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The topology of a semisimple Lie group is essentially unique

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Abstract

We study locally compact group topologies on simple and semisimple Lie groups. We show that the Lie group topology on such a group S is very rigid: every "abstract" isomorphism between S and a locally compact and σ-compact group Γ is automatically a homeomorphism, provided that S is absolutely simple. If S is complex, then noncontinuous field automorphisms of the complex numbers have to be considered, but that is all. We obtain similar results for semisimple groups.

Keywords: Lie group; Locally compact group; Continuity of homomorphisms

Abstract isomorphisms between Lie groups have been studied by several authors. In particular, É. Cartan [2] and B. van der Waerden [19] proved that an abstract isomorphism between compact semisimple Lie groups is automatically continuous. This was generalized by H. Freudenthal [3] to isomorphisms between absolutely simple real Lie groups. A model-theoretic proof of Freudenthal's result was given later by Y. Peterzil, A. Pillay and S. Starchenko [14]. More generally, abstract isomorphisms between simple algebraic groups were studied by A. Borel and J. Tits [1].

However, all of these results deal with rigidity within the class of Lie groups. In the present paper, we study a more general problem: what can be said about abstract isomorphisms between locally compact groups and simple Lie groups. To put it differently, we want to determine to what extent the group topology of a simple Lie group is unique. Some restrictions on the topology are

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obviously necessary. We have to exclude the discrete topology, and also all topologies that are not locally compact (the field of real numbers admits many field topologies which are not locally compact). A simplified version of our main result can be stated as follows. For compact S, this was proved by R. Kallman [9].

Theorem. *Let S be an absolutely simple real Lie group. Then the Lie group topology is the unique locally compact and σ-compact group topology on S.*

More general results are given in Theorems 8, 11, 18, and 20.

1 Notation. By a *simple Lie group* we mean a connected centerless real Lie group S whose Lie algebra $\mathrm{Lie}(S) = \mathfrak{s}$ is simple. Such a Lie group is simple as an abstract group, see Salzmann et al. [15, 94.21], and conversely, every nondiscrete Lie group which is simple as an abstract group is a simple Lie group in the sense above. We say that S (or $\mathfrak{s}$) is *absolutely simple* if $\mathfrak{s} \otimes_{\mathbb{R}} \mathbb{C}$ is a complex simple Lie algebra.

By $\mathrm{AutLie}(S)$ we denote the Lie group of all Lie group automorphisms of S. This is the same group as $\mathrm{Aut}_{\mathbb{R}}(\mathfrak{s})$, the group of all $\mathbb{R}$-linear Lie algebra automorphisms. The group S acts faithfully by conjugation on itself (from the left). In this way we may view it in a natural way as an open subgroup of $\mathrm{AutLie}(S)$. In fact, $S = \mathrm{AutLie}(S)^\circ$ and S has finite index in $\mathrm{AutLie}(S)$, see Helgason [5, IX, Thm. 5.4] for the compact (and complex) case and Murakami [13, Cor. 2] for the noncompact real case. In particular, $\mathrm{AutLie}(S)$ is second countable.

2 Definition. We call a Hausdorff space *σ-compact* if it is a countable union of compact subsets (some authors require that σ-compact spaces are also locally compact, but the present definition is more suitable for our purposes). Finite products of σ-compact spaces are σ-compact. If $f : X \longrightarrow Y$ is a continuous map between Hausdorff spaces and if X is σ-compact, then $f(X)$ is also σ-compact. Closed subspaces of σ-compact spaces are again σ-compact.

All topological groups are assumed to be Hausdorff. We recall the following results about locally compact groups.

3 Theorem *(Open Mapping Theorem). Let $\psi : G \longrightarrow H$ be a surjective continuous homomorphism between locally compact groups. If H is σ-compact, then ψ is an open map.*

Proof. See Hewitt and Ross [6, II.5.29] or Stroppel [17, 6.19]. □

4 Theorem *(Automatic continuity). Suppose that G is a locally compact group and that H is a σ-compact group. Assume that $\psi : G \longrightarrow H$ is a group homomorphism which is also a Borel map, i.e. that the preimage of every open set $U \subseteq H$ is a Borel set. Then ψ is continuous.*

Proof. This is a special case of Hewitt and Ross [6, V.22.18]; see also Kleppner [11, Thm. 1]. □

5 Lemma. *Let S be a simple m-dimensional Lie group and $G \subseteq \mathrm{AutLie}(S)$ an open subgroup. Suppose that $C \subseteq G$ is a compact subset that contains a nonconstant smooth curve. Then there exist elements $g_0, \ldots, g_m \in G$ such that $g_0 C g_1 C c_2 \cdots g_{r-1} C g_m$ is a compact neighborhood of the identity.*

Proof. Let $c : (-1, 1) \longrightarrow C$ be a smooth curve with tangent vector $\dot{c}(0) = X \neq 0$. Translating by $c(0)^{-1}$, we may assume that $c(0) = e$ and that $X \in T_eG = \mathrm{Lie}(G)$. Since G acts via Ad irreducibly on $\mathrm{Lie}(G)$, we find elements $h_1, \dots, h_m \in G$ such that the vectors $\mathrm{Ad}(h_1)(X), \dots, \mathrm{Ad}(h_m)(X)$ span $\mathrm{Lie}(G)$. From the inverse function theorem we see that $D = h_1Ch_1^{-1}h_2Ch_2^{-1} \cdots h_mCh_m^{-1}$ is a neighborhood of the identity. □

The next lemma is essentially due to van der Waerden; see [19, p. 783] and Freudenthal [3, Satz 8]. We denote the commutator by $[a, b] = aba^{-1}b^{-1}$.

