On the local structure and the homology
of $\mathrm{CAT}(\kappa)$ spaces and euclidean buildings

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Abstract. We prove that every open subset of a euclidean building is a finite-dimensional absolute neighborhood retract. This implies in particular that such a set has the homotopy type of a finite dimensional simplicial complex. We also include a proof for the rigidity of homeomorphisms of euclidean buildings. A key step in our approach to this result is the following: the space of directions $\Sigma_o X$ of a $\mathrm{CAT}(\kappa)$ space X is homotopy equivalent to a small punctured disk $B_\varepsilon(X, o) - o$. The second ingredient is the local homology sheaf of X. Along the way, we prove some results about the local structure of $\mathrm{CAT}(\kappa)$-spaces which may be of independent interest.

Introduction

A $\mathrm{CAT}(\kappa)$ space X is a metric space where geodesic triangles of perimeter less than $2D_\kappa$ exist and are not thicker than in the comparison space M_κ of constant sectional curvature κ and diameter D_κ (where $D_\kappa = \pi/\sqrt{\kappa}$ for $\kappa > 0$ and $D_\kappa = \infty$ for $\kappa \leq 0$). Bridson–Haefliger [5] and Burago–Burago–Ivanov [7] give a thorough introduction to these spaces. More generally, a metric space is said to have curvature bounded above if every point has a neighborhood which is $\mathrm{CAT}(\kappa)$, for some κ. We prove the following results.

Theorem A. *Let X be a metric space with curvature bounded above. For every point $o \in X$, there is a number $\varepsilon > 0$ such that the punctured ball $B_\varepsilon(o) - o$ is homotopy equivalent to the space of directions $\Sigma_o X$.*

The proof uses at some point that spaces with curvature bounded above are ANRs. See Theorem 3.2 and the remarks preceding it.

Theorem B. *Let X be a euclidean building with n-dimensional apartments. Then X is an n-dimensional absolute retract. Every nonempty open subset of X is an n-dimensional absolute neighborhood retract.*

As an application, we give a proof of the following result.

Theorem C. *Let $f : X \to X'$ be a homeomorphism between metrically irreducible euclidean buildings of dimension at least 2. Then f maps apartments in the maximal atlas of X to apartments in the maximal atlas of X'. Possibly after rescaling X', there exists a unique isometry $\tilde{f} : X \to X'$ with $f(A) = \tilde{f}(A)$ for all apartments $A \subseteq X$. If X has more than one thick point, then f has finite distance from $\tilde{f}$.*

After the paper was completed, I learned that Theorem A had also been obtained by Lytchak and Nagano some time ago; see [29]. Bruce Kleiner informed me that he had also obtained this result (unpublished). Also, Lang and Schlichenmaier had computed the Nagata dimension of euclidean buildings in [26]. Their result implies Theorem B. The result that $\mathrm{CAT}(\kappa)$ spaces are ANRs has been obtained by various authors (in different degrees of generality). Theorem C was proved by Kleiner and Leeb in [22] for the case of complete euclidean buildings. More references and details are given below.

Outline of the paper. In Section 1 we collect some general facts about spaces with upper curvature bounds. We introduce in Section 2 the space of directions and the 'blow-up' at a given point. Section 3 contains a proof of Theorem A. Our approach is essentially homotopy theoretic. Some simple sheaf theoretic notions are introduced in Section 4. In Section 5, these are applied to the homology of spherical buildings. Section 6 combines the previous results into a proof of Theorem C. In Section 7, we prove Theorem B. The Appendix serves as a reference for some topological results on ANRs and the covering dimension.

The definition of a spherical building that we use is 'simplicial' as in Tits' Lecture Note [37, 3.1] or [38, 1.1], except that we allow weak (non-thick) spherical buildings; see 5.4 below. We define euclidean buildings essentially as Tits does in [38, 1.4], see also [22] and in particular [35]. Euclidean buildings are $\mathrm{CAT}(0)$ spaces which can be highly singular (they need not be simplicial and they may branch everywhere); see 6.1.

1   Spaces with curvature bounded above

We first introduce some notation. Suppose that (X, d) is a metric space, that $o \in X$ and that $r > 0$. We put

$$B_r(o) = \{x \in X \mid d(x, o) < r\} \quad \text{and} \quad \bar{B}_r(o) = \{x \in X \mid d(x, o) \leq r\}.$$

We say that X has *radius at most* r if $X \subseteq \bar{B}_r(o)$ for some $o \in X$, and that X has *diameter at most* r if $X \subseteq \bar{B}_r(o)$ for every $o \in X$. Two maps $f, g : Z \to X$ are *r-near* if $d(f(z), g(z)) \leq r$ holds for all $z \in Z$. A *geodesic* in X is an isometry $J \to X$, where $J \subseteq \mathbb{R}$ is a closed interval. We call the image of J a *geodesic segment* in X.

We now collect some facts about spaces with curvature bounded above. The books by Bridson–Haefliger [5] and Burago–Burago–Ivanov [7] are excellent introductions to this area of geometry.

1.1   The model spaces M_κ. The *model spaces* M_{-1}, M_0 and M_1 are the 2-dimensional[1] hyperbolic space $\mathbb{H}^2$, euclidean space $\mathbb{E}^2$ and the 2-sphere $\mathbb{S}^2$, respectively. These spaces are endowed with the standard hyperbolic, euclidean and spherical metrics d_H, d_E, d_S, respectively; see [5, I.2]. For general $\kappa < 0$ the model space M_κ is defined as $(M_{-1}, \sqrt{-\kappa}d_H)$, and for $\kappa > 0$ as $(M_1, \sqrt{\kappa}d_S)$. These are precisely the complete simply connected surfaces of constant sectional curvature κ and diameter D_κ, where $D_\kappa = \pi/\sqrt{\kappa}$ for $\kappa > 0$ and $D_\kappa = \infty$ for $\kappa \leq 0$.

1.2   CAT(κ) spaces. Let κ be a real number and let M_κ and D_κ be as in 1.1. A metric space is called a CAT(κ) *space* if any two points at distance less than D_κ can be joined by a geodesic, and if geodesic triangles of perimeter less than $2D_\kappa$ are not thicker than the comparison triangles in M_κ; see [5, II.1.1] for more details. This condition on triangles is called the CAT(κ) *inequality*.

The metric completion of a CAT(κ) space is again CAT(κ); see [5, II.3.11]. If X is CAT(κ), then X is also CAT(κ'), for all $\kappa' \leq \kappa$; see [5, II.1.12]. A metric space has *curvature bounded above* if every point has a neighborhood which is CAT(κ), for some κ (which may depend on the point).

The CAT(κ) inequality implies that points u, v at distance $d(u, v) < D_\kappa$ can be joined by a unique geodesic. The corresponding geodesic segment is denoted $[u, v]$. It varies continuously with u and v; see [5, II.1.4]. If Z is a topological space and if $f, g : Z \to X$ are continuous and r-near, for some $r < D_\kappa$, then f and g are homotopic via the *geodesic homotopy* that moves $f(z)$ along $[f(z), g(z)]$ to $g(z)$.

1.3   Convex sets and convex hulls. A subspace C of a CAT(κ) space of diameter less than D_κ is *convex* if $[u, v] \subseteq C$ holds for all $u, v \in C$. For $o \in X$, the balls $B_r(o)$ and $\bar{B}_r(o)$ are convex if $r < D_\kappa/2$; see [5, II.1.4]. If $A \subseteq X$ has radius $r < D_\kappa/2$, then the *convex hull* of A is defined to be the intersection of all convex sets containing A. The convex hull contains A, is convex, CAT(κ), and has radius at most r. A convex set of radius less than $D_\kappa/2$ is contractible by a geodesic homotopy; see [5, II.1.5].

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[1]Sometimes it is convenient to have higher-dimensional model spaces. For our purposes the surfaces suffice.

1.4   The Alexandrov angle. Suppose that o is a point in the $\mathrm{CAT}(\kappa)$ space X and that $c : [0, a] \to X$ and $c' : [0, a'] \to X$ are two geodesics starting at o. The *Alexandrov angle* between c and c' is

$$\angle(c, c') = 2 \lim_{s \to 0} \arcsin(d(c(s), c'(s))/2s);$$

see [5, II.3.1][2]. For the endpoints $u = c(a)$ and $v = c'(a')$ we put

$$\angle_o(u, v) = \angle(c, c').$$

The angle $(u, v) \mapsto \angle_o(u, v)$ is a continuous pseudometric on $B_{D_\kappa}(o) - o$; see [5, I.1.14, II.3.3]. The Alexandrov angles in a triangle in X are not greater than the angles in the comparison triangle in M_κ; see [5, II.1.7(4)].

1.5   The Law of Sines in M_κ. Let a, b, c be a triangle of diameter less than $2D_\kappa$ in M_κ. Let α, β, γ denote the angles at a, b, c. We will use the following two formulas.

