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Asymptotic cones of finitely presented groups☆

Linus Kramer,[a] Saharon Shelah,[b] Katrin Tent,[a] and Simon Thomas[c,*]

[a] Mathematisches Institut, Universität Würzburg, Am Hubland, D-97074 Würzburg, Germany
[b] Mathematics Department, The Hebrew University, Jerusalem, Israel
[c] Mathematics Department, Rutgers University, 110 Frelinghuysen Road, Piscataway, NJ 08854-8019, USA

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Abstract

Let G be a connected semisimple Lie group with at least one absolutely simple factor S such that $\mathbb{R}\text{-rank}(S) \geqslant 2$ and let Γ be a uniform lattice in G.

(a) If CH holds, then Γ has a unique asymptotic cone up to homeomorphism.
(b) If CH fails, then Γ has $2^{2^{\omega}}$ asymptotic cones up to homeomorphism.

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1. Introduction

Let Γ be a finitely generated group equipped with a fixed finite generating set and let d be the corresponding word metric. Consider the sequence of metric spaces

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*Corresponding author.
E-mail addresses: kramer@mathematik.tu-darmstadt.de (L. Kramer), shelah@math.huji.ac.il (S. Shelah), tent@mathematik.uni-wuerzburg.de (K. Tent), sthomas@math.rutgers.edu (S. Thomas).

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$X_n = (\Gamma, d_n)$ for $n \geqslant 1$, where $d_n(g,h) = d(g,h)/n$. In [15], Gromov proved that if Γ has polynomial growth, then the sequence $(X_n \mid n \geqslant 1)$ of metric spaces converges in the pointed Gromov–Hausdorff topology to a complete geodesic space $\mathrm{Con}_\infty(\Gamma)$, the asymptotic cone of Γ. In [12], van den Dries and Wilkie generalised the construction of asymptotic cones to arbitrary finitely generated groups. However, their construction involved the choice of a nonprincipal ultrafilter $\mathcal{D}$ over the set ω of natural numbers, and it was initially not clear whether the resulting asymptotic cone $\mathrm{Con}_\mathcal{D}(\Gamma)$ depended on the choice of the ultrafilter $\mathcal{D}$. In [30], answering a question of Gromov [16], Thomas and Velickovic constructed an example of a finitely generated group Γ and two nonprincipal ultrafilters $\mathcal{A}$, $\mathcal{B}$ such that the asymptotic cones $\mathrm{Con}_\mathcal{A}(\Gamma)$ and $\mathrm{Con}_\mathcal{B}(\Gamma)$ were not homeomorphic. But this still left open the interesting question of whether there exists a finitely presented group with more than one asymptotic cone up to homeomorphism. It seems almost certain that such a group exists; and, in fact, it seems natural to conjecture that there exists a finitely presented group with 2^{2^ω} asymptotic cones up to homeomorphism. (Recall that there are exactly 2^{2^ω} distinct nonprincipal ultrafilters over ω.) The main result of this paper provides a confirmation of this conjecture, under the assumption that CH fails.

Suppose that G is a connected semisimple Lie group and let Γ be a uniform lattice in G, i.e. a discrete subgroup such that G/Γ is compact. (For the existence of such a subgroup Γ, see [2].) Then it is well-known that Γ is finitely presented. (For example, see [17, Chapter V, 35, Chapter 3].) Furthermore, Γ is quasi-isometric to G; and hence for each ultrafilter $\mathcal{D}$, the asymptotic cones $\mathrm{Con}_\mathcal{D}(\Gamma)$ and $\mathrm{Con}_\mathcal{D}(G)$ are homeomorphic. In [16, Section 2.2.B_1], Gromov suggested that " it seems that these groups G have pretty looking finite dimensional cones $\mathrm{Con}_\mathcal{D}(G)$ which are (essentially) independent of the choice of the ultrafilter $\mathcal{D}$." It turns out that the situation is more interesting.

Theorem 1.1. *Suppose that G is a connected semisimple Lie group with at least one absolutely simple factor S such that $\mathbb{R}$-rank$(S) \geqslant 2$ and let Γ be a uniform lattice in G.*

(a) *If* CH *holds, then Γ has a unique asymptotic cone up to homeomorphism.*
(b) *If* CH *fails, then Γ has 2^{2^ω} asymptotic cones up to homeomorphism.*

Here CH denotes the *continuum hypothesis*, i.e. the statement that $2^\omega = \omega_1$. Of course, it is well-known that CH can neither be proved nor disproved using the usual ZFC axioms of set theory. (For example, see [18].)

In the remainder of this section, we shall sketch the main points of the proof for the special case when $G = \mathrm{SL}_m(\mathbb{R})$ for some $m \geqslant 3$. In particular, we shall explain the unexpected appearance of the Robinson field ${}^\rho\mathbb{R}_\mathcal{D}$ as a topological invariant of the asymptotic cone $\mathrm{Con}_\mathcal{D}(\Gamma)$. (As we shall explain below, the Robinson field ${}^\rho\mathbb{R}_\mathcal{D}$ is a valued field closely related to the corresponding field ${}^*\mathbb{R}_\mathcal{D}$ of nonstandard real numbers.) We shall begin by recalling the definition of an asymptotic cone of an arbitrary metric space.

Definition 1.2. A *nonprincipal ultrafilter* over the set ω of natural numbers is a collection $\mathcal{D}$ of subsets of ω satisfying the following conditions:

(i) If A, $B \in \mathcal{D}$, then $A \cap B \in \mathcal{D}$.
(ii) If $A \in \mathcal{D}$ and $A \subseteq B \subseteq \omega$, then $B \in \mathcal{D}$.
(iii) For all $A \subseteq \omega$, either $A \in \mathcal{D}$ or $\omega \backslash A \in \mathcal{D}$.
(iv) If F is a finite subset of ω, then $F \notin \mathcal{D}$.

Equivalently, if $\mu : \mathcal{P}(\omega) \to \{0,1\}$ is the function such that $\mu(A) = 1$ if and only if $A \in \mathcal{D}$, then μ is a finitely additive probability measure on ω such that $\mu(F) = 0$ for all finite subsets F of ω. It is easily checked that if $\mathcal{D}$ is a nonprincipal ultrafilter and (r_n) is a bounded sequence of real numbers, then there exists a unique real number ℓ such that

$$\{n \in \omega \mid |r_n - \ell| < \epsilon\} \in \mathcal{D}$$

for all $\epsilon > 0$. We write $\ell = \lim_{\mathcal{D}} r_n$.

Definition 1.3. Suppose that $\mathcal{D}$ is a nonprincipal ultrafilter over ω. Let (X,d) be a metric space and for each $n \geqslant 1$, let d_n be the rescaled metric defined by $d_n(x,y) = d(x,y)/n$. Let $e \in X$ be a fixed base point. Then X_∞ is the set of all sequences (x_n) of elements of X such that there exists a constant c with

$$d_n(x_n, e) \leqslant c$$

for all $n \geqslant 1$. Define an equivalence relation $\sim$ on X_∞ by

$$(x_n) \sim (y_n) \quad \text{if and only if} \quad \lim_{\mathcal{D}} d_n(x_n, y_n) = 0,$$

and for each $(x_n) \in X_\infty$, let $(x_n)_{\mathcal{D}}$ be the corresponding equivalence class.

Then the asymptotic cone of X is

$$\mathrm{Con}_{\mathcal{D}}(X) = \{(x_n)_{\mathcal{D}} \mid (x_n) \in X_\infty\}$$

endowed with the metric

$$d_{\mathcal{D}}((x_n)_{\mathcal{D}}, (y_n)_{\mathcal{D}}) = \lim_{\mathcal{D}} d_n(x_n, y_n).$$

If (X,d) and (X',d') are metric spaces, then a map $f : X \to X'$ is a *quasi-isometry* iff there exist constants $L \geqslant 1$ and $C \geqslant 0$ such that for all x, $y \in X$

- $\frac{1}{L} d(x,y) - C \leqslant d'(f(x), f(y)) \leqslant L d(x,y) + C$;
  and for all $z \in X'$
- $d'(z, f[X]) \leqslant C$.

The following result is well-known. (For example, see Proposition 2.4.6 of Kleiner-Leeb [19].)

Proposition 1.4. *Suppose that $\mathcal{D}$ is a nonprincipal ultrafilter over ω and that $f: X \to X'$ is a quasi-isometry of nonempty metric spaces. Then f induces a bilipschitz homeomorphism $\varphi : \mathrm{Con}_{\mathcal{D}}(X) \to \mathrm{Con}_{\mathcal{D}}(X')$* defined by $\varphi((x_n)_{\mathcal{D}}) = (f(x_n))_{\mathcal{D}}$.

