hence we know also the correct image of u in G_q. This determines the map $K_q \times A \to G_q$ uniquely.

A completely analogous discussion shows that there is a unique possibility for G_d and the map $K_d \times A \to G_d$. In particular, there is a unique possibility for $G_d \leftarrow K_{d,q} \times A \to G_q$. The diagram $\mathcal{G}$ is now uniquely determined. $\square$

It remains to consider the case $n = 1$. We have seen in 4.32 that there are precisely two possibilities for $\mathrm{Lie}(\mathcal{G})$. One is realized in the polar space associated to the symmetric bilinear form $(f_{-3}) \oplus f_4$ on $\mathbb{R}^{3+4}$ and the associated polar representation of $\mathrm{SO}(3) \times \mathrm{SO}(4)$ on $\mathbb{R}^{3 \times 4}$. It is analogous to the case $n > 1$ considered before, and we call this the *orthogonal situation*.

The other possibility is associated to the polar space corresponding to the standard symplectic form on $\mathbb{R}^6$. The associated polar representation is $\mathrm{U}(3)/\{\pm 1\}$ acting on the space of complex symmetric 3×3-matrices, via $(g, X) \longmapsto gXg^T$. In this action the group G_p is the stabilizer of $\begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}$, the group G_q is the stabilizer of $\begin{pmatrix} 1 & & \\ & 0 & \\ & & 0 \end{pmatrix}$, and the group G_d is the stabilizer of $\begin{pmatrix} 1 & & \\ & 1 & \\ & & 0 \end{pmatrix}$. We call this the *symplectic situation*.

The two generalized quadrangles that may appear as the link at q are dual to each other (isomorphic under a not type-preserving simplicial isomorphism). The connected component of $L = G_q/B$ is

$$L^\circ = \mathrm{SO}(2) \times \mathrm{SO}(3).$$

In the orthogonal situation, $(L^\circ)_p \cong \mathrm{SO}(2)$ acts with a two-element kernel on $\mathrm{lk}(\{p, q\})$, while $(L^\circ)_d \cong \mathrm{O}(2)$ acts faithfully on $\mathrm{lk}(\{d, q\})$. The L°-stabilizer of $\gamma = \{p, d, q\}$ acts trivially on $\mathrm{lk}(\{p, q\})$, and as a reflection on $\mathrm{lk}(\{d, q\})$.

In the symplectic situation, it is the other way around.

We note that in both cases u becomes trivial in $\pi_0(G_{p,d})$, $\pi_0(G_p)$ and $\pi_0(G_d)$, and that v becomes trivial in $\pi_0(G_{p,q})$, $\pi_0(G_p)$ and $\pi_0(G_q)$. The element v is not trivial in $\pi_0(G_d)$, because its action on $\mathbb{S}^1 \times \mathbb{S}^1$ is not orientation preserving. Since we have a product action of M° on $\mathrm{lk}(d)$, the element u acts trivially on $\mathrm{lk}(d)$, and in particular $C = \{1, u\} \subseteq (G_d)^\circ$.

4.35. (In the symplectic situation $\mathcal{G}$ is unique). The circle group $(K_q)^\circ$ acts with kernel $\{1, v\}$ on $\mathrm{lk}(\{p, q\})$. The group $(L_p)^\circ$, which must be its image, acts faithfully on $\mathrm{lk}(\{p, q\})$. Therefore we have $B = \{1, v\} \subseteq (G_q)^\circ$.

We claim that $A = 1$. Suppose to the contrary that $1 \neq a \in A$. Then a is nontrivial in $\pi_0(G_p)$ by 4.29. It is nontrivial in $\pi_0(G_d)$, since its action on $\mathbb{S}^1 \times \mathbb{S}^1$ is not orientation preserving. Also, it acts differently than the L°-stabilizer of γ, hence a is nontrivial in $\pi_0(G_q)$. It follows now easily that $\mathcal{G}$ admits a simple homomorphism onto $\mathbb{Z}/2$, hence G is by 2.29 and the remark preceding 4.4 not connected, a contradiction.

Therefore $A = 1$ and $\pi_0(G_\gamma) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$ has u, v as a $\mathbb{Z}/2$-basis. From the action of $\langle u, v \rangle = G_\gamma$ on $\mathrm{lk}(\{p, q\}) \cup \mathrm{lk}(\{d, q\})$ we see that $G_q/B = L = \mathrm{SO}(3) \times$

SO(2) is connected. Since $B \subseteq (G_q)^\circ$, the group G_q is also connected. Since $\mathfrak{q}_p$ is contained in the simple part $\mathfrak{q}' = [\mathfrak{q},\mathfrak{q}] \cong \mathfrak{so}(3)$ and since the corresponding connected circle group contains the nontrivial kernel B, we have $[G_q, G_q] \cong \mathrm{SU}(2)$ and we may identify $(K_q)^\circ$ with the subgroup $\mathrm{SO}(2) \subseteq \mathrm{SU}(2)$. The group $\mathrm{SU}(2) \times \mathrm{SO}(2)$ contains no involution u that normalizes SO(2) and acts by inversion. Thus $G_q = \mathrm{U}(2)$ and $G_{p,q} = \mathrm{O}(2) \subseteq \mathrm{U}(2)$, embedded in the standard way as the group of elements fixed by complex conjugation. The group $(G_{p,d})^\circ$ is determined by its Lie algebra, and $G_{p,d} = G_{p,d,q}(G_{p,d})^\circ$. This group can be identified with $\begin{pmatrix} \mathrm{O}(1) & \\ & \mathrm{U}(1) \end{pmatrix} \subseteq \mathrm{U}(2)$. The image of v in G_q, being in the kernel B, is $\begin{pmatrix} -1 & \\ & -1 \end{pmatrix}$. The action of $(G_{p,d})^\circ$ on $\mathrm{lk}(\{d,q\})$ has in its kernel the element $\begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$. Since $(G_{p,d})^\circ$ acts trivially on $\mathrm{lk}(\{p,d\})$, we conclude that this matrix is the image of u (and not of uv, which acts nontrivially on $\mathrm{lk}(\{d,q\})$) in G_q. Thus we know G_q and its stabilizers, and the homomorphism $K_q \to G_q$. It remains to determine G_d. But G_d is the quotient of $G_{p,d} \times G_{d,q}$, where we identify the respective images of u, v and uv. The diagram $\mathcal{G}$ is now completely determined. □