6 Lemma. *Let S be a simple m-dimensional Lie group and $G \subseteq \mathrm{AutLie}(S)$ an open subgroup. Assume that $D \subseteq G$ is a compact neighborhood of the identity. Let $U \subseteq G$ be an arbitrary neighborhood of the identity. Then there exist elements $a_1, \dots, a_m \in G$ such that $[a_1, D][a_2, D] \cdots [a_m, D]$ is a neighborhood of the identity which is contained in U.*

Proof. Let $a \in G - \{e\}$. Then $\mathrm{Ad}(a) \neq \mathrm{id}$, so there exists $X \in \mathrm{Lie}(G)$ with $\mathrm{Ad}(a)(X) - X \neq 0$. Since S acts irreducibly on $\mathrm{Lie}(G)$, we can find elements $a_1, \dots, a_m$ in any neighborhood of the identity, and vectors $X_1, \dots, X_m \in \mathrm{Lie}(G)$ such that $\mathrm{Ad}(a_1)(X_1) - X_1, \dots, \mathrm{Ad}(a_m)(X_m) - X_m$ is a basis of $\mathrm{Lie}(G)$. It follows readily from the inverse function theorem that $[a_1, D] \cdots [a_m, D]$ is a compact neighborhood of the identity.

Let now U be an open neighborhood of the identity. Consider the continuous map $h : G^m \times G^m \longrightarrow G$, $(x_1, \dots, x_m, y_1, \dots, y_m) \longmapsto [x_1, y_1][x_2, y_2] \cdots [x_m, y_m]$. We have that $h(\{e\}^m \times D^m) \subseteq U$. By Wallace's Lemma, see Kelley [10, 5.12], there is an open neighborhood V of the identity such that $h(V^m \times D^m) \subseteq U$. The claim follows if we choose $a_1, \dots, a_m \in V$. □

The following technical result is the main ingredient in our continuity proofs. It generalizes Kallman's method [9].

7 Theorem. *Let Γ be a locally compact and σ-compact group. Let S be a simple Lie group, let G be an open subgroup of* $\mathrm{AutLie}(S)$ *and suppose that*

$$\varphi : \Gamma \longrightarrow G$$

is an abstract surjective group homomorphism. Suppose that there is a compact subset $C \subseteq G$ which contains a nonconstant smooth curve and whose φ-preimage $\varphi^{-1}(C)$ is σ-compact. Then φ is continuous and open.

Proof. By Lemma 5, there are elements $g_0, \dots, g_r \in G$ such that $D = g_0Cg_1Cg_2 \cdots g_{r-1}Cg_r$ is a compact neighborhood of the identity. If we choose φ-preimages g_i' of the g_i, then we have $\varphi^{-1}(D) = g_0'\varphi^{-1}(C)g_1'\varphi^{-1}(C)g_2' \cdots g_{r-1}'\varphi^{-1}(C)g_r'$. In particular, $\varphi^{-1}(D)$ is σ-compact. Let $U \subseteq G$ be an open neighborhood of the identity. By Lemma 6 we find elements $a_1, \dots, a_m \in G$ such that $E_{a_1,\dots,a_m} = [a_1, D] \cdots [a_m, D] \subseteq U$ is a neighborhood of the identity. Moreover, the set $\varphi^{-1}(E_{a_1,\dots,a_m})$ is σ-compact and in particular a Borel set in Γ. If $W \subseteq G$ is an arbitrary open subset, then we find a countable collection of elements $b_j, a_{1,j}, \dots, a_{m,j} \in G$ such that $W = \bigcup_{j=0}^{\infty} b_j E_{a_{1,j},\dots,a_{m,j}}$, because G is second countable. Each set $\varphi^{-1}(b_j E_{a_{1,j},\dots,e_{m,j}})$ is Borel, so $\varphi^{-1}(W)$ is a Borel set in Γ. Therefore φ is a Borel map and by Theorem 4 continuous. By Theorem 3, φ is open. □

If S is a compact simple Lie group, then every open subgroup G of $\mathrm{AutLie}(S)$ is compact by the remarks in Notation 1. Therefore we have the following consequence of Theorem 7. In the case that φ is bijective, this result was proved by Kallman in [9], and for compact Γ in Hofmann and Morris [7, Thm. 5.64].

8 Corollary. *Suppose that S is a compact simple Lie group and that $G \subseteq \mathrm{AutLie}(S)$ is open. Let Γ be a locally compact and σ-compact group and assume that $\varphi : \Gamma \longrightarrow G$ is an abstract surjective group homomorphism. Then φ is continuous and open. In particular,* $\mathrm{Aut}(S) = \mathrm{AutLie}(S)$.

In order to extend this result to noncompact simple Lie groups, we need some structure theory. We will avoid the classification of real simple Lie groups. The following case, however, requires special considerations in our approach.

