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# The real quadrangle of type $E_6$

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**Abstract.** Based on the first author's diploma thesis [11] we use the theories of Lie groups and of Tits buildings in order to describe a Veronese embedding of the real quadrangle of type $E_6$, i.e., the $C_2$ sub-building of the complex $E_6$ building corresponding to the real form $E_{6(-14)}$ of the semisimple complex Lie group of type $E_6$.

## Introduction

The complex Lie group of type $E_6$ has four noncompact real forms, denoted $E_{6(6)}$, $E_{6(2)}$, $E_{6(-26)}$, and $E_{6(-14)}$ in [9]. The first of these groups is the split real form of $\mathbb{R}$-rank 6, the second has $\mathbb{R}$-rank 4 and belongs to a building of type $F_4$, and the third noncompact real form is the automorphism group of the real projective Cayley plane. This group $E_{6(-26)}$ has been studied in detail by Freudenthal [5]; a modern account is given in [17].

This paper is concerned with the last group, $E_{6(-14)}$. Veldkamp studied it in his papers on real forms of groups of type $E_6$ in detail; however, this was in the "pre-building" days and was carried out largely in the language of Hjelmslev planes [22], [27]. The building for the group $E_{6(-14)}$ is a polar space of type $C_2$, i.e. a generalized quadrangle. In an abstract building-theoretic setting, it is studied in [25], [28], [29] in detail (over arbitrary fields). It is our aim to describe a projective embedding (called a Veronese embedding) of this generalized quadrangle over the reals, which is similar in spirit to Freudenthal's description of the Cayley plane.

Similarly as the Cayley plane, the $E_6$ generalized quadrangle has a smaller and somewhat less mysterious relative, the dual of a classical hermitian generalized quadrangle. We first describe a Veronese embedding for this "toy model" before getting to $E_6$. This dual classical generalized quadrangle exhibits in fact many of the properties (and difficulties) we encounter afterwards with $E_6$.

In the first three sections of this article we collect information about incidence geometries of rank two in general and generalized quadrangles in particular (Section 1), Cayley and Jordan algebras (Section 2), and real and complex Lie groups of types $E_6$ and $F_4$ (Section 3). This information is combined in Section 4 in order to describe the

Veronese embedding of the real quadrangle of type $E_6$ (Theorem 4.14); in an Appendix A we provide a concrete list of equations that coordinatize this embedding. The "toy model" dual classical hermitian generalized quadrangle mentioned in the introduction is described in 1.15 and 1.16.

**Acknowledgements.** This paper is based on the first author's diploma thesis [11], written in 2000 in Würzburg under Theo Grundhöfer's and the third author's supervision. In the present form, it was written up by the last two authors. We are indebted to Arjeh Cohen, Theo Grundhöfer, Bernhard Mühlherr and Richard Weiss. We also thank an anonymous referee for helping us to improve the exposition of the material covered in this article.

## 1 Incidence geometries and some linear algebra

We collect some basic notions from incidence geometry. A general reference is the book [26].

**1.1 Incidence geometries.** An *incidence geometry* (*of rank* 2) is a triple $\Gamma = (\mathcal{P}, \mathcal{L}, \mathbf{I})$ consisting of two non-empty disjoint sets $\mathcal{P}$ and $\mathcal{L}$ and a reflexive symmetric relation

$$\mathbf{I} \subseteq (\mathcal{P} \cup \mathcal{L}) \times (\mathcal{P} \cup \mathcal{L}),$$

the *incidence relation*, satisfying $\mathbf{I}_{|\mathcal{P}\times\mathcal{P}} = \mathrm{id}$ and $\mathbf{I}_{|\mathcal{L}\times\mathcal{L}} = \mathrm{id}$. The elements of $\mathcal{P}$ and $\mathcal{L}$ are called *points* and *lines*, respectively. A point $p$ and a line $L$ are *incident* if $p\mathbf{I}L$. The elements of $\mathcal{V} = \mathcal{P} \cup \mathcal{L}$ are also called *vertices*. We say that $\Gamma$ is *thick* if every vertex $x$ of $\Gamma$ is incident with at least three other vertices. Two points $p, q \in \mathcal{P}$ are called *collinear* if there exists a line $L \in \mathcal{L}$ such that $p\mathbf{I}L\mathbf{I}q$, in which case we write $p\perp q$. Similarly we call two lines $L$ and $M$ *confluent*, in symbols $L\perp M$, if there exists a point $p \in \mathcal{P}$ satisfying $L\mathbf{I}p\mathbf{I}M$. If $\Gamma = (\mathcal{P}, \mathcal{L}, \mathbf{I})$ is an incidence geometry, then the incidence geometry $\Gamma^D = (\mathcal{L}, \mathcal{P}, \mathbf{I})$ is called the *dual incidence geometry of* $\Gamma$.

Collineations (isomorphisms) between incidence geometries are type-preserving and incidence-preserving permutations of the set $\mathcal{P} \cup \mathcal{L}$ whose inverses are also incidence-preserving. The automorphism group of $\Gamma$ is denoted by $\mathrm{Aut}(\Gamma)$.

**1.2 Generalized quadrangles.** A *projective plane* is a thick geometry where any two distinct points are joined by a unique line, and any two distinct lines meet in a unique point. A thick incidence geometry $\Gamma = (\mathcal{P}, \mathcal{L}, \mathbf{I})$ is called a *generalized quadrangle* if any two distinct lines have at most one point in common, and if for any non-incident point-line pair $(p, M) \in \mathcal{P} \times \mathcal{L}$ there exists a unique point-line pair $(q, L)$ such that $p\mathbf{I}L\mathbf{I}q\mathbf{I}M$.

Clearly, the dual incidence geometry of a projective plane (a generalized quadrangle) is also a projective plane (a generalized quadrangle).

For details on generalized quadrangles we refer to [15], [26].

**1.3 Collinearity relations.** Let $S$ be a non-empty set and $R \subseteq S \times S$ a reflexive and symmetric relation. For $x, y \in S$ let $L(x,y) = \{z \in S \mid zRx, zRy\}$. Then

$$\Gamma_R = (S, \{L(x,y) \mid x, y \in S, x \neq y, xRy\}, \in)$$

is an incidence geometry. If $\Gamma$ is a thick incidence geometry and if $\Gamma$ contains no digons and triangles, then obviously $\Gamma_\perp \cong \Gamma$, where $\perp \subseteq \mathcal{P} \times \mathcal{P}$ is the collinearity relation of $\Gamma$. This applies in particular to generalized quadrangles (whereas the collinearity relation of a projective plane carries no information).

Assume now that $K$ is a field of characteristic $\mathrm{char}(K) \neq 2$ and $V$ a vector space over $K$ of finite dimension $n$.

**1.4 Hermitian forms.** Let $\sigma : K \to K$ be an automorphism of the field $K$ satisfying $\sigma^2 = \mathrm{id}_K$. A *$\sigma$-hermitian form* is a map $h : V \times V \to K$ such that for all $x, y, x_1, x_2, y_1, y_2 \in V$ and $a, b \in K$ we have

(i) $h(ax, by) = ab^\sigma h(x,y)$,
(ii) $h(x_1 + x_2, y_1 + y_2) = h(x_1, y_1) + h(x_1, y_2) + h(x_2, y_1) + h(x_2, y_2)$, and
(iii) $h(x,y) = h(y,x)^\sigma$.

For a subset $M \subseteq V$ we define the linear subspace

$$M^{\perp_h} = M^\perp = \{y \in V \mid h(x,y) = 0 \text{ for all } x \in M\}.$$

We call $h$ *non-degenerate* if $V^\perp = \{0\}$. A linear subspace $U \subseteq V$ is *non-degenerate* if $U^\perp \cap U = \{0\}$, *degenerate* if $U^\perp \cap U \neq \{0\}$, and *totally isotropic* if $U \subseteq U^\perp$. The maximal dimension of a totally isotropic subspace of $V$ is called the *Witt index* of $h$. The *unitary group* of a non-degenerate hermitian form is defined as

$$\mathrm{U}(V,h) = \{\varphi \in \mathrm{GL}(V) \mid h(\varphi(x), \varphi(y)) = h(x,y) \text{ for all } x, y \in V\}.$$

**1.5 Classical quadrangles.** Assume that $n = \dim(V) \geq 5$ and let $h : V \times V \to K$ be a non-degenerate $\sigma$-hermitian form of Witt index 2. Let $\mathcal{P}$ be the set of all one-dimensional totally isotropic subspaces of $V$ and $\mathcal{L}$ be the set of all two-dimensional totally isotropic subspaces of $V$. We call the incidence geometry

$$Q(V,h) = (\mathcal{P}, \mathcal{L}, \subseteq)$$

a *classical quadrangle* (*over* $K$).

**Proposition 1.6.** *The incidence geometry $Q(V,h)$ defined above is a generalized quadrangle.*

*Proof.* See [23, Chapter 7] or [26, Section 2.3]. □

Clearly the unitary group $\mathrm{U}(V,h)$ acts on this generalized quadrangle. Now we want to describe the dual of this quadrangle, using the Plücker embedding.

**Definition 1.7.** The *Grassmannian of $r$-dimensional subspaces of $V$* is the set

$$G_r(V) = \{U \mid U \text{ is a linear subspace of } V, \dim_K U = r\}.$$

The elements of $G_1(V) = P(V)$ are called points and the elements of $G_2(V)$ are called lines.

To give an algebraic description of the quadrangle $Q(V,h)^D$ we need some facts on exterior powers and the exterior algebra. For details see [6] and [12, Chapter XIX]. Recall that the second exterior power $\bigwedge^2 V$ of $V$ is a vector space of dimension $\binom{n}{2}$.

**1.8 The Plücker embedding.** Given linearly independent vectors $x, y \in V$, we put $p(Kx + Ky) = K(x \wedge y)$. This is a well-defined injective map $p : G_2(V) \to P(\bigwedge^2 V)$, see [7, Chapter 1, Section 5]. We set $\mathcal{K} = p(G_2(V)) \subseteq P(\bigwedge^2 V)$. Then one has

$$\mathcal{K} = \{K(x \wedge y) \mid x, y \in V \text{ are linearly independent}\} = \{Ku \in P(\textstyle\bigwedge^2 V) \mid u \wedge u = 0\}$$

(cf. [7, Chapter 1, Section 5] or [18, Chapter 1, Section 4.1]). Note that the second equality depends on our assumption that $\mathrm{char}(K) \neq 2$.