The euclidean case $\kappa = 0$. If $\kappa = 0$, then we have the well-known euclidean *Law of Sines:*

$$\frac{\sin(\alpha)}{d(b, c)} = \frac{\sin(\beta)}{d(c, a)} = \frac{\sin(\gamma)}{d(a, b)}.$$

The case $\kappa > 0$. For $\kappa > 0$, the model space M_κ is the unit sphere $\mathbb{S}^2$ in $\mathbb{E}^3$ with metric $d(u, v) = \frac{1}{\sqrt{\kappa}} \arccos(\langle u, v \rangle)$. The spherical *Law of Sines* in M_κ is then

$$\frac{\sin(\alpha)}{\sin(\sqrt{\kappa}d(b, c))} = \frac{\sin(\beta)}{\sin(\sqrt{\kappa}d(c, a))} = \frac{\sin(\gamma)}{\sin(\sqrt{\kappa}d(a, b))}.$$

Proof. To see this, let u, v be unit tangent vectors at a pointing towards b and c, respectively. We may assume that a, u, v is right-handed. Then $u \times v = a \sin(\alpha)$, whence

$$\det(a, u, v) = \langle a, u \times v \rangle = \sin(\alpha).$$

We also have $b = a \cos(\sqrt{\kappa}d(a, b)) + u \sin(\sqrt{\kappa}d(a, b))$ and $c = a \cos(\sqrt{\kappa}d(a, c)) + v \sin(\sqrt{\kappa}d(a, c))$. Thus

$$\begin{aligned}
\det(a, b, c) &= \langle a, b \times c \rangle \\
&= \langle a, u \times v \rangle \sin(\sqrt{\kappa}d(a, b)) \sin(\sqrt{\kappa}d(a, c)) \\
&= \sin(\alpha) \sin(\sqrt{\kappa}d(a, b)) \sin(\sqrt{\kappa}d(a, c)).
\end{aligned}$$

Permuting a, b, c cyclically we obtain three times the same value and the formula above follows.   □

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[2]The Alexandrov angle in M_κ coincides with the usual Riemannian angle.

2   The space of directions and the blow-up at a point

In this section we introduce the space of directions $\Sigma_o X$ at a point (the 'unit tangent space' at o). We also show that by a change of the metric, $\Sigma_o X$ can be glued into the puncture o of $X - o$. We call the resulting space the blow-up of X at o.

The following terminology will be fixed throughout this section. We assume that X is a $\mathrm{CAT}(\kappa)$ space with $o \in X$ such that

$$X \subseteq \bar{B}_{D_\kappa/2}(o).$$

We put $Y = X - o$. For $x \in X$ and $s \in [0, 1]$, we denote by sx the unique point in the geodesic segment $[o, x]$ with $d(sx, o) = sd(x, o)$.

2.1   The space of directions.   We noted in 1.4 that the Alexandrov angle induces a pseudometric $(u, v) \mapsto \angle_o(u, v)$ on Y. The metric completion of Y with respect to $\angle_o$ is the *space of directions* $\Sigma_o X$ (where points x, y with $\angle_o(x, y) = 0$ are identified); see [5, II.3.18 and 19]. The space $(\Sigma_o X, \angle_o)$ is always $\mathrm{CAT}(1)$; see [5, II.3.19]. We denote the canonical map $Y \to \Sigma_0 X$ by ρ.

In order to make this map Lipschitz, we change the metric on Y. For $x, y \in Y$ we put

$$d_o(x, y) = \sqrt{d(x, y)^2 + \angle_o(x, y)^2}.$$

The identity map $(Y, d_o) \to (Y, d)$ is obviously 1-Lipschitz, and so is $\rho : (Y, d_o) \to (\Sigma_o X, \angle_o)$.

2.2 Lemma.   *The identity $(Y, d) \to (Y, d_o)$ is locally uniformly continuous.*

Proof. Let $p \in Y$. Suppose first that $\kappa \leq 0$. We choose $r < d(o, p)/4$. For $u, v \in B_r(p)$ we have by the euclidean Law of Sines $\sin(\angle_o(u, v)) \leq \frac{d(u,v)}{d(o,v)} < \frac{d(u,v)}{d(o,p)-r}$ and $\angle_o(u, v) < \pi/2$. For $\kappa > 0$ we choose $r < D_\kappa/4$ and we obtain similarly that $\sin(\angle_o(u, v)) \leq \frac{\sin(\sqrt{\kappa}d(u,v))}{\sin(\sqrt{\kappa}d(o,v))} < \frac{\sin(\sqrt{\kappa}d(u,v))}{\sin(\sqrt{\kappa}(d(o,p)-r))}$ and $\angle_o(u, v) < \pi/2$. $\square$

2.3   The blow-up at o.   Let $(\bar{X}, d)$ denote the metric completion of (Y, d) (equivalently, of (X, d)). We denote the completion of (Y, d_o) by $(\hat{X}, d_o)$ and call it the *blow-up*[3] of X at o. From the 1-Lipschitz maps $(Y, d_o) \to (Y, d)$ and $(Y, d_o) \to (Y, \angle_o)$ we obtain 1-Lipschitz maps

$$q : (\hat{X}, d_o) \to (\bar{X}, d) \quad \text{and} \quad \hat{\rho} : (\hat{X}, d_o) \to (\Sigma_o X, \angle_o).$$

We note also that d and $\angle_o$ extend to continuous pseudometrics on $\hat{X}$, which we denote by the same symbols.

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[3]This is not the blow-up in the sense of Lytchak [28].

2.4 Proposition. *The preimage $q^{-1}(o)$ is the space of directions $\Sigma_o X$. The restriction $q : \hat{X} - q^{-1}(o) \to \bar{X} - o$ is a homeomorphism. Thus we have*

$$\hat{X} = (\bar{X} - o) \dot{\cup} \Sigma_o X$$

and $\Sigma_o X \hookrightarrow \hat{X}$ is a closed isometric embedding.

Proof. Let $(x_n)_{n \in \mathbb{N}}$ and $(y_n)_{n \in \mathbb{N}}$ be Cauchy sequences in (Y, d_o) with $\lim_n d(x_n, o) = \lim_n d(y_n, o) = 0$. Then $\lim d_o(x_n, y_n) = \angle_o(x_n, y_n)$. Therefore their limits x and y in $\hat{X}$ have distance $d_o(x, y) = \angle_o(x, y)$. Both sequences are also Cauchy sequences with respect to $\angle_o$. So $q^{-1}(o)$ is a complete subset of $\Sigma_o X$. If $y \in Y$ represents a point in $\Sigma_o X$, then the d_o-Cauchy sequence $(\frac{1}{n} y)_{n \in \mathbb{N}}$ represents the same point. Thus $q^{-1}(o)$ is dense in $\Sigma_o X$. $\square$

2.5 Lemma. *For $s \in [0, 1]$ and $x \in \hat{X}$ we put*

$$\hat{h}_s(x) = \begin{cases} sx & \text{for } x \in \bar{X} - o \text{ and } s > 0 \\ \hat{\rho}(x) & \text{else.} \end{cases}$$

Then $(x, s) \mapsto \hat{h}_s(x)$ is a 1-Lipschitz strong deformation retraction of $\hat{X}$ onto $\Sigma_o X$.

Proof. We have $\angle_o(u, v) = \angle_o(\hat{h}_s(u), \hat{h}_{s'}(v))$ for all $u, v \in \hat{X}$ and $s, s' \in [0, 1]$. From the comparison triangle in M_κ we see that $d(su, s'v)^2 \le d(u, v)^2 + (s - s')^2$, whence $d_o(su, s'v)^2 \le d_o(u, v)^2 + (s - s')^2$. Thus $(x, s) \mapsto \hat{h}_s(x)$ is 1-Lipschitz. $\square$

We remark that this homotopy can be restricted to the subset $Y \cup \Sigma_o X \subseteq \hat{X}$.

3   The homotopy type of the space of directions

In this section we prove that the space of directions at a point o in a $\mathrm{CAT}(\kappa)$ space X has the homotopy type of a small punctured neighborhood of o. We first collect some facts from homotopy theory. Unless specified otherwise, all simplicial complexes are endowed with the weak topology (the CW topology). This allows us to define continuous maps simplex-wise. We start with a fact from obstruction theory.

3.1 Lemma (Acyclic carriers). *Let K be a simplicial complex and Y a topological space. Let $f : K^{(0)} \to Y$ be a map from the 0-skeleton (the vertex set) of K to Y. Suppose that there is a map C that assigns to every simplex $A \subseteq K$ a subset $C_A \subseteq Y$ with the following properties.*

(1) *If $A \subseteq K$ is a simplex, then C_A contains $f(A \cap K^{(0)})$.*
(2) *If $A \subseteq B$ are simplices, then $C_A \subseteq C_B$.*
(3) *$\pi_*(C_A, p) = 0$ for all $p \in C_A$.*

Then f has a continuous extension $f : K \to Y$ such that $f(A) \subseteq C_A$ holds for all simplices $A \subseteq K$.