4.36. (In the orthogonal situation $\mathcal{G}$ is unique). The circle group $(K_q)^\circ$ acts with kernel $\{1, v\}$ on $\mathrm{lk}(\{p,q\})$. The group $(L_p)^\circ$, which must be its image, acts also with a 2-element kernel on $\mathrm{lk}(\{p,q\})$. The element v acts therefore as a reflection on $\mathrm{lk}(\{d,q\})$ and on $\mathrm{lk}(\{p,d\})$.

Let $Q \subseteq G_q$ denote the connected normal subgroup with Lie algebra $\mathfrak{so}(3)$. We claim that $Q = \mathrm{SO}(3)$. Suppose to the contrary that $Q = \mathrm{SU}(2)$, with center $\{1, z\}$. The circle group $(G_{d,q})^\circ \subseteq Q$ contains z. Since we have a product action on $\mathrm{lk}(d)$, the element z acts trivially on $\mathrm{lk}(d)$, and it acts trivially on $\mathrm{lk}(q)$. This contradicts 2.8. Thus $Q = \mathrm{SO}(3)$ has trivial center. It follows that $(G_q)^\circ = \mathrm{SO}(3) \times \mathrm{SO}(2)$. We have $G_q/G_{p,q} \cong \mathbb{R}\mathrm{P}^3$. From the exact sequence

$$1 \to \underbrace{\pi_1(G_{p,q})}_{\mathbb{Z}} \to \underbrace{\pi_1(G_q)}_{\mathbb{Z}\oplus\mathbb{Z}/2} \to \underbrace{\pi_1(\mathbb{R}\mathrm{P}^3)}_{\mathbb{Z}/2} \to \pi_0(G_{p,q}) \to \pi_0(G_q) \to 1$$

we have an isomorphism $\pi_0(G_{p,q}) \cong \pi_0(G_q)$. If $A \neq 1$, then we see from the action on $\mathcal{E}_1(\{p,d,q\})$ that $G_{p,d,q}$ has 8 elements, and from the action of $G_{p,q}$ on $\mathrm{lk}(\{p,q\})$ that $\pi_0(G_{p,q}) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$. As in the symplectic case, we conclude that $\pi_0(\mathcal{G})$ has a nontrivial simple homomorphism to $\mathbb{Z}/2$, which is impossible. Therefore $A = 1$ and $G_{p,d,q} = \langle u, v \rangle$ has 4 elements. From the action of $G_{p,d,q}$ on $\mathcal{E}_1(\{p,d,q\})$ we see that $B = 1$. Also, we know $(K_q)^\circ \to G_q$. Since $B = 1$, there is a unique target for u in G_q. This determines $K_q \to G_q$ completely. Also, $G_{d,q}$ is now determined by its Lie algebra and by $K_{d,q} \to G_q$.

Finally, $(G_{p,d})^\circ \cap (G_{d,q})^\circ = 1$, since u is not contained in $G_{d,q}$, hence $(G_d)^\circ = (G_{p,d})^\circ \times (G_{d,q})^\circ$. Also, we have $G_d \subseteq G_{p,d} \times G_{d,q}$ and we know the image of v in the first factor. The image in the second factor is uniquely determined by the action of v on $\mathrm{lk}(\{d,q\})$, a reflection, since $G_{d,q}$ acts faithfully on $\mathrm{lk}(\{d,q\})$. Thus, the target of v in G_d is uniquely determined, and this determines the remaining homomorphisms in $\mathcal{G}$. The diagram $\mathcal{G}$ is now completely determined. □

Proof of Theorem 4.1. In the previous sections we have determined, up to isomorphism, all possibilities for a simple complex of compact groups $\mathcal{G}$ arising from a homogeneous compact geometry in $\mathbf{HCG}(\mathsf{C}_3)$, with G being minimal. Each example that we found is either realized by a rank 3 polar space (a building), or by the exceptional C_3 geometry. □

5. Consequences and applications

In this last section we show first that in homogeneous compact geometries of higher rank, no exceptions occur.

Lemma 5.1. *Suppose that* (G, Δ) *is a homogeneous compact geometry in the category* $\mathbf{HCG}(\mathsf{F}_4)$*. Then* Δ *is continuously and equivariantly* 2*-covered by a compact connected Moufang building of type* F_4*.*

Proof. In the exceptional geometry of type C_3 from Section 3B, the panels have dimensions 2 and 3. In the geometry Δ, however, all panels belong to compact Moufang planes and have therefore dimensions 1, 2, 4, or 8 by 2.6. By 4.1, every link of type C_3 in Δ is covered by a C_3 building. By Tits' Theorem 1.16, there exists a building $\widetilde{\Delta}$ and a 2-covering $\rho : \widetilde{\Delta} \to \Delta$. By 2.22, the building $\tilde{\Delta}$ can be topologized in such a way that ρ becomes equivariant and continuous, and $\widetilde{\Delta}$ is the compact Moufang building associated to a simple noncompact Lie group. □

Lemma 5.2. *Suppose that* (G, Δ) *is a homogeneous compact geometry in the category* $\mathbf{HCG}(\mathsf{C}_4)$*. Then* Δ *is continuously and equivariantly* 2*-covered by a compact connected Moufang building of type* C_4*.*

Proof. We label the vertex types as follows:

Let $\gamma = \{v_1, v_2, v_3, v_4\}$ be a chamber, where v_i has type i. We have to show that $\mathrm{lk}(v_1)$ cannot be the exceptional C_3 geometry from Section 3B. Assume to the contrary that this is the case. For $\varnothing \neq \alpha \subseteq \gamma$, we put $\mathfrak{g}_\alpha = \mathrm{Lie}(G_\alpha)$ and we let $\mathfrak{n}_\alpha \trianglelefteq \mathfrak{g}_\alpha$ denote the Lie algebra of the kernel of the action on $\mathrm{lk}(\alpha)$. Finally, we put $\mathfrak{h}_\alpha = \mathfrak{g}_\alpha/\mathfrak{n}_\alpha$. This is the Lie algebra of the group induced by G_α on $\mathrm{lk}(\alpha)$.