**Lemma 1.9.** *Suppose that $L_1, L_2 \in G_2(V)$ and $p(L_i) = Ku_i$ where $i = 1, 2$. Then $L_1 \cap L_2 \neq \{0\}$ holds if and only if $u_1 \wedge u_2 = 0$.*

*Proof.* For $i = 1, 2$, let $u_i = x_i \wedge y_i$ where $x_i$ and $y_i$ is a basis for the two-dimensional subspace $L_i$. Then we have

$$\begin{aligned} L_1 \cap L_2 \neq \{0\} \quad &\iff \quad x_1, y_1, x_2, y_2 \text{ are linearly dependent} \\ &\iff \quad x_1 \wedge y_1 \wedge x_2 \wedge y_2 = 0 \\ &\iff \quad u_1 \wedge u_2 = 0 \end{aligned}$$

which proves the lemma. □

Suppose that $h : V \times V \to K$ is a $\sigma$-hermitian form. Then there exists a unique $\sigma$-hermitian form $h_2 : \bigwedge^2 V \times \bigwedge^2 V \to K$ satisfying

$$h_2(x_1 \wedge x_2, y_1 \wedge y_2) = \det((h(x_i, y_j)_{i,j=1,2})) = h(x_1, y_1)h(x_2, y_2) - h(x_1, y_2)h(x_2, y_1)$$

for all $x_1, x_2, y_1, y_2 \in V$ (see [6, 5.9]). We remark that if $h$ is non-degenerate on $V$, then $h_2$ is also non-degenerate.

**Lemma 1.10.** *Assume $Ku \in \mathcal{K}$ and let $h : V \times V \to K$ be a $\sigma$-hermitian form. Then one has $h_2(u, u) = 0$ if and only if the subspace $p^{-1}(Ku) \in G_2(V)$ is degenerate.*

*Proof.* Let $u = x \wedge y$ such that $x$ and $y$ span $L = p^{-1}(Ku)$. Then

$$h_2(u, u) = \det \begin{pmatrix} h(x, x) & h(x, y) \\ h(y, x) & h(y, y) \end{pmatrix} = h(x, x)h(y, y) - h(x, y)h(y, x)$$

is a scalar multiple of the discriminant of $L$ (with respect to $h$) (compare [8, p. 299]). Now the lemma follows from [8, 6.1.9 (i)]. □

Therefore we introduce the following terminology.

**Definition 1.11.** Let $Ku \in \mathcal{K}$ and let $h : V \times V \to K$ be an arbitrary non-degenerate $\sigma$-hermitian form. Then we call $Ku$

(i) *weakly isotropic* if $h_2(u, u) = 0$ and
(ii) *strongly isotropic* if $p^{-1}(Ku)$ is totally isotropic.

Note that a strongly isotropic subspace is also weakly isotropic.

**Lemma 1.12.** *A space $Ku \in \mathcal{K}$ is strongly isotropic if and only if for each $Kv \in \mathcal{K}$ satisfying $u \wedge v = 0$ we have $h_2(u, v) = 0$.*

*Proof.* Assume $p^{-1}(Ku)$ is totally isotropic, let $x_1$, $x_2$ be a basis of $p^{-1}(Ku)$, and let $y_1$, $y_2$ be a basis of $p^{-1}(Kv)$. Since $u \wedge v = 0$, we can choose $x_2 = y_1$ by Lemma 1.9. Then we have $h(x_1, y_1) = h(x_1, x_2) = 0$ and $h(x_2, y_1) = h(x_2, x_2) = 0$. Hence $h_2(x_1 \wedge x_2, y_1 \wedge y_2) = h(x_1, y_1)h(x_2, y_2) - h(x_1, y_2)h(x_2, y_1) = 0$.

If $p^{-1}(Ku)$ is not totally isotropic, there exists $Kv \in \mathcal{K}\backslash\{Ku\}$ with $u \wedge v = 0$ such that $p^{-1}(Kv) \cap p^{-1}(Ku)$ is not contained in the radical of $p^{-1}(Ku)$ and such that $p^{-1}(Kv)^{\perp_h} \cap p^{-1}(Ku) = \{0\}$. Choose $0 \neq x_1 \in p^{-1}(Ku)$ with $x_1 \not\perp_h p^{-1}(Kv) \cap p^{-1}(Ku)$, choose $0 \neq x_2 = y_1 \in p^{-1}(Kv) \cap p^{-1}(Ku)$ and choose $0 \neq y_2 \in x_1^{\perp_h} \cap p^{-1}(Kv)$. Then $h(x_1, y_2) = 0$ and $h(x_1, y_1) \neq 0 \neq h(x_2, y_2)$. Therefore $h_2(x_1 \wedge x_2, y_1 \wedge y_2) = h(x_1, y_1)h(x_2, y_2) - h(x_1, y_2)h(x_2, y_1) \neq 0$. □

**1.13.** We are now in the position to give a description of dual classical quadrangles using suitable exterior algebras. So let $Q(V, h)^D$ be such a quadrangle where $h : V \times V \to K$ is a non-degenerate $\sigma$-hermitian form of Witt index 2. Then put

$$\mathcal{H} = \{Ku \in \mathcal{K} \mid Ku \text{ is strongly isotropic}\}.$$

The elements of $\mathcal{H}$ obviously correspond to the points of $Q^D(V, h)$. Define on $\mathcal{H}$ a collinearity relation $\perp$ by

$$Ku \perp Kv \quad \Longleftrightarrow \quad u \wedge v = 0.$$

By 1.9 the relation $\perp$ is isomorphic to the collinearity relation in $Q^D(V, h)$, so $\Gamma_\perp$ is isomorphic to $Q(V, h)^D$, that is, we have derived the desired description.

**1.14.** Now we specialize to the case where $K = \mathbb{C}$ and $V = \mathbb{C}^6$. Let $\sigma$ denote complex conjugation on $\mathbb{C}$ and define a non-degenerate $\sigma$-hermitian form $h$ of Witt index 2 on $V$ and a non-degenerate symmetric bilinear form $(\,\cdot\,,\,\cdot\,)$ by setting

$$\begin{aligned} h(x, y) &= -x_1\overline{y_1} - x_2\overline{y_2} + x_3\overline{y_3} + x_4\overline{y_4} + x_5\overline{y_4} + x_5\overline{y_5} + x_6\overline{y_6}, \\ (x, y) &= x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4 + x_5y_5 + x_6y_6 \end{aligned}$$

for all $x = (x_1, \ldots, x_6), y = (y_1, \ldots, y_6) \in \mathbb{C}^6$. Clearly, $h$ and $(\,\cdot\,,\,\cdot\,)$ are non-degenerate on $\mathbb{C}^6$. Hence their induced forms on $\bigwedge^2 \mathbb{C}^6$ are also non-degenerate. Note that $h_2$ (cf. the paragraph after Lemma 1.9) has Witt index 7 on the 15-dimensional space $\bigwedge^2 \mathbb{C}^6$.

**1.15.** We now give a description of the quadrangle $Q(\mathbb{C}^6, h)^D$ in the vector space $\bigwedge^2 \mathbb{C}^6$ using a cross-product. Fix a vector space isomorphism $\varphi : \bigwedge^6 \mathbb{C}^6 \to \mathbb{C}$ and define a trilinear form $(\,\cdot\,,\,\cdot\,,\,\cdot\,)$ on $\bigwedge^2 \mathbb{C}^6$ by

$$(u, v, w) = \varphi(u \wedge v \wedge w)$$

for all $u, v, w \in \bigwedge^2 \mathbb{C}^6$. Note that this form is symmetric (i.e. invariant under all permutations of the variables) and non-degenerate. Now we can define a bilinear and symmetric cross-product $\times$ on $\bigwedge^2 \mathbb{C}^6$ by

$$(u \times v, w) = (u, v, w) \quad \text{for all } w \in \textstyle\bigwedge^2 \mathbb{C}^6$$

for $u, v \in \bigwedge^2 \mathbb{C}^6$. Note also that

$$u \times v = 0 \quad \Longleftrightarrow \quad u \wedge v = 0$$

for all $u, v \in \bigwedge^2 \mathbb{C}^6$.

**1.16.** To summarize, we have

$$\begin{aligned}
&\mathcal{K} = \{Ku \in P(\textstyle\bigwedge^2 V) \mid u \times u = 0\},\\
&\mathcal{H} = \{Ku \in \mathcal{K} \mid h_2(u, v) = 0 \text{ for all } Kv \in \mathcal{K} \text{ with } u \times v = 0\},\\
&\perp = \{(Ku, Kv) \in \mathcal{H} \times \mathcal{H} \mid u \times v = 0\}.
\end{aligned}$$

So we have derived a description of the dual classical quadrangle $Q(\mathbb{C}^6, h)^D$ in the 15-dimensional vector space $\bigwedge^2 \mathbb{C}^6$ in terms of a symmetric trilinear form, a symmetric bilinear form, a cross-product (defined by these forms) and a $\sigma$-hermitian form $h_2$. Note also that $\bigwedge^2 V$ is a $\mathrm{U}(V, h)$-module, so $\mathrm{U}(V, h)$ acts on $Q(\mathbb{C}^6, h)^D$.

## 2 Cayley and Jordan algebras

One of the purposes of this section is to remind the reader of the constructions of the algebras of real and complex quaternions and octonions by using the so-called Cayley–Dickson process. A general reference for the material covered below is e.g. [10], [12], [17]. Later in this section we introduce Jordan algebras which we will then use to give a description of the $E_6$ building and the Cayley plane.

We continue to assume that $K$ is a field of characteristic $\mathrm{char}(K) \neq 2$.

**2.1 Algebras.** An *algebra over $K$* or *$K$-algebra* is a $K$-vector space $A$ equipped with a $K$-bilinear map $(x, y) \mapsto x \cdot y = xy$. An element $1_A \in A$ is called an *identity element of $A$* if $1_A x = x 1_A = x$ holds for all $x \in A$. If $xy = 1_A = yx$, then $x, y$ are called invertible. If the maps $x \mapsto ax$ and $x \mapsto xa$ are bijective for every $a \in A \setminus \{0\}$, then $A$ is called a *division algebra*. An algebra is called *alternative* if it satisfies the following weak form of associativity for all $x, y \in A$

$$x^2 y = x(xy) \quad \text{and} \quad yx^2 = (yx)x.$$

A map $\varphi \in \mathrm{GL}(A)$ is called an *automorphism* if $\varphi(1_A) = 1_A$ and if $\varphi(xy) = \varphi(x)\varphi(y)$ holds for all $x, y \in A$, and an *anti-automorphism* if instead $\varphi(xy) = \varphi(y)\varphi(x)$ holds for all $x, y \in A$. An anti-automorphism whose square is the identity is also called an *involution*.

Recall that the real Cayley algebra $\mathbb{O}$ is an 8-dimensional alternative division algebra over the reals (in fact, it is the unique non-associative alternative division algebra over $\mathbb{R}$). We briefly recall the construction of $\mathbb{O}$.