From now on, fix some $m \geq 3$ and let Γ be a uniform lattice in $\mathrm{SL}_m(\mathbb{R})$, i.e. Γ is a discrete subgroup of $\mathrm{SL}_m(\mathbb{R})$ such that $\mathrm{SL}_m(\mathbb{R})/\Gamma$ is compact. (For example, let $K = k(\sqrt{\epsilon})$, where $\epsilon = 1 + \sqrt{2}$ and $k = \mathbb{Q}(\sqrt{2})$. Let σ be the automorphism of K over k such that $\sigma(\sqrt{\epsilon}) = -\sqrt{\epsilon}$ and let f be the Hermitian form defined by

$$f(x, y) = x_1 y_1^{\sigma} + \cdots + x_m y_m^{\sigma}.$$

Finally let J be the ring of algebraic integers in K. By Vinberg et al. [3, Chapter 3], $SU(f, J)$ is a uniform lattice in $\mathrm{SL}_m(\mathbb{R})$.) By Theorem IV.23 [17], Γ is quasi-isometric to $\mathrm{SL}_m(\mathbb{R})$, viewed as a metric space with respect to some left-invariant Riemannian metric. Thus if $\mathcal{D}$ is a nonprincipal ultrafilter over ω, then $\mathrm{Con}_{\mathcal{D}}(\Gamma)$ is bilipschitz homeomorphic to $\mathrm{Con}_{\mathcal{D}}(\mathrm{SL}_m(\mathbb{R}))$. However, instead of working directly with the Riemannian manifold $\mathrm{SL}_m(\mathbb{R})$, it turns out to be more convenient to work with the corresponding symmetric space, obtained by factoring out the maximal compact subgroup $\mathrm{SO}_m(\mathbb{R})$. In more detail, recall that $\mathrm{SL}_m(\mathbb{R})$ acts transitively as a group of isometries on the symmetric space $P(m, \mathbb{R})$ of positive-definite symmetric $m \times m$ matrices with determinant 1, via the action

$$g \cdot A = gAg^t.$$

Clearly the stabiliser of the identity matrix I under this action is the subgroup $\mathrm{SO}_m(\mathbb{R})$; and since $\mathrm{SO}_m(\mathbb{R})$ is compact, it follows that Γ also acts cocompactly on $P(m, \mathbb{R})$. (For example, see [34, Chapter 1].) Hence, applying Theorem IV.23 [17] once again, it follows that Γ is also quasi-isometric to the symmetric space $P(m, \mathbb{R})$. The invariant Riemannian metric d on $P(m, \mathbb{R})$ can be described as follows. First recall that if $A, B \in P(m, \mathbb{R})$, then there exists $g \in \mathrm{SL}_m(\mathbb{R})$ such that gAg^t and gBg^t are simultaneously diagonal. Hence it suffices to consider the case when

$$A = (a_{i,j}) \quad \text{and} \quad B = (b_{i,j})$$

are both diagonal matrices; in which case,

$$d(A, B) = \sqrt{\sum (\log a_{i,i} - \log b_{i,i})^2}.$$

Fix some nonprincipal ultrafilter $\mathcal{D}$ over ω. Let $X = P(m, \mathbb{R})$ and let $I \in X$ be the base point. We next consider the question of which sequences of diagonal matrices lie in X_∞. For each $n \geq 1$, let

$$A_n = (a_{i,j}^{(n)}) \in X$$

be a diagonal matrix. Then

$$d_n(I, A_n) = \frac{\sqrt{\sum (\log a_{i,i}^{(n)})^2}}{n}$$

and so $(A_n) \in X_\infty$ if and only if there exists $k \geqslant 1$ such that

$$e^{-kn} < a_{i,i}^{(n)} < e^{kn} \tag{Exp}$$

for each $1 \leqslant i \leqslant m$ and $n \geqslant 1$.

Before we can describe the structure of the asymptotic cone $\mathrm{Con}_{\mathcal{D}}(X)$, we first need to recall the definition of the corresponding field ${}^*\mathbb{R}_{\mathcal{D}}$ of nonstandard reals.

Definition 1.5. Let $\mathbb{R}^\omega$ be the set of all sequences (x_n) of real numbers. Define an equivalence relation $\equiv$ on $\mathbb{R}^\omega$ by

$$(x_n) \equiv (y_n) \quad \text{if and only if} \quad \{n \in \omega \mid x_n = y_n\} \in \mathcal{D},$$

and for each $(x_n) \in \mathbb{R}^\omega$, let $[x_n]_{\mathcal{D}}$ be the corresponding equivalence class. Then the field of *nonstandard reals* is defined to be

$${}^*\mathbb{R}_{\mathcal{D}} = \{[x_n]_{\mathcal{D}} \mid (x_n) \in \mathbb{R}^\omega\}$$

equipped with the operations

$$[x_n]_{\mathcal{D}} + [y_n]_{\mathcal{D}} = [x_n + y_n]_{\mathcal{D}},$$

$$[x_n]_{\mathcal{D}} \cdot [y_n]_{\mathcal{D}} = [x_n \cdot y_n]_{\mathcal{D}}$$

and the ordering

$$[x_n]_{\mathcal{D}} < [y_n]_{\mathcal{D}} \quad \text{if and only if} \quad \{n \in \omega \mid x_n < y_n\} \in \mathcal{D}.$$

In order to simplify notation, during the next few paragraphs, we shall write ${}^*\mathbb{R}$ instead of the more precise ${}^*\mathbb{R}_{\mathcal{D}}$. It is well-known that ${}^*\mathbb{R}$ is a nonarchimedean real closed field and that $\mathbb{R}$ embeds into ${}^*\mathbb{R}$ via the map $r \mapsto [r]_{\mathcal{D}}$, where (r) denotes the sequence with constant value r. (For the basic properties of ${}^*\mathbb{R}$, see [21].)

Now suppose that $(A_n) \in X_\infty$ is a sequence of diagonal matrices, where each $A_n = (a_{i,j}^{(n)})$. Then we can define a corresponding diagonal matrix

$$(A_n)_{\mathcal{D}} = (\alpha_{i,j}) \in P(m, {}^*\mathbb{R})$$

by setting $\alpha_{i,j} = [a_{i,j}^{(n)}]_{\mathcal{D}}$. Using equation (Exp), we see that there exists $k \geqslant 1$ such that each $\alpha_{i,i}$ satisfies the inequality

$$\rho^k < \alpha_{i,i} < \rho^{-k},$$

where $\rho \in {}^*\mathbb{R}$ is the positive infinitesimal defined by

$$\rho = (e^{-n})_{\mathcal{D}}.$$

This suggests that $\mathrm{Con}_{\mathcal{D}}(X)$ should "essentially" be $P(m, K)$ for some field defined in terms of ${}^*\mathbb{R}$ and ρ.

Definition 1.6. Let M_0 be the subring of ${}^*\mathbb{R}$ defined by

$$M_0 = \{t \in {}^*\mathbb{R} \mid |t| < \rho^{-k} \quad \text{for some } k \geqslant 1\}.$$

Then M_0 has a unique maximal ideal

$$M_1 = \{t \in {}^*\mathbb{R} \mid |t| < \rho^{k} \quad \text{for all } k \geqslant 1\}$$

and the *Robinson field* ${}^\rho\mathbb{R}$ is defined to be the residue field M_0/M_1. By Lightstone and Robinson [21], ${}^\rho\mathbb{R}$ is also a real closed field.

If $(A_n) \in X_\infty$ is a sequence of diagonal matrices, then

$$(A_n)_{\mathcal{D}} = (\alpha_{i,j}) \in P(m, M_0),$$

and so we can define a corresponding matrix

$$\overline{(A_n)}_{\mathcal{D}} = (\bar{\alpha}_{i,j}) \in P(m, {}^\rho\mathbb{R}),$$

where each $\bar{\alpha}_{i,j} \in {}^\rho\mathbb{R}$ is the element naturally associated with $\alpha_{i,j} \in {}^*\mathbb{R}$. Unfortunately, it is *not* always the case that if (A_n), $(A'_n) \in X_\infty$ correspond to the same element of $\mathrm{Con}_{\mathcal{D}}(X)$, then $\overline{(A_n)}_{\mathcal{D}} = \overline{(A'_n)}_{\mathcal{D}}$. Thus, in order to obtain $\mathrm{Con}_{\mathcal{D}}(X)$, we must first factor $P(m, {}^\rho\mathbb{R})$ by a suitable equivalence relation.

Definition 1.7. If $0 \neq \alpha \in M_0$, then $\log_\rho|\alpha|$ is a finite possibly nonstandard real and hence is infinitesimally close to a unique standard real denoted by $\mathrm{st}(\log_\rho|\alpha|)$. By Lightstone-Robinson [21, Section 3.3], if $t \in M_0 \backslash M_1$ and $i \in M_1$, then

$$\mathrm{st}(\log_\rho|t|) = \mathrm{st}(\log_\rho|t + i|).$$

Hence we can define a valuation $\upsilon : {}^\rho\mathbb{R} \to \mathbb{R} \cup \{\infty\}$ by

$$\upsilon(\alpha) = \mathrm{st}(\log_\rho|\alpha|).$$

Let

$$|\alpha|_{\upsilon} = e^{-\upsilon(\alpha)}$$

be the associated absolute value.

Following Leeb and Parreau [22], the asymptotic cone $\mathrm{Con}_{\mathcal{D}}(X)$ can now be described as follows. Let $V(m, {}^{\rho}\mathbb{R})$ be the vector space of $m \times 1$ column vectors over the field ${}^{\rho}\mathbb{R}$. For each $A \in P(m, {}^{\rho}\mathbb{R})$, define the corresponding norm

$$\varphi_A : V(m, {}^{\rho}\mathbb{R}) \to \mathbb{R}$$

by

$$\varphi_A(x) = |\sqrt{x^t A x}|_{\upsilon}.$$

Then the map from X_∞ defined by $(A_n) \mapsto \varphi_A$, where $A = \overline{(A_n)_{\mathcal{D}}}$, induces an isometry

$$\mathrm{Con}_{\mathcal{D}}(X) \cong \{\varphi_A \mid A \in P(m, {}^{\rho}\mathbb{R})\}.$$

(When $A = (\alpha_i) \in P(m, {}^{\rho}\mathbb{R})$ and $B = (\beta_i) \in P(m, {}^{\rho}\mathbb{R})$ are both diagonal matrices, then the corresponding distance in the space of norms is given by

$$d(\varphi_A, \varphi_B) = \sqrt{\sum (\upsilon(\alpha_i) - \upsilon(\beta_i))^2}.$$