We have $\mathfrak{h}_{v_1} = \mathfrak{su}(3) \oplus \mathfrak{su}(3)$ and, by 4.24, we have $\mathfrak{h}_{v_1,v_3} = \mathfrak{so}(4)$, corresponding to the exceptional action of $\mathrm{SO}(4)$ on $\mathbb{S}^2 \times \mathbb{S}^3$.

Now we consider $\mathfrak{h}_{v_3} \subseteq \mathfrak{su}(3) \oplus \mathfrak{so}(4)$. The projection $\mathrm{pr}_1 : \mathfrak{h}_{v_3} \to \mathfrak{su}(3)$ to the first factor is onto, since $\mathrm{PSU}(3)$ has no chamber-transitive closed subgroups. Let $\mathfrak{h}_2$ denote the kernel of this projection, and let $\mathfrak{h}_1 \cong \mathfrak{su}(3)$ be a supplement of the kernel, $\mathfrak{h}_{v_3} = \mathfrak{h}_1 \oplus \mathfrak{h}_2$. Every homomorphism from $\mathfrak{su}(3)$ to $\mathfrak{so}(4)$ is trivial, hence $\mathfrak{h}_1$ is the kernel of the projection $\mathrm{pr}_2 : \mathfrak{h}_{v_3} \to \mathfrak{so}(4)$. The Lie algebra $\mathfrak{h}_{v_3}$ splits therefore in its action on $\mathrm{lk}(v_3)$ as a direct sum. It follows that the Lie algebra $\mathfrak{h}_{v_1,v_3}$ splits also in its action on $\mathrm{lk}(\{v_1, v_3\})$. We have reached a contradiction.

As in the previous lemma, we conclude from 1.16 and 2.22 that there exists a compact building $\widetilde{\Delta}$ associated to a simple noncompact Lie group and a continuous equivariant covering $\rho : \widetilde{\Delta} \to \Delta$. □

The following two theorems summarize the main results of our classification.

Theorem 5.3. *Let M be a spherical irreducible Coxeter matrix of rank at least 4 and suppose that (G,Δ) is a homogeneous compact geometry in* $\mathbf{HCG}(M)$*. Then there exists a compact connected spherical building $\widetilde{\Delta}$ and a continuous 2-covering $\rho:\widetilde{\Delta}\to\Delta$.*

Proof. By the previous two lemmata and by induction we see that the link of every vertex is 2-covered by a building. The claim follows now as in the proof of 5.1. □

The next theorem is an immediate consequence of 2.4, 5.3 and 4.1. It contains the Theorem A of the introduction as a special case.

Theorem 5.4. *Let M be a Coxeter matrix of spherical type and let (G,Δ) be a homogeneous compact geometry in* $\mathbf{HCG}(M)$*. Suppose that the Coxeter diagram of M has no isolated nodes. Then there exists a homogeneous compact geometry $(K,\widetilde{\Delta})$ in* $\mathbf{HCG}(M)$ *which is a join of buildings associated to simple noncompact Lie groups and geometries of type* C_3 *which are isomorphic to the exceptional geometry in Section* 3B*, and a continuous* 2*-covering $\widetilde{\Delta}\to\Delta$, which is equivariant with respect to a compact connected Lie group K acting transitively on the chambers of $\widetilde{\Delta}$.*

Proof. We decompose Δ as a join $\Delta=\Delta_1*\Delta_2*\cdots*\Delta_m$ of irreducible factors, and we let H_i denote the group induced by G on Δ_i. Now we apply 4.1 and 5.3 to the homogeneous compact geometries (H_i,Δ_i). We obtain equivariant 2-coverings $\widetilde{\Delta}_i\to\Delta_i$, where $\widetilde{\Delta}_i$ is either a compact building or the exceptional C_3 geometry from Section 3B. Taking the join of these 2-coverings, we obtain the result that we claimed. We note that the group induced by K on Δ may be strictly larger than the group G we started with. □

Recall from Section 3 that an isometric group action $G\times X\to X$ on a complete Riemannian manifold X is called *polar* if it admits a *section* $\Sigma\subseteq X$, i.e., a complete totally geodesic submanifold that intersects every orbit perpendicularly. A polar action is called *hyperpolar* if the section Σ is flat. One motivation for the present work is a recent result by the second author [Lyt]. Relying on the classification of buildings by Tits and Burns–Spatzier [BuSp], the main theorem of [Lyt] contains a classification of polar foliations on symmetric spaces of compact type under the assumption that the irreducible parts of the foliation have codimension at least 3. We refer to [Lyt] for the history of the subject and an extensive list of literature about this problem. Our main result now covers the remaining cases of codimension 2, provided that the polar foliation arises from a polar action. The result is as follows.