Proof. The map f is defined inductively on the m-skeleton. Suppose that $f : K^{(m-1)} \to Y$ is already defined. Let $A \subseteq K$ be an m-simplex and put $\partial A = K^{(m-1)} \cap A$. Choose $p \in f(\partial A) \subseteq C_A$. Since $\pi_{m-1}(C_A, p) = 0$, there exists an extension of the map $f|_{\partial A} : \partial A \to C_A \subseteq Y$ over A; see [18, VI.6.5] or [41, V.5.14]. These maps fit together to a continuous map $f : K^{(m)} \to Y$. $\qquad\square$

We call a function C satisfying (1), (2), (3) in the previous lemma a *topological acyclic carrier* for f. Our first application is as follows[4].

3.2 Theorem. *Let Z be a metric space with curvature bounded above. Then every open subset of Z is an ANR (an absolute neighborhood retract).*

Proof. It suffices to show that Z is locally an AR (an absolute retract); see [19, III.8.1]. Let $o \in Z$ and choose $R > 0$ in such a way that $\bar{B}_R(o)$ is a $\mathrm{CAT}(\kappa)$ space of radius less than $D_\kappa/2$. We show that X is an AR (absolute retract). Let K denote the simplicial complex on all finite subsets of X. Suppose that $A \subseteq K$ is a simplex spanned by the points $a_0, \ldots, a_m \in X$. Let C_A denote the convex hull of $a_0, \ldots, a_m$ in X. Then C_A is contractible. In this way we obtain a topological acyclic carrier for the identity map $X = K^{(0)} \to X$. By Lemma 3.1, there exists a continuous map $q : K \to Z$ such that the image of every simplex $A \subseteq K$ is contained in C_A. Since the radius of C_A is bounded by the diameter of the set $\{a_0, \ldots, a_m\}$, the map q has the following two properties:

(1) q is surjective.
(2) If $(A_j)_{j \in \mathbb{N}}$ is a sequence of simplices in K such that $(q(A_j \cap K^{(0)}))_{j \in \mathbb{N}}$ converges to $x \in X$, then $(q(A_j))_{j \in \mathbb{N}}$ also converges to x.

By Wojdysławski's Theorem 8.2, X is an AR. $\qquad\square$

Next, we recall the following version of the Whitehead Theorem.

3.3 Theorem (Whitehead). *Let $f : X \to Y$ be a continuous map between topological spaces. Then f is a weak equivalence if and only if for every finite simplicial complex K the induced map*

$$f_* : [K, X] \to [K, Y]$$

between free homotopy sets is bijective.

Proof. The bijectivity is sufficient by [41, IV.7.16]. There, the result is stated under the stronger assumption that bijectivity holds for every CW complex K. However, the only spaces K used in the proof in *loc.cit.* are spheres. The bijectivity is a necessary condition by [41, IV.7.17]. $\qquad\square$

Armed with 3.2 and Whitehead's Theorem, we now proceed to the proof of Theorem A. The following Lifting Lemma is the crucial step. Related results can be found in [21]. However, our topological approach avoids all completeness assumptions (for example, we do not use barycentric simplices).

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[4]After the present paper was completed, Urs Lang informed me that [16, Theorem 7] contains this result for complete $\mathrm{CAT}(\kappa)$ spaces. See also the remarks in [15, 4.2] and [5, I.7A.15, II.5.13]. The result is also proved in [34, Lemma 1.1] and in [30, Section 6] — essentially by the same characterization of ANRs.

3.4 Lemma (The Lifting Lemma). *Let X be a $\mathrm{CAT}(\kappa)$ space and suppose that $o \in X$ is a point with $X \subseteq \bar{B}_{D_\kappa/2}(o)$. Put $Y = X - o$ and let $\rho : Y \to \Sigma_o X$ be as in 2.1. Assume that K is a finite simplicial complex and that $g : K \to \Sigma_o X$ is a continuous map. Then there exists for every $\varepsilon > 0$ a continuous map $f : K \to Y$ such that $\rho \circ f$ is ε-near to g.*

Proof. We may assume that $\kappa > 0$, since we are interested in a local question. We may also assume that $\varepsilon < \pi/2$. We choose $\alpha > 0$ such that $\alpha < \max\{1/2, \sin(\varepsilon/4)\}$. We subdivide K in such a way that the g-image of every simplex $A \subseteq K$ has diameter less than $\alpha/3$. Next, we choose for every vertex a of K a point $y_a \in Y$ such that $\angle_o(\rho(y_a), g(a)) < \alpha/3$ (this is possible because $\rho(Y)$ is dense in $\Sigma_o X$). The vertex set $K^{(0)}$ is finite, so moving points along geodesic segments towards o, we can arrange that all y_a have the same positive distance r from o. If the vertices a, b belong to the same simplex A in K, then

$$\angle_o(y_a, y_b) < \angle_0(y_a, g(a)) + \angle_o(g(a), g(b)) + \angle_o(g(b), y_b) < \alpha.$$

Since

$$\sin(\angle_o(y_a, y_b)/2) = \lim_{s \to 0} d(sy_a, sy_b)/(2rs) < \sin(\alpha/2) < \alpha/2,$$

we have $d(sy_a, sy_b) < sr\alpha$ for all sufficiently small $s > 0$. Since $K^{(0)}$ is finite, there exists an $s_0 > 0$ such that the following holds for all $s \in (0, s_0)$:

(*) If $\{a_0, \ldots, a_m\}$ span a simplex A in K, then the convex hull C_A of $\{sy_{a_0}, \ldots, sy_{a_m}\}$ has radius less than $sr\alpha$.

Since $\alpha < 1/2$ and $d(sy_{a_j}, o) = sr$, this implies in particular that C_A does not contain o, i.e. $C_A \subseteq Y$. We put $f(a) = sy_a$. In this way we have constructed a topological acyclic carrier and we may apply Lemma 3.1. We obtain a continuous extension $f : K \to Y$ such that $f(A) \subseteq C_A$ for every simplex $A \subseteq K$. Such an extension exists for every $s < s_0$. In particular, we may assume that $\sqrt{\kappa}sr\alpha < \pi/2$. If $z \in Y$ is any point with $d(z, sy_a) < sr\alpha$, then we have by the Law of Sines 1.5

$$\sin(\angle_o(z, sy_a)) < \frac{\sin(\sqrt{\kappa}sr\alpha)}{\sin(\sqrt{\kappa}sr)}.$$

Since $\alpha < \sin(\varepsilon/4)$, we see (using l'Hôpital's rule) that for all sufficiently small s, we have $\frac{\sin(\sqrt{\kappa}sr\alpha)}{\sin(\sqrt{\kappa}sr)} < \sin(\varepsilon/4)$. Thus we can choose s in such a way that $\angle_o(z, sy_a) < \varepsilon/4$ holds for all vertices a and all $z \in B_{sr\alpha}(sy_a)$. For such an s we consider the map $f : K \to Y$ constructed before. If x is in the simplex $A \subseteq K$ and $a \in A$ is a vertex, then

$$\angle_o(\rho(f(x)), g(x)) \leq \angle_o(\rho(f(x)), \rho(f(a))) + \angle_o(\rho(f(a)), g(a)) + \angle_o(g(a), g(x))$$
$$< \varepsilon/4 + \alpha/3 + \alpha/3 + \alpha/3$$
$$< \varepsilon. \qquad \square$$

3.5 Theorem. *Let X be a $\mathrm{CAT}(\kappa)$ space and suppose that $o \in X$ is a point with $X \subseteq \bar{B}_{D_\kappa/2}(o)$. Put $Y = X - o$ and let $\rho : Y \to \Sigma_o X$ be as in 2.1. Then ρ is a homotopy equivalence.*

Proof. Let K be a finite simplicial complex. We show that $\rho_* : [K, Y] \cong [K, \Sigma_o X]$ is a bijection. Suppose that $g : K \to \Sigma_o X$ is continuous. By the Lifting Lemma 3.4 there is a continuous map $f : K \to Y$ such that $\rho \circ f$ is $\pi/2$-near to g. Therefore $\rho \circ f$ is homotopic to g by the geodesic homotopy in $\Sigma_o X$. This shows that ρ_* is surjective.