For more details, see [22].) This space of norms is an instance of a classical construction of Bruhat–Tits [8,9] and has a rich geometric structure. (The various geometric notions discussed in the remainder of this paragraph will be defined and discussed in more detail in Section 2.) More precisely, $\mathrm{Con}_{\mathcal{D}}(X)$ is an affine $\mathbb{R}$-building; and the set $\mathcal{F}$ of apartments of $\mathrm{Con}_{\mathcal{D}}(X)$ is precisely the collection of subspaces of $\mathrm{Con}_{\mathcal{D}}(X)$ which are isometric to $\mathbb{R}^{m-1}$. Furthermore, the elements of $\mathcal{F}$ correspond naturally to the unordered frames $\{{}^{\rho}\mathbb{R}v_1, \dots, {}^{\rho}\mathbb{R}v_m\}$ of the m-dimensional vector space $V(m, {}^{\rho}\mathbb{R})$ over ${}^{\rho}\mathbb{R}$. Let $\partial\,\mathrm{Con}_{\mathcal{D}}(X)$ be the associated spherical building at infinity of $\mathrm{Con}_{\mathcal{D}}(X)$. Then the apartments of $\partial\,\mathrm{Con}_{\mathcal{D}}(X)$ are the boundaries at infinity of the apartments of $\mathrm{Con}_{\mathcal{D}}(X)$; and so the apartments of $\partial\,\mathrm{Con}_{\mathcal{D}}(X)$ also correspond naturally to the unordered frames $\{{}^{\rho}\mathbb{R}v_1, \dots, {}^{\rho}\mathbb{R}v_m\}$ of $V(m, {}^{\rho}\mathbb{R})$. As this observation suggests, $\partial\,\mathrm{Con}_{\mathcal{D}}(X)$ is the usual spherical building associated with the natural BN-pair of $\mathrm{SL}_m({}^{\rho}\mathbb{R})$, i.e. $\partial\,\mathrm{Con}_{\mathcal{D}}(X)$ is the flag complex of the vector space $V(m, {}^{\rho}\mathbb{R})$. In particular, since $m \geqslant 3$, the isomorphism type of the Robinson field ${}^{\rho}\mathbb{R}$ is determined by the isomorphism type of the spherical building $\partial\,\mathrm{Con}_{\mathcal{D}}(X)$. By the Kleiner–Leeb topological rigidity theorem for affine $\mathbb{R}$-buildings [19], the isomorphism type of the spherical building $\partial\,\mathrm{Con}_{\mathcal{D}}(X)$ is a topological invariant of $\mathrm{Con}_{\mathcal{D}}(X)$. Consequently, the isomorphism type of the Robinson field ${}^{\rho}\mathbb{R}$ is also a topological invariant of $\mathrm{Con}_{\mathcal{D}}(X)$.

From now on, we shall write ${}^{\rho}\mathbb{R}_{\mathcal{D}}$ to indicate the possible dependence on $\mathcal{D}$ of the Robinson field. Since ${}^{\rho}\mathbb{R}_{\mathcal{D}}$ is a topological invariant of $\mathrm{Con}_{\mathcal{D}}(X)$, the following result implies that if CH fails, then $X = P(m, \mathbb{R})$ has 2^{2^ω} asymptotic cones up to homeomorphism. Consequently, as the uniform lattice Γ is quasi-isometric to $P(m, \mathbb{R})$, it follows that Γ also has 2^{2^ω} asymptotic cones up to homeomorphism.

Theorem 1.8. *If CH fails, then there exists a set* $\{\mathcal{D}_\alpha | \alpha < 2^{2^\omega}\}$ *of nonprincipal ultrafilters over* ω *such that*

$$^\rho\mathbb{R}_{\mathcal{D}_\alpha} \ncong {}^\rho\mathbb{R}_{\mathcal{D}_\beta}$$

for all $\alpha < \beta < 2^{2^\omega}$.

Now suppose that CH holds. In this case, it is well-known that if $\mathcal{A}$, $\mathcal{B}$ are nonprincipal ultrafilters, then the corresponding fields $^*\mathbb{R}_\mathcal{A}$, $^*\mathbb{R}_\mathcal{B}$ of nonstandard reals are isomorphic. In [31], Thornton used the Diarra–Pestov [11,23] representation of $^*\mathbb{R}_\mathcal{A}$ as a Hahn field to prove that if CH holds, then the Robinson fields $^\rho\mathbb{R}_\mathcal{A}$ and $^\rho\mathbb{R}_\mathcal{B}$ are isomorphic as valued fields. It follows that $X = P(m, \mathbb{R})$ has a unique asymptotic cone up to isometry. In fact, Thornton proved that the analogous result holds for arbitrary symmetric spaces. Hence the following result holds for uniform lattices in arbitrary connected semisimple Lie groups. (Recall that the word metric d on Γ depends on the choice of a finite generating set and so the metric space (Γ, d) is determined only up to quasi-isometry. Consequently, the following result is optimal.)

Theorem 1.9 (Thornton [31]). *Assume CH. If* $\mathcal{A}$, $\mathcal{B}$ *are nonprincipal ultrafilters, then*

$$^\rho\mathbb{R}_\mathcal{A} \cong {}^\rho\mathbb{R}_\mathcal{B}.$$

Furthermore, if G is a connected semisimple Lie group and Γ *is a uniform lattice in G, then* Γ *has a unique asymptotic cone up to bilipchitz homeomorphism.*

Finally, we should stress that it remains an open problem whether it can be proved in ZFC that there exists a finitely presented group with more than one asymptotic cone up to homeomorphism. However, when it comes to the question of whether there exists a finitely presented group with 2^{2^ω} asymptotic cones up to homeomorphism, then the case is altered.

Theorem 1.10. *If CH holds, then every finitely generated group* Γ *has at most* 2^ω *asymptotic cones up to isometry.*

Corollary 1.11. *The following statements are equivalent.*

(a) *CH fails.*
(b) *There exists a finitely presented group* Γ *which has* 2^{2^ω} *asymptotic cones up to homeomorphism.*
(c) *There exists a finitely generated group* Γ *which has* 2^{2^ω} *asymptotic cones up to homeomorphism.*

The rest of this paper is organised as follows. In Section 2, we shall discuss the notions of an affine $\mathbb{R}$-building and its spherical building at infinity; and we shall explain why ${}^{\rho}\mathbb{R}_{\mathcal{D}}$ is a topological invariant of $\mathrm{Con}_{\mathcal{D}}(\Gamma)$, whenever Γ is a uniform lattice in a connected semisimple Lie group G with at least one absolutely simple factor S such that $\mathbb{R}\text{-rank}(S)\geqslant 2$. Sections 3 and 4 will be devoted to the proof of Theorem 1.8. Finally we shall prove Theorem 1.10 in Section 5.

Throughout this paper, unless otherwise stated, the term "Lie group" will always mean a real Lie group.

2. Spherical and euclidean buildings

In this section, we shall discuss the notions of an affine $\mathbb{R}$-building and its spherical building at infinity; and we shall explain why ${}^{\rho}\mathbb{R}_{\mathcal{D}}$ is a topological invariant of $\mathrm{Con}_{\mathcal{D}}(\Gamma)$, whenever Γ is a uniform lattice in a connected semisimple Lie group G with at least one absolutely simple factor S such that $\mathbb{R}\text{-rank}(S)\geqslant 2$.

Suppose that $V\cong\mathbb{R}^n$ is a finite-dimensional Euclidean vector space, with inner product $\langle -,-\rangle : V\times V\to\mathbb{R}$. For a nonzero vector $v\in V$, let $\sigma_v : V\to V$ denote the Euclidean reflection $u\mapsto u-2\frac{\langle u,v\rangle}{\langle v,v\rangle}v$. A finite spanning set $\Phi\subseteq V$ of (nonzero) vectors is called a root system if $\sigma_u v\in\Phi$ and $2\frac{\langle x,v\rangle}{\langle v,v\rangle}\in\mathbb{Z}$ for all $u,v\in\Phi$, see [5, Chapter VI]. The group W generated by the σ_v is called the Weyl group of the root system. Associated to such a finite reflection group W is a certain simplicial complex $\Sigma(W)$, its *Coxeter complex* [5, Chapter IV]: roughly speaking, $\Sigma(W)$ is a W-invariant triangulation of the unit sphere $\mathbb{S}^{n-1}\subseteq V$. (The triangulation is obtained from the intersections of $\mathbb{S}^{n-1}$ with the reflection hyperplanes $v^{\perp}$ for $v\in\Phi$.)

For example, the root system of type A_n and its Coxeter complex are defined as follows. Let $\{e_1,\dots,e_{n+1}\}$ denote the standard orthonormal basis for $\mathbb{R}^{n+1}$ and let V be the subspace of $\mathbb{R}^{n+1}$ defined by

$$V=\{x\in\mathbb{R}^{n+1}\mid x_1+\cdots+x_{n+1}=0\}.$$

Then $\Phi=\{e_i-e_j\mid i\neq j\}\subseteq V$ is a root system of type A_n. As an abstract group, the Weyl group W is the symmetric group on $n+1$ letters, acting by coordinate permutations; and as a poset, $\Sigma(W)$ is the set of all $\subseteq$-ordered chains consisting of nontrivial subsets of $\{1,\dots,n+1\}$ (i.e. $\Sigma(W)$ is the first barycentric subdivision of the boundary of an $n+1$-simplex).

A *spherical building* is an abstract simplicial complex (a poset) $(\Delta,\leqslant)$ with a distinguished collection of subcomplexes Σ, called *apartments*, satisfying the following axioms:

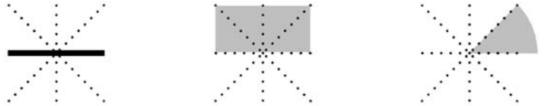
(B_1) Each apartment $\Sigma\subseteq\Delta$ is isomorphic to a (fixed) Coxeter complex $\Sigma(W)$.
(B_2) Any two simplices in Δ are contained in some apartment.
(B_3) Given two apartments $\Sigma_1,\Sigma_2\subseteq\Delta$, there exists an isomorphism $\Sigma_1\to\Sigma_2$ fixing $\Sigma_1\cap\Sigma_2$ simplex-wise.

For more details, we refer to [7,26,32]. The standard examples of buildings are obtained from algebraic groups as follows.

Let **G** be a reductive algebraic group defined over a field k [3,4,29] and let Δ denote the poset of all k-parabolic subgroups of **G**, ordered by reversed inclusion. Then $\Delta = \Delta(\mathbf{G}, k)$ is a spherical building, the canonical building associated to **G** over k, and the group $\mathbf{G}(k)$ of k-points of **G** acts strongly transitively on $\Delta(\mathbf{G}, k)$, see Tits [32, Chapter 5]. If **G** is absolutely simple (i.e. if **G** is simple over the algebraic closure $\bar{k}$) and adjoint, then the building uniquely determines the field of definition k.