Theorem 5.5. *Suppose that $G\times X\to X$ is a polar action of a compact connected Lie group G on a symmetric space X of compact type. Then, possibly after replacing G by a larger orbit equivalent group, we have splittings $G=G_1\times\cdots\times G_m$ and $X=X_1\times\cdots\times X_m$, such that the action of G_i on X_i is either trivial or hyperpolar or the space X_i has rank* 1*, for $i=1,\ldots m$.*

We indicate briefly the connection between our main theorem and 5.5 and refer the interested reader to [Lyt]. Given a polar action on a symmetric space of compact type, the de Rham decomposition of a section of the action gives rise to an equivariant product decomposition of the whole space. Removing the hyperpolar pieces (corresponding to the flat factor of the section) and the pieces that do not admit reflection groups (corresponding to trivial actions), one is left with polar actions whose sections have constant positive curvature. Moreover, one can split off another factor with a trivial action of our group G, unless any two points of the manifold can be connected by a sequence of points $p_1, \ldots, p_r$, where each consecutive pair p_i, p_{i+1} is contained in a section. All these decomposition results rely heavily on results by Wilking [Wilk]. Then it remains to show that in this case the symmetric space has rank 1.

In order to do this, one observes that each section is a sphere or a projective space, and that the quotient space of the action is isometric to the quotient of the universal covering of a section by a finite Coxeter group. If the Coxeter group is reducible, the decomposition of the Coxeter group implies the existence of very special "polar" submanifolds of our symmetric space and from this one deduces that the rank must be 1. In the irreducible case, the quotient is a Coxeter simplex and each section is decomposed by such Coxeter simplices. Taking all these simplices from all sections together one finds a huge polyhedral complex. This polyhedral complex turns out to be a geometry of spherical type, each link of which is a spherical building (defined by the corresponding slice representations).

Thus we have found a homogeneous compact geometry of spherical type. If the geometry is covered by a building (which is always the case if the geometry is not of type C_3) then the covering complex is a Moufang building Δ belonging to a simple noncompact Lie group and the manifold we started with is the base of a principal bundle with total space homeomorphic to a sphere (the geometric realization $|\Delta|_K$ in the coarse topology). Thus X turns out to be of rank 1 in this case. The C_3 case cannot be handled in this way, since the Cayley plane $\mathbb{O}\mathrm{P}^2$ is not the quotient of a free action of a compact group on a sphere. Indeed, the exceptional geometry described in Section 3B cannot arise in this way. However, 4.1 says that the Podestà–Thorbergsson example is the only exception. □

Using the methods of [Koll], the above extension of [Lyt] was obtained in [KoLy] under the additional assumption that X is irreducible. Kollross has announced an independent extension of [KoLy] to the reducible case, with the methods of [Koll] (unpublished).

Key ideas and methods of [Lyt] were discovered and used independently by Fang–Grove–Thorbergsson in their classification of polar actions in positive curvature [FGTh]. In particular, the classification of compact Moufang buildings by Tits and Burns–Spatzier, and Tits' local approach to buildings play crucial roles. With a few exceptions, the classification in cohomogeneity two involving C_3 geometries is based on an axiomatic characterization by Tits. Of course our main result here can also be used for this purpose.

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ERRATUM TO "HOMOGENEOUS COMPACT GEOMETRIES"

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Abstract. We correct an error in our article [6], completing the classification of compact homogeneous geometries.

The present note contains a correction for our article [6] and some additional remarks. In [6] we classified compact homogeneous geometries. Kollross and Gorodski [2] discovered in 2015 that our classification as well as many related classifications, for instance [7], were incomplete: there exists a further example of a polar action on the Cayley plane which gives rise to an exceptional compact homogeneous geometry of type C_3. In the present note we correct the mistake in our classification by showing that this is the only other example of a compact homogeneous geometry. The final result is that there exist exactly two homogeneous compact geometries of type C_3 which are not covered by buildings. These two geometries have been studied in a purely algebraic setting by Schillewaert and Struyve [9], who determined in particular their automorphism groups and showed that the underlying simplicial complexes are simply connected. The last claim was independently obtained by Pasini [11]. The main results of [6] should be modified as follows.

Theorem A in [6], revised. *Let Δ be a compact geometry of irreducible spherical type and rank at least 2, with connected panels. Assume that a compact group acts continuously and transitively on the chambers of Δ.*

If Δ is not of type C_3, then there exists a simple noncompact Lie group S, a compact chamber-transitive subgroup $K \subseteq S$ and a K-equivariant 2-covering $\widetilde{\Delta} \to \Delta$, where $\widetilde{\Delta}$ is the canonical spherical building associated to S.

If Δ is of type C_3, then either there exists a building $\widetilde{\Delta}$ and a 2-covering $\widetilde{\Delta} \to \Delta$ as in the previous case, or Δ is isomorphic to one of the two exceptional

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homogeneous compact C_3 *geometries which cannot be* 2*-covered by any building.*

Theorem 5.4 in [6], revised. *Let* M *be a Coxeter matrix of spherical type and let* (G, Δ) *be a homogeneous compact geometry in* $\mathbf{HCG}(M)$*. Suppose that the Coxeter diagram of* M *has no isolated nodes. Then there exists a homogeneous compact geometry* $(K, \widetilde{\Delta})$ *in* $\mathbf{HCG}(M)$ *which is a join of buildings associated to simple noncompact Lie groups and geometries of type* C_3 *which are isomorphic to the exceptional geometry in Section* 3B *in* [6] *or to the exceptional geometry described in this note, and a continuous* 2*-covering* $\widetilde{\Delta} \to \Delta$*, which is equivariant with respect to a compact connected Lie group* K *acting transitively on the chambers of* $\widetilde{\Delta}$*.*

Theorem 4.1 in [6], revised. *Let* (G, Δ) *be a homogeneous compact geometry of type* C_3 *with connected panels. Assume that* G *is compact and acts faithfully, and let* $(\widehat{G}, \widehat{\Delta})$ *denote the corresponding universal compact homogeneous geometry, as in* 2.27 *in* [6]*. Then* $\widehat{\Delta}$ *is either a building or the exceptional geometry described in Section* 3B *in* [6] *or the exceptional geometry described below in* 6.8*.*

Remark 3.20 in [6] should also be modified accordingly: the alternative approach to the exceptional geometries would require in addition the classification results in [2].