Now suppose that $h_0, h_1 : K \to Y$ are continuous and that $\rho \circ h_0$ and $\rho \circ h_1$ are homotopic in $\Sigma_o X$. Thus there is a homotopy $g : K \times [0, 1] \to \Sigma_o X$ with $\rho \circ h_i = g_i$, for $i = 0, 1$. By the Lifting Lemma 3.4, there exists a continuous map $f : K \times [0, 1] \to Y$ such that $\rho \circ f$ is $\pi/2$-near to g. For $x \in K$ and $i = 0, 1$ we have

$$\begin{aligned}
\angle_o(h_i(x), f_i(x)) &= \angle_o(\rho(h_i(x)), \rho(f_i(x))) \\
&= \angle_o(g_i(x), \rho(f_i(x))) \\
&\leq \pi/2.
\end{aligned}$$

In particular, o is not in the geodesic segment $[h_i(x), f_i(x)] \subseteq X$, since otherwise we would have $\angle_o(h_i(x), f_i(x)) = \pi$. Thus the geodesic homotopy in X between h_i and f_i takes its values in Y. Therefore $h_0 \simeq f_0 \simeq f_1 \simeq h_1$ in Y. This shows that ρ_* is injective. By Whitehead's Theorem 3.3, ρ is a weak homotopy equivalence. Since Y and $\Sigma_o X$ are ANRs by 3.2, both spaces have the homotopy types of CW complexes; see 8.1. A weak homotopy equivalence between such spaces is a homotopy equivalence by Whitehead's (other) Theorem; see [41, V.3.5]. □

This finishes the proof of Theorem A in the introduction[5].

4   The local homology sheaf of a space

Let X be a topological space. Recall that a *presheaf* on X is a cofunctor[6] on the category of open sets of X. An important example is the map which assigns to an open set $U \subseteq X$ the relative (singular) homology group $H_*(X, X - U)$. The sheaf generated by this presheaf is the *local homology sheaf* $\mathscr{H}_*(X)$ of X. Its stalk at $p \in X$ is the local singular homology

$$\mathscr{H}_*(X)_p = \lim_{U \ni p} H_*(X, X - U) = H_*(X, X - p);$$

see [4, p. 7]. Every relative k-cycle $\sigma \in H_k(X, X - U)$ induces a section $p \mapsto \sigma_p \in \mathscr{H}_k(X)_p$ over U via the restriction map $H_k(X, X - U) \to H_k(X, X - p)$. The *support* of σ is

$$\text{supp}(\sigma) = \{p \in U \mid \sigma_p \neq 0\}.$$

This set is closed in U (by general facts about sheaves, or more directly because singular homology satisfies the axiom of compact supports). We note also that $\text{supp}(\sigma)$ is contained in the image of any relative singular cycle representing σ.

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[5]Bruce Kleiner informed me that he had obtained the following result some time ago (unpublished): for any open set $W \subseteq \Sigma_o X$ and $\varepsilon > 0$ small enough, the map $(B_\varepsilon(o) - o) \cap \rho^{-1}(W) \to W$ is a homotopy equivalence. In fact, our proof above could easily be modified in order to obtain this stronger result.

[6]With values in a some given category.

On X we have also the set-valued presheaf

$$Closed : U \mapsto \{A \cap U \mid A \subseteq X \text{ is closed}\}.$$

The sheaf $\mathscr{Closed}$ generated by $Closed$ has as its stalk $\mathscr{Closed}_p$ the set of germs of closed sets near p. This stalk $\mathscr{Closed}_p$ is a poset and a distributive lattice[7]; see [3, I §6.9, I §6.10] or [20]. It is clear that supp is a natural transformation of presheaves

$$\text{supp} : H_*(X, X - U) \to Closed(U)$$

and sheaves, in particular we have an induced map on the stalks

$$\text{supp} : \mathscr{H}_*(X)_p \to \mathscr{Closed}_p.$$

5   Cycles in spherical buildings

We start with some elementary observations about simplicial complexes. (We recall our convention that we endow simplicial complexes with the weak topology.) Let K be an m-dimensional simplicial complex and let σ be an m-cycle in the homology $H_m(K)$. We are interested in the support of σ over K.

5.1 Lemma. *The support* $\text{supp}(\sigma) \subseteq K$ *is a pure[8] m-dimensional subcomplex.*

Proof. We represent σ as a finite sum $\beta_1 C_1 + \cdots + \beta_k C_k$, where the C_j are distinct m-simplices in K and the β_j are nonzero coefficients. For each interior point $p \in C_j$, the cycle σ maps in $H_m(C_j, C_j - p) \cong H_m(K, K - p) \cong \mathbb{Z}$ to β_j times a generator (by excision). Thus $C_1 \cup \cdots \cup C_k \subseteq \text{supp}(\sigma)$. On the other hand, the support set cannot be bigger than the union of the simplices representing σ. □

The *join* of two simplicial complexes K, L is denoted $K * L$. The complex $K * \mathbb{S}^0$ is the same as the unreduced suspension of K.

5.2 Lemma. *Under the Mayer–Vietoris isomorphism* $H_{m+1}(K * \mathbb{S}^0) \to H_m(K)$, *the preimage of* $\sigma \in H_m(K)$ *has* $\text{supp}(\sigma) * \mathbb{S}^0$ *as its support set.*

Proof. Let $\mathbb{S}^0 = \{u, v\}$. As in the previous proof we represent σ as a finite sum $b = \beta_1 C_1 + \cdots + \beta_k C_k$, where the C_j are distinct m-simplices and the β_j are nonzero coefficients. Consider the two $m + 1$-chains $u * b = \beta_1(u * C_1) + \cdots + \beta_k(u * C_k)$ and $b * v = \beta_1(C_1 * v) + \cdots + \beta_k(C_k * v)$. Then $\partial(u * b) = -b$ and $\partial(b * v) = b$, so $u * b + b * v$ is a cycle in $K * S^0$, with support $\text{supp}(\sigma) * \mathbb{S}^0$. We claim that it maps to b. Indeed, the map $H_{m+1}(K * \mathbb{S}^0) \to H_m(K)$ is the composite

$$H_{m+1}(K * \mathbb{S}^0) \xrightarrow{\cong} H_{m+1}(K * \mathbb{S}^0, K * u) \xleftarrow{\cong} H_{m+1}(K * v, K) \xrightarrow{\cong} H_m(K);$$

see [13, §15]. Tracing the cycle $u * b + b * v$ along these arrows, we end up at b. □

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[7] It is a 'pointless topological space'.

[8] A complex is pure if all maximal simplices have the same dimension.

5.3 Corollary. *The preimage of* $\sigma \in H_m(K)$ *under the isomorphism* $H_{m+n+1}(K * \mathbb{S}^n) \to H_m(K)$ *has as its support* $\mathrm{supp}(\sigma) * \mathbb{S}^n$.

Proof. This is an iterated application of the previous lemma, since $\mathbb{S}^0 * \mathbb{S}^k \cong \mathbb{S}^{k+1}$.   □

5.4   Spherical buildings. A *spherical building* Δ is a simplicial complex together with a collection $Apt(\Delta)$ of subcomplexes, called *apartments*, which are isomorphic to a fixed spherical Coxeter complex Σ (a certain triangulated sphere). The apartments have to satisfy the following compatibility conditions.

(B1) For any two simplices $C, D \subseteq \Delta$, there is an apartment $A \subseteq \Delta$ containing C, D.
(B2) If $A, A' \subseteq \Delta$ are apartments containing the simplices C, D, then there is a combi-
        natorial (type preserving) isomorphism $A \to A'$ fixing C and D.

The maximal simplices are also called *chambers*. If every non-maximal simplex is con-
tained in at least three apartments, then the building is called *thick*. Buildings in the sense of the present definition are sometimes called *weak buildings*. We refer to [37] and to [1, 38, 40] for details and to [10] for a nice characterization from a metric viewpoint.
    Let now Δ be a (possibly weak) spherical building of dimension m (viewed as a simplicial complex and endowed with the CW topology). By the Solomon–Tits Theorem, Δ has the homotopy type of a sum of m-spheres. This can be made more precise as follows. Let C_0 be a chamber (an m-simplex) and let M denote the collection of all chambers opposite C_0. Each chamber $C \in M$ determines together with C_0 a unique apartment $A \simeq \mathbb{S}^m$. The complex $N = \Delta - \bigcup\{\mathrm{int}(C) \mid C \in M\}$ consisting of all simplices which are not in M is contractible and thus

$$\Delta \simeq \Delta/N \cong \bigvee_M \mathbb{S}^m.$$

See [1, 4.12] for details. We now inspect the cycles in the top-dimensional singular ho-
mology of Δ. For each apartment $A \subseteq \Delta$ containing C_0 we have the cycle $[A] \in H_m(\Delta)$ given by the fundamental class[9] of A. These cycles $[A]$ form a basis for the free $\mathbb{Z}$-module $H_m(\Delta)$ and the support of $[A]$ is A, since the fundamental class of A consists of all sim-
plices in A.