**2.2.** The Cayley–Dickson process yields a family of $K$-algebras $\mathbb{F}_m^K$ with unit and with a canonical involutions $x \mapsto x^*$. We begin with $\mathbb{F}_0^K = K$, $x \mapsto x^* := x$. Assuming that the $K$-algebra $\mathbb{F}_m^K$ is defined for $0 \leq m$, we put $\mathbb{F}_{m+1}^K = \mathbb{F}_m^K \oplus \mathbb{F}_m^K$ with the product

$$(x_1, x_2) \cdot (y_1, y_2) = (x_1 y_1 - y_2^* x_2, x_1 y_2 + x_2 y_1^*).$$

The involution $x \mapsto x^*$ extends to $\mathbb{F}_{m+1}^K$ via

$$(x_1, x_2)^* = (x_1^*, -x_2).$$

Clearly, $\dim_K \mathbb{F}_m^K = 2^m$. Via the embedding $x \mapsto (x, 0)$ we can view $\mathbb{F}_m^K$ as a subalgebra of $\mathbb{F}_{m+1}^K$, the involutions commute with this embedding. In this way, we have embeddings $K = \mathbb{F}_0^K \subseteq \mathbb{F}_1^K \subseteq \cdots \subseteq \mathbb{F}_m^K$ and $1 \in K$ is a unit element. Moreover, $K = \mathbb{F}_0^K$ is a central subalgebra in $\mathbb{F}_m^K$ and consists precisely of the fixed elements of the involution $x \mapsto x^*$.

**2.3.** As usual, we put

$$\mathrm{Re}_K(x) = \frac{1}{2}(x + x^*)$$

for $x \in \mathbb{F}_m^K$. Note that $\mathrm{Re}_K(x) = \mathrm{Re}_K(x^*) \in K$ and that $\mathrm{Re}_{\mathbb{R}}$ commutes with the inclusions $\mathbb{F}_m \subseteq \mathbb{F}_{m+1}$. We also define the *Norm form* by

$$N_K(x) = xx^*$$

and its polarization

$$\langle x|y \rangle = N_K(x+y) - N_K(x) - N_K(y) = 2\,\mathrm{Re}_K(xy^*).$$

Note that every element $x$ with $N_K(x) \neq 0$ is invertible, since $xx^* = x^*x = N_K(x)$. For $z = (x, y) \in \mathbb{F}_{m+1}^K$ we have $N_K(z) = N_K(x) + N_K(y)$.

Obviously the Cayley–Dickson process is functorial, an inclusion of fields $K \subseteq L$ yields algebra inclusions $\mathbb{F}_m^K \subseteq \mathbb{F}_m^L$, and there is a natural isomorphism $\mathbb{F}_m^K \otimes_K L = \mathbb{F}_m^L$. In particular, the complex conjugation $z \longmapsto \overline{z}$ on $\mathbb{C}$ extends to an automorphism of $\mathbb{F}_m^{\mathbb{C}}$ (which we denote by the same symbol).

**2.4.** We continue with some more observations. Since $x^2 = 2\,\mathrm{Re}_K(x)x - N_K(x)$, we see that every element is contained in an associative and commutative subalgebra. Also, $\mathbb{F}_1^K$ is obviously commutative. Direct inspection shows: if $\mathbb{F}_m^K$ is commutative, then $\mathbb{F}_{m+1}^K$

is associative, and if $\mathbb{F}_m^K$ is associative, then $\mathbb{F}_{m+1}^K$ is alternative. So we see that $\mathbb{F}_2^K$ is associative and $\mathbb{F}_3^K$ is alternative. The algebras $\mathbb{F}_m^K$ for $m \geq 4$ are of no further interest to us. Note that the algebra $\mathbb{F}_2^K$ is in fact the quaternion algebra associated to the quaternion symbol $\left(\frac{-1,-1}{K}\right)$. The non-associative alternative algebra $\mathbb{F}_3^K$ is a Cayley algebra over $K$. Any two elements $x, y \in \mathbb{F}_3^K$ are contained in an associative subalgebra; in particular, $\mathbb{F}_3^K$ is a division algebra if and only if the quadratic form $N_K$ is anisotropic. This is the case for $K = \mathbb{R}$; the real Cayley division algebra is

$$\mathbb{O} = \mathbb{F}_3^{\mathbb{R}}.$$

Note that $\mathbb{F}_1^{\mathbb{R}} = \mathbb{C}$, and $\mathbb{F}_2^{\mathbb{R}}$ is the (unique) real quaternion division algebra. For $K = \mathbb{C}$, already $\mathbb{F}_1^{\mathbb{C}} \cong \mathbb{C}[\varepsilon]/(\varepsilon^2 + 1)$ has zero divisors. We put $\mathbb{H}^{\mathbb{C}} = \mathbb{F}_2^{\mathbb{C}} = \mathbb{H} \otimes_{\mathbb{R}} \mathbb{C}$ and $\mathbb{O}^{\mathbb{C}} = \mathbb{F}_3^{\mathbb{C}} = \mathbb{O} \otimes_{\mathbb{R}} \mathbb{C}$. This is the complex Cayley algebra; as we noted, it has zero divisors.

The following identities hold for all $x, y, z \in \mathbb{F}_m^K$ for $m \leq 3$.

- $N_K(xy) = N_K(x)N_K(y)$,
- $\langle xy|z\rangle = \langle x|y^*z\rangle$,
- $\langle xy|z\rangle = \langle yx|z\rangle$, $\langle x|yz\rangle = \langle x|zy\rangle$,
- $\mathrm{Re}_K(xy) = \mathrm{Re}_K(yx)$,
- $\mathrm{Re}_K(x(yz)) = \mathrm{Re}_K((xy)z)$, so we may write $\mathrm{Re}_K(xyz)$.

**2.5.** Next we introduce the Jordan algebra $\mathfrak{H}_3^K$ of hermitian $(3 \times 3)$-matrices with entries in the Cayley algebra. Then on this Jordan algebra we introduce a symmetric bilinear form and a symmetric trilinear form which we use to define a cross-product on $\mathfrak{H}_3^K$ and the notion of a Veronese vector (see 2.10, 2.11, 2.12 and 2.17). This cross-product and the Veronese vectors turn out to be important ingredients in the description of the so-called $E_6$ building over $\mathbb{C}$ (compare 2.18). The section closes with a short description of the real projective octonion plane.

From now on we assume that $\mathrm{char}(K) \neq 2, 3$.

**2.6.** Consider the matrix algebra $\mathfrak{M}_n(K) = K^{n \times n}$ of $(n \times n)$-matrices over $K$. The matrix transposition $X \mapsto X^T$ is an involution on this algebra. Given any $K$-algebra $A$, let $\mathfrak{M}_n(A) = A \otimes_K \mathfrak{M}_n(K)$. The elements of this algebra are $(n \times n)$-matrices with entries in $A$, with the usual matrix multiplication. If $A$ admits an involution $x \mapsto x^*$, then $a \otimes X \mapsto a^* \otimes X^T$ is an involution on $\mathfrak{M}_n(A)$ which we denote by the same symbol.

We apply this remark with $A = \mathbb{F}_m^K$. Let

$$\mathfrak{H}_n^K = \{X \in \mathfrak{M}_n(\mathbb{F}_3^K) \mid X = X^*\}.$$

As a $K$-vector space, $\dim \mathfrak{H}_n^K = (4n - 3)n$. We define a product $\circ$ on this vector space by

$$X \circ Y = \frac{1}{2}(XY + YX).$$

Obviously, this makes $\mathfrak{H}_n^K$ into a commutative algebra with unit element $I$ (the $n \times n$ identity matrix).

**Theorem 2.7.** *The $K$-vector space $\mathfrak{H}_3^K$ equipped with the product $\circ$ is a commutative 27-dimensional Jordan algebra over $K$ (and hence power-associative).*

*Proof.* See [2, Chapter 6, §4 and Chapter 7, §6]. □

We write a typical element $X \in \mathfrak{H}_3^K$ as

$$X = \begin{pmatrix} \xi_1 & x_3 & x_2^* \\ x_3^* & \xi_2 & x_1 \\ x_2 & x_1^* & \xi_3 \end{pmatrix}$$

or, shorter, as

$$X = (\xi_1, \xi_2, \xi_3; x_1, x_2, x_3) = ((\xi_i)_i; (x_i)_i)$$

where $\xi_i \in K$ and $x_i \in \mathbb{F}_3^K$ for $i = 1, 2, 3$.

**Proposition 2.8.** *For $X = ((\xi_i)_i; (x_i)_i)$ and $Y = ((\eta_i)_i; (y_i)_i)$, the product $X \circ Y = Z = ((\zeta_i)_i; (z_i)_i)$ is given by the formula*

$$\begin{aligned} \zeta_i &= \xi_i \eta_i + \frac{1}{2}\langle x_j | y_j \rangle + \frac{1}{2}\langle x_k | y_k \rangle, \\ z_i &= \frac{1}{2}((\xi_j + \xi_k) y_i + (\eta_j + \eta_k) x_i + (y_j x_k)^* + (x_j y_k)^*) \end{aligned}$$

*for $(i, j, k) \in \{(1, 2, 3), (2, 3, 1), (3, 1, 2)\}$.*

*Proof.* Write

$$X = \begin{pmatrix} \xi_1 & x_3 & x_2^* \\ x_3^* & \xi_2 & x_1 \\ x_2 & x_1^* & \xi_3 \end{pmatrix} \quad \text{and} \quad Y = \begin{pmatrix} \eta_1 & y_3 & y_2^* \\ y_3^* & \eta_2 & y_1 \\ y_2 & y_1^* & \eta_3 \end{pmatrix}$$

and compute

$$XY = \begin{pmatrix} \xi_1\eta_1 + x_3 y_3^* + x_2^* y_2 & x_1^* y_3 + x_3 \eta_2 + x_2^* y_1^* & \xi_1 y_2^* + x_3 y_1 + x_2^* \eta_3 \\ x_3^* \eta_1 + \xi_2 y_3^* + x_1 y_2 & x_3^* y_3 + \xi_2 \eta_2 + x_1 y_1^* & x_3^* y_2^* + \xi_2 y_1 + x_1 \eta_3 \\ x_2 \eta_1 + x_1^* y_3^* + \xi_3 y_2 & x_2 y_3 + x_1^* \eta_2 + \xi_3 y_1^* & x_2 y_2^* + x_1^* y_1 + \xi_3 \eta_3 \end{pmatrix}.$$

The results follows, since $Z = \frac{1}{2}(XY + YX)$. □

**Corollary 2.9.** *For $X = ((\xi_i)_i; (x_i)_i)$, we have $X \circ X = XX = Z = ((\zeta_i)_i; (z_i)_i)$, with*

$$\begin{aligned} \zeta_i &= \xi_i^2 + N_K(x_j) + N_K(x_k), \\ z_i &= (\xi_j + \xi_k) x_i + (x_j x_k)^* \end{aligned}$$