We note also that Problem 5 in [6] is answered in [9]: the two exceptional compact homogeneous geometries of type C_3 are simply connected, and their automorphism groups are compact. Similar geometries may be defined at least over other real closed fields. We finally mention that an affirmative answer to Problem 1 in [6] has been given in [10]. In particular, [10] shows that without additional topological assumptions there exist free constructions of homogeneous geometries of type C_3.

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6. The corrections

We will freely use the terminology and the results from our article, which we will refer to as Part I. We first indicate where an error occurs in Part I. At the beginning of Section 4D, on the bottom of p. 841 in Part I we write: “By 2.5 we have a product action of M° on $\mathbb{S}^1 \times \mathbb{S}^n$.” But this conclusion is not implied by Lemma 2.5 in Part I.

In the relevant section we consider a compact homogeneous geometry (G, Δ) of type C_3 with parameters $(1, n)$. The group G is a compact connected Lie group. We recall the notions introduced in 4.10 in Part I. We fix a chamber $\{p, d, q\}$ in the geometry Δ. Then

$$\mathrm{lk}(\{p, d\}) \cong \mathbb{S}^1 \cong \mathrm{lk}(\{p, q\}) \text{ and } \mathrm{lk}(\{d, q\}) \cong \mathbb{S}^n.$$

The link of $\{p\}$ is the projective geometry of the real projective plane, the link of q is a generalized quadrangle with parameters $(1, n)$ and the link of $\{d\}$ is a generalized digon of type $\mathbb{S}^1 \times \mathbb{S}^m$. We denote the kernels of the actions of G_p, G_q and G_d on the respective links by A, B, C, and we put

$$K = G_p/A, \quad L = G_q/B \quad \text{and} \quad M = G_d/C.$$

We know that $K = \mathrm{SO}(3)$ and that the stabilizers K_d and K_q act as $\mathrm{O}(2)$ on the 1-spheres $\mathrm{lk}(\{p,d\})$ and $\mathrm{lk}(\{p,q\})$. For $n > 1$, the group G_p is connected by Lemma 4.4 in Part I. We recall also from Lemma 2.8 in Part I that $A \cap B = \{1\}$. The relevant case of Lemma 2.5 in Part I reads as follows.

Lemma 6.1. *Consider the standard orthogonal action of* $\mathrm{SO}(2) \times \mathrm{SO}(n+1)$ *on* $\mathbb{S}^1 \times \mathbb{S}^n$. *Suppose that* $H \subseteq \mathrm{SO}(2) \times \mathrm{SO}(n+1)$ *is a transitive compact connected subgroup. Then* H *splits as a product* $H = H_1 \times H_2$, *where* H_j *is the image of* H *under the projection* pr_j *for the diagram*

$$\mathrm{SO}(2) \xleftarrow{\mathrm{pr}_1} \mathrm{SO}(2) \times \mathrm{SO}(n+1) \xrightarrow{\mathrm{pr}_2} \mathrm{SO}(n+1),$$

provided that either $H_2 = \mathrm{SO}(n+1)$ *or that* H_2 *is a compact simple group.*

Corollary 6.2. *If the group* N *induced by* $(G_{d,q})^\circ$ *on* $\mathrm{lk}(\{d,q\}) \cong \mathbb{S}^n$ *is either* $\mathrm{SO}(n+1)$ *or if* N *is simple and maximal among compact connected subgroups in* $\mathrm{SO}(n+1)$, *then the action of* M° *on* $\mathrm{lk}(\{d\})$ *is a product action. Under this additional hypothesis, all the classification results in Section* 4D *of Part* I *are valid.*

Proof. For the projection H_2 in Lemma 6.1 we have $N \subseteq H_2 \subseteq \mathrm{SO}(n+1)$. □

Inspecting the classification of compact connected groups acting transitively on compact generalized quadrangles with parameters $(1,n)$ in [4], [5], we see that the only cases where the connected group induced on $\mathrm{lk}(\{d,q\})$ is not $\mathrm{SO}(n+1)$ are $n = 5, 6$, with $L^\circ = \mathrm{SO}(2) \times \mathrm{G}_2$ and $L^\circ = \mathrm{SO}(2) \cdot \mathrm{Spin}(7)$, respectively. In the case $n = 6$, the group induced by $(G_{d,q})^\circ$ on $\mathbb{S}^6$ is G_2, which is simple and a maximal connected subgroup in $\mathrm{SO}(7)$, see, e.g., 95.12 in [8]. So this case is covered by Lemma 6.1 and hence Corollary 4.31 in Part I shows that this case cannot occur in the geometry Δ.

In the case $n = 5$, the group induced by $(G_{d,q})^\circ$ on $\mathbb{S}^5$ is $\mathrm{SU}(3)$. Here, we can embed a 1-torus T diagonally into $\mathrm{SO}(2) \times \mathrm{U}(3)$. Then it may happen that $H_2 = \mathrm{U}(3)$ and then the action of $K = T \cdot \mathrm{SU}(3)$ is not split. This case was overlooked in Part I and leads to a new compact homogeneous C_3-geometry.

Auxiliary results

We introduce some groups. We put

$$C_2 = \{\pm 1\}$$

and we let $Q \subseteq \mathrm{Spin}(3) \subseteq \mathbb{H}$ denote the quaternion group of order 8, i.e., the group generated by $\boldsymbol{i}, \boldsymbol{j} \in \mathbb{H}$. We put

$$J = (Q \times \mathrm{SU}(2))/\{\pm(1,1)\}$$

and we note that $\pi_0(J) \cong C_2^2$. The group J can be realized as a subgroup of $\mathrm{SO}(4)$.