5.5 Lemma. *Let Δ be a thick spherical m-dimensional building and let $n \geq -1$. Put $\mathbb{S}^{-1} = \emptyset$. Let $\mathcal{S}$ denote the lattice consisting of all finite intersections and unions of support sets of cycles in $H_{m+n+1}(\Delta * \mathbb{S}^n)$. Then $\mathcal{S}$ has a unique minimal element, $\emptyset * \mathbb{S}^n$. If a is a simplex in Δ, then $a * \mathbb{S}^n$ is in $\mathcal{S}$.*

Proof. By 5.3, every support set of a cycle in $H_{m+n+1}(\Delta * \mathbb{S}^n)$ is of the form $\mathrm{supp}(\sigma) * \mathbb{S}^n$, for some cycle $\sigma \in H_m(\Delta)$. Therefore the first claim follows. Since Δ is thick, every simplex $a \in \Delta$ can be written as the intersection of finitely many apartments in Δ. This shows the second claim.   □

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[9]We fix an orientation of the simplex C_0 and choose the orientation of the m-sphere A accordingly.

5.6 We can recover the combinatorial structure of $\Delta * \mathbb{S}^n$ from $\mathcal{S}$ as follows. Let us call an element of a poset lattice *indecomposable* if it is not a union of finitely many strictly smaller elements. All elements of $\mathcal{S}$ are of the form $K * \mathbb{S}^n$, where K is a finite subcomplex of Δ. We have seen above that every simplex a of Δ occurs in this way in $\mathcal{S}$ as $a * \mathbb{S}^n$. Obviously, these sets $a * \mathbb{S}^n$ are precisely the indecomposable elements of $\mathcal{S}$. The combinatorial structure of the underlying simplicial complex of a thick spherical building determines the buildings uniquely: there is precisely one set of apartments which turns Δ into a spherical building; see [37, 3.15].

5.7 Thick reductions. The reason why we consider joins of spherical buildings and spheres is as follows. If Δ is a weak spherical building, then there exists a thick spherical building Δ_0 and a number $n \geq -1$ such that a simplicial subdivision of Δ is isomorphic to $\Delta_0 * \mathbb{S}^n$; see [9, 1.3], [10, 3.8. or 7.1], [22, 3.7], [36]. (Strictly speaking, we have to allow here that $\Delta_0 = \emptyset = \mathbb{S}^{-1}$.)

5.8 So far, we have considered spherical buildings as simplicial complexes endowed with the weak (simplicial) topology. A spherical building carries also another topology coming from its natural $\mathrm{CAT}(1)$ metric, see [5, II.10A.4]. The identity is a continuous map from the weak topology to the metric topology. If Δ is infinite, then this is not a homeomorphism. However, it is always a homotopy equivalence. For a proof, see Section I.7 in [5], in particular [5, Exception I.7A.11(2)], combined with [11, Theorem 1] or [27, 5.4.6]. The metric topology yields the same cycles and support sets in Δ as the weak topology. That the support sets remain unchanged can be seen as follows: the metric topology and the CW topology coincide in the interior of every chamber.

6   Topological rigidity of euclidean buildings

We recall briefly the definition of a euclidean building. We refer to [38] and to [22, 25, 35] for details and further results.

6.1 Euclidean buildings. Let W be a spherical Coxeter group and $W\mathbb{R}^n$ the corresponding affine Weyl group (we extend W by the full translation group $(\mathbb{R}^n, +)$). From the reflection hyperplanes of W we obtain a decomposition of $\mathbb{R}^n$ into *walls*, *half spaces*, *Weyl chambers* (a Weyl chamber is a fundamental domain for W — these are Tits' *chambres vectorielles*) and *Weyl simplices* (Tits' *facettes vectorielles*).

Let X be a metric space. A *chart* is an isometric embedding $\varphi : \mathbb{E}^n \to X$, and its image is called an *affine apartment*. We call two charts φ, ψ W-*compatible* if $Y = \varphi^{-1}(\psi(\mathbb{E}^n))$ is convex (in the Euclidean sense) and if there is an element $w \in W\mathbb{R}^n$ such that $\psi \circ w|_Y = \varphi|_Y$ (this condition is void if $Y = \emptyset$). We call a metric space X together with a collection $\mathcal{A}$ of charts a *Euclidean building* if it has the following properties.

(A1) For all $\varphi \in \mathcal{A}$ and $w \in W\mathbb{R}^n$, the composition $\varphi \circ w$ is in $\mathcal{A}$.
(A2) The charts are W-compatible.
(A3) Any two points $x, y \in X$ are contained in some affine apartment.

The charts allow us to map Weyl chambers, walls and half spaces into X; their images are also called Weyl chambers, walls and half spaces. The first three axioms guarantee that these notions are coordinate independent.

(A4) If $C, D \subseteq X$ are Weyl chambers, then there is an affine apartment A such that the intersections $A \cap C$ and $A \cap D$ contain Weyl chambers.

(A5') For every apartment $A \subseteq X$ and every $p \in A$ there is a 1-Lipschitz retraction $h : X \to A$ with $h^{-1}(p) = \{p\}$.

Condition (A5') may be replaced by the following condition:

(A5) If A_1, A_2, A_3 are affine apartments which intersect pairwise in half spaces, then $A_1 \cap A_2 \cap A_3 \neq \emptyset$.

See [35] for a thorough discussion of different sets of axioms.

We assume now that X is a euclidean building with n-dimensional apartments.

6.2 Lemma. *Let $A \subseteq X$ be an apartment and let $p \in A$. Then $H_n(A, A - p) \to H_n(X, X - p)$ is a split injection.*

Proof. This is clear from Axiom (A5').   □

6.3 Corollary. *Let $r > 0$ and let σ be a generator of $H_n(A, A - B_r(p)) \cong \mathbb{Z}$. The support of the image of σ in $H_n(X, X - B_r(p))$ is $A \cap B_r(p)$.*

Proof. We represent σ as the fundamental class of the closed n-disk $\bar{B}_r(p) \cap A$ relative to its boundary, an $n - 1$-sphere. This class generates $H_n(A, A - B_r(p)) \cong \mathbb{Z}$. We conclude that $\mathrm{supp}(\sigma) \subseteq A \cap B_r(p)$. For every $o \in A \cap B_r(p)$ we have a commutative diagram

$$
\begin{array}{ccc}
H_n(A, A - B_r(p)) & \longrightarrow & H_n(X, X - B_r(p)) \\
\cong \downarrow & & \downarrow \\
H_n(A, A - o) & \xrightarrow{\text{injective}} & H_n(X, X - o).
\end{array}
$$

Thus o is in the support.   □

6.4 Let $o \in X$. We recall the following facts (essentially, all due to Tits); see [35, 2.10], [22, 4.2.2 and 4.4.] or [35, 1.14 and p. 20] for proofs.

(a) A euclidean building is a (not necessarily complete) $\mathrm{CAT}(0)$ space.

(b) The space of directions $\Sigma_o X$ at any point $o \in X$ is the $\mathrm{CAT}(1)$ realization of an $(n - 1)$-dimensional (possibly weak) spherical building $\Delta = \Delta_o$ of type (W_0, I).

(c) Let $\mathcal{W}_o$ denote the collection of all o-based Weyl simplices in X. The image of $\mathcal{W}_o$ in $\mathscr{Closed}_p$ is (as a poset) precisely the simplicial complex Δ_o.

The next result is essentially [22, 6.2.3].

6.5 Proposition. *Let $\sigma \in H_n(X, X - B_r(o))$. Then there exists $\varepsilon \in (0, r)$ such that the following holds. There are o-based Weyl chambers $C_1, \ldots, C_k$ such that the support of σ in $B_\varepsilon(o)$ is $\mathrm{supp}(\sigma) = B_\varepsilon(o) \cap (C_1 \cup \cdots \cup C_k)$.*

Proof. Since X is contractible, we have by 3.5

$$H_n(X, X - B_r(o)) \longrightarrow H_n(X, X - o) \xrightarrow[\cong]{\partial} H_{n-1}(X - o) \xrightarrow[\cong]{\rho_*} H_{n-1}(\Sigma_o X).$$

Let $K \subseteq \Sigma_o X$ denote the support of the image of σ in $\Sigma_o X$. By 5.1 and (b) above, K is a finite pure subcomplex of the spherical building Δ_o. We fix a finite set $\mathcal{K}_o \subseteq \mathcal{W}_o$ of o-based Weyl simplices representing the simplices in K. Since K is finite, we can find by (c) a number $s \in (0, r)$ such that for all simplices $A', B' \subseteq K$ with corresponding Weyl simplices $A, B \in \mathcal{K}_o$, we have

$$A' \subseteq B' \quad \text{if and only if} \quad \bar{B}_s(o) \cap A \subseteq \bar{B}_s(o) \cap B.$$

Thus $K_s = \{x \in \bigcup \mathcal{K}_o \mid d(x, o) = s\}$ is a simplicial complex which is isomorphic to K under the canonical map $\rho : X - o \to \Sigma_o X$. The set $K_{\leq s} = \{x \in \bigcup \mathcal{K}_o \mid d(x, o) \leq s\}$ is a finite n-dimensional simplicial complex isomorphic to the cone over K_s.