9 Lemma. *Let $\mathfrak{s}$ be an absolutely simple real Lie algebra of real rank $\mathrm{rk}_{\mathbb{R}}(\mathfrak{s}) = 1$. Let $\mathfrak{s} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$ be an Iwasawa decomposition and put $\mathfrak{m} = \mathrm{Cen}_{\mathfrak{k}}(\mathfrak{a})$. If $\mathfrak{m}$ is abelian, then $\mathfrak{s} = \mathfrak{sl}_2(\mathbb{R})$ or $\mathfrak{s} = \mathfrak{su}_{2,1}(\mathbb{C})$ (the Lie algebra of the special unitary group of a 3-dimensional nondegenerate complex hermitian form of Witt index 1).*

Proof. We note that $\dim(\mathfrak{a}) = \mathrm{rk}_{\mathbb{R}}(\mathfrak{s}) = 1$. If $\mathfrak{m}$ is abelian, then $(\mathfrak{m} \oplus \mathfrak{a}) \otimes_{\mathbb{R}} \mathbb{C}$ is a Cartan subalgebra in $\mathfrak{s} \otimes_{\mathbb{R}} \mathbb{C}$; see Knapp [12, 6.47] or Helgason [5, p. 259]. Thus $\mathfrak{s}$ is quasi-split: all nodes in the Tits diagram are white/encircled (see Tits [18], and also the “Satake diagrams” in [5, Table VI] or in Warner [20]). Because $\mathfrak{s}$ has real rank 1, all white/encircled nodes of the underlying Dynkin diagram are in one orbit of the $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$-action. Since $\mathfrak{s}$ is absolutely simple, the underlying Dynkin diagram is connected and hence a tree. Thus the Dynkin diagram is either A_1 or A_2. The corresponding quasi-split real Lie algebras are $\mathfrak{sl}_2(\mathbb{R})$ and $\mathfrak{su}_{2,1}(\mathbb{C})$ (this follows either from the tables in Helgason [5, p. 259] or directly from the classification of involutions on $\mathfrak{sl}_3(\mathbb{C})$; see also Tits [18, p. 55]). $\square$

We need the following characterization of real absolutely simple Lie algebras which we could not find in the literature. The result follows of course also from the classification of the real simple Lie algebras, see Helgason [5, pp. 532–534]. We remark that the complex Kac–Moody algebra of type $\tilde{\mathsf{A}}_{2k+1}$ has, for $k \geqslant 1$, a real form where every rank 1 Levi factor is of type $\mathfrak{sl}_2(\mathbb{C})$, so the result is not completely trivial.

10 Lemma. *Let $\mathfrak{s}$ be an absolutely simple real Lie algebra. Then there exists a next-to-minimal parabolic subalgebra whose semisimple Levi algebra is not isomorphic to $\mathfrak{sl}_2(\mathbb{C})$.*

Proof. Assume that the semisimple Levi algebra of every next-to-minimal parabolic subalgebra is isomorphic to $\mathfrak{sl}_2(\mathbb{C})$. Then the underlying Dynkin diagram of each corresponding complexified Levi algebra is $\mathsf{A}_1 \times \mathsf{A}_1$ and the Galois group $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ permutes the two nodes. In the Tits diagram, the two nodes are white/encircled.

Now in general, the Tits diagram of a semisimple Levi group of a parabolic is obtained by removing from the Tits diagram of $\mathfrak{s}$ a collection of $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$-orbits consisting of white/encircled nodes. Thus, in our situation all nodes are white/encircled (hence $\mathfrak{s}$ is quasi-split) and all $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$-orbits consist of two white/encircled nodes which do not form an edge in the Dynkin diagram. It follows that $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ acts freely on the Dynkin diagram.

On the other hand, the Dynkin diagram of $\mathfrak{s} \otimes_{\mathbb{R}} \mathbb{C}$ is connected (because $\mathfrak{s}$ is absolutely simple) and therefore a tree. This is a contradiction: the Galois group $\mathrm{Gal}(\mathbb{C}/\mathbb{R}) \cong \mathbb{Z}/2$ cannot act freely on a tree. $\square$

The following theorem contains and extends Freudenthal's Continuity Theorem [3].

11 Theorem. *Suppose that S is an absolutely simple Lie group and that $G \subseteq \mathrm{AutLie}(S)$ is open. Let Γ be a locally compact and σ-compact group and assume that $\varphi : \Gamma \longrightarrow G$ is an abstract group isomorphism. Then φ is a homeomorphism. In particular,* $\mathrm{Aut}(S) = \mathrm{AutLie}(S)$.

Before we embark on the proof, we note the following. We may assume that the real rank $\ell = \mathrm{rk}_{\mathbb{R}}(\mathfrak{s})$ of S is at least 1, since we dealt already with compact simple groups (groups of real rank $\ell = 0$) in Corollary 8. We first consider the case $\ell = 1$ in a slightly more general situation. Suppose that H is a (not necessarily connected) reductive real Lie group of real rank 1 (i.e. the semisimple part of H° has real rank 1). Let $\mathrm{Lie}(H) = \mathfrak{h}$ denote its Lie algebra. The semisimple part $\mathfrak{h}_{ss}$ of $\mathfrak{h}$ decomposes as a sum of a simple ideal $\mathfrak{h}_s$ of real rank 1 and a compact semisimple ideal $\mathfrak{h}_c$ (which may be trivial). We assume that there is a maximal compact subgroup $K \subseteq H$ corresponding to a Cartan involution and correspondingly an Iwasawa decomposition

$$\mathfrak{h} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}.$$

From our assumptions, the identity component H° has finite index in H. Let $\mathfrak{m} = \mathrm{Cen}_{\mathfrak{k}}(\mathfrak{a})$ and $A = \exp(\mathfrak{a})$. The compact group $M = \mathrm{Cen}_K(A)$ has $\mathfrak{m}$ as its Lie algebra and $\mathfrak{m}$ contains $\mathfrak{h}_c$. We distinguish the following cases.