Lemma 6.3. *The group J is not isomorphic to $C_2^2 \times \mathrm{SU}(2)$.*

Proof. The center of J is C_2 and the center of $C_2^2 \times \mathrm{SU}(2)$ is C_2^3. □

Lemma 6.4. *Consider the action of the group $\mathrm{G}_2 = \mathrm{Aut}(\mathbb{O})$ on the 7-dimensional space of all pure elements in $\mathbb{O}$. The subgroup of G_2 consisting of all elements h with $h(\boldsymbol{i}) = \pm\boldsymbol{i}$ and $h(\boldsymbol{j}) = \pm\boldsymbol{j}$ is isomorphic to the group J.*

Proof. The quaternion subalgebra $\mathbb{H} \subseteq \mathbb{O}$ is generated by $\boldsymbol{i}, \boldsymbol{j}$ and thus invariant under this group. Let $\boldsymbol{\ell} \in \mathbb{O}$ be a pure element with $\boldsymbol{\ell}^2 = -1$ which is orthogonal to $\mathbb{H}$. Then $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{\ell}$ is a Cayley triple in $\mathbb{O}$ and $\mathbb{O} = \mathbb{H} \oplus \mathbb{H}\boldsymbol{\ell}$. By Lemma 11.30 in [8], the subgroup of $\mathrm{Aut}(\mathbb{O})$ of all automorphisms which leave $\mathbb{H}$ invariant consists of all maps of the form

$$[a, b] : \mathbb{H} \oplus \mathbb{H}\boldsymbol{\ell} \to \mathbb{H} \oplus \mathbb{H}\boldsymbol{\ell}, \quad u + v\boldsymbol{\ell} \mapsto au\bar{a} + (bv\bar{a})\boldsymbol{\ell},$$

where $a, b \in \mathrm{Spin}(3) \subseteq \mathbb{H}$ are unit quaternions. The claim follows. □

Lemma 6.5. *Consider the polar representation of $\mathrm{O}(2) \times \mathrm{G}_2 \subseteq \mathrm{O}(2) \times \mathrm{SO}(7)$ on $\mathbb{R}^{2\times 7}$. The principal isotropy group is isomorphic to J.*

Proof. Let $e_1, \ldots, e_7$ denote the standard basis of $\mathbb{R}^7$, and consider the 2×7-matrices $P_1 = (2e_1, e_2)$, $P_2 = (e_1, 0)$ and $P_3 = (e_1, e_2)$. The $\mathrm{O}(2) \times \mathrm{SO}(7)$-stabilizer of P_1 is principal and fixes P_2 and P_3. It consist of all elements of the form

$$\begin{pmatrix} \varepsilon_1 & 0 \\ 0 & \varepsilon_2 \end{pmatrix} \times \begin{pmatrix} \varepsilon_1 & 0 & 0 \\ 0 & \varepsilon_2 & 0 \\ 0 & 0 & \varepsilon_1\varepsilon_2 A \end{pmatrix}$$

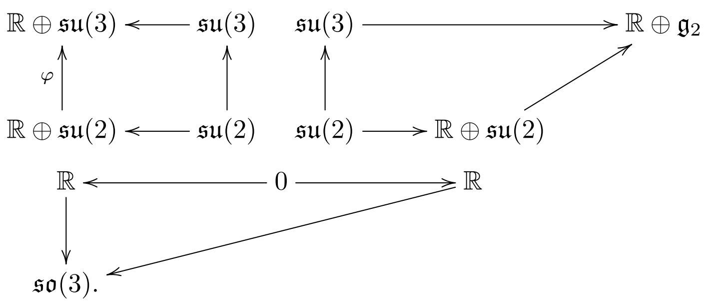
with $\varepsilon_1, \varepsilon_2 = \pm 1$ and $A \in \mathrm{SO}(5)$. We may put $\boldsymbol{i} = e_1$ and $\boldsymbol{j} = e_2$ and then apply Lemma 6.4. □

We remark that Lemma 6.5 contradicts the formula for the principal isotropy group given on p. 54 in [2].

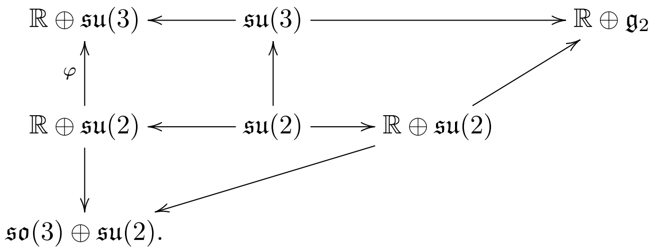
The missing case with $n = 5$ and $\mathfrak{q} = \mathbb{R} \oplus \mathfrak{g}_2$

The maps $L_p \longrightarrow L$ and $M_p \longrightarrow M$ are $(n-1)$-equivalences. Since the groups induced by $K = G_p/A$ on the 1-spheres $\mathrm{lk}(\{p, q\})$ are in both cases $\mathrm{O}(2)$, we conclude that neither L nor M are connected. The group $K_{d,q}$ is finite and hence $\mathfrak{g}_{p,d,q} = \mathfrak{a} = \mathfrak{q}_{p,d}$, and $\mathfrak{b} = \mathfrak{c} = 0$. It follows that $A^\circ = (G_{p,d,q})^\circ = \mathrm{SU}(2)$.