Theorem 2.1 (Tits [32, 5.8]). *Let* **G** *and* **G**$'$ *be adjoint absolutely simple algebraic groups of rank at least* 2 *defined over the fields* k *and* k'. *If there is a building isomorphism* $\Delta(\mathbf{G}, k) \cong \Delta(\mathbf{G}', k')$, *then the fields* k *and* k' *are isomorphic.*

We next introduce the notion of an affine $\mathbb{R}$-building (also called a *Euclidean building*). Below we shall give Tits' definition [33], as corrected in [26, App. 3]. (A different set of axioms was proposed in Kleiner–Leeb [19], based on nonpositive curvature and geodesics; in [22], Parreau showed that these two approaches are equivalent.) Let W be the Weyl group of a root system $\Phi \subseteq V$, and let W_{aff} denote the semidirect product of W and the vector group $(V, +)$. Then in V, we obtain the corresponding *reflection hyperplanes* (the fixed point sets of reflections), *half-spaces* (determined by reflection hyperplanes), and *Weyl chambers* (the fundamental domains for W), see [5].



Definition 2.2. Fix W and V as above. A pair $(\mathcal{I}, \mathcal{F})$ consisting of a nonempty set $\mathcal{I}$ and a family $\mathcal{F}$ of injections $\phi : V \to \mathcal{I}$ is called an *affine* $\mathbb{R}$-*building* if it has the following properties:

(A$\mathbb{R}$B$_1$) If $w \in W_{\text{aff}}$ and $\phi \in \mathcal{F}$, then $\phi \circ w \in \mathcal{F}$.

(A$\mathbb{R}$B$_2$) For $\phi, \psi \in \mathcal{F}$, the preimage $X = \phi^{-1}\psi(V)$ is closed and convex (possibly empty), and there exists $w \in W_{\text{aff}}$ such that ϕ and $\psi \circ w$ agree on X.

(A$\mathbb{R}$B$_3$) Given $x, y \in \mathcal{I}$, there exists $\phi \in \mathcal{F}$ with $\{x, y\} \subseteq \phi(V)$.

Any ϕ-image of V is called an *apartment*; the ϕ-image of a reflection hyperplane, a half-space, and a Weyl chamber is called a *wall*, a *half-apartment*, and a *sector*, respectively. A wall is *thick* if it bounds three distinct half-apartments, and a point is *thick* if every wall passing through it is thick.

(A$\mathbb{R}$B$_4$) Given two sectors $S_1, S_2 \subseteq \mathcal{I}$, there exist subsectors $S_i' \subseteq S_i$ and an apartment $F \subseteq \mathcal{I}$ with $S_1' \cup S_2' \subseteq F$.

(A$\mathbb{R}$B$_5$) If F_1, F_2, F_3 are apartments having pairwise a half-apartment in common, then $F_1 \cap F_2 \cap F_3 \neq \emptyset$.

The *dimension* of an affine $\mathbb{R}$-building is the vector space dimension of V.

It follows that $\mathcal{I}$ admits a unique metric d which pulls back to the Euclidean metric on V for every $\phi : V \to \mathcal{I}$, and that $(\mathcal{I}, d)$ is a CAT(0)-space. (See [6] for the notion of a CAT(0)-space.) We say that $\mathcal{I}$ is *complete* if $(\mathcal{I}, d)$ is complete as a metric space (every Cauchy sequence converges) and that $\mathcal{F}$ is *complete* if every injection $\phi : V \to \mathcal{I}$ which is compatible with the axioms (A$\mathbb{R}$B$_1$)–(A$\mathbb{R}$B$_5$) is already in $\mathcal{F}$. It can be shown that every $\mathcal{F}$ admits a unique completion $\widehat{\mathcal{F}}$ such that $(\mathcal{I}, \widehat{\mathcal{F}})$ is an affine $\mathbb{R}$-building; the metric completion of $\mathcal{I}$, however, is in general not an affine $\mathbb{R}$-building.

Proposition 2.3. *Suppose that $(\mathcal{I}, \mathcal{F})$ and $(\mathcal{I}', \mathcal{F}')$ are affine $\mathbb{R}$-buildings, and that $f : \mathcal{I} \to \mathcal{I}'$ is a homeomorphism. Let $F \subseteq \mathcal{I}$ be an apartment. If $\mathcal{I}'$ and $\mathcal{F}'$ are complete, then $f(F)$ is an apartment in $\mathcal{I}'$.*

Proof. This was first proved by Kleiner–Leeb [19, Proposition 6.4.1]. A sheaf-theoretic proof was given in [20], using the fact that apartments can be viewed as certain global sections in the orientation sheaf of the topological space $\mathcal{I}$, which can be characterised topologically. From this, one deduces that both buildings have the same dimension, and that $f(F)$ is locally isometric to Euclidean space. Since $\mathcal{I}'$ is complete, $f(F)$ is also complete and thus—being contractible—globally isometric to Euclidean space. Such a subspace is always an apartment in the completion of $\mathcal{F}'$. □

Propositions 2.3 and 2.4 are false if $\mathcal{I}'$ is not assumed to be complete.

An affine $\mathbb{R}$-building $(\mathcal{I}, \mathcal{F})$ has a *spherical building at infinity*, denoted by $\partial(\mathcal{I}, \mathcal{F})$. The chambers of this building are equivalence classes of sectors, where two sectors are equivalent if the Hausdorff distance between them is finite. If a basepoint $x_0 \in \mathcal{I}$ is fixed, then every chamber of $\partial(\mathcal{I}, \mathcal{F})$ has a unique x_0-based sector as its representative.

The *draft* $D(\mathcal{I}, \mathcal{F})$ of an affine $\mathbb{R}$-building $(\mathcal{I}, \mathcal{F})$ is the pair $(\mathcal{I}, \mathcal{A})$ consisting of all points and all apartments of $(\mathcal{I}, \mathcal{F})$. As a direct consequence of 2.3, we have the following result.

Proposition 2.4. *Let $(\mathcal{I}, \mathcal{F})$ and $(\mathcal{I}', \mathcal{F}')$ be affine $\mathbb{R}$-buildings such that $\mathcal{I}, \mathcal{I}', \mathcal{F}$ and $\mathcal{F}'$ are complete. Then each homeomorphism $f : \mathcal{I} \to \mathcal{I}'$ induces an isomorphism of drafts $D(\mathcal{I}, \mathcal{F}) \xrightarrow{\cong} D(\mathcal{I}', \mathcal{F}')$.*

The draft disregards the metric structure and the Weyl group of the affine $\mathbb{R}$-building. In general, nonisomorphic affine $\mathbb{R}$-buildings can have isomorphic drafts. However, we have the following result.

Proposition 2.5. *Suppose that $\mathcal{I}$ contains a thick point x. If*

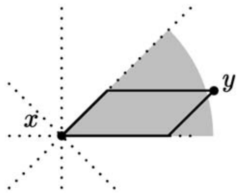
$$f : D(\mathcal{I}, \mathcal{F}) \xrightarrow{\cong} D(\mathcal{I}', \mathcal{F}')$$

is an isomorphism of drafts of affine $\mathbb{R}$-buildings, then f induces an isomorphism

$$\partial f : \partial(\mathcal{I}, \mathcal{F}) \to \partial(\mathcal{I}', \mathcal{F}')$$

between the respective spherical buildings at infinity.

Proof. Fix some thick point x of $\mathcal{I}$. For each $y \in \mathcal{I}$, let $\mathrm{cvx}\{x, y\}$ denote the intersection of all apartments containing x and y. This set should be pictured as a diamond-shaped set with x as one tip. Since x has thick walls, $\mathrm{cvx}\{x, y\}$ is always contained in one of the x-based sectors of $\mathcal{I}$.



Let $F \subseteq \mathcal{I}$ be an apartment containing x. In the poset $(\{\mathrm{cvx}\{x, y\} \mid y \in F\}, \subseteq)$, the unions over the maximal chains are precisely the x-based sectors in F. Thus any isomorphism of drafts preserves x-based sectors. The intersections of the x-based sectors in F naturally form a poset isomorphic to $\Sigma(W)$; and the union of these posets, where F runs through all apartments containing x, is canonically isomorphic to the spherical building $\partial(\mathcal{I}, \mathcal{F})$. The result now follows from the fact that the underlying poset of a spherical building completely determines the building itself. □

Let $\mathbf{G}$ be a semisimple algebraic group defined over the real closure $\mathbb{Q}_{\mathbb{R}}$ of $\mathbb{Q}$. Then the $\mathbb{R}$-*rank* of $\mathbf{G}$ is the maximal dimension of a $\mathbb{Q}_{\mathbb{R}}$-split torus $\mathbf{S}$ of $\mathbf{G}$. If $\mathcal{R}$ is any real closed field (such as $\mathcal{R} = \mathbb{R}$ or $\mathcal{R} = {}^{\rho}\mathbb{R}$), then $\mathbb{Q}_{\mathbb{R}} \subseteq \mathcal{R}$ and $\mathbf{S}$ is also a maximal $\mathcal{R}$-split torus over $\mathcal{R}$. In fact, $\mathbf{G}$ has the same structure, rank, Tits diagram and building type over $\mathcal{R}$ as over its field of definition $\mathbb{Q}_{\mathbb{R}}$, see [15, Section 4]. Note that the group $G = \mathbf{G}(\mathbb{R})$ of $\mathbb{R}$-points of $\mathbf{G}$, endowed with the Hausdorff topology, is a real Lie group; and its (Hausdorff) connected component G° is a semisimple Lie group G such that $[G : G^\circ] < \infty$. Furthermore, if G is any connected semisimple Lie group, then there exists a semisimple algebraic group defined over $\mathbb{Q}_{\mathbb{R}}$ such that $G/Z(G)$ and $\mathbf{G}(\mathbb{R})^\circ$ are isomorphic as Lie groups; for example, see [14, 1.14.6]. (In fact, $\mathbf{G}$ can even be taken to be defined over $\mathbb{Q}$; but in our setting, it is more convenient to work with real closed fields.) We then define $\mathbb{R}\text{-rank}(G)$ to be the $\mathbb{R}$-rank of the algebraic group $\mathbf{G}$; and we define G to be absolutely simple if and only if $\mathbf{G}$ is absolutely simple. (Equivalently, G is absolutely simple if and only if the complexification $\mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$ of the Lie algebra $\mathfrak{g}$ of G remains simple.) Since $Z(G)$ is finite, the

Riemannian manifolds G and $G/Z(G)$ are quasi-isometric. Hence, in the remainder of this section, we can restrict our attention to connected semisimple Lie groups of the form $\mathbf{G}(\mathbb{R})^\circ$.