*for $(i, j, k) \in \{(1, 2, 3), (2, 3, 1), (3, 1, 2)\}$.*

**2.10 The bilinear form.** For $X, Y, Z \in \mathfrak{H}_3^K$ put $\operatorname{tr}(Z) = \zeta_1 + \zeta_2 + \zeta_3$ and

$$(X, Y) = \operatorname{tr}(X \circ Y).$$

Note that $(X, I) = \operatorname{tr}(X)$. This is obviously a symmetric bilinear form on the vector space $\mathfrak{H}_3^K$. From 2.8 we get

$$(X, Y) = \xi_1\eta_1 + \xi_2\eta_2 + \xi_3\eta_3 + \langle x_1|y_1\rangle + \langle x_2|y_2\rangle + \langle x_3|y_3\rangle,$$

hence $(\,\cdot\,,\,\cdot\,)$ is non-degenerate. It can be shown that $(X, Y \circ Z) = (X \circ Y, Z)$ (see [2, Chapter 7, §5]), so we may write

$$\operatorname{tr}(X \circ (Y \circ Z)) = \operatorname{tr}((X \circ Y) \circ Z) = \operatorname{tr}(X \circ Y \circ Z).$$

**2.11 The trilinear form.** For $X, Y, Z \in \mathfrak{H}_3^K$ we define a symmetric trilinear form $(\,\cdot\,,\,\cdot\,,\,\cdot\,) : \mathfrak{H}_3^K \times \mathfrak{H}_3^K \times \mathfrak{H}_3^K \to K$ by

$$\begin{aligned} 3(X, Y, Z) = \operatorname{tr}(X \circ Y \circ Z) &- \frac{1}{2}\operatorname{tr}(X)\operatorname{tr}(Y \circ Z) - \frac{1}{2}\operatorname{tr}(Y)\operatorname{tr}(X \circ Z) \\ &- \frac{1}{2}\operatorname{tr}(Z)\operatorname{tr}(X \circ Y) + \frac{1}{2}\operatorname{tr}(X)\operatorname{tr}(Y)\operatorname{tr}(Z). \end{aligned}$$

Furthermore, the *determinant of* $X$ is defined as

$$\det X = (X, X, X) = \frac{1}{3}\operatorname{tr}(X \circ X \circ X) - \frac{1}{2}\operatorname{tr}(X)\operatorname{tr}(X^2) + \frac{1}{6}(\operatorname{tr}(X))^3.$$

One can check that

$$\begin{aligned} 6(X, Y, Z) = \det(X + Y + Z) &- \det(X + Y) - \det(X + Z) - \det(Y + Z) \\ &+ \det X + \det Y + \det Z. \end{aligned}$$

**2.12 The cross product.** We define a symmetric bilinear map $\times : \mathfrak{H}_3^K \times \mathfrak{H}_3^K \to \mathfrak{H}_3^K$ through

$$(X \times Y, Z) = 3(X, Y, Z).$$

Then

$$\begin{aligned} &(X \times Y, Z) \\ &= \operatorname{tr}(X \circ Y \circ Z) - \frac{1}{2}\operatorname{tr}(X)\operatorname{tr}(Y \circ Z) - \frac{1}{2}\operatorname{tr}(Y)\operatorname{tr}(X \circ Z) \\ &\quad - \frac{1}{2}\operatorname{tr}(Z)\operatorname{tr}(X \circ Y) + \frac{1}{2}\operatorname{tr}(X)\operatorname{tr}(Y)\operatorname{tr}(Z) \\ &= \operatorname{tr}\Big(\Big(X \circ Y - \frac{1}{2}\operatorname{tr}(X)Y - \frac{1}{2}\operatorname{tr}(Y)X - \frac{1}{2}\operatorname{tr}(X \circ Y)I + \frac{1}{2}\operatorname{tr}(X)\operatorname{tr}(Y)I\Big) \circ Z\Big) \\ &= \Big(X \circ Y - \frac{1}{2}\operatorname{tr}(X)Y - \frac{1}{2}\operatorname{tr}(Y)X - \frac{1}{2}\operatorname{tr}(X \circ Y)I + \frac{1}{2}\operatorname{tr}(X)\operatorname{tr}(Y)I, Z\Big) \end{aligned}$$

whence

$$X \times Y = X \circ Y - \frac{1}{2}(Y, I)X - \frac{1}{2}(X, I)Y - \frac{1}{2}(X, Y)I + \frac{1}{2}(X, I)(Y, I)I.$$

**Proposition 2.13.** *For $Z = X \times Y$ we have*

$$2\zeta_i = \xi_j \eta_k + \xi_k \eta_j - \langle x_i | y_i \rangle,$$
$$2z_i = (y_j x_k)^* + (x_j y_k)^* - \xi_i y_i - \eta_i x_i,$$

*where* $(i, j, k) \in \{(1, 2, 3), (2, 3, 1), (3, 1, 2)\}$.

*Proof.* A direct computation using the last formula. □

**Corollary 2.14.** *For $Z = X \times X$ we have*

$$\zeta_i = \xi_j \xi_k - N_K(x_i),$$
$$z_i = (x_j x_k)^* - \xi_i x_i,$$

*where* $(i, j, k) \in \{(1, 2, 3), (2, 3, 1), (3, 1, 2)\}$.

**Corollary 2.15.** *We have*

$$\det X = \xi_1 \xi_2 \xi_3 - \xi_1 N_K(x_1) - \xi_2 N_K(x_2) - \xi_3 N_K(x_3) + 2\,\mathrm{Re}(x_1 x_2 x_3).$$

*Proof.* Expand $3 \det X = (X \times X, X)$. □

**Corollary 2.16.** *We have*

$$X \det X = (X \times X) \times (X \times X).$$

*Proof.* Expand both sides. □

**Definition 2.17.** A non-zero element $X = ((\xi_i)_i; (x_i)_i) \in \mathfrak{H}_3^K$ is called a *Veronese vector* if $X \times X = 0$. Note that this implies $\det(X) = 0$ (see 2.11 and 2.12). By 2.14 an element $X \neq 0$ is a Veronese vector if and only if the *Veronese conditions*

$$N_K(x_i) = \xi_j \xi_k,$$
$$\xi_i x_i^* = x_j x_k$$

are satisfied for all $(i, j, k) \in \{(1, 2, 3), (2, 3, 1), (3, 1, 2)\}$. Finally, we define the set

$$\mathcal{V} = \{KX \mid X \text{ is a Veronese vector}\}$$

of all one-dimensional subspaces of $\mathfrak{H}_3^K$ which are generated by Veronese vectors.

**2.18 The building of type $E_6$.** Suppose now that the 8-dimensional norm form $N_K$ is isotropic over $K$ (for example that $K$ is quadratically or algebraically closed, e.g. $K = \mathbb{C}$). Then $\mathbb{F}_3^K$ is the split Cayley algebra, and it follows that the product $\times$ given in 2.12 is identical with the cross-product defined in [4, p. 691ff]; this follows from the results proved in [19], [20]. Then the set $\mathcal{V}$ given in 2.17 is the point set of an incidence geometry, the $E_6$ *building over* $\mathbb{C}$ which is defined in [4, p. 691ff]; see also 4.7 of the present paper. Moreover, two points $KX, KY \in \mathcal{V}$ of the $E_6$ building are *collinear* if and only if $X \times Y = 0$ (compare also [4, p. 692]).

**2.19 The Cayley plane.** On the other hand, assume that the norm form on $\mathbb{F}_3^K$ is anisotropic; this holds for example if $K$ is formally real, eg. $K = \mathbb{R}$. Then $\mathbb{F}_3^K$ is a Cayley division algebra. We define

$$\begin{aligned}\mathcal{P} &= \mathcal{V},\\ \mathcal{L} &= \{X^{\perp(\cdot,\cdot)} \mid KX \in \mathcal{P}\}.\end{aligned}$$

For $K = \mathbb{R}$, the incidence geometry

$$\Gamma = (\mathcal{P}, \mathcal{L}, \subseteq)$$

is the so-called (and well-known) projective Cayley plane which is studied in [17, Sections 16, 17, 18] and Freudenthal's classical paper [5], and Adams' book [1].

## 3 Some exceptional Lie groups

In this section we define several noteworthy groups of automorphisms of the real and complex Jordan algebras defined in the preceding section.

**3.1 The invariance group and automorphisms.** Let $K \in \{\mathbb{R}, \mathbb{C}\}$. We consider the Jordan algebra $\mathfrak{H}_3^K$ with its bilinear form $(\,\cdot\,,\,\cdot\,)$ (see 2.10) and determinant $\det$ (see 2.11). The set

$$\begin{aligned}\mathrm{Inv}_K(\det) &= \{\varphi \in \mathrm{GL}(\mathfrak{H}_3^K) \mid \det \varphi(X) = \det X \text{ for all } X \in \mathfrak{H}_3^K\}\\ &= \{\varphi \in \mathrm{GL}(\mathfrak{H}_3^K) \mid (\varphi(X), \varphi(Y), \varphi(Z)) = (X, Y, Z) \text{ for all } X, Y, Z \in \mathfrak{H}_3^K\}\end{aligned}$$

forms a subgroup of $\mathrm{GL}(\mathfrak{H}_3^K)$. (See 2.11 for the latter equality.) Clearly, $\mathrm{Inv}_{\mathbb{R}}(\det) \leq \mathrm{Inv}_{\mathbb{C}}(\det)$.

By [5], [17, Section 17] the group $\mathrm{Inv}_{\mathbb{R}}(\det)$ is a simple real Lie group of type $E_{6(-26)}$. The group $E_{6(-26)}$ is the collineation group of the real projective Cayley plane described in 2.19 (see [17], Section 17). Moreover, by [21, Section 7] the group $\mathrm{Inv}_{\mathbb{C}}(\det)$ is an almost simple complex Lie group of type $E_6$.

It can be shown (cf. [5]) that $\mathrm{Aut}(\mathfrak{H}_3^K) = \mathrm{Inv}_K(\det) \cap \mathrm{O}(\mathfrak{H}_3^K, (\,\cdot\,,\,\cdot\,))$. Since $\mathrm{Inv}_{\mathbb{R}}(\det) \leq \mathrm{Inv}_{\mathbb{C}}(\det)$ we have $\mathrm{Aut}(\mathfrak{H}_3^{\mathbb{R}}) \leq \mathrm{Aut}(\mathfrak{H}_3^{\mathbb{C}})$. By [5] and [17, Section 18] the group $\mathrm{Aut}(\mathfrak{H}_3^{\mathbb{R}})$ is a simple real Lie group of type $F_{4(-52)}$. By [21, Section 7] the group $\mathrm{Aut}(\mathfrak{H}_3^{\mathbb{C}})$ is a simple complex Lie group of type $F_4$.