Case (A). $\mathfrak{m}$ *is not abelian*. Then we find an element $h \in M^\circ$ such that the conjugacy class $C = \{ghg^{-1} \mid g \in M\}$ is compact and of positive dimension. Put $L = \mathrm{Cen}_H(A)$. Then $L = MA$ is a central product and therefore

$$C = \{ghg^{-1} \mid g \in M\} = \{ghg^{-1} \mid g \in L\}.$$

Case (B). $\mathfrak{m}$ *is abelian*. Then $\mathfrak{h}_c = 0$ and thus the semisimple part $\mathfrak{h}_{ss} = \mathfrak{h}_s$ is in fact simple of real rank 1. Thus we may use Lemma 9. If $\mathfrak{h}_s \neq \mathfrak{sl}_2(\mathbb{C})$, then $\mathfrak{k} \cap \mathfrak{h}_s$ is isomorphic to $\mathbb{R}$ or $\mathbb{R} \oplus \mathfrak{su}(2)$; in particular, $\mathfrak{k} \cap \mathfrak{h}_s$ has a 1-dimensional center $\mathfrak{z}$. Assume that this is the case. Let $Z \subseteq K$ denote the corresponding connected subgroup. Since $\mathfrak{k} \cap \mathfrak{h}_s$ is an ideal in $\mathfrak{k}$, the group K° centralizes Z. Let h be an element in the analytic subgroup H_s corresponding to $\mathfrak{h}_s$ whose Z-conjugacy class has positive dimension. Let $L = \mathrm{Cen}_H(Z)$. The Lie algebra $\mathrm{Lie}(L) = \mathfrak{l}$ decomposes as $\mathfrak{l} \cong \mathrm{Cen}(\mathfrak{h}) \oplus \mathfrak{z}$. Since L is a finite extension of L°, the set

$$C = \{ghg^{-1} \mid g \in L\}$$

is compact and of positive dimension.

The remaining case, where $\mathfrak{h}_s = \mathfrak{sl}_2(\mathbb{C})$, will not be important.

Proof of Theorem 11. We use the structure theory of the (not necessarily connected) group G. See Warner [20, p. 85] for some remarks on the nonconnected case. By Lemma 10 we can find a next-to-minimal parabolic $P \subseteq G$ whose semisimple Levi group is not of type $\mathfrak{sl}_2(\mathbb{C})$. Let $H \subseteq P$ denote the reductive Levi group of P. The group H can be written as a centralizer of a torus; see Warner [20, p. 73]. From this description it is clear that $\varphi^{-1}(H)$ is closed in Γ.

We now use our results above about reductive groups of real rank 1. In Cases (A) and (B) above, we see from the respective definitions of the subgroup $L \subseteq H$ that $\varphi^{-1}(L)$ is also closed and hence σ-compact. Therefore $\varphi^{-1}(C)$ is in both cases σ-compact. The claim follows now from Theorem 7. □

It may seem that the previous proof with the two Cases (A) and (B) is too complicated. However, Theorem 11 is false if G happens to be a complex Lie group. In this case, the general construction of the subset C with the properties required in Theorem 7 will, in general, not be possible. In such a complex Lie group, the subgroup M° that we used in our construction of C will always be abelian. Nevertheless, we can prove something in the complex case. Our methods are, however, somewhat different. We use the following results.

12 Theorem. *Let G be a locally compact group. If $G/\mathrm{Cen}(G)$ is compact, then the algebraic commutator group DG of G has compact closure.*

Proof. See Grosser and Moskowitz [4, Cor. 1, p. 331]. □

The following is well known; actually, we need it only for the group $G = \mathrm{SU}(2)$, where it is easily verified by hand.

13 Theorem. *Let G be a compact semisimple Lie group. Then G consists of commutators, $G = \{[a,b] \mid a, b \in G\}$.*

Proof. See Hofmann and Morris [7, Thm. 6.55]. □

Finally, we use the following fact about the complex numbers. We recall that $\mathbb{C}$ has $2^{2^{\aleph_0}}$ (noncontinuous) field automorphisms.

14 Theorem. *Let $\mathcal{T}$ be a nondiscrete locally compact Hausdorff topology on the set $\mathbb{C}$. Suppose that for every $a \in \mathbb{C}$, the maps $z \longmapsto a + z$ and $z \longmapsto az$ are continuous with respect to $\mathcal{T}$. Then there is a field automorphism $\alpha \in \mathrm{Aut}(\mathbb{C})$ such that $\alpha(\mathcal{T})$ is the standard topology on $\mathbb{C}$.*

Proof. By Warner [21, Thm. 11.17], $(\mathbb{C}, \mathcal{T})$ is a topological field. By Weil [22, I, §3, Thm. 5], there is, up to topological isomorphism, only one nondiscrete locally compact algebraically closed field; see also Salzmann et al. [16, 58.8]. □

15 Complex simple Lie groups. Suppose that S is a complex simple Lie group with Lie algebra $\mathrm{Lie}(S) = \mathfrak{s}$. The group $\mathrm{Aut}_{\mathbb{C}}(\mathfrak{s})$ of all $\mathbb{C}$-linear automorphisms is a complex Lie group containing S. We denote by $\mathrm{Aut}_{\mathbb{Q}}(\mathfrak{s})$ the group of all semilinear automorphisms of $\mathfrak{s}$ (with respect to arbitrary field automorphisms of $\mathbb{C}$).