Fix a semisimple algebraic group $\mathbf{G}$ defined over $\mathbb{Q}_\mathbb{R}$ of $\mathbb{R}$-rank $m \geqslant 1$. Let $\mathbf{S} \subseteq \mathbf{G}$ be a maximal $\mathbb{Q}_\mathbb{R}$-split torus and $\mathbf{N} = \mathrm{Nor}_\mathbf{G}(\mathbf{S})$ its normaliser. The quotient $W = \mathbf{N}/\mathbf{S}$ is the relative Weyl group for $\mathbf{G}$ (once again, over any real closed field $\mathcal{R}$). Let $\mathcal{D}$ be a nonprincipal ultrafilter, let $\varepsilon = (1/n)_\mathcal{D}$, and let $\rho = e^\varepsilon = (e^{-n})_\mathcal{D}$. The Robinson field ${}^\rho\mathbb{R}$ has a unique maximal o-convex subring ${}^\rho\mathcal{O} \subseteq {}^\rho\mathbb{R}$, corresponding to the canonical o-valuation $v : {}^\rho\mathbb{R} \to \mathbb{R} \cup \{\infty\}$, which was defined in Definition 1.7. (For general results about real closed fields and o-valuations, see [24].) Since ${}^\rho\mathcal{O}$ is o-convex, we have that $\mathbb{Q}_\mathbb{R} \subseteq {}^\rho\mathcal{O} \subseteq {}^\rho\mathbb{R}$. Therefore the groups $\mathbf{G}({}^\rho\mathcal{O}) \subseteq \mathbf{G}({}^\rho\mathbb{R})$ and $\mathbf{S}({}^\rho\mathcal{O}) \subseteq \mathbf{S}({}^\rho\mathbb{R})$ are defined, as well as the coset spaces

$$\mathcal{I} = \mathbf{G}({}^\rho\mathbb{R})/\mathbf{G}({}^\rho\mathcal{O}) \quad \text{and} \quad V = \mathbf{S}({}^\rho\mathbb{R})/\mathbf{S}({}^\rho\mathcal{O}).$$

Moreover, V can be regarded in a natural way as an m-dimensional real vector space, equipped with a natural action of W as a finite reflection group. For example, if $\mathbf{G} = \mathrm{SL}_{m+1}$ and $\mathbf{S}$ is the group of diagonal matrices in SL_{m+1}, then $\mathbf{N}$ consists of permutation matrices acting by coordinate permutations; and the resulting root system is the one of type A_m described at the beginning of this section.

If we extend the W-action by the translations, then we obtain an affine Weyl group W_{aff}. For $g \in \mathbf{G}({}^\rho\mathbb{R})$ and $v = n\mathbf{S}({}^\rho\mathcal{O}) \in \mathbf{S}({}^\rho\mathbb{R})/\mathbf{S}({}^\rho\mathcal{O}) = V$, let $\phi_g(v) = gn\mathbf{G}({}^\rho\mathcal{O}) \in \mathcal{I}$ and put

$$\mathcal{F} = \{\phi_g \mid g \in \mathbf{G}({}^\rho\mathbb{R}_\mathcal{D})\}.$$

Theorem 2.6. *The pair* $(\mathcal{I}, \mathcal{F}) = \Delta_{\mathrm{aff}}(\mathbf{G}, {}^\rho\mathbb{R}, {}^\rho\mathcal{O})$ *is an affine* $\mathbb{R}$*-building such that* $\mathcal{I}$ *and* $\mathcal{F}$ *are complete. The building at infinity is the spherical building* $\partial\Delta_{\mathrm{aff}}(\mathbf{G}, {}^\rho\mathbb{R}, {}^\rho\mathcal{O}) = \Delta(\mathbf{G}, {}^\rho\mathbb{R})$.

Proof. This is a special case of a much more general result on the affine Λ-buildings associated to arbitrary real closed valued fields, which was proved in [20]. The fact that $\mathcal{I}$ and $\mathcal{F}$ are complete follows from the ω_1-saturatedness of countable ultrapowers. $\square$

Now we shall explain how asymptotic cones fit into the picture. Let G be a semisimple Lie group of rank $m \geqslant 1$, endowed with a left-invariant Riemannian metric d. Let $K \leqslant G$ be a maximal compact subgroup. The Riemannian symmetric space $X = G/K$ carries a natural metric (unique up to homothety), and it is not difficult to show that the natural map $G \to G/K$ is a quasi-isometry. Thus X and G have bilipschitz homeomorphic asymptotic cones. As above, let $\mathbf{G}$ be an algebraic group over $\mathbb{Q}_\mathbb{R}$ with $\mathbf{G}(\mathbb{R})^\circ = G$.

Proposition 2.7. *The asymptotic cone* $\mathrm{Con}_\mathcal{D}(X)$ *is isometric to the point space* $\mathcal{I}$ *of the building* $\Delta_{\mathrm{aff}}(\mathbf{G}, {}^\rho\mathbb{R}_\mathcal{D}, {}^\rho\mathcal{O}_\mathcal{D})$.

Proof. This was proved in [20] and also independently in [31]. □

The fact that $\mathrm{Con}_{\mathcal{D}}(X)$ is an affine $\mathbb{R}$-building was proved first by Kleiner–Leeb [19]. (A vague conjecture pointing in this direction was made by Gromov in [16, p. 54]). However, Kleiner and Leeb did not determine the building which one obtains. As we mentioned above, the fact that $\mathrm{Con}_{\mathcal{D}}(X)$ can be identified as a metric space with the quotient $\mathbf{G}({}^{\rho}\mathbb{R})/\mathbf{G}({}^{\rho}\mathcal{O})$ was proved independently by Thornton [3], but he did not identify the affine building or the spherical building at infinity. This was done by Leeb and Parreau [22] for the special case of $\mathbf{G} = \mathrm{SL}_{m+1}$; and one should also mention Bennett's general result [1] on the affine Λ-buildings related to $\mathrm{SL}_{m+1}(F)$ for arbitrary valued fields F.

Theorem 2.8. *Suppose that G is a connected absolutely simple Lie group such that $\mathbb{R}\text{-rank}(G) \geqslant 2$ and let Γ be a uniform lattice in G. Let $\mathcal{D}$ and $\mathcal{D}'$ be nonprincipal ultrafilters. If $\mathrm{Con}_{\mathcal{D}}(\Gamma)$ and $\mathrm{Con}_{\mathcal{D}'}(\Gamma)$ are homeomorphic, then the Robinson fields ${}^{\rho}\mathbb{R}_{\mathcal{D}}$ and ${}^{\rho}\mathbb{R}_{\mathcal{D}'}$ are isomorphic.*

Proof. Let $\mathbf{G}$ be an absolutely simple algebraic group defined over $\mathbb{Q}_{\mathbb{R}}$ such that $\mathbb{R}\text{-rank}(\mathbf{G}) \geqslant 2$ and $G = \mathbf{G}(\mathbb{R})^{\circ}$. Let $K \leqslant G$ be a maximal compact subgroup. Since Γ is quasi-isometric to the symmetric space $X = G/K$, the asymptotic cones $\mathrm{Con}_{\mathcal{D}}(\Gamma)$ and $\mathrm{Con}_{\mathcal{D}'}(\Gamma)$ are bilipschitz homeomorphic to $\mathrm{Con}_{\mathcal{D}}(X)$, $\mathrm{Con}_{\mathcal{D}'}(X)$ respectively. Consequently, if $\mathrm{Con}_{\mathcal{D}}(\Gamma)$ and $\mathrm{Con}_{\mathcal{D}'}(\Gamma)$ are homeomorphic, then $\mathrm{Con}_{\mathcal{D}}(X)$ and $\mathrm{Con}_{\mathcal{D}'}(X)$ are also homeomorphic. Hence, by Propositions 2.7 and 2.4, the affine $\mathbb{R}$-buildings $\Delta_{\mathrm{aff}}(\mathbf{G}, {}^{\rho}\mathbb{R}_{\mathcal{D}}, {}^{\rho}\mathcal{O}_{\mathcal{D}})$ and $\Delta_{\mathrm{aff}}(\mathbf{G}, {}^{\rho}\mathbb{R}_{\mathcal{D}'}, {}^{\rho}\mathcal{O}_{\mathcal{D}'})$ have isomorphic drafts. Applying Proposition 2.5, it follows that the corresponding buildings at infinity $\Delta(\mathbf{G}, {}^{\rho}\mathbb{R}_{\mathcal{D}})$ and $\Delta(\mathbf{G}, {}^{\rho}\mathbb{R}_{\mathcal{D}'})$ are also isomorphic. Finally, by Theorem 2.1, the building $\Delta(\mathbf{G}, {}^{\rho}\mathbb{R}_{\mathcal{D}})$ determines the field ${}^{\rho}\mathbb{R}_{\mathcal{D}}$ up to isomorphism, and so ${}^{\rho}\mathbb{R}_{\mathcal{D}} \cong {}^{\rho}\mathbb{R}_{\mathcal{D}'}$. □