**Proposition 3.2.** *Let $\varphi \in \mathrm{Inv}_{\mathbb{C}}(\det)$ and $X, Y \in \mathfrak{H}_3^{\mathbb{C}}$. Then $X \times Y = 0$ if and only if $\varphi(X) \times \varphi(Y) = 0$.*

*Proof.* Since $\varphi$ lets the trilinear form $(\,\cdot\,,\,\cdot\,,\,\cdot\,)$ invariant (see 3.1), we have

$$\begin{aligned}(\varphi(X) \times \varphi(Y), T) &= 3(\varphi(X), \varphi(Y), T) = 3(\varphi(X), \varphi(Y), \varphi(\varphi^{-1}(T)))\\ &= 3(X, Y, \varphi^{-1}(T)) = (X \times Y, \varphi^{-1}(T))\end{aligned}$$

for all $T \in \mathfrak{H}_3^{\mathbb{C}}$. Because $T$ is arbitrary and $\varphi$ is a bijection, the non-degeneracy of $(\,\cdot\,,\,\cdot\,)$ implies the desired equivalence. $\square$

**3.3 Real forms of $E_6$ and $F_4$.** Recall that $z \longmapsto \overline{z}$ denotes the involutive automorphism of the complex Cayley algebra $\mathbb{O}_{\mathbb{C}} = \mathbb{O} \otimes \mathbb{C}$ induced by complex conjugation on the scalars. The map

$$H : \mathfrak{H}_3^{\mathbb{C}} \to \mathfrak{H}_3^{\mathbb{C}} : (\xi_1, \xi_2, \xi_3; x_1, x_2, x_3) \mapsto (\overline{\xi}_1, \overline{\xi}_2, \overline{\xi}_3; -\overline{x}_1, -\overline{x}_2, \overline{x}_3)$$

is $\mathbb{R}$-linear, $\mathbb{C}$-semilinear, and bijective. Furthermore, $H$ preserves the Veronese conditions, so that it maps Veronese vectors onto Veronese vectors. Moreover, given $X, Y \in \mathfrak{H}_3^{\mathbb{C}}$, one has $X \times Y = 0$ if and only if $H(X) \times H(Y) = 0$.

The form $h : \mathfrak{H}_3^{\mathbb{C}} \times \mathfrak{H}_3^{\mathbb{C}} \to \mathbb{C}$ defined by

$$h(X, Y) = (X, H(Y)) = \xi_1\overline{\eta}_1 + \xi_2\overline{\eta}_2 + \xi_3\overline{\eta}_3 - \langle x_1|\overline{y}_1\rangle - \langle x_2|\overline{y}_2\rangle + \langle x_3|\overline{y}_3\rangle$$

for all $X = ((\xi_i)_i; (x_i)_i), Y = ((\eta_i)_i; (y_i)_i) \in \mathfrak{H}_3^{\mathbb{C}}$ is $^{-}$-hermitian. By [27, Sections 5, 6, 7] the group $\mathrm{Inv}_{\mathbb{C}}(\det) \cap \mathrm{U}(\mathfrak{H}_3^{\mathbb{C}}, h)$ is an almost simple real Lie group of type $E_6$ (in fact $E_{6(-14)}$ by Theorem 3.7 below). Moreover, by [17, Section 18] the group $\mathrm{Inv}_{\mathbb{R}}(\det) \cap \mathrm{O}(\mathfrak{H}_3^{\mathbb{R}}, h_{|\mathfrak{H}_3^{\mathbb{R}} \times \mathfrak{H}_3^{\mathbb{R}}})$ is a simple real Lie group of type $F_{4(-20)}$.

**Lemma 3.3.** *An almost simple real Lie group of type $E_6$ which contains an almost simple real Lie group of type $F_{4(-20)}$ and the group $\mathrm{Spin}(10)$ as subgroups is of type $E_{6(-14)}$.*

*Proof.* This follows by inspection of the ranks and dimensions of the real forms of the complex Lie group of type $E_6$. □

**3.5.** We turn our attention to $\mathfrak{H}_3^{\mathbb{R}}$. We define an automorphism $B \in \mathrm{GL}(\mathfrak{H}_3^{\mathbb{R}})$ by restricting $H$ to this real subspace,

$$B((\xi_1, \xi_2, \xi_3; x_1, x_2, x_3)) = (\xi_1, \xi_2, \xi_3; -x_1, -x_2, x_3)$$

for all $((\xi_i)_i; (x_i)_i) \in \mathfrak{H}_3^{\mathbb{R}}$, and a symmetric bilinear form $\beta$ on $\mathfrak{H}_3^{\mathbb{R}}$ by putting

$$\beta(X, Y) = (X, B(Y)) = \xi_1\eta_1 + \xi_2\eta_2 + \xi_3\eta_3 - \langle x_1|y_1\rangle - \langle x_2|y_2\rangle + \langle x_3|y_3\rangle$$

for all $X = ((\xi_i)_i; (x_i)_i), Y = ((\eta_i)_i; (y_i)_i) \in \mathfrak{H}_3^{\mathbb{R}}$. Then we have

$$H(X \otimes \xi) = H((\xi^{(1)}X, \xi^{(2)}X)) = B(X) \otimes \overline{\xi}$$

and therefore

$$h(X \otimes \xi, Y \otimes \eta) = \beta(X, Y)\xi\overline{\eta},$$

where $h$ is the form defined in 3.3. From this we obtain $\mathrm{O}(\mathfrak{H}_3^{\mathbb{R}}, \beta) \leq \mathrm{U}(\mathfrak{H}_3^{\mathbb{C}}, h)$. Hence we can consider the group $\mathrm{Inv}_{\mathbb{R}}(\det) \cap \mathrm{O}(\mathfrak{H}_3^{\mathbb{R}}, \beta)$ as a subgroup of $\mathrm{Inv}_{\mathbb{C}}(\det) \cap \mathrm{U}(\mathfrak{H}_3^{\mathbb{C}}, h)$.

**3.6 The groups Spin(8), Spin(9), Spin(10).** Let $a \in \mathbb{O}$ with $N_{\mathbb{R}}(a) = 1$. We define a map $T_a \in \mathrm{GL}(\mathfrak{H}_3^{\mathbb{C}})$ by

$$\begin{aligned} T_a(X) &= \begin{pmatrix} a & 0 & 0 \\ 0 & a^* & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \xi_1 & x_3 & x_2^* \\ x_3^* & \xi_2 & x_1 \\ x_2 & x_1^* & \xi_3 \end{pmatrix} \begin{pmatrix} a^* & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} \xi_1 & ax_3a & (x_2a^*)^* \\ (ax_3a)^* & \xi_2 & a^*x_1 \\ x_2a^* & (a^*x_1)^* & \xi_3 \end{pmatrix} \end{aligned}$$

for all $X = ((\xi_i)_i; (x_i)_i) \in \mathfrak{H}_3^{\mathbb{C}}$. The group generated by the maps $T_a$ is the group $\mathrm{Spin}(8)$ (see [16, p. 267]). Furthermore, for an ordered pair $(c, s) \in \mathbb{R}^2$ where $c^2 + s^2 = 1$ one defines a map $R_{(c,s)} \in \mathrm{GL}(\mathfrak{H}_3^{\mathbb{C}})$ by setting

$$\begin{aligned} R_{(c,s)}(X) &= \begin{pmatrix} c & s & 0 \\ -s & c & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \xi_1 & x_3 & x_2^* \\ x_3^* & \xi_2 & x_1 \\ x_2 & x_1^* & \xi_3 \end{pmatrix} \begin{pmatrix} c & -s & 0 \\ s & c & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} \begin{matrix} c^2\xi_1 + s^2\xi_2 \\ +2cs\,\mathrm{Re}(x_3) \end{matrix} & \begin{matrix} c^2 x_2 - s^2 x_3^* \\ +cs(\xi_2 - \xi_1) \end{matrix} & (cx_2 + sx_1^*)^* \\[3ex] \begin{matrix} (c^2 x_2 - s^2 x_3^*)^* \\ +cs(\xi_2 - \xi_1) \end{matrix} & \begin{matrix} s^2\xi_1 + c^2\xi_2 \\ -2cs\,\mathrm{Re}(x_3) \end{matrix} & -sx_2^* + cx_1 \\[3ex] cx_2 + sx_1^* & (-sx_2^* + cx_1)^* & \xi_3 \end{pmatrix} \end{aligned}$$

for every $X = ((\xi_i)_i; (x_i)_i) \in \mathfrak{H}_3^{\mathbb{C}}$. The group generated by the maps $T_a$ and $R_{(c,s)}$ is the group $\mathrm{Spin}(9)$ (compare [16, p. 275]). It is well known that $\mathrm{Spin}(9)$ is a subgroup of $F_{4(-52)}$ and hence of the complex Lie group $\mathrm{Aut}(\mathfrak{H}_3^{\mathbb{C}}) = \mathrm{Inv}_{\mathbb{C}}(\det) \cap \mathrm{O}(\mathfrak{H}_3^{\mathbb{C}}, (\,\cdot\,,\,\cdot\,))$ (see 3.1). The maps $T_a$ and $R_{(c,s)}$ obviously commute with the map $H$ defined in 3.3, that is, we have $H \circ T_a = T_a \circ H$ and $H \circ R_{(c,s)} = R_{(c,s)} \circ H$. Because of $\mathrm{Spin}(9) \leq \mathrm{O}(\mathfrak{H}_3^{\mathbb{C}}, (\,\cdot\,,\,\cdot\,))$ this implies $\mathrm{Spin}(9) \leq \mathrm{U}(\mathfrak{H}_3^{\mathbb{C}}, h)$. Thus $\mathrm{Spin}(9)$ is a subgroup of $\mathrm{Inv}_{\mathbb{C}}(\det) \cap \mathrm{U}(\mathfrak{H}_3^{\mathbb{C}}, h)$. A similar argument shows by restricting every $\varphi \in \mathrm{Spin}(9)$ to $\mathfrak{H}_3^{\mathbb{R}}$ and seeing that $B$ (cf. 3.5) commutes with each generator of $\mathrm{Spin}(9)$ that $\mathrm{Spin}(9)$ is a subgroup of $F_{4(-20)}$.