We now write the image of σ in $H_{n-1}(\Sigma_o X)$ as a linear combination of distinct chambers $b = \beta_1 C'_1 + \cdots + \beta_k C'_k$ with nonzero coefficients β_j. Let $C_j \in \mathcal{K}_o$ denote the Weyl chamber representing C'_j, put $\tilde{C}_j = C_j \cap \bar{B}_s(o)$ and consider the relative cycle $\beta_1 \tilde{C}_1 + \cdots + \beta_k \tilde{C}_k$ in $H_m(K_{\leq s}, K_s)$. Let τ denote its image in $H_m(X, X - B_s(o))$. If p is an interior point in the m-simplex $\tilde{C}_j$, then τ restricts by excision and 6.3 to β_j times a generator of $H_m(\tilde{C}_j, \tilde{C}_j - p)$. Thus $\operatorname{supp}(\tau) = K_{\leq s} \cap B_s(o)$. Moreover, τ has (by construction) the same image in $H_{n-1}(\Sigma_o X)$ as σ. Therefore there exists $\varepsilon \in (0, s)$ such that τ and σ have the same support set in $B_\varepsilon(o)$ (because the complements of supports are open). □

6.6 Corollary. *Consider the image of* $\operatorname{supp} : \mathscr{H}_*(X)_o \to \mathscr{C}losed_o$. *Let* $\mathcal{S}_o \subseteq \mathscr{C}losed_o$ *denote the lattice consisting of all finite intersections and unions of germs of support sets. The indecomposable elements in $\mathcal{S}_o$ form the simplices of the thick part of the reduction of Δ_o, see 5.6. The germ of a support set comes from the fundamental class of an apartment A containing o if and only if it can be written as a union of indecomposables representing an apartment in the thick reduction of Δ_o.*

In particular, we obtain the following result.

6.7 Theorem. *Let $f : X \to X'$ be a homeomorphism between two euclidean buildings with apartments of dimensions n and n', respectively. Then $n = n'$. If $A \subseteq X$ is an apartment and $o \in A$, then there exists a small neighborhood $U \subseteq A$ of o so that $f(U)$ is contained in an apartment A' of X'.*

Proof. By 6.2, the number n is characterized by the fact that $H_j(X, X - o) = 0$ for $j \neq n$ (we can of course also use the equality $n = \dim(X) = \dim(X') = n'$ established in 7.1). Thus $n = n'$. From 6.6 we see that Δ_o is a sub-poset of $\mathscr{C}losed_o$ that can be read off from $\mathscr{H}_n(X)_o$. In particular, we can read off which cycles in $\mathscr{H}_n(X)_o$ arise from fundamental classes of apartments. The claim follows from 6.5. □

An immediate corollary is the following result due to Kleiner and Leeb [22, 6.4.2].

6.8 Corollary (Kleiner–Leeb). *Let X, X' be euclidean buildings and suppose that f : $X \to X'$ is a homeomorphism. Let $A \subseteq X$ be an apartment. If X' is complete, then $f(A)$ is an apartment in the maximal atlas $\mathcal{A}_{\max}$ of X'.*

Proof. For each $o \in A$ there is a small neighborhood $U \subseteq A$ of o that is contained in some apartment $A' \subseteq X'$. Thus $f(A)$ is a complete simply connected metric space that is locally isometric to euclidean space $\mathbb{E}^n$. In particular, $f(A)$ is a flat complete simply connected Riemannian manifold. Such a manifold is isometric to euclidean space; see [23, V.4.1]. It follows that $f(A)$ is an apartment in $\mathcal{A}_{\max}$; see [22, 4.6] or [35, 2.25]. □

The following example shows that the completeness assumption in 6.8 is, apparently, not superfluous. We also remark that there exist higher-dimensional non-complete Bruhat–Tits buildings; some algebraic conditions are discussed in [6, 7.5].

6.9 Example. Consider the $\mathbb{R}$-trees $T_1 = (-1, 1) \times \mathbb{R}$ and $T_2 = \mathbb{R} \times \mathbb{R}$, both endowed with the metric

$$d((x, y), (x', y')) = \begin{cases} |y - y'| & \text{if } x = x' \\ |y| + |x - x'| + |y'| & \text{if } x \neq x'. \end{cases}$$

Both trees are 1-dimensional euclidean buildings and T_2 is complete (but T_1 is not complete). Stretching the x-axis, we obtain a homeomorphism $T_2 \to T_1$. In T_2, the x-axis is an apartment in the maximal atlas. However $(-1, 1) \times 0 \subseteq T_1$ is not even contained in an apartment of T_1.

Nevertheless, we can do better than 6.8. The completeness assumption can be removed if the dimension is high enough. We need some more facts about the local structure of euclidean buildings which can be found in [25]. A wall in a euclidean building is called *thick* if it is the intersection of three apartments; see [25, Section 10].

6.10 Lemma. *Let $f : X \to X'$ be a homeomorphism between euclidean buildings. Let $A \subseteq X$ be an apartment and $U \subseteq A$ an open subset. Suppose that $f(U)$ is in some apartment A' in X'. If $M \subseteq A$ is a thick wall, then there is a thick wall $M' \subseteq A'$ with $f(U \cap M) = f(U) \cap M'$.*

Proof. Let $p \in M \cap U$. The germ of $f(A)$ at $f(p)$ is an apartment and the germ of $f(M)$ at p is a thick wall in the spherical building $\Delta_{f(p)}$. So $f(U \cap M)$ is locally a thick wall in $f(U)$, and therefore part of a thick wall; see [25, 10.2]. □

A point in a euclidean building is *thick* if every wall passing through the point is thick. Thick points exist if and only if the spherical building at infinity is thick; see [25, 10.5].

6.11 Theorem. *Let X, X' be irreducible euclidean buildings and suppose that $f : X \to X'$ is a homeomorphism. Assume that X contains a thick point and that X is not an $\mathbb{R}$-tree (equivalently, that $\dim(X) \geq 2$). Then f maps apartments to apartments.*

Proof. First, we note that f maps the thick points in X onto the thick points in X'. By [25, 10.6, 10.7, 10.8] there are three possibilities: (I) X and X' are euclidean cones over thick spherical buildings. (II) X and X' are thick simplicial buildings. (III) The thick points are dense in every apartment. In case (I) and (II), both buildings are complete, so we are done by 6.8. In the remaining case (III), the thick walls are dense in every apartment $A \subseteq X$. Exactly by the same argument as in [2, p. 172], we see the following: If $U \subseteq A$ is an open neighborhood of $p \in A$ such that $f(U)$ is contained in an apartment $A' \subseteq X'$, then $f : U \to A'$ is near p an affine-linear map. From the irreducibility of the Weyl group and the fact that f preserves thick walls near p we conclude that f is locally a homothety. The homothety factor is locally constant on A and therefore constant on A. Thus $f(A)$ is isometric to $\mathbb{E}^n$. By [35, 2.25], $f(A)$ is an apartment in the maximal atlas of X'. $\square$

Finally, we obtain Theorem C from the introduction. For complete euclidean buildings with infinitely many thick points, this is proved in [22, Section 6]. We call a euclidean building *metrically irreducible* if it is not a product of two metric spaces (with the euclidean product of the metrics). Equivalently, X is irreducible as a building and contains a thick point, or $X = \mathbb{R}$.

6.12 Theorem. *Let X and X' be metrically irreducible euclidean buildings. Assume that $\dim(X) \geq 2$. Suppose that $f : X \to X'$ is a homeomorphism. Possibly after rescaling the metric on X', there exists a unique isometry $\tilde{f} : X \to X'$ such that $\tilde{f}(A) = f(A)$ holds for all apartments $A \subseteq X$ of the maximal atlas of X. If X has more than one thick point, then f has finite distance from $\tilde{f}$.*

Proof. We have already seen that f induces a bijection between the apartments in the maximal atlases. We use again the trichotomy from [25, 10.6, 10.7, 10.8]. If X is of type (I) or (II), this determines the combinatorial structure of the euclidean buildings. In case (II), we see also that f has finite distance from a combinatorial isomorphism $\tilde{f}$. If X is of type (III), then we proved in 6.11 that f is apartmentwise a homothety. It follows readily that f is a homothety, i.e. that the metric on X' can be rescaled in such a way that $f = \tilde{f}$ is an isometry. Every isometry is determined completely by its action on the apartments, so $\tilde{f}$ is unique. $\square$

Using similar arguments as in [22, 6.4.3, 6.4.5], it is not difficult to extend 6.12 to products of euclidean buildings and euclidean spaces, as long as no tree factors occur. We leave this to the reader.