We should make a few comments concerning the hypotheses on the Lie group G in the statement of Theorem 2.8. If G has $\mathbb{R}$-rank 1, then $\mathrm{Con}_{\mathcal{D}}\Gamma$ is a homogeneous $\mathbb{R}$-tree with uncountable branching at every point. Furthermore, the isometry type of this $\mathbb{R}$-tree is independent of the choice of the ultrafilter $\mathcal{D}$ (and even of the Lie type of G.) For example, see [13]. Thus the hypothesis on the $\mathbb{R}$-rank of G is certainly necessary. To understand the reason for the hypothesis that G is absolutely simple, consider the complex Lie group $G = \mathrm{SL}_n(\mathbb{C})$ for some $n \geqslant 3$. By embedding $\mathrm{SL}_n(\mathbb{C})$ as an algebraic subgroup $\mathbf{G}(\mathbb{R})$ of $GL_{2n}(\mathbb{R})$, we can regard G as a real simple Lie group of $\mathbb{R}$-rank $n-1$. However, G is not absolutely simple, since $\mathbf{G}(\mathbb{C}) \cong \mathrm{SL}_n(\mathbb{C}) \times \mathrm{SL}_n(\mathbb{C})$. (More generally, it turns out that a real simple Lie group G is absolutely simple if and only if $G/Z(G)$ is not isomorphic to a complex Lie group.) When we consider the spherical building at infinity of the corresponding affine $\mathbb{R}$-building, then we are only able to recover the algebraic closure ${}^{\rho}\mathbb{C}_{\mathcal{D}} = {}^{\rho}\mathbb{R}_{\mathcal{D}}(\sqrt{-1})$ rather than the Robinson field ${}^{\rho}\mathbb{R}_{\mathcal{D}}$ itself. Since ${}^{\rho}\mathbb{C}_{\mathcal{D}}$ is an algebraically closed field of cardinality 2^{ω}, it follows that ${}^{\rho}\mathbb{C}_{\mathcal{D}} \cong \mathbb{C}$ for every nonprincipal ultrafilter $\mathcal{D}$ over ω. Thus the fields ${}^{\rho}\mathbb{C}_{\mathcal{D}}$ cannot be used to distinguish between the (possibly different) asymptotic cones of $\mathrm{SL}_n(\mathbb{C})$; and it remains an open question whether the asymptotic cones of complex simple Lie groups depend on the chosen ultrafilter. However, there is an obvious generalisation of Theorem 2.8 to semisimple Lie groups.

Corollary 2.9. *Let G be a connected semisimple Lie group with at least one absolutely simple factor S such that* $\mathbb{R}$-rank$(S) \geqslant 2$ *and let* Γ *be a uniform lattice in G. Let* $\mathcal{D}$ *and* $\mathcal{D}'$ *be nonprincipal ultrafilters. If* $\mathrm{Con}_{\mathcal{D}}(\Gamma)$ *and* $\mathrm{Con}_{\mathcal{D}'}(\Gamma)$ *are homeomorphic, then the corresponding Robinson fields* ${}^{\rho}\mathbb{R}_{\mathcal{D}}$ *and* ${}^{\rho}\mathbb{R}_{\mathcal{D}'}$ *are isomorphic.*

This follows from Theorem 2.8, together with the fact that the buildings at infinity of $\mathrm{Con}_{\mathcal{D}}(\Gamma)$ and $\mathrm{Con}_{\mathcal{D}'}(\Gamma)$ decompose into products of the buildings corresponding to the simple factors of G.

3. Invariants of linear orders

The next two sections will be devoted to the proof of Theorem 1.8. We will begin by reducing Theorem 1.8 to an analogous statement concerning the linearly ordered sets $\omega^{\omega}/\mathcal{D}$, where $\mathcal{D}$ is a nonprincipal ultrafilter over ω.

Definition 3.1. A *filter* over ω is a collection D of subsets of ω satisfying the following conditions:

(i) $\omega \in D$.
(ii) If $A, B \in D$, then $A \cap B \in D$.
(iii) If $A \in D$ and $A \subseteq B \subseteq \omega$, then $B \in D$.

The filter D is *nontrivial* if and only if

(iv) $\emptyset \notin D$.

Let D be a nontrivial filter over ω. Then $\equiv_D$ is the equivalence relation defined on $\omega^{\omega} = \{f \mid f : \omega \to \omega\}$ by

$$f \equiv_D g \quad \text{if and only if} \quad \{n \in \omega \mid f(n) = g(n)\} \in D.$$

For each $f \in \omega^{\omega}$, we denote the corresponding $\equiv_D$-equivalence class by f/D; and we let

$$\omega^{\omega}/D = \{f/D \mid f \in \omega^{\omega}\}$$

equipped with the partial order defined by

$$f/D < g/D \quad \text{if and only if} \quad \{n \in \omega \mid f(n) < g(n)\} \in D.$$

As usual, we identify each natural number $\ell \in \omega$ with the corresponding element $c_{\ell}/D \in \omega^{\omega}$, defined by $c_{\ell}(n) = \ell$ for all $n \in \omega$. If $\mathcal{D}$ is a nonprincipal ultrafilter over ω, then $\omega^\omega/\mathcal{D}$ is a linear order. In this case, we define

$$(\omega^\omega/\mathcal{D})^* = \{g/\mathcal{D} \in \omega^\omega/\mathcal{D} \mid \ell < g/\mathcal{D} \text{ forall } \ell \in \omega\}.$$

As we shall now explain, Theorem 1.8 is an easy consequence of Theorem 3.3, which we shall prove in Section 4.

Definition 3.2. Let L_1, L_2 be linear orders.

(a) $L_1 \approx_f L_2$ if and only if L_1 and L_2 have nonempty isomorphic final segments.
(b) $L_1 \approx_i L_2$ if and only if L_1 and L_2 have nonempty isomorphic initial segments.

Theorem 3.3. *If CH fails, then there exists a set* $\{\mathcal{D}_\alpha \mid \alpha < 2^{2^\omega}\}$ *of nonprincipal ultrafilters over* ω *such that*

$$(\omega^\omega/\mathcal{D}_\alpha)^* \not\approx_i (\omega^\omega/\mathcal{D}_\beta)^*$$

for all $\alpha < \beta < 2^{2^\omega}$.

Proof of Theorem 1.8. For each nonprincipal ultrafilter $\mathcal{D}$ over ω, let

$${}^\rho\mathbb{R}^\infty_\mathcal{D} = \{a \in {}^\rho\mathbb{R}_\mathcal{D} \mid a > \ell \quad \text{for every } \ell \in \omega\};$$

and let $E_\mathcal{D}$ be the convex equivalence relation defined on ${}^\rho\mathbb{R}^\infty_\mathcal{D}$ by

$$aE_\mathcal{D}b \quad \text{if and only if} \quad |a-b| < \ell \quad \text{for some } \ell \in \omega.$$

Let $L_\mathcal{D} = {}^\rho\mathbb{R}^\infty_\mathcal{D}/E_\mathcal{D}$, equipped with the quotient linear ordering; and regard $L_\mathcal{D} \times \mathbb{Z}$ as a linear ordering with respect to the usual lexicographical ordering, defined by $(a_1, z_1) < (a_2, z_2)$ if and only if either:

- $a_1 < a_2$; or
- $a_1 = a_2$ and $z_1 < z_2$.

Then it is easily checked that

$$L_\mathcal{D} \times \mathbb{Z} \cong \{g/\mathcal{D} \in (\omega^\omega/\mathcal{D})^* | g/\mathcal{D} < \rho^{-n} \quad \text{for some } n \geqslant 1\}.$$

Now suppose that $\mathcal{A}$, $\mathcal{B}$ are nonprincipal ultrafilters over ω and that $f : {}^\rho\mathbb{R}_\mathcal{A} \to {}^\rho\mathbb{R}_\mathcal{B}$ is a field isomorphism. Since ${}^\rho\mathbb{R}_\mathcal{A}$, ${}^\rho\mathbb{R}_\mathcal{B}$ are real closed, it follows that f is also order-preserving. It is also clear that $f[{}^\rho\mathbb{R}^\infty_\mathcal{A}] = {}^\rho\mathbb{R}^\infty_\mathcal{B}$ and that f maps the equivalence relation $E_\mathcal{A}$ to the equivalence relation $E_\mathcal{B}$. It follows that the ordered sets $L_\mathcal{A}$ and $L_\mathcal{B}$ are isomorphic; and hence that

$$(\omega^\omega/\mathcal{A})^* \approx_i (\omega^\omega/\mathcal{B})^*.$$

Consequently, if $\{\mathcal{D}_\alpha \mid \alpha < 2^\kappa\}$ is the set of nonprincipal ultrafilters over ω given by Theorem 3.3, then

$$^\rho\mathbb{R}_{\mathcal{D}_\alpha} \not\cong {}^\rho\mathbb{R}_{\mathcal{D}_\beta}$$

for all $\alpha < \beta < 2^\kappa$. $\square$

A similar argument shows that the corresponding fields $^*\mathbb{R}_{\mathcal{D}_\alpha}$, $\alpha < 2^\kappa$, of nonstandard reals are also pairwise nonisomorphic. This improves a result of Roitman [25], who proved that it is consistent that there exist 2^ω pairwise nonisomorphic fields of nonstandard reals, each of the form $^*\mathbb{R}_{\mathcal{D}}$ for some nonprincipal ultrafilter $\mathcal{D}$ over ω.

Most of the remainder of this section will be devoted to the construction of a collection of extremely nonisomorphic linear orders, which will later be used as suitable "invariants" in the proof of Theorem 3.3. This construction is a special case of the more general techniques which are developed in Chapter III of Shelah [28]. In order to make this paper relatively self-contained, we have provided proofs of the relevant results. We assume that the reader is familiar with the basic properties of regular cardinals, singular cardinals, and stationary subsets of regular cardinals. (For example, see Sections 6 and 7 of Jech [18].)

Definition 3.4. Suppose that I is a linear order and that $\emptyset \neq A \subseteq I$.