We now consider maps $S_\omega \in \mathrm{GL}(\mathfrak{H}_3^{\mathbb{C}})$ where $\omega \in \mathbb{C}$ satisfies $\omega\overline{\omega} = 1$ which are defined by

$$S_\omega(X) = \begin{pmatrix} \omega & 0 & 0 \\ 0 & \overline{\omega} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \xi_1 & x_3 & x_2^* \\ x_3^* & \xi_2 & x_1 \\ x_2 & x_1^* & \xi_3 \end{pmatrix} \begin{pmatrix} \omega & 0 & 0 \\ 0 & \overline{\omega} & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \omega^2\xi_1 & x_3 & \omega x_2^* \\ x_3^* & \overline{\omega}^2\xi_2 & \overline{\omega}x_1 \\ \omega x_2 & \overline{\omega}x_1^* & \xi_3 \end{pmatrix}$$

for all $X = ((\xi_i)_i; (x_i)_i) \in \mathfrak{H}_3^{\mathbb{C}}$. The group generated by $\mathrm{Spin}(9)$ and the maps $S_\omega$ is the group $\mathrm{Spin}(10)$ (see [16, p. 282f]). From 2.15 and 3.3 one directly sees that each map $S_\omega$ is an element of $\mathrm{Inv}_{\mathbb{C}}(\det) \cap \mathrm{U}(\mathfrak{H}_3^{\mathbb{C}}, h)$. Hence $\mathrm{Spin}(10)$ (and also $F_{4(-20)}$) is a subgroup of $\mathrm{Inv}_{\mathbb{C}}(\det) \cap \mathrm{U}(\mathfrak{H}_3^{\mathbb{C}}, h)$. Thus 3.4 allows us to determine the type of the group $\mathrm{Inv}_{\mathbb{C}}(\det) \cap \mathrm{U}(\mathfrak{H}_3^{\mathbb{C}}, h)$.

**Theorem 3.7.** *The group* $\mathrm{Inv}_{\mathbb{C}}(\det) \cap \mathrm{U}(\mathfrak{H}_3^{\mathbb{C}}, h)$ *is an almost simple Lie group over the reals of type* $E_{6(-14)}$.

## 4 The real quadrangle of type $E_6$

Recall from 2.17 that $\mathcal{V} = \{\mathbb{C}X \mid X \text{ is a Veronese vector}\}$. As remarked in 2.18, the set $\mathcal{V}$ is the point set of the complex building of type $E_6$. In 4.7 we will describe an incidence geometry related to this building in detail.

**Proposition 4.1.** *The group* $\mathrm{Inv}_{\mathbb{C}}(\det)$ *acts transitively on the set* $\mathcal{V}$*, as does any co-central quotient.*

*Proof.* See [22, Section 3]. □

**4.2 Isotropic points.** Let $p = \mathbb{C}X \in \mathcal{V}$. Recall the definition of $H$ and $h$ from 3.3. We call $p$ *weakly isotropic* if $h(X, X) = 0$ and *strongly isotropic* if additionally $4H(X) \times (X \times T) = h(T, X)X$ for all $T \in \mathfrak{H}_3^{\mathbb{C}}$ (compare [27, Definition (1.3)], [22, Section 1]). Denote the set of strongly isotropic points by $\mathcal{H}$.

**Theorem 4.3.** *The real Lie group* $\mathrm{Inv}_{\mathbb{C}}(\det) \cap \mathrm{U}(\mathfrak{H}_3^{\mathbb{C}}, h)$ *of type* $E_{6(-14)}$ *acts transitively on the set* $\mathcal{H}$ *of all strongly isotropic points.*

*Proof.* See [27, Section 5]. □

**4.4.** Let $p, q \in \mathcal{H}$ be strongly isotropic points. Write $p = \mathbb{C}X$ and $q = \mathbb{C}Y$ where $X$ and $Y$ are Veronese vectors. Put $p \perp q :\Longleftrightarrow X \times Y = 0$ (see 2.12). Clearly, this defines a collinearity relation $\perp$ on $\mathcal{H}$. The aim of this section is to prove that the incidence geometry $\Gamma_\perp$ associated to $\perp \subset \mathcal{H} \times \mathcal{H}$ is a generalized quadrangle, namely the real $E_6$ quadrangle $Q(E_6, \mathbb{R})$. To this end we prove that the geometry $\Gamma_\perp$ is isomorphic to the sub-building of type $C_2$ fixed by the involution $\iota$ defined in 4.8 of the complex $E_6$ building; see 4.12 and 4.14.

**4.5 Singular subspaces of $P(\mathfrak{H}_3^{\mathbb{C}})$.** Let $S = P(U)$ be a subspace of $P(\mathfrak{H}_3^{\mathbb{C}})$ where $U$ is a linear subspace of $\mathfrak{H}_3^{\mathbb{C}}$. Call $S$ *singular* provided that $X \times Y = 0$ holds for all $X, Y \in U$. Note that the vectors in $U$ are Veronese vectors. A singular subspace $S$ is called *maximal* if for every singular subspace $S'$ satisfying $S \subseteq S'$ we have $S = S'$. It can be shown that each maximal singular subspace of $P(\mathfrak{H}_3^{\mathbb{C}})$ has rank 4 or 5, and both cases occur; see [22, p. 259f] and [4, p. 693].

**4.6 Symplecta.** Suppose that $X$ is a Veronese vector. Then

$$X \times \mathfrak{H}_3^{\mathbb{C}} = \{X \times Y \mid Y \in \mathfrak{H}_3^{\mathbb{C}}\}$$

is a linear subspace of $\mathfrak{H}_3^{\mathbb{C}}$. Call a subset $Q$ of $\mathcal{V}$ a *symplecton* provided that $Q$ is of the form $P(X \times \mathfrak{H}_3^{\mathbb{C}}) \cap \mathcal{V}$ where $X$ is a Veronese vector. Note that the symplecton $Q$ generates the projective space $P(X \times \mathfrak{H}_3^{\mathbb{C}})$, and, vice versa, every projective space of the form $P(X \times \mathfrak{H}_3^{\mathbb{C}})$, where $X$ is a Veronese vector, determines a unique symplecton $Q$ which is given by $Q = P(X \times \mathfrak{H}_3^{\mathbb{C}}) \cap \mathcal{V}$. So in the sequel we shall not distinguish between symplecta and such projective spaces. If we put

$$\mathcal{L}_1 = \mathcal{V} = \{\mathbb{C}X \mid X \text{ is a Veronese vector}\} \quad \text{and} \quad \mathcal{L}_6 = \{Q \mid Q \text{ is a symplecton}\},$$

then

$$\mathbb{C}X \mapsto P(X \times \mathfrak{H}_3^{\mathbb{C}}) \cap \mathcal{V}$$

defines a bijective map from $\mathcal{L}_1$ onto $\mathcal{L}_6$ such that the following holds: If one has $\mathbb{C}X$, $\mathbb{C}Y \in \mathcal{L}_1$ where $X$ and $Y$ are Veronese vectors, then $X \times Y = 0$ holds (that is, $\mathbb{C}X$ and

$\mathbb{C}Y$ are collinear) if and only if $P(X \times \mathfrak{H}_3^{\mathbb{C}}) \cap \mathcal{V}$ and $P(Y \times \mathfrak{H}_3^{\mathbb{C}}) \cap \mathcal{V}$ meet in a maximal singular subspace of rank 4 (cf. [4, p. 694]). The intersection of two distinct symplecta is either empty or a single point or a maximal singular subspace (in each symplecton); see [4, p. 694].

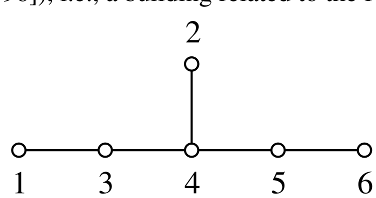
**4.7 The complex $E_6$ building.** We define an incidence geometry $\Gamma$ of rank 6. For this, we put

$$\begin{aligned}
\mathcal{L}_3 &= \{l \mid l \text{ is a singular subspace of rank } 1\},\\
\mathcal{L}_4 &= \{E \mid E \text{ is a singular subspace of rank } 2\},\\
\mathcal{L}_5 &= \{y \mid y \text{ is a maximal singular subspace of rank } 4\},\\
\mathcal{L}_2 &= \{v \mid v \text{ is a maximal singular subspace of rank } 5\}
\end{aligned}$$

and define incidence relations $\mathbf{I}_{ij} \subseteq \mathcal{L}_i \times \mathcal{L}_j$ where $1 \leq i < j \leq 6$ in the following way. Let $x \in \mathcal{L}_i$ and $y \in \mathcal{L}_j$. If $(i,j) \neq (2,5),(2,6)$, define $x\mathbf{I}_{ij}y$ if and only if $x \in y$. Moreover, $x\mathbf{I}_{25}y$ if and only if $x$ and $y$ meet in a singular subspace of rank 3, and $x\mathbf{I}_{26}y$ if and only if $x$ and $y$ meet in a non-maximal singular subspace of rank 4. We usually suppress the indices and write $x\mathbf{I}y$ instead of $x\mathbf{I}_{ij}y$. Then

$$\Gamma = (\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3, \mathcal{L}_4, \mathcal{L}_5, \mathcal{L}_6, (\mathbf{I}_{ij})_{1 \leq i < j \leq 6})$$

is an incidence geometry of rank six, whose chamber system is a complex spherical building of type $E_6$ (see [4, p. 696]), i.e., a building related to the following diagram:



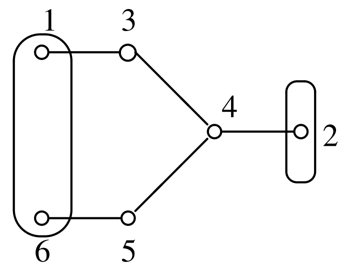

The complex Lie group of type $E_6$ acts as a chamber-transitive group of automorphisms on this building. As usual we call the elements of $\mathcal{L}_3$ *lines*. We recall that two points $\mathbb{C}X$ and $\mathbb{C}Y$ are collinear if and only if $X \times Y = 0$ is satisfied (compare 2.18). Moreover, we define for a point $p = \mathbb{C}X \in \mathcal{L}_1$ the subspace $p^\perp = P(\{Y \in \mathfrak{H}_3^{\mathbb{C}} \mid X \times Y = 0\})$. Finally, we remark that for $2 \leq i \leq 6$ each element $x \in \mathcal{L}_i$ is uniquely determined by the set of all points incident with $x$.

**4.8 The involution $\iota$.** We consider the map $\mathbb{C}X \mapsto P(X \times \mathfrak{H}_3^{\mathbb{C}}) \cap \mathcal{V}$ from $\mathcal{L}_1$ onto $\mathcal{L}_6$ as described in 4.6. This map can be extended to an involution $\iota_0$ of the building $\Gamma$ such that $\iota_0$ satisfies the following conditions (see [3], [4, 5.3]):

1. $\iota_0(\mathcal{L}_2) = \mathcal{L}_2$, $\iota_0(\mathcal{L}_4) = \mathcal{L}_4$.
2. $\iota_0(\mathcal{L}_1) = \mathcal{L}_6$, $\iota_0(\mathcal{L}_6) = \mathcal{L}_1$.
3. $\iota_0(\mathcal{L}_3) = \mathcal{L}_5$, $\iota_0(\mathcal{L}_5) = \mathcal{L}_3$.