The group $\mathrm{Aut}_{\mathbb{C}}(\mathfrak{s})$ is a complex linear algebraic group. It can be realized as a matrix group which is defined by a (finite) set of polynomial equations on the entries. In this way, one obtains an action of $\mathrm{Aut}(\mathbb{C})$ on $\mathrm{Aut}_{\mathbb{C}}(\mathfrak{s})$ (and on $\mathfrak{s}$), where the field automorphisms are applied entry-wise to the matrices.[2] In particular, there are split short exact sequences

$$\begin{tikzcd} 1 \arrow[r] & \mathrm{Aut}_{\mathbb{C}}(\mathfrak{s}) \arrow[r] \arrow[d, equal] & \mathrm{Aut}_{\mathbb{R}}(\mathfrak{s}) \arrow[r, shift left, "j"] \arrow[d, hook] & \mathrm{Gal}(\mathbb{C}/\mathbb{R}) \arrow[r] \arrow[l, shift left] \arrow[d, hook] & 1 \\ 1 \arrow[r] & \mathrm{Aut}_{\mathbb{C}}(\mathfrak{s}) \arrow[r] & \mathrm{Aut}_{\mathbb{Q}}(\mathfrak{s}) \arrow[r, shift left, "j"] & \mathrm{Aut}(\mathbb{C}) \arrow[r] \arrow[l, shift left] & 1. \end{tikzcd}$$

The group $\mathrm{Aut}_{\mathbb{Q}}(\mathfrak{s})$ is thus contained in the group $\mathrm{Aut}(S)$ of all "abstract" automorphisms of S. We will see below that both groups are equal. For $\alpha \in \mathrm{Aut}_{\mathbb{Q}}(\mathfrak{s})$ we put

$$c_\alpha = \left[g \longmapsto {}^{\alpha}g = \alpha g \alpha^{-1}\right].$$

As in the real case, we first consider groups of rank 1. We remark that $\mathrm{PSL}_2(\mathbb{C})$ has index 2 in $\mathrm{AutLie}(\mathrm{PSL}_2(\mathbb{C}))$; the quotient is $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$.

16 Lemma. *Suppose that $S = \mathrm{PSL}_2(\mathbb{C})$ and $\mathfrak{s} = \mathfrak{sl}_2(\mathbb{C})$. Let $G \subseteq \mathrm{Aut}_{\mathbb{Q}}(\mathfrak{s})$ be a subgroup containing S and let Γ be a locally compact and σ-compact group. Suppose that $\varphi : \Gamma \longrightarrow G$ is an abstract group isomorphism. Then there exists an element $\alpha \in \mathrm{Aut}_{\mathbb{Q}}(\mathfrak{s})$ such that $c_\alpha \circ \varphi$ is a homeomorphism onto an open subgroup of* $\mathrm{AutLie}(S)$. *In particular,* $\mathrm{Aut}(S) = \mathrm{Aut}_{\mathbb{Q}}(\mathfrak{s})$.

Proof. We represent the elements of $\mathrm{SL}_2(\mathbb{C})$ in the standard way as complex 2×2 matrices. This gives us a canonical action of $\mathrm{Aut}(\mathbb{C})$ on $S = \mathrm{PSL}_2(\mathbb{C})$. Let $V \subseteq S$ denote the unipotent subgroup represented by all upper triangular matrices with ones on the diagonal. Then $V = \mathrm{Cen}_G(V)$ is isomorphic to the additive group $(\mathbb{C}, +)$. Let $T \subseteq S$ denote the group presented by all diagonal matrices. Then $T = \mathrm{Cen}_G(T)$ acts on V as multiplication by (squares of) nonzero complex numbers.

The group $V' = \varphi^{-1}(V) = \mathrm{Cen}_\Gamma(V')$ is a closed and therefore locally compact and σ-compact copy of the abstract group $(\mathbb{C}, +)$. Moreover, the multiplication by any complex scalar is continuous on V', since each element of $T' = \varphi^{-1}(T)$ acts continuously on V'. By Theorem 14, there is a field automorphism $\alpha \in \mathrm{Aut}(\mathbb{C})$ such that the restriction $c_\alpha \circ \varphi|_{V'} : V' \longrightarrow V$ is a homeomorphism of topological groups.