(a) A subset $B \subseteq A$ is said to be *cofinal* in A if for all $a \in A$, there exists an element $b \in B$ such that $a \leqslant b$. The *cofinality* of A is defined to be

$$\mathrm{cf}(A) = \min\{|B| \mid B \text{ is a cofinal subset of } A\}.$$

(b) A subset $B \subseteq A$ is said to be *coinitial* in A if for all $a \in A$, there exists an element $b \in B$ such that $b \leqslant a$. The *coinitiality* of A is defined to be

$$\mathrm{coi}(A) = \min\{|B| \mid B \text{ is a coinitial subset of } A\}.$$

Definition 3.5. Suppose that I is a linear order and that $\lambda, \theta \geqslant \omega$ are regular cardinals. Then (I_1, I_2) is a (λ, θ)-cut of I if the following conditions hold:

(a) $I = I_1 \cup I_2$ and $s < t$ for all $s \in I_1$, $t \in I_2$.
(b) $\mathrm{cf}(I_1) = \lambda$.
(c) $\mathrm{coi}(I_2) = \theta$.

Definition 3.6. Suppose that J, L are linear orders. Then the order-preserving map $\varphi : J \to L$ is an *invariant embedding* if whenever (J_1, J_2) is a (λ, θ)-cut of J for some λ, $\theta > \omega$, then there does *not* exist an element $x \in L$ such that $\varphi(s) < x < \varphi(t)$ for all $s \in J_1$, $t \in J_2$.

In this case, φ is said to be an *invariant cofinal embedding* if $\varphi[J]$ is cofinal in L and φ is said to be an *invariant coinitial embedding* if $\varphi[J]$ is coinitial in L

Lemma 3.7. *If $\lambda > \omega_1$ is a regular cardinal, then there exists a set $\{I_\alpha | \alpha < 2^\lambda\}$ of linear orders satisfying the following conditions:*

(a) $\mathrm{cf}(I_\alpha) = |I_\alpha| = \lambda$.
(b) *If $\alpha \neq \beta$ and $\varphi_\alpha : I_\alpha \to L$, $\varphi_\beta : I_\beta \to L'$ are invariant cofinal embeddings, then $L \not\approx_f L'$.*

Proof. Applying Solovay's Theorem, let $\{S_\tau \mid \tau < \lambda\}$ be a partition of the stationary set

$$S = \{\delta < \lambda \mid \mathrm{cf}(\delta) = \omega_1\}$$

into λ pairwise disjoint stationary subsets. (For example, see [18, 7.6].) Fix some subset $X \subset \lambda$. Then for each $\alpha < \lambda$, we define

$$\lambda_\alpha^X = \begin{cases} \omega_2 & \text{if } \alpha \in \bigcup_{\tau \in X} S_\tau, \\ \omega_1 & \text{otherwise;} \end{cases}$$

and we define the linear order

$$I_X = \{(\alpha, \beta) \mid \alpha < \lambda \quad \text{and} \quad \beta < \lambda_\alpha^X\}$$

by setting $(\alpha_1, \beta_1) < (\alpha_2, \beta_2)$ if and only if either:

- $\alpha_1 < \alpha_2$, or
- $\alpha_1 = \alpha_2$ and $\beta_1 > \beta_2$.

Suppose that $X \neq Y \subseteq \lambda$. Let L, L' be linear orders and let $\varphi_X : I_X \to L$, $\varphi_Y : I_Y \to L'$ be invariant cofinal embeddings. Suppose that $L \approx_f L'$ and let $\psi : M \to M'$ be an isomorphism between the final segments M, M' of L, L' respectively. For each $\delta < \lambda$, let

$$M_\delta = \{m \in M \mid m < \varphi_X(\gamma, 0) \quad \text{for some } \gamma < \delta\}$$

and

$$M'_\delta = \{m' \in M' \mid m' < \varphi_Y(\gamma, 0) \quad \text{for some } \gamma < \delta\}.$$

Then there exists a club $C \subseteq \lambda$ such that $\psi[M_\delta] = M'_\delta$ for all $\delta \in C$. Without loss of generality, we can suppose that there exists an ordinal $\tau \in X \setminus Y$. Choose $\delta \in C \cap S_\tau$ such that $M_\delta \neq \emptyset$. Since φ_X is an invariant embedding, it follows that $\mathrm{coi}(M \setminus M_\delta) = \omega_2$. Similarly, $\mathrm{coi}(M' \setminus M'_\delta) = \omega_1$. But this is impossible since $\psi[M \setminus M_\delta] = M' \setminus M'_\delta$. □

Note that in the statement of Theorem 3.8, κ is not necessarily regular. In Section 4, we shall apply Theorem 3.8 in the case when $\kappa = 2^\omega > \omega_1$.

Theorem 3.8. *If* $\kappa > \omega_1$, *then there exists a set* $\{J_\alpha \mid \alpha < 2^\kappa\}$ *of linear orders satisfying the following conditions*:

(a) $|J_\alpha| = \kappa$.
(b) $\mathrm{coi}(J_\alpha) = \mathrm{cf}(\kappa) + \omega_2$.
(c) *If* $\alpha \neq \beta$ *and* $\varphi_\alpha : J_\alpha \to L$, $\varphi_\beta : J_\beta \to L'$ *are invariant coinitial embeddings, then* $L \not\approx_i L'$.

Proof. Let $\langle \kappa_i \mid i < \mathrm{cf}(\kappa) \rangle$ be a sequence of regular cardinals with each $\kappa_i > \omega_1$ such that:

(a) if κ is singular, then $\kappa = \sup_{i < \mathrm{cf}(\kappa)} \kappa_i$; and
(b) if κ is regular, then $\kappa_i = \kappa$ for all $i < \mathrm{cf}(\kappa) = \kappa$.

In either case, we have that $\prod_{i < \mathrm{cf}(\kappa)} 2^{\kappa_i} = 2^\kappa$. (For the case when κ is singular, see [18, 6.5].) Let $\theta = \mathrm{cf}(\kappa) + \omega_2$ and let $\{S_\tau \mid \tau < \mathrm{cf}(\kappa)\}$ be a partition of the stationary set

$$S = \{\delta < \theta \mid \mathrm{cf}(\delta) = \omega_1\}$$

into $\mathrm{cf}(\kappa)$ pairwise disjoint stationary subsets. Let $h : \theta \to \mathrm{cf}(\kappa)$ be the function defined by:

- $\delta \in S_{h(\delta)}$ for all $\delta \in S$; and
- $h(\xi) = 0$ for all $\xi \in \theta \setminus S$.

For each $i < \mathrm{cf}(\kappa)$, let $\{I_{i,\alpha} \mid \alpha < 2^{\kappa_i}\}$ be the set of linear orders of cardinality κ_i given by Lemma 3.7. For any $v \in \prod_{i < \mathrm{cf}(\kappa)} 2^{\kappa_i}$, we define the linear order

$$J_v = \{(\alpha, x) \mid \alpha < \theta \quad \text{and} \quad ux \in I_{h(\alpha), v(h(\alpha))}\}$$

by setting $(\alpha_1, x_1) < (\alpha_2, x_2)$ if and only if either:

- $\alpha_1 > \alpha_2$, or
- $\alpha_1 = \alpha_2$ and $x_1 < x_2$.

Arguing as in the proof of Lemma 3.7, we see that the set $\{J_v \mid v \in \prod_{i < \mathrm{cf}(\kappa)} 2^{\kappa_i}\}$ of linear orders satisfies our requirements. □

Now suppose that $2^\omega = \kappa > \omega_1$ and let $\{J_\alpha \mid \alpha < 2^\kappa\}$ be the set of linear orders given by Theorem 3.8. Then clearly Theorem 3.3 would follow if we could construct a set $\{\mathcal{D}_\alpha \mid \alpha < 2^\kappa\}$ of nonprincipal ultrafilters over ω such that for each $\alpha < 2^\kappa$, there exists an invariant coinitial embedding

$$\varphi_\alpha : J_\alpha \to (\omega^\omega / \mathcal{D}_\alpha)^*.$$

Unfortunately, this direct approach leads to serious technical difficulties which we have not yet been able to overcome. In order to avoid these difficulties, in the next section, we shall instead construct a set $\{\mathcal{D}_\alpha \mid \alpha < 2^\kappa\}$ of nonprincipal ultrafilters over ω such that the following condition is satisfied:

- For each $\alpha < 2^\kappa$ and each initial segment L of $(\omega^\omega/\mathcal{D}_\alpha)^*$, there exists an invariant embedding $\varphi : \omega_1 + J_\alpha \to L$.

(Here $\omega_1 + J_\alpha$ is the linear order consisting of a copy of the ordinal ω_1 followed by a copy of J_α. In particular, (ω_1, J_α) is an $(\omega_1, \mathrm{cf}(\kappa) + \omega_2)$-cut of $\omega_1 + J_\alpha$.) Of course, this is not enough to ensure that

$$(\omega^\omega/\mathcal{D}_\alpha)^* \not\approx_i (\omega^\omega/\mathcal{D}_\beta)^*$$

for all $\beta \neq \alpha$, since the above condition does not rule out the possibility that there also exists an invariant embedding

$$\psi \omega_1 + J_\beta \to (\omega^\omega/\mathcal{D}_\alpha)^*.$$

Fix some $\alpha < 2^\kappa$ and let C_α be the set of $\beta < 2^\kappa$ such that there exists an invariant embedding

$$\psi_\beta : \omega_1 + J_\beta \to (\omega^\omega/\mathcal{D}_\alpha)^*.$$

For each $\beta \in C_\alpha$, let (A_β, B_β) be the $(\omega_1, \mathrm{cf}(\kappa) + \omega_2)$-cut of $(\omega^\omega/\mathcal{D}_\alpha)^*$ defined by

$$A_\beta = \{g/\mathcal{D}_\alpha \in (\omega^\omega/\mathcal{D}_\alpha)^* | g/\mathcal{D}_\alpha < \psi_\beta(t) \quad \text{for some } t \in \omega_1\}$$

and

$$B_\beta = \{g/\mathcal{D}_\alpha \in (\omega^\omega/\mathcal{D}_\alpha)^* | g/\mathcal{D}_\alpha > \psi_\beta(t) \quad \text{for some } t \in J_\beta\}.$$

Then Theorem 3.8 implies that $(A_\beta, B_\beta) \neq (A_\gamma, B_\gamma)$ for all $\beta \neq \gamma \in C_\alpha$. Since the following result implies that the number of $(\omega_1, \mathrm{cf}(\kappa) + \omega_2)$-cuts of $(\omega^\omega/\mathcal{D}_\alpha)^*$ is at most $2^\omega = \kappa$, it follows that $|C_\alpha| \leqslant \kappa$. This implies that there exists a subset $W \subseteq 2^\kappa$ of cardinality 2^κ such that

$$(\omega^\omega/\mathcal{D}_\alpha)^* \not\approx_i (\omega^\omega/\mathcal{D}_\beta)^*$$

for all $\alpha \neq \beta \in W$.