The function $H$ from 3.3 maps $\mathcal{L}_1$ bijectively onto itself and preserves collinearity. Since $\Gamma$ is determined by its point set and the collinearity of points (compare 4.7), this map $H$

induces a collineation of $\Gamma$, denoted by $\varphi_H$. Then $\iota = \varphi_H \circ \iota_0$ is a permutation of $\Gamma$ satisfying the above properties; since $\varphi_H$ is an involution and since $\varphi_H$ and $\iota_0$ commute, the permutation $\iota$ is in fact an involution. By [27, Sections 5, 6] the centralizer of the restriction $\iota_{16}$ of $\iota$ to $\mathcal{L}_1 \cup \mathcal{L}_6$ in the complex Lie group $\mathrm{Inv}_{\mathbb{C}}(\det)$ of type $E_6$ is a real form of type $E_{6(-14)}$. Since every element $x \in \mathcal{L}_i$, $2 \leq i \leq 5$, is uniquely determined by the set of points incident with $x$, this real form also centralizes $\iota$. By [9, p. 534, p. 518] this means that $\iota$ is an involution which is related to the following diagram:




(We prefer to use the Mühlherr diagram instead of the Satake diagram because of our application of results from [14] in 4.9 below.) Consequently, the involution $\iota$ additionally satisfies the following conditions:

4. There exist elements $x \in \mathcal{L}_2$ such that $\iota(x) = x$ but no elements of $\mathcal{L}_4$ having this property.
5. There exist $\iota$-invariant flags in $\mathbf{I}_{16}$ but no flags in $\mathbf{I}_{35}$ having this property.

Here a flag $(x, y) \in \mathbf{I}_{ij}$ is called *$\iota$-invariant* if $\iota(x, y) = (x, y)$ holds.

**4.9 The real $E_6$ quadrangle.** Let $\iota$ be the involution discussed in 4.8. Define

$$\begin{aligned} \mathcal{P} &= \{p \in \mathcal{L}_1 \mid (p, \iota(p)) \text{ is a } \iota\text{-invariant flag}\}, \\ \mathcal{L} &= \{L \in \mathcal{L}_2 \mid \iota(L) = L\}. \end{aligned}$$

By [14, Theorems 1.7.27 and 1.8.22] the incidence geometry $Q(E_6, \mathbb{R}) = (\mathcal{P}, \mathcal{L}, \mathbf{I}_{12})$ is a spherical building of type $C_2$. (We refer the reader to [14, Chapter 2] for details on how to determine the type of a fixed building using Mühlherr diagrams.) In other words, the geometry $Q(E_6, \mathbb{R}) = (\mathcal{P}, \mathcal{L}, \mathbf{I}_{12})$ is the *real quadrangle of type $E_6$*. Unfortunately, [14] is not easily accessible; an alternative reference dealing with fixed buildings is [3, Proposition 14.6.1], which at the time of writing of this article at least can be accessed via the internet.

We will now prepare the proof that the geometry $\Gamma_\perp$ defined in 4.4 is in fact isomorphic to the real $E_6$ quadrangle $Q(E_6, \mathbb{R}) = (\mathcal{P}, \mathcal{L}, \mathbf{I}_{12})$.

**Proposition 4.10.** *Let $p \in \mathcal{L}_1$ and $P \in \mathcal{L}_6$. Assume that $p$ is not incident with $P$. Then either $p^\perp \cap P = \varnothing$ or there exists a unique $v \in \mathcal{L}_2$ such that $p\mathbf{I}v\mathbf{I}P$. This unique $v$ is generated by $p$ and the subspace $p^\perp \cap P$.*

*Proof.* Assume $p^\perp \cap P \neq \varnothing$. Then the subspace $p^\perp \cap P$ is a singular subspace of rank 4 (see [4, p. 694]). Hence the subspace $v$ generated by $p$ and $p^\perp \cap P$ is a maximal singular subspace of rank 5. Clearly, $p$ is contained in $v$ and $v \cap P = p^\perp \cap P$ holds. Hence we obtain $p\mathbf{I}v\mathbf{I}P$. The uniqueness of $v$ is shown in [13, p. 581]. $\square$

**Proposition 4.11.** *Let $\iota$ be the involution defined in* 4.8 *and suppose that* $(p, P), (q, Q) \in \mathbf{I}_{16}$ *are $\iota$-invariant flags where $p \neq q$. Then $p$ is not incident with $Q$, and $q$ is not incident with $P$.*

*Proof.* We assume that $p\mathbf{I}Q$ holds. Applying $\iota$ yields $\iota(Q) = q\mathbf{I}P = \iota(p)$. Hence $p, q \in P \cap Q$. Since $P \cap Q$ is a (maximal) singular subspace (compare 4.6), $p$ and $q$ must be collinear. Let $l$ be the line joining $p$ and $q$ and put $y = \iota(l)$. Because of $p, q\mathbf{I}l$ we have $y\mathbf{I}P, Q$, and therefore we obtain $y = P \cap Q$ which yields $l\mathbf{I}y = \iota(l)$. Hence $(l, y) \in \mathbf{I}_{35}$ is a $\iota$-invariant flag which contradicts the properties of $\iota$ established in 4.8. This completes the proof. $\square$

**Theorem 4.12.** *Let $\iota$ be the involution defined in* 4.8 *and assume that $p$ and $q$ are points of $Q(E_6, \mathbb{R})$. Then $p$ and $q$ are collinear in $Q(E_6, \mathbb{R})$ if and only if $p$ and $q$ are collinear in* $\Gamma$.

*Proof.* "$\Longrightarrow$": Assume that $p$ and $q$ are collinear in $Q(E_6, \mathbb{R})$. Then there exists a $v \in \mathcal{L}_2$ such that $p, q\mathbf{I}v$. Hence $p, q \in v$. Since $v$ is a maximal singular subspace, it follows that $p$ and $q$ are collinear in $\Gamma$.

"$\Longleftarrow$": Let $p$ and $q$ be collinear in $\Gamma$. Put $P = \iota(p)$ and $Q = \iota(q)$. Then $(p, P)$ and $(q, Q)$ are $\iota$-invariant flags. Suppose that $p \neq q$ (in the case $p = q$ the assertion is clear). From 4.11 we get that $p$ is not incident with $Q$ and $q$ is not incident with $P$. Since $q \in p^\perp \cap Q$ holds, there exists a unique $v \in \mathcal{L}_2$ such that $p\mathbf{I}v\mathbf{I}Q$ is satisfied (see 4.10). Similarly, one obtains a unique $v' \in \mathcal{L}_2$ where $q\mathbf{I}v'\mathbf{I}P$. Applying of $\iota$ yields $\iota(Q) = q\mathbf{I}\iota(v)\mathbf{I}P = \iota(p)$ and hence $\iota(v) = v'$. The theorem is proved if $v = v'$ is shown. For this, we remark that $P$ can be considered as a polar space if one takes $\mathcal{L}_1 \cap P$ as point set, $\mathcal{L}_3 \cap P$ as line set and $\mathbf{I}_{13}$ as incidence relation (because the residue of a symplecton is a diagram geometry of type $D_5$ and hence a polar space). Since $p$ and $q$ are collinear, $P \cap Q$ must be a maximal singular subspace of rank 4 in $P$. Thus $p^\perp \cap P \cap Q$ is a singular subspace of rank 3. Because of $p \notin Q$ the singular subspace generated by $p$ and $p^\perp \cap P \cap Q$ has rank 4. Since $v$ is generated by $p$ and $p^\perp \cap Q$, this subspace is the intersection of $v$ and $P$ which implies $v\mathbf{I}P$. Moreover, from $q \in p^\perp \cap Q$ it follows $q\mathbf{I}v$. Hence we have $q\mathbf{I}v\mathbf{I}P$ and $q\mathbf{I}v'\mathbf{I}P$, and the uniqueness of $v'$ yields $v = v'$. $\square$

**4.13.** Let $X$ and $Y$ be Veronese vectors. We call the point $\mathbb{C}X$ and the symplecton $P(Y \times \mathfrak{H}_3^{\mathbb{C}}) \cap \mathcal{V}$ *V-incident*, in symbols $\mathbb{C}X\mathbf{I}_V P(Y \times \mathfrak{H}_3^{\mathbb{C}})$, provided that $(X, Y) = 0$ and

$$4Y \times (X \times T) = (T, Y)X$$

holds for all $T \in \mathfrak{H}_3^{\mathbb{C}}$ (see [22, Sections 1 and 3]). Note that a point $\mathbb{C}X$ is strongly isotropic if and only if $\mathbb{C}X$ is V-incident with $P(H(X) \times \mathfrak{H}_3^{\mathbb{C}}) \cap \mathcal{V}$. The complex Lie group of type $E_6$ preserves the relation $\mathbf{I}_V$ (cf. [22, Sections 2, 3]).

**Claim.** $\mathbb{C}X\mathbf{I}_V P(Y \times \mathfrak{H}_3^{\mathbb{C}}) \cap \mathcal{V} \quad \Longleftrightarrow \quad \mathbb{C}X\mathbf{I}_{16} P(Y \times \mathfrak{H}_3^{\mathbb{C}}) \cap \mathcal{V}.$

*Proof.* Consider the Veronese vector $U = (1,0,0;0,0,0)$ and let $X = ((\xi_i)_i;(x_i)_i)$, $Z = ((\zeta_i)_i;(z_i)_i)$. Assume $Z = X \times U$. Then we compute using 2.13

$$\begin{aligned}
\zeta_1 &= \xi_2 \cdot 0 + \xi_3 \cdot 0 - \langle 0|0\rangle = 0, & z_1 &= (x_2 \cdot 0)^* + (0 \cdot x_3)^* - \xi_1 \cdot 0 - 1 \cdot x_1 = -x_1,\\
\zeta_2 &= \xi_3 \cdot 1 + \xi_1 \cdot 0 - \langle 0|0\rangle = \xi_3, & z_2 &= (x_3 \cdot 0)^* + (0 \cdot x_1)^* - \xi_2 \cdot 0 - 0 \cdot x_2 = 0,\\
\zeta_3 &= \xi_1 \cdot 0 + \xi_2 \cdot 1 - \langle 0|0\rangle = \xi_2, & z_3 &= (x_1 \cdot 0)^* + (0 \cdot x_2)^* - \xi_3 \cdot 0 - 0 \cdot x_3 = 0.
\end{aligned}$$

Hence we obtain $U \times \mathfrak{H}_3^{\mathbb{C}} = \{(0,\xi_2,\xi_3;x_1,0,0)|\xi_2,\xi_3 \in \mathbb{C}, x_1 \in \mathbb{O}^{\mathbb{C}}\}$. Moreover, we have

$$\begin{aligned}
X \circ U = \frac{1}{2}(XU + UX) &= \frac{1}{2}\left(\begin{pmatrix}\xi_1 & 0 & 0\\ x_3^* & 0 & 0\\ x_2 & 0 & 0\end{pmatrix} + \begin{pmatrix}\xi_1 & x_3 & x_2^*\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\right)\\
&= \frac{1}{2}\begin{pmatrix}2\xi_1 & x_3 & x_2^*\\ x_3^* & 0 & 0\\ x_2 & 0 & 0\end{pmatrix}.
\end{aligned}$$

Hence the set of all $X \in \mathfrak{H}_3^{\mathbb{C}}$ satisfying $X \circ U = 0$ equals $U \times \mathfrak{H}_3^{\mathbb{C}}$. From [22, Proposition (1.3) (ii)] it follows that a point $\mathbb{C}X$ is V-incident with $P(U \times \mathfrak{H}_3^{\mathbb{C}}) \cap \mathcal{V}$ if and only if $X \circ U = 0$ holds. Hence $\mathbb{C}X\mathbf{I}_V P(U \times \mathfrak{H}_3^{\mathbb{C}}) \cap \mathcal{V}$ is equivalent to $\mathbb{C}X\mathbf{I}_{16} P(U \times \mathfrak{H}_3^{\mathbb{C}}) \cap \mathcal{V}$.