We now consider the action of Γ on the complex projective line $\mathbb{C}\mathrm{P}^1$ via $c_\alpha \circ \varphi$. The Γ-stabilizer of a suitable point is $\mathrm{Nor}_\Gamma(V')$. Since S acts transitively on $\mathbb{C}\mathrm{P}^1$, we have a factorization $\Gamma = \mathrm{Nor}_\Gamma(V')(c_\alpha \circ \varphi)^{-1}(S)$. Thus Γ and $\mathrm{Nor}_\Gamma(V')$ have the same images in $\mathrm{Aut}(\mathbb{C})$ under $j \circ c_\alpha \circ \varphi$. But $\mathrm{Nor}_\Gamma(V')$ acts continuously on $V' \cong \mathbb{C}$, hence it maps into $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$. Thus ${}^{\alpha}G = c_\alpha(\varphi(\Gamma)) \subseteq \mathrm{AutLie}(S)$. Now we can apply Theorem 7. Let $C \subseteq V$ denote the closed unit disk. Then $(c_\alpha \circ \varphi)^{-1}(C)$ is closed in V' and hence closed in Γ. By Theorem 7, the composite $c_\alpha \circ \varphi : \Gamma \longrightarrow {}^{\alpha}G \subseteq \mathrm{AutLie}(S)$ is a homeomorphism. $\square$

17. For groups of higher rank, we recall a few combinatorial facts. Associated to a complex simple Lie group S of rank ℓ there is an $\ell - 1$-dimensional simplicial complex, the spherical building $\Delta(S)$. The groups S and $\mathrm{Aut}_{\mathbb{Q}}(\mathfrak{s})$ act on Δ (the latter by not necessarily type-preserving

[2] This follows also if S is viewed as a group scheme defined over $\mathbb{Q}$, but the present down-to-earth approach with matrix groups suffices for our purposes.

automorphisms). The group $\mathrm{Aut}_{\mathbb{Q}}(\mathfrak{s})$ acts on the Dynkin diagram of S. In this action, $\mathrm{Aut}_{\mathbb{Q}}(\mathfrak{s})$ cannot exchange long and short roots, that is, the action is trivial for the diagrams C_n, F_4 and G_2. To see this, we note that $\mathrm{Aut}_{\mathbb{Q}}(\mathfrak{s})$ can be split as a semidirect product of $\mathrm{Aut}_{\mathbb{C}}(\mathfrak{s})$ and $\mathrm{Aut}(\mathbb{C})$, with $\mathrm{Aut}(\mathbb{C})$ acting trivially on the Dynkin diagram. This reduces the claim to the group $\mathrm{Aut}_{\mathbb{C}}(\mathfrak{s})$, where it is a well-known fact; see Jacobson [8, IX, Thm. 4].

Pairs of opposite roots determine walls in $\Delta(S)$; these walls are combinatorial $\ell-2$-spheres. The result that we will need below is that $\mathrm{Aut}_{\mathbb{Q}}(\mathfrak{s})$ in its action on $\Delta(S)$ has the same orbits on the walls as S: one orbit if the Dynkin diagram is simply laced, and two orbits for the Dynkin diagrams C_n, $n \geqslant 2$, F_4 and G_2.

18 Theorem. *Let S be a complex simple Lie group with Lie algebra $\mathfrak{s}$, let $G \subseteq \mathrm{Aut}_{\mathbb{Q}}(\mathfrak{s})$ be a subgroup containing S and let Γ be a locally compact and σ-compact group. Suppose that $\varphi : \Gamma \longrightarrow G$ is an abstract group isomorphism. Then there exists an element $\alpha \in \mathrm{Aut}_{\mathbb{Q}}(\mathfrak{s})$ such that the composite $c_\alpha \circ \varphi : \Gamma \longrightarrow \mathrm{Aut}_{\mathbb{Q}}(\mathfrak{s})$ is a homeomorphism onto an open subgroup of* $\mathrm{AutLie}(S)$. *In particular,* $\mathrm{Aut}(S) = \mathrm{Aut}_{\mathbb{Q}}(\mathfrak{s})$.

Proof. Let $H \subseteq S$ be a reductive Levi subgroup of rank 1 in a next-to-minimal parabolic $P \subseteq S$. Thus $H \cong H_0 T$, where H_0 is isomorphic to $\mathrm{PSL}_2(\mathbb{C})$ or $\mathrm{SL}_2(\mathbb{C})$ and $T \cong (\mathbb{C}^*)^{\ell-1}$ where ℓ is the complex rank of S. In view of Lemma 16 we may assume that $\ell \geqslant 2$, so T is nontrivial. We can arrange the matrix representation of $\mathrm{Aut}_{\mathbb{C}}(\mathfrak{s})$ in such a way that T is a group of diagonal matrices which is invariant under $\mathrm{Aut}(\mathbb{C})$.

Let $L = \mathrm{Cen}_G(T)$. We claim that L is contained in $\mathrm{Aut}_{\mathbb{C}}(\mathfrak{s})$. The group $\mathrm{Aut}(\mathbb{C})$ normalizes T, hence every element $g \in L$ is a product $g = h\eta$, with $\eta \in \mathrm{Aut}(\mathbb{C})$ and $h \in \mathrm{Nor}_{\mathrm{Aut}_{\mathbb{C}}(\mathfrak{s})}(T)$. Since T is nontrivial, we find a nontrivial algebraic character $\lambda : T \longrightarrow \mathbb{C}^*$ which commutes with $\mathrm{Aut}(\mathbb{C})$ (by evaluating a suitable matrix entry on the diagonal). Then h^{-1} has to act in the same way on $\mathbb{C}^*$ as η. However, the only nontrivial algebraic automorphism of $\mathbb{C}^*$ is inversion, and this map is not induced by a field automorphism of $\mathbb{C}$ (because it is not additive). It follows that $\eta = 1$ and thus $g = h \in \mathrm{Aut}_{\mathbb{C}}(\mathfrak{s})$.