6.13 Remark. Theorem 6.12 is an important ingredient in proofs of quasi-isometric rigidity. A quasi-isometry $f : X \to Y$ between metric spaces always induces a homeomorphism between their asymptotic cones[10]. If the spaces X and Y are euclidean buildings or Riemannian symmetric spaces of noncompact type, then their asymptotic cones are homeomorphic complete euclidean buildings. From 6.12 and the ω_1-saturatedness of

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[10]Asymptotic cones are truncated ultrapowers; see [24] for a model theoretic viewpoint.

ultraproducts one can conclude that the quasi-isometry maps apartments (or, in the Riemannian symmetric case, maximal flats) Hausdorff-close to apartments; see [22, 7.1.5]. A more general result about quasi-isometries between euclidean buildings is proved in [25].

7   Euclidean buildings have finite dimension

We recall some notions from dimension theory. See [14] or [33] for a thorough introduction. An *open covering* $\mathfrak{U}$ of a space X is a collection of open subsets, with $\bigcup \mathfrak{U} = X$. The covering has *order* $\leq n + 1$ if every point $x \in X$ is contained in at most $n + 1$ elements of $\mathfrak{U}$. Equivalently, the dimension of the nerve of $\mathfrak{U}$ is not greater than n. The open covering $\mathfrak{U}'$ *refines* $\mathfrak{U}$ if every $U' \in \mathfrak{U}'$ is contained in some $U \in \mathfrak{U}$. The space X has *covering dimension* $\dim(X) \leq n$ if the following holds: every (finite) open covering $\mathfrak{U}$ of X has a refinement of order $\leq n + 1$. In this section we prove Theorem B in the introduction. For the special case of $\mathbb{R}$-trees, this is also proved in [8] and [32][11].

7.1 Theorem. *Let X be a euclidean building with n-dimensional apartments. Then the covering dimension of X is*

$$\dim(X) = n.$$

There are various other topological dimension functions, e.g. the (small and large) *inductive dimension* ind and Ind. Kleiner defined in [21] a *geometric dimension* for $\mathrm{CAT}(\kappa)$ spaces which we denote by gdim. We will show that

$$\dim(X) = \mathrm{Ind}(X) = \mathrm{ind}(X) = \mathrm{gdim}(X) = n.$$

For metric spaces, one has always $\dim(X) = \mathrm{Ind}(X) \geq \mathrm{ind}(X)$. If $B \subseteq X$ is a subspace, then $\mathrm{ind}(X) \geq \mathrm{ind}(B)$. For these facts, see [14, 33]. Kleiner proves in [21] that $\mathrm{gdim}(X) = \sup\{\dim(C) \mid C \subseteq X \text{ is compact.}\}$. Thus $\dim(X) \geq \mathrm{gdim}(X) \geq n$. Since X contains closed sets homeomorphic to $\mathbb{R}^n$ and since $\mathrm{ind}(\mathbb{R}) = \dim(\mathbb{R}^n) = n$, we have $\dim(X) \geq n$. In order to prove the theorem, we have to show that $\dim(X) \leq n$. We use a theorem due to Burillo; see 8.4. In order to apply his result, we need that from certain directions, euclidean buildings look 'tree-like'. We fix an apartment $A \subseteq X$ and a regular point $\xi \in \partial A$ at infinity. Associated to these data we have the Iwasawa retraction $\varphi_{A,\xi} = \varphi : X \rightarrow A$ which is defined as follows. If $A' \subseteq X$ is an apartment with $\xi \in \partial A$, then $A \cap A'$ containts a Weyl chamber. Therefore there is a unique isometry $A' \rightarrow A$ fixing $A \cap A'$ pointwise. Since X is the union of all apartments containing ξ in their respective boundaries, these isometries fit together to a well-defined 1-Lipschitz retraction $p : X \rightarrow A$; see [35, 1.20]. We denote the fiber over $a \in A$ by $X_a = p^{-1}(a)$. We define an ultrametric δ on X_a as follows. For $b, c \in X_a$ the geodesic rays $(\xi, b]$ and $(\xi, c]$ branch in a point e, $(\xi, b] \cap (\xi, c] = (\xi, e]$. We put

$$\delta(b, c) = d(e, b) + d(e, c) = 2d(e, b).$$

---

[11] When I wrote this paper, I was not aware that Lang and Schlichenmaier proved in [26, Prop. 3.3] that the Nagata dimension of a euclidean building with n-dimensional apartments is n. Their result implies Theorem 7.1. Their proof is virtually the same as ours, which follows [8].

It is clear that this is an ultrametric on X_a, and that

$$d(b, c) \leq \delta(b, c).$$

7.2 Lemma. *There exists a positive constant L, depending only on ξ, such that*

$$d(b, c) \leq \delta(b, c) \leq L \cdot d(b, c)$$

holds for all $a \in A$ and all $b, c \in X_a$.

Proof. Suppose that $b \neq c$. Let $(\xi, e] = (\xi, b] \cap (\xi, c]$ as above. The segments $[e, b]$ and $[e, c]$ determine two different points in the spherical building $\Sigma_e X$. These two points have the same type, and this type depends only on ξ. Now

$$d(b, c)^2 \geq 2d(e, b)^2(1 - \cos(\angle_e(b, c)))$$
$$= \frac{1}{2}\delta(b, c)^2(1 - \cos(\angle_e(b, c)));$$

see [5, II.1.]. There are only finitely many values that $\angle_e(b, c)$ can assume, since the two points determined by $[be]$ and $[ce]$ in the spherical building $\Sigma_e X$ are of the same fixed type (depending only on ξ, not on e). Moreover, $\pi \geq \angle_e(b, c) > 0$. Thus $1 - \cos(\angle_e(b, c))$ is bounded away from 0. □

Theorem 7.1 follows now from Burillo's Theorem 8.4 below.

Proof of Theorem B. By 3.2 and 7.1, X is a finite-dimensional ANR. An open subset of an ANR is again an ANR; see [19, III.7.9]. Therefore, every open subset of a euclidean building is a finite dimensional ANR. Since X is contractible, X is an AR; see [19, III.7.2]. The homotopy properties follow from 8.1 below. □

8 Appendix

We first state a result about the homotopy types of ANRs. Appendix 1 §2.2 in Mardešić–Segal [31] contains more results in this direction.

8.1 Theorem (Wall, Whitehead). *Every ANR X has the homotopy type of a CW complex. If X has finite covering dimension $\dim(X) = n$, then X has the homotopy type of an n-dimensional CW complex, except possibly in low dimensions $n \leq 2$, where the dimension of the CW complex may be 3.*

Proof. Every ANR X is dominated by a simplicial complex K; see [19, IV.6.1]. This means that there are continuous maps

$$X \to K \to X$$

whose composite is homotopic to the identity. It follows that X has the homotopy type of a CW complex; see [27, IV.3.8].

Assume now that X has covering dimension n. Let $m = \max\{3, n\}$. The space X is dominated by an n-dimensional simplicial complex K; see [31, Appendix 1 §2.2 Theorem 6]. Then K satisfies Wall's Condition Dm in [39, p. 62].

Dm: For all $k > m$, the groups $H_k(K)$ vanish, and $H^{m+1}(K; \mathcal{B}) = 0$ for every coefficient bundle $\mathcal{B}$ on K.

As Wall remarks on p. 57 in *loc.cit.*, it follows that X (or its CW approximation) also satisfies condition Dm (since K dominates X — all higher dimensional homology and cohomology of X has to vanish). By Theorem E on p. 63 in *loc.cit.*, X has the homotopy type of an m-dimensional CW complex.   □

We remark that for $n = 1$, the CW complex can also be chosen to be 1-dimensional by Wall's results. The 2-dimensional case is related to the $D(2)$-problem.

The following result is proved for separable metric spaces in [42]. In the present form, it is stated in [17], but unfortunately without a proof. Our proof follows essentially the proof of Dugundji's Extension Theorem; see [12, IX.6.1].

8.2 Theorem (Wojdysławski). *Let Z be a metric space and let K be the complete simplicial complex on a set S (i.e. K is the geometric realization of the poset of all finite subsets of S). Suppose that there exists a continuous map $q : K \to Z$ with the following properties.*

(1) *The q-image of the 0-skeleton S of K is dense in Z.*
(2) *If $(A_j)_{j \in \mathbb{N}}$ is a sequence of simplices in K such that $(q(A_j \cap S))_{j \in \mathbb{N}}$ converges to $z \in Z$, then $(q(A_j))_{j \in \mathbb{N}}$ also converges to z.*

Then Z is an absolute extensor (AE) and in particular an absolute retract (AR) for the class of metric spaces; see [19, III.3.1].