Now assume that $X$ and $Y$ are arbitrary Veronese vectors. Put $p = \mathbb{C}X$, $P = P(Y \times \mathfrak{H}_3^{\mathbb{C}}) \cap \mathcal{V}$ and $Q = P(U \times \mathfrak{H}_3^{\mathbb{C}}) \cap \mathcal{V}$. Assume that $p\mathbf{I}_V P$ holds. Since the complex Lie group of type $E_6$ acts transitively on $\mathcal{L}_6$, it contains an element $\varphi$ such that $P^\varphi = Q$. Thus $p^\varphi \mathbf{I}_V Q$, because the complex Lie group of type $E_6$ preserves the relation $\mathbf{I}_V$ (see 4.13). Hence we have $p^\varphi \mathbf{I}_{16} Q$ and therefore $p\mathbf{I}_{16} P$. In the same way one gets that $p\mathbf{I}_{16} P$ implies $p\mathbf{I}_V P$. This proves the claim. □

**Theorem 4.14.** *Let $\mathcal{H}$ be the set of strongly isotropic points. Then the relation $\perp \subseteq \mathcal{H} \times \mathcal{H}$ defined by*

$$\mathbb{C}X \perp \mathbb{C}Y \quad \Longleftrightarrow \quad X \times Y = 0$$

*is a collinearity relation, and the incidence geometry $\Gamma_\perp$ associated to $\perp$ is isomorphic to the real $E_6$ quadrangle $Q(E_6,\mathbb{R})$.*

*Proof.* The point set $\mathcal{P}$ of $Q(E_6,\mathbb{R})$ consists of all points $\mathbb{C}X$ ($X$ is a Veronese vector) which are incident with $P(H(X) \times \mathfrak{H}_3^{\mathbb{C}})$. Hence the set $\mathcal{P}$ equals the set $\mathcal{H}$ of strongly isotropic points (compare 4.13). By 4.12 the collinearity relation $\perp$ in $\Gamma$ describes the collinearity in $Q(E_6,\mathbb{R})$. Using 1.3 we derive the claim. □

## A Equations for strongly isotropic points

In this appendix we list concrete equations that describe the Veronese embedding of the $E_6$ quadrangle given in Theorem 4.14.

**Proposition A.1.** *Suppose $p = \mathbb{C}X \in \mathcal{V}$ is a point where $X = ((\xi_i)_i;(x_i)_i)$ is a Veronese vector. Then $p$ is strongly isotropic if and only if the equations*

$$\begin{aligned}
|\xi_1|^2+|\xi_2|^2+|\xi_3|^2 &= \langle x_1|\overline{x}_1\rangle+\langle x_2|\overline{x}_2\rangle-\langle x_3|\overline{x}_3\rangle\\
|\xi_j|^2+|\xi_k|^2+\sigma_i\langle x_i|\overline{x}_i\rangle &= |\xi_i|^2\\
\overline{\xi}_k x_j+\sigma_i(x_k\overline{x}_i)^* &= -\sigma_j\xi_i\overline{x}_j\\
\overline{\xi}_j x_k+\sigma_i(\overline{x}_i x_j)^* &= -\sigma_k\xi_i\overline{x}_k\\
\sigma_i\xi_k\overline{x}_i+\sigma_k(x_j\overline{x}_k)^* &= -\overline{\xi}_j x_i\\
\sigma_i\xi_j\overline{x}_i+\sigma_j(\overline{x}_j x_k)^* &= -\overline{\xi}_k x_i\\
\sigma_j(tx_j)\overline{x}_j^*+\sigma_k\overline{x}_k^*(x_kt)+|\xi_i|^2t+\sigma_i\langle x_i|t\rangle\overline{x}_i &= \sigma_i\langle\overline{x}_i|t\rangle x_i\\
\sigma_j(x_it)\overline{x}_j^*-\sigma_k\xi_j(t\overline{x}_k)^*-\overline{\xi}_i(tx_k)^* &= \sigma_j\langle\overline{x}_j|t\rangle x_i\\
-\sigma_j\xi_k(\overline{x}_jt)^*+\sigma_k\overline{x}_k^*(tx_i)-\overline{\xi}_i(x_jt)^* &= \sigma_k\langle\overline{x}_k|t\rangle x_i
\end{aligned}$$

*hold for all $t \in \mathbb{O}^{\mathbb{C}}$ and $(i,j,k) \in \{(1,2,3),(2,3,1),(3,1,2)\}$.*

*Proof.* Let $T = ((\tau_i)_i;(t_i)_i) \in \mathfrak{H}_3^{\mathbb{C}}$ and put $Y = ((\eta_i)_i;(y_i)_i) = 2X \times T$. Then we obtain from 2.13

$$\eta_i = \xi_j\tau_k+\xi_k\tau_j-\langle x_i|t_i\rangle \quad \text{and} \quad y_i = (x_jt_k)^*+(t_jx_k)^*-\xi_it_i-\tau_ix_i$$

for all $(i,j,k) \in \{(1,2,3),(2,3,1),(3,1,2)\}$. Define $(\sigma_1,\sigma_2,\sigma_3) = (-1,-1,1)$. Then we have

$$H(X) = (\overline{\xi}_1,\overline{\xi}_2,\overline{\xi_3};\sigma_1\overline{x}_1,\sigma_2\overline{x}_2,\sigma_3\overline{x}_3).$$

If we put

$$Z = 2H(X)\times Y = 4H(X)\times(X\times T)$$

where $Z = ((\zeta_i)_i;(z_i)_i)$, then we compute

$$\begin{aligned}
\zeta_i &= \overline{\xi}_j\eta_k+\overline{\xi}_k\eta_j-\langle\sigma_i\overline{x}_i|y_i\rangle\\
&= \overline{\xi}_j(\xi_i\tau_j+\xi_j\tau_i-\langle x_k|t_k\rangle)+\overline{\xi}_k(\xi_k\tau_i+\xi_i\tau_k-\langle x_j|t_j\rangle)\\
&\quad -\langle\sigma_i\overline{x}_i|(x_jt_k)^*+(t_jx_k)^*-\xi_it_i-\tau_ix_i\rangle\\
&= (\overline{\xi}_j\xi_j+\overline{\xi}_k\xi_k+\langle\sigma_i\overline{x}_i|x_i\rangle)\tau_i+\overline{\xi}_j\xi_i\tau_j+\overline{\xi}_k\xi_i\tau_k\\
&\quad +\langle\xi_i\sigma_i\overline{x}_i|t_i\rangle-\langle\overline{\xi}_kx_j+(x_k\sigma_i\overline{x}_i)^*|t_j\rangle-\langle\overline{\xi}_jx_k+(\sigma_i\overline{x}_ix_j)^*|t_k\rangle
\end{aligned}$$

and

$$\begin{aligned}
z_i &= (\sigma_j\overline{x}_jy_k)^*+(y_j\sigma_k\overline{x}_k)^*-\overline{\xi}_iy_i-\eta_i\sigma_i\overline{x}_i\\
&= \Big(\sigma_j\overline{x}_j\big((x_it_j)^*+(t_ix_j)^*-\xi_kt_k-\tau_kx_k\big)\Big)^*\\
&\quad +\Big(\big((x_kt_i)^*+(t_kx_i)^*-\xi_jt_j-\tau_jx_j\big)\sigma_k\overline{x}_k\Big)^*\\
&\quad -\overline{\xi}_i\big((x_jt_k)^*+(t_jx_k)^*-\xi_it_i-\tau_ix_i\big)-\big(\xi_j\tau_k+\xi_k\tau_j-\langle x_i|t_i\rangle\big)\sigma_i\overline{x}_i
\end{aligned}$$

$$
\begin{aligned}
&= \overline{\xi}_i\tau_i x_i - \tau_j\big((x_j\sigma_k\overline{x}_k)^* + \xi_k\sigma_i\overline{x}_i\big) - \tau_k\big((\sigma_j\overline{x}_j x_k)^* + \xi_j\sigma_i\overline{x}_i\big)\\
&\quad + (t_i x_j)\sigma_j\overline{x}_j^* + \sigma_k\overline{x}_k^*(x_k t_i) + \overline{\xi}_i\xi_i t_i + \langle x_i|t_i\rangle\sigma_i\overline{x}_i\\
&\quad + (x_i t_j)\sigma_j\overline{x}_j^* - \xi_j(t_j\sigma_k\overline{x}_k)^* - \overline{\xi}_i(t_j x_k)^*\\
&\quad - \xi_k(\sigma_j\overline{x}_j t_k)^* + \sigma_k\overline{x}_k^*(t_k x_i) - \overline{\xi}_i(x_j t_k)^*
\end{aligned}
$$

for all $(i,j,k) \in \{(1,2,3),(2,3,1),(3,1,2)\}$. On the other hand, if we put $Z = h(T,X)X$, then we have

$$\zeta_i = \tau_i\overline{\xi}_i\xi_i + \tau_j\overline{\xi}_j\xi_i + \tau_k\overline{\xi}_k\xi_i + \langle t_i|\sigma_i\overline{x}_i\rangle\xi_i + \langle t_j|\sigma_j\overline{x}_j\rangle\xi_i + \langle t_k|\sigma_k\overline{x}_k\rangle\xi_i$$

and

$$z_i = \tau_i\overline{\xi}_i x_i + \tau_j\overline{\xi}_j x_i + \tau_k\overline{\xi}_k x_i + \langle t_i|\sigma_i\overline{x}_i\rangle x_i + \langle t_j|\sigma_j\overline{x}_j\rangle x_i + \langle t_k|\sigma_k\overline{x}_k\rangle x_i$$

for each $(i,j,k) \in \{(1,2,3),(2,3,1),(3,1,2)\}$. Now a comparison of both sides of the Equation $4H(X) \times (X \times T) = h(T,X)X$ yields the claim. □

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