The group L is thus an algebraic finite extension of H and acts algebraically on $H/\mathrm{Cen}(H) \cong \mathrm{PSL}_2(\mathbb{C})$, with kernel $\mathrm{Cen}(L)$. Thus

$$L/\mathrm{Cen}(L) \cong H/\mathrm{Cen}(H) \cong \mathrm{PSL}_2(\mathbb{C})$$

(here we use that $\mathrm{Aut}_{\mathbb{C}}(\mathfrak{sl}_2(\mathbb{C})) = \mathrm{PSL}_2(\mathbb{C})$).

Now we consider the preimage $L' = \varphi^{-1}(L) = \mathrm{Cen}_\Gamma(\varphi^{-1}(T))$. This is a closed subgroup of Γ. The map φ induces an abstract group isomorphism

$$\tilde{\varphi} : L'/\mathrm{Cen}\big(L'\big) \longrightarrow L/\mathrm{Cen}(L) \cong \mathrm{PSL}_2(\mathbb{C})$$

between the locally compact and σ-compact group $L'/\mathrm{Cen}(L')$ and the Lie group $L/\mathrm{Cen}(L)$. By Lemma 16 there is an element $\alpha \in \mathrm{Aut}(\mathbb{C})$ such that $c_\alpha \circ \tilde{\varphi}$ is a homeomorphism.

We claim that ${}^\alpha G \subseteq \mathrm{AutLie}(S)$. We note that the Levi group $H \subseteq S$ is the pointwise stabilizer of a unique wall $M \cong \mathbb{S}^{\ell-2}$ in the spherical building $\Delta(S)$. (The wall is the boundary of the flat subspace corresponding to the vector part of $\mathrm{Lie}(T)$ in the symmetric space of S.) As we remarked above, $\mathrm{Aut}_{\mathbb{Q}}(\mathfrak{s})$ acts on $\Delta(S)$ (by simplicial, but not necessarily type-preserving maps). The group $\mathrm{Aut}(\mathbb{C})$ fixes this wall M pointwise and acts by type-preserving automorphisms on Δ. The group $\mathrm{Nor}_{{}^\alpha G}(T) = \mathrm{Nor}_{{}^\alpha G}(L)$ is precisely the setwise ${}^\alpha G$-stabilizer of this wall M. Now ${}^\alpha G$

has the same orbits on the walls of Δ as S, whence ${}^{\alpha}G = \mathrm{Nor}_{{}^{\alpha}G}(T)S$. The group $\mathrm{Nor}_{\Gamma}(L')$ acts continuously on $L'/\mathrm{Cen}(L')$. Pushing this action forward with $c_\alpha \circ \varphi$, we see that $\mathrm{Nor}_{{}^{\alpha}G}(L)$ acts by homeomorphisms on $L/\mathrm{Cen}(L)$. Thus $\mathrm{Nor}_{{}^{\alpha}G}(L)$ maps into $\mathrm{Gal}(\mathbb{C}/\mathbb{R}) \subseteq \mathrm{Aut}(\mathbb{C})$, because $\mathrm{Aut}(\mathbb{C})$ acts faithfully on $L/\mathrm{Cen}(L)$. It follows from the factorization of ${}^{\alpha}G$ above that ${}^{\alpha}G$ also maps into $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$, that is, ${}^{\alpha}G \subseteq \mathrm{AutLie}(S)$.

Now we want to apply Theorem 7, and we have to find a set $C \subseteq {}^{\alpha}G$. Let $K \subseteq H/\mathrm{Cen}(H)$ be a maximal compact subgroup, $K \cong \mathrm{SO}(3)$. Let $J' \subseteq L'$ denote the preimage of K under the continuous map

$$L' \longrightarrow L'/\mathrm{Cen}\big(L'\big) \xrightarrow{c_\alpha \circ \tilde{\varphi}} L/\mathrm{Cen}(L) = H/\mathrm{Cen}(H).$$

We note that as an abstract group, J' is a central product of a group C' which is isomorphic to SO(3) or SU(2) and an abelian group. Moreover $J'/\mathrm{Cen}(J')$ is (via $c_\alpha \circ \tilde{\varphi}$) homeomorphic to SO(3) and thus compact. By Theorem 12, the algebraic commutator group DJ' has compact closure $\overline{DJ'}$. Algebraically, C' is the set of commutators $C' = \{[g,h] \mid g,h \in J'\}$ by Theorem 13. Thus $C' = \{[g,h] \mid g,h \in \overline{DJ'}\}$ is a compact subset of Γ. Now $c_\alpha \circ \varphi$ maps C' onto a closed subgroup $C \subseteq {}^{\alpha}G$. We now may apply Theorem 7 to the map $\Gamma \xrightarrow{c_\alpha \circ \varphi} {}^{\alpha}G \subseteq \mathrm{AutLie}(S)$ and conclude that $c_\alpha \circ \varphi$ is continuous. $\square$

We finally consider semisimple groups. If S is an absolutely simple real Lie group with Lie algebra $\mathfrak{s}$, then $\mathrm{Aut}_{\mathbb{Q}}(\mathfrak{s}) = \mathrm{Aut}_{\mathbb{R}}(\mathfrak{s}) = \mathrm{AutLie}(S)$. We need the following fact.