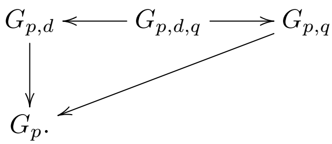
We first inspect the diagrams for the Lie algebras of M, L and K. They look as follows:



Except for φ, all arrows are canonical or determined by the diagrams for $\mathrm{lk}(p)$ and $\mathrm{lk}(q)$. For φ, there are two fundamentally different possibilities. If φ restricts an isomorphism between the $\mathbb{R}$-factors, then the action of M° on $\mathbb{S}^1 \times \mathbb{S}^5$ is split. This case is covered by Section 4D in Part I. So we have to consider the case where the φ-image of $\mathbb{R}$ is diagonally embedded into $\mathbb{R} \times \mathfrak{su}(3)$. Up to automorphism, there is just one such map. (The φ-image of $\mathbb{R}$ cannot be contained in $\mathfrak{su}(3)$, since $M/M_p \cong \mathbb{S}^5$.) There is just one way to glue these data to a diagram $\mathrm{Lie}(\mathcal{G})$, which looks as follows:



In all instances, the maps between the $\mathfrak{su}(2)$ factors are the identity maps. (The Lie algebras $\mathfrak{so}(3)$ and $\mathfrak{su}(2)$ are of course isomorphic, but naming them differently helps keeping track of the maps.) By Lemma 4.13 in Part I the diagram $\mathcal{G}^\circ$ of the connected components is uniquely determined by the diagram $\mathrm{Lie}(\mathcal{G})$ and the subdiagram



(1)

We now study the stabilizers in more detail. The group G_p is connected, and hence

$$G_p = (\mathrm{Spin}(3) \times \mathrm{SU}(2))/E,$$

and $E \subseteq C_2 \times C_2$ is a central subgroup. Therefore $G_{p,d,q} = (Q \times \mathrm{SU}(2))/E$, where $Q \subseteq \mathrm{Spin}(3)$ denotes again the quaternion group of order 8.

We noted above that the group L is not connected. By [5] we have $[L : L^\circ] = 2$ and thus $L = \mathrm{O}(2) \times \mathrm{G}_2$, in its natural polar action on $\mathbb{R}^{2\times 7}$. By Lemma 6.5 we have $L_{p,d} \cong J$. Since $L_{p,d} = G_{p,d,q}/B$, we deduce that $|E| \cdot |B| = 2$.

If $|B| = 2$, then we have by Lemma 6.3 that $B = \{\pm(1,1)\}$, because $(Q \times \mathrm{SU}(2))/B \cong J$. But $(-1,-1)$ acts also trivially on $\mathrm{lk}(p)$, contradicting the fact that $A \cap B$ is trivial. Hence B is trivial.

It follows from Lemma 6.3 that $E = \{\pm(1,1)\}$ and thus $G_p = (\mathrm{Spin}(3) \times \mathrm{SU}(2))/E \cong \mathrm{SO}(4)$. This determines the diagram (1) uniquely, and hence by Lemma 4.13 in Part I the diagram $\mathcal{G}^\circ$ of connected components.

It also follows that $G_q = L$ and that

$$\mathbb{Z}/2 \cong \pi_0(G_q) \cong \pi_0(G_{p,q}) \cong \pi_0(G_{d,p}) \cong \pi_0(G_d). \tag{2}$$

At this stage, the following subdiagram of $\mathcal{G}$ is uniquely determined:

$$\begin{array}{ccccccc}
 & & G_{d,q} & & \longrightarrow & & G_q \\
 & & \uparrow & & & \nearrow & \\
G_{p.d} & \longleftarrow & G_{p,d,q} & \longrightarrow & G_{p,q} & & \\
\downarrow & & & \swarrow & & & \\
G_p. & & & & & &
\end{array} \tag{3}$$

Let $\mathrm{O}(2) \subseteq G_{p,d} \subseteq G_p = \mathrm{SO}(4)$ denote the subgroup which maps onto $K_d \cong \mathrm{O}(2)$. We choose an element h in $\mathrm{O}(2)$ which is a reflection and fixes $\{p, d, q\}$ (there are two such elements in $\mathrm{O}(2)$). Then h is an involution and generates each of the cyclic groups in Equation (2). In particular we have

$$G_{d,q} = (G_{d,q})^\circ \rtimes \langle h \rangle \quad \text{and} \quad G_{p,d} = (G_{p,d})^\circ \rtimes \langle h \rangle.$$

This determines the conjugation action of h on $(G_d)^\circ = \langle (G_{p,d})^\circ \cup (G_{d,q})^\circ \rangle$, and hence the structure of $G_d = (G_d)^\circ \rtimes \langle u \rangle$ as well as the missing upper left corner in diagram (3). We have established the following result.

Proposition 6.6. *In the case where $\mathfrak{q} = \mathbb{R} \oplus \mathfrak{g}_2$ and where the action of M° is not a product action, there is at most one possibility for the diagram $\mathcal{G}$ and hence at most one universal compact homogeneous geometry of type C_3.*

It remains to show the existence of the geometry. There are two ways for this. In Section 3.4 in [2], the construction of a polar action on the Cayley plane is shown, in such a way that the resulting geometry is of the type described above. Similarly to Section 3B in Part I, one could work out the precise orbit structure for this polar action. There is a second and very explicit way to construct the geometry in a purely algebraic way, as shown in [9]. We indicate it briefly at the end of this note.

Proposition 6.7. *Besides the geometry described in Section* 3B *in Part* I *there exist, up to isomorphism, exactly one compact homogeneous geometry of type* C_3 *which is not* 2*-covered by a building.*

Proof. Let (G, Δ) denote the exceptional compact homogeneous geometry of type C_3 which arises from the polar action on the Cayley plane as described in [2]. We claim first that Δ is not covered by a building Δ'. Otherwise, there would (by Theorem 2.22 in Part I and the subsequent remarks) exist a fiber bundle $\mathbb{S}^m \cong |\Delta'| \to |\Delta| \cong \mathbb{O}\mathrm{P}^2$, which arises from a free action of a compact Lie group E on the building Δ'. However, such a fiber bundle $\mathbb{S}^m/E \to \mathbb{O}\mathrm{P}^2$ does not exist. Its mapping cone would be a finite 7-connected CW complex with cohomology ring $\mathbb{Z}[x]/(x^4)$ for a generator x in degree 8, which is impossible by Adams' results on the Hopf invariant. Theorem 5.1. and the subsequent remarks in [1] show this explicitly.