Proof. Let (X, d) be a metric space and $A \subseteq X$ closed. Let $f : A \to Z$ be continuous. We have to show that f extends to a continuous map $F : X \to Z$. We proceed similarly as in [12, IX.6.1]. For each $z \in X - A$ put $r_z = d(z, A)$ and $U_z = B_{r_z/2}(z)$. These sets form an open covering $\mathfrak{U}$ of $X - A$. We also choose $a_z \in A$ such that $d(z, a_z) \leq 2r_z$. Finally, we choose $s_z \in S$ such that $d(f(a_z), q(s_z)) \leq r_z$ (here we use the assumption (1)).

For $x \in U_z \in \mathfrak{U}$ and $a \in A$ we have $r_z \leq d(z, a) \leq d(z, x) + d(x, a) \leq \frac{1}{2}r_z + d(x, a)$, whence

$$r_z \leq 2d(x, a) \quad \text{for all } a \in A, x \in U_z.$$

Moreover we have $d(a, a_z) \leq d(a, x) + d(x, z) + d(z, a_z) \leq d(a, x) + \frac{1}{2}r_z + 2r_z$ and thus

$$d(a, a_z) \leq 6d(a, x) \quad \text{for all } a \in A, x \in U_z.$$

Let Φ be a partition of unity on $X - A$ subordinate to $\mathfrak{U}$. We choose for each $\varphi \in \Phi$ a point $z_\varphi \in X - A$ such that the support of φ is in $U_\varphi = U_{z_\varphi}$. Let $a_\varphi = a_{z_\varphi}$, $r_\varphi = r_{z_\varphi}$ and $s_\varphi = s_{z_\varphi}$. We now define $F : X \to Z$ as follows. For $a \in A$ we put $F(a) = f(a)$. For $x \in X - A$, let

$$F(x) = q\left(\sum_\Phi \varphi(x) s_\varphi\right).$$

Clearly, F is continuous on $X - A$. We claim that F is also continuous at each point $a \in A$. Let $a \in A$ and $x \in U_z$. Then

$$d(F(a), q(s_z)) \leq d(f(a), f(a_z)) + d(f(a_z), q(s_z)) \leq d(f(a), f(a_z)) + 2d(x, a).$$

Suppose that $(x_j)_{j \in \mathbb{N}}$ is a sequence in X that converges to $a \in A$. We wish to show that $\lim_j F(x_j) = F(a)$. Since $F|_A = f$ is continuous on A, it suffices to consider the case where all x_j are in $X - A$. Suppose that $x_j \in U_{z_j} \in \mathfrak{U}$. Then $\lim_j q(s_{U_{z_j}}) = F(a)$ by the inequalities above. From Condition (2) and $F(x_j) = q\left(\sum_\Phi \varphi(x_j) s_\varphi\right)$ we see that $\lim_j F(x_j) = F(a)$. This shows that F is continuous at a. $\qquad\square$

The next result shows that certain maps with ultrametric fibers cannot lower the dimension. Our proof follows closely [8, Theorem 16]. The main difference is that we allow a Lipschitz condition on the fibers. We use the following characterization of the covering dimension. Recall that an open covering has *order* $\leq n + 1$ if its nerve has dimension $\leq n$, or equivalently, if every point is in at most $n + 1$ members of the covering. We say the *mesh* is $\leq r$ if every member of the covering has diameter less than r. The criterion that we use is the following.

8.3 Proposition. *Let X be a metric space. Then $\dim(X) \leq n$ holds if an and only if there exists a sequence of open coverings $(\mathfrak{U}_k)_{k \in \mathbb{N}}$ of X of order $\leq n + 1$ and mesh $\leq r_k$, with $\lim_k r_k = 0$, such that $\mathfrak{U}_{k+1}$ refines $\mathfrak{U}_k$.*

Proof. See [33, V.1]. $\qquad\square$

8.4 Theorem (Burillo). *Let X and A be metric spaces and let $p : X \to A$ be a continuous map. We denote the fiber over $a \in A$ by $X_a = p^{-1}(a)$. Suppose that p has the following two properties.*
(1) *Pairs of points lift isometrically: For every $a \in A$ and $x \in X$, there exists $y \in X_a$ with $d(x, y) = d(p(x), p(y))$.*
(2) *Fibers are bi-Lipschitz ultrametric: There exists a constant $L \geq 0$ and, on each fiber X_a, an ultrametric $\delta = \delta_a : X_a \times X_a \to \mathbb{R}$ such that $d(x, y) \leq \delta(x, y) \leq L d(x, y)$ holds for all $x, y \in X_a$.*
If $\dim(A)$ is finite, then $\dim(X) \leq \dim(A)$.

Proof. Let $\mathfrak{U}$ be an open covering of A of mesh $\leq r$ and order $\leq n + 1$. For each $U \in \mathfrak{U}$ we chose a point $a = a_U \in U$. For $x \in X_a$ we put

$$W_{U,x} = \{y \in p^{-1}(U) \mid d(y, y') \leq r \text{ for some } y' \in X_a \text{ with } \delta(y', x) \leq 3rL\}.$$

Claim 1. If $y \in W_{U,x}$ and if $z \in p^{-1}(U)$ is a point with $d(z, y) \leq r$, then $z \in W_{U,x}$. In particular, $W_{U,x}$ is open.

Proof. Let y' be as in the definition of $W_{U,x}$ above. Using (1), choose $z' \in X_a$ with $d(z, z') \leq r$. Then $d(z', y') \leq 3r$, so $\delta(z', y') \leq 3rL$. Therefore $\delta(x, z') \leq 3rL$ (because δ is an ultrametric).

Claim 2. For all $y, z \in W_{U,x}$ we have $d(y,z) \leq r(2 + 3L)$.

Proof. Let $y, z \in W_{U,x}$ and let $y', z' \in X_a$ be points with $d(y, y'), d(z, z') \leq r$ and $\delta(x, z'), \delta(x, y') \leq 3rL$. Then $d(y, z) \leq r + 3rL + r = r(2 + 3L)$.

Claim 3. Every $y \in p^{-1}(U)$ is in some $W_{U,x}$.

Proof. Using (1), choose $x \in X_a$ with $d(x, y) = d(p(x), a) \leq r$. Then $y \in W_{U,x}$.

Claim 4. If $W_{U,x} \cap W_{U,z} \neq \emptyset$, then $W_{U,x} = W_{U,z}$.

Proof. Suppose that $y \in W_{U,x} \cap W_{U,z}$. Let $y' \in X_a$ with $d(y, y') \leq r$. By Claim 1 we have $y' \in W_{U,x} \cap W_{U,z}$. Since δ is ultrametric, we have $\delta(x, z) \leq 3rL$ and therefore $W_{U,x} = W_{U,z}$.

Claim 5. The set $\mathfrak{W} = \{W_{U,x} \mid U \in \mathfrak{U} \text{ and } p(x) = a_U\}$ is an open covering of X of order $\leq n + 1$ and mesh $\leq r(2 + 3L)$.

Proof. Only the claim about the order remains to be shown. Suppose that $W_{U_0,x_0}, \dots, W_{U_m,x_m}$ are pairwise distinct, with $W_{U_0,x_0} \cap \cdots \cap W_{U_m,x_m} \neq \emptyset$. By Claim 4, we have $U_i \neq U_j$ for $i \neq j$. Moreover, $U_0 \cap \cdots \cap U_m \neq \emptyset$. It follows that $m \leq n$.

Claim 6. Let $\tilde{\mathfrak{U}}$ be an open covering of A of mesh $\leq \tilde{r} \leq r/(2 + 3L)$ and order $\leq n + 1$ that refines $\mathfrak{U}$. Let $\tilde{\mathfrak{W}}$ be constructed from $\tilde{\mathfrak{U}}$ and $\tilde{r}$ as above. Then $\tilde{\mathfrak{W}}$ refines $\mathfrak{W}$.

Proof. Let $\tilde{W}_{V,v} \in \tilde{\mathfrak{W}}$. We choose $U \in \mathfrak{U}$ with $V \subseteq U$ and $x \in X_{a_U}$ such that $v \in W_{U,x}$. For $y \in \tilde{W}_{V,v}$ we have (by Claim 2) that $d(y, v) \leq \tilde{r}(2 + 3L) \leq r$. By Claim 1 we have $y \in W_{U,x}$.

The claim of the theorem now follows. We choose a sequence of coverings $\mathfrak{U}_k$ of A of order $\leq n + 1$ and mesh $\leq r_k$, with $(2 + 3L)r_{k+1} \leq r_k$. The resulting sequence of coverings $\mathfrak{W}_k$ of X then has order $\leq n + 1$ and mesh $\leq r_k(2 + 3L)$. □

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