19. Let S be a connected centerless semisimple Lie group with Lie algebra $\mathfrak{s}$. Let $\mathfrak{s} = \mathfrak{s}_1 \oplus \cdots \oplus \mathfrak{s}_n$ be its decomposition into simple ideals, and $S = S_1 \times \cdots \times S_n$ the corresponding factorization into simple groups. Then $\mathrm{Aut}_{\mathbb{R}}(\mathfrak{s}) = \mathrm{AutLie}(S)$ is a semidirect product of a subgroup $\Pi \subseteq \mathrm{Sym}(n)$ and the direct product $\mathrm{Aut}_{\mathbb{R}}(\mathfrak{s}_1) \times \cdots \times \mathrm{Aut}_{\mathbb{R}}(\mathfrak{s}_n)$. The group Π consists of all permutations π of $\{1,\ldots,n\}$ that preserve isomorphy of the simple ideals, i.e. $\mathfrak{s}_i \cong \mathfrak{s}_{\pi(i)}$ for all i. Similarly, the group $\mathrm{Aut}(S)$ of all abstract automorphisms of S decomposes as a semidirect product of the same group Π and the direct product $\mathrm{Aut}(S_1) \times \cdots \times \mathrm{Aut}(S_n)$. In particular, there is a split exact sequence

$$1 \longrightarrow \mathrm{Aut}(S_1) \times \cdots \times \mathrm{Aut}(S_n) \longrightarrow \mathrm{Aut}(S) \longrightarrow \Pi \longrightarrow 1.$$

20 Theorem. *Let S be a connected centerless semisimple Lie group with Lie algebra* $\mathrm{Lie}(S) = \mathfrak{s}$. *Let $S = S_1 \times \cdots \times S_n$ denote its decomposition into simple factors. Let $G \subseteq \mathrm{Aut}(S)$ be a subgroup containing S. Suppose that Γ is a locally compact and σ-compact group and that $\varphi : \Gamma \longrightarrow G$ is an abstract group isomorphism. Then there exist elements $\alpha_i \in \mathrm{Aut}(S_i)$ such that $\alpha = \alpha_1 \times \cdots \times \alpha_n$ conjugates G to an open subgroup ${}^{\alpha}G \subseteq \mathrm{AutLie}(S)$, and $c_\alpha \circ \varphi : \Gamma \longrightarrow {}^{\alpha}G$ is a homeomorphism. If S_i is absolutely simple, then α_i may be chosen to be the identity.*

Proof. Let $H_i = \prod_{k \neq i} S_k$. The $\mathrm{Aut}(S)$-centralizer of H_i is $\mathrm{Cen}_{\mathrm{Aut}(S)}(H_i) = \mathrm{Aut}(S_i)$. Thus we have $\prod_i \mathrm{Aut}(S_i) = \bigcap_i \mathrm{Nor}_{\mathrm{Aut}(S)}(\mathrm{Cen}_{\mathrm{Aut}(S)}(H_i))$. From this description it is clear that the subgroup $\Gamma_0 = \varphi^{-1}(\prod_i \mathrm{Aut}(S_i))$ is closed (and open, since it has finite index). By Theorem 11 and Theorem 18, we find $\alpha_i \in \mathrm{Aut}(S_i)$ such that $\alpha = \alpha_1 \times \cdots \times \alpha_n$ conjugates $\varphi(\Gamma_0)$ into an open subgroup of $\mathrm{AutLie}(S_1) \times \cdots \times \mathrm{AutLie}(S_n)$, and such that $c_\alpha \circ \varphi : \Gamma_0 \longrightarrow {}^{\alpha}G$ is a homeomorphism onto its image. The identity component Γ° of Γ is contained in the open subgroup Γ_0.

Therefore $c_\alpha \circ \varphi$ maps Γ° homeomorphically onto S. Since Γ acts by automorphisms on Γ°, the map $c_\alpha \circ \varphi$ extends continuously to $\Gamma \longrightarrow \mathrm{AutLie}(S)$, that is, $c_\alpha(\varphi(\Gamma)) \subseteq \mathrm{AutLie}(S)$. $\square$

So far, all the groups that we considered were centerless. It is, however, easy to see that we have the following consequence of Theorem 20.

21 Corollary. *Let S and G be as in Theorem 20. Suppose that $\tilde{\Gamma}$ is a locally compact and σ-compact group and that $\varphi : \tilde{\Gamma} \longrightarrow G$ is a central surjective homomorphism. If $\mathrm{Cen}(\tilde{\Gamma})$ is discrete (for example, finite or countable), then there exist elements $\alpha_i \in \mathrm{Aut}(S_i)$ such that $\alpha = \alpha_1 \times \cdots \times \alpha_n$ conjugates G to an open subgroup ${}^\alpha G \subseteq \mathrm{AutLie}(S)$, and $c_\alpha \circ \varphi : \tilde{\Gamma} \longrightarrow {}^\alpha G$ is an open map. If S_i is absolutely simple, then α_i may be chosen to be the identity.*

Proof. We just note that $\tilde{\Gamma} \longrightarrow \Gamma = \tilde{\Gamma}/\mathrm{Cen}(\tilde{\Gamma})$ is an open map. $\square$

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