Hence (G, Δ) is a compact homogeneous geometry of type C_3 which is not 2-covered by a building. By our classification of possible diagrams for such geometries, this geometry has to be in the category $\mathbf{HCG}_{\mathcal{G}}(\mathsf{C}_3)$, where $\mathcal{G}$ is the unique simple complex of groups determined above. Let $(\widehat{G}, \widehat{\Delta}, \{\hat{p}, \hat{d}, \hat{q}\}, \widehat{\psi})$ denote the universal homogeneous compact geometry in $\mathbf{HCG}_{\mathcal{G}}(\mathsf{C}_3)$. The actions of K and L° on $\mathrm{lk}(\hat{p})$ and $\mathrm{lk}(\hat{q})$ are minimal by the classification in [4, 5]. Therefore the $\widehat{G}$-action on $\widehat{\Delta}$ is minimal, i.e., $\widehat{G}$ has no proper compact connected chamber transitive subgroup. Hence there exists a continuous surjective group homomorphism $\widehat{G} \to G$ which induces an equivariant morphism $\widehat{\Delta} \to \Delta$. We have to show that the kernel $F \trianglelefteq \widehat{G}$ of the group homomorphism is trivial. We first show that the Lie algebra $\mathfrak{f}$ of F is trivial. We have $\mathrm{Lie}(\widehat{G}) \cong \mathrm{Lie}(G) \oplus \mathfrak{f}$, and $\mathrm{Lie}(G) \cong \mathfrak{so}(3) \oplus \mathfrak{g}_2$ by the results in [2]. From the polar action on the Cayley plane as given in [2] we know that $|\Delta| \cong \mathbb{O}\mathrm{P}^2$ in the coarse topology.

We put $H_1 = \exp(\mathfrak{so}(3)) \subseteq \widehat{G}_p$ and $H_2 = \exp(\mathfrak{g}_2) \subseteq \widehat{G}_q$ and we note the following. The group H_1 and H_2 induce transitive groups on the links $\mathrm{lk}(\{\hat{p}, \hat{q}\})$, $\mathrm{lk}(\{\hat{p}, \hat{d}\})$ and $\mathrm{lk}(\{\hat{d}, \hat{q}\})$ in $\widehat{\Delta}$ and hence the abstract group $H = \langle H_1 \cup H_2 \rangle$ acts transitively on the chambers of $\widehat{\Delta}$. Moreover, H_1 and H_2 consist of commutators, hence H is a perfect group. Being path-connected, $H \hookrightarrow \widehat{G}$ is an analytic subgroup with a Lie algebra $\mathfrak{h}$, see [3]. Since H is perfect, $\mathfrak{h}$ is a compact semisimple Lie algebra and hence $H \subseteq \widehat{G}$ is compact. Since $\widehat{G}$ is minimal, $\widehat{G} = H$ and $\mathrm{Lie}(\widehat{G})$ is generated as a Lie algebra by $\mathrm{Lie}(H_1) + \mathrm{Lie}(H_2)$.

By the formula given before 4.3 in Part I we have

$$\dim \widehat{G} - \dim \widehat{G}_{\hat{p}, \hat{d}, \hat{q}} \leq 6 \cdot 1 + 3 \cdot 5 = 21$$

and thus $\dim(F) \leq 7$. The projection

$$\mathrm{Lie}(\hat{G}) \cong \mathfrak{so}(3) \oplus \mathfrak{g}_2 \oplus \mathfrak{f} \longrightarrow \mathfrak{so}(3) \oplus \mathfrak{f}$$

is surjective and annihilates $\mathrm{Lie}(H_2) \cong \mathfrak{g}_2$. Therefore $\mathrm{Lie}(H_1)$ maps onto $\mathfrak{so}(3) \oplus \mathfrak{f}$. But $\dim(\mathrm{Lie}(H_1)) = 3$ and thus $\mathfrak{f} = 0$. It follows that $|\widehat{\Delta}| \to |\Delta|$ is in the coarse topology a covering. Since $|\Delta| \cong \mathbb{O}\mathrm{P}^2$ is simply connected, $|\widehat{\Delta}| = |\Delta|$ and $F = \{1\}$. Finally, the Cayley plane admits no fixed-point free homeomorphism. Therefore no group can act continuously and freely on $\mathbb{O}\mathrm{P}^2$ and therefore the geometry (G, Δ) has no equivariant quotients. □

Construction 6.8 (The exceptional geometry)**.** Let $\mathbb{H}$ denote the quaternion algebra, and $\mathbb{O}$ the Cayley algebra. For $a \in \mathbb{H}$ with $a^2 = -1$ put $[a] = \{\pm a\}$. For $b \in \mathbb{H}$ and $c \in \mathbb{O}$ with $b^2 = c^2 = -1$ put $[b, c] = \{\pm(b, c)\}$. Let Δ be the simplicial complex whose maximal simplices are the sets $\{[a], [b, c], \delta\}$, where $\delta : \mathbb{H} \longrightarrow \mathbb{O}$ is an algebra homomorphism, with the property that

$$a \perp b \quad \text{and} \quad \delta(b) = c.$$

The group $G = \mathrm{Aut}(\mathbb{H}) \times \mathrm{Aut}(\mathbb{O}) = \mathrm{SO}(3) \times \mathrm{G}_2$ acts in the obvious way on Δ. It is shown in [9] that Δ is a simply connected geometry of type C_3 and one can check that (G, Δ) fits the diagram $\mathcal{G}$ above. For example, we could put $a = \boldsymbol{j}$, $b = \boldsymbol{i} = c$ and $\delta : \mathbb{H} \subseteq \mathbb{O}$. In this case it is not difficult to work out the stabilizers of $p = \delta$, $d = [\boldsymbol{i}, \boldsymbol{i}]$ and $q = [\boldsymbol{j}]$.

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