While the basic idea of Theorem 3.8 is implicitly contained in Section VIII.0 of Shelah [27] and Chapter III of Shelah [28], the result does not seem to have been explicitly stated anywhere in the literature.

Theorem 3.9. *Suppose that I is a linear order and that $\theta \neq \lambda$ are regular cardinals. Then the number of (λ, θ)-cuts of I is at most $|I|$.*

Proof. We shall just consider the case when $\lambda < \theta$. Suppose that I is a counter-example of minimal cardinality and let $\{(A_i, B_i) \mid i < |I|^+\}$ be a set of $|I|^+$ distinct

(λ, θ)-cuts of I. Let $\mathrm{cf}(|I|) = \kappa$ and express $I = \cup_{\gamma<\kappa} I_\gamma$ as a smooth strictly increasing union of substructures such that $|I_\gamma| < |I|$ for all $\gamma < \kappa$.

First suppose that $\kappa \neq \theta, \lambda$. Then for each $i < |I|^+$, there exists an ordinal $\gamma_i < \kappa$ such that $A_i \cap I_{\gamma_i}$ is cofinal in A_i and $B_i \cap I_{\gamma_i}$ is coinitial in B_i. It follows that there exists a subset $X \subseteq |I|^+$ of cardinality $|I|^+$ and a fixed ordinal $\gamma < \kappa$ such that $\gamma_i = \gamma$ for all $i \in X$. But this means that $\{(A_i \cap I_\gamma, B_i \cap I_\gamma) \mid i < X\}$ is a set of $|I|^+$ distinct (λ, θ)-cuts of I_γ, which contradicts the minimality of $|I|$.

Next suppose that $\kappa = \lambda$. Once again, for each $i < |I|^+$, there exists an ordinal $\gamma_i < \kappa$ such that $B_i \cap I_{\gamma_i}$ is coinitial in B_i; and there exists a subset $X \subseteq |I|^+$ of cardinality $|I|^+$ and a fixed ordinal $\gamma < \kappa$ such that $\gamma_i = \gamma$ for all $i \in X$. Arguing as in the previous paragraph, we can suppose that for each $i \in X$, $A_i \cap I_\gamma$ is *not* cofinal in A_i. For each $i \in X$, choose an element $a_i \in A_i \backslash I_\gamma$ such that $s < a_i < t$ for all $s \in A_i \cap I_\gamma$ and $t \in B_i$. Suppose that $i \neq j \in X$. Then we can suppose that $B_i \subsetneq B_j$. Since $B_j \cap I_\gamma$ is coinitial in B_j, it follows that there exists an element

$$c \in (B_j \backslash B_i) \cap I_\gamma \subseteq A_i \cap I_\gamma.$$

But this means that $a_j < c < a_i$ and so $\{a_i \mid i \in X\}$ is a set of $|I|^+$ distinct elements of I, which is a contradiction. A similar argument handles the case when $\kappa = \theta$. This completes the proof of Theorem 3.9. $\square$

4. Constructing ultrafilters

In this section, we shall prove Theorem 3.3. Our construction of the required set $\{\mathcal{D}_\alpha \mid \alpha < 2^{2^\omega}\}$ of nonprincipal ultrafilters makes use of the techniques developed in Section VI.3 of Shelah [27].

Definition 4.1. Let D be a filter over ω and let

$$I_D = \{X \subseteq \omega \mid \omega \backslash X \in D\}$$

be the corresponding dual ideal. If A, $B \subseteq \omega$, then we define

$$A \subset B \bmod D \quad \text{if and only if} \quad A \backslash B \in I_D$$

and

$$A = B \bmod D \quad \text{if and only if} \quad (A \backslash B) \cup (B \backslash A) \in I_D.$$

Definition 4.2. Suppose that D is a filter over ω and that $\mathcal{G} \subseteq \omega^\omega$ is a family of surjective functions. Then $\mathcal{G}$ is *independent* mod D if for all distinct $g_1, \ldots, g_\ell \in \mathcal{G}$ and all (not necessarily distinct) $j_1, \dots, j_\ell \in \omega$,

$$\{n \in \omega \mid g_k(n) = j_k \quad \text{forall } 1 \leqslant k \leqslant \ell\} \neq \emptyset \text{ mod } D.$$

(Of course, this condition implies that D is a nontrivial filter.)

Suppose that $\mathcal{G}$ is independent mod D and that $|\mathcal{G}| = \kappa$. Let I be any linear order of cardinality κ and suppose that $\mathcal{G} = \{f_t \mid t \in I\}$ is indexed by the elements of I. For each $s < t \in I$, let

$$B_{s,t} = \{n \in \omega \mid f_s(n) < f_t(n)\}.$$

Then it is easily checked that $D \cup \{B_{s,t} \mid s < t \in I\}$ generates a nontrivial filter D^+. (For example, see the proof of Lemma 4.7.) It follows that if $\mathcal{D} \supseteq D^+$ is an ultrafilter, then we can define an order-preserving map $\varphi : I \to \omega^\omega/\mathcal{D}$ by $\varphi(t) = f_t/\mathcal{D}$. However, if we wish φ to be an invariant embedding, then we need to be able to control the behaviour of arbitary elements $g/\mathcal{D} \in \omega^\omega/\mathcal{D}$. The next few paragraphs will introduce the techniques which will enable us to accomplish this.

Definition 4.3. Suppose that $\mathcal{G} \subseteq \omega^\omega$ is a family of surjective functions.

(a) $FI(\mathcal{G})$ is the set of functions h satisfying the following conditions:
 (i) $\operatorname{dom} h$ is a finite subset of $\mathcal{G}$;
 (ii) $\operatorname{ran} h \subset \omega$.
(b) For each $h \in FI(\mathcal{G})$, let

$$A_h = \{n \in \omega \mid g(n) = h(g) \quad \text{for all } g \in \operatorname{dom} h\}.$$

(c) $\mathrm{FI}_s(\mathcal{G}) = \{A_h \mid h \in FI(\mathcal{G})\}$.

Lemma 4.4. *Suppose that D is a filter over ω and that $\mathcal{G} \subseteq \omega^\omega$ is a family of surjective functions.*

(a) *$\mathcal{G}$ is independent* mod D *if and only if $A_h \neq \emptyset$* mod D *for every $h \in FI(\mathcal{G})$.*
(b) *If $\mathcal{G}$ is independent* mod D, *then there exists a maximal filter $D^* \supseteq D$ modulo which $\mathcal{G}$ is independent.*
(c) *If $\mathcal{G}$ is independent* mod D *and $X \subseteq \omega$, then there exists a finite subset $\mathcal{F} \subseteq \mathcal{G}$ such that $\mathcal{G} \setminus \mathcal{F}$ is independent modulo either the filter generated by $D \cup \{X\}$ or the filter generated by $D \cup \{\omega \setminus X\}$.*

Proof. These are the statements of Claims 3.15(4), 3.15(3) and 3.3 from Shelah [27, Chapter VI]. □

Now suppose that $\mathcal{G} \subseteq \omega^\omega$ is a family of surjective functions and that D is a *maximal* filter over ω modulo which $\mathcal{G}$ is independent. Then $\mathcal{A}$ is said to be a *partition* mod D if the following conditions are satisfied:

- $A \neq \emptyset$ mod D for all $A \in \mathcal{A}$;

- $A \cap A' = \emptyset \bmod D$ for all $A \neq A' \in \mathcal{A}$;
- for all $B \in \mathcal{P}(\omega)$ with $B \neq \emptyset \bmod D$, there exists $A \in \mathcal{A}$ such that $A \cap B \neq \emptyset \bmod D$.

The subset $B \subseteq \omega$ is said to be *based on* $\mathcal{A}$ if for every $A \in \mathcal{A}$, either $A \subseteq B \bmod D$ or $A \cap B = \emptyset \bmod D$. By Claim 3.17(1) [27, Chapter VI], for every subset $B \subseteq \omega$, there exists a partition $\mathcal{A}$ mod D such that

(i) B is based on $\mathcal{A}$; and
(ii) $\mathcal{A} \subseteq \mathrm{FI}_s(\mathcal{G})$.

Furthermore, by Claim 3.17(5) [27, Chapter VI], $\mathcal{A}$ is necessarily countable and so there exists a countable subset $\mathcal{G}_0 \subseteq \mathcal{G}$ such that $\mathcal{A} \subseteq \mathrm{FI}_s(\mathcal{G}_0)$. In this case, we say that B is *supported by* $\mathrm{FI}_s(\mathcal{G}_0)$ mod D.

The next lemma summarises the properties of supports that we shall require later in this section.

Lemma 4.5. *Suppose that $\mathcal{G} \subseteq \omega^\omega$ is a family of surjective functions and that D is a maximal filter over ω modulo which $\mathcal{G}$ is independent.*

(a) *$\mathrm{FI}_s(\mathcal{G})$ is dense mod D, i.e. for every $B \subseteq \omega$ with $B \neq \emptyset \bmod D$, there exists $A_h \in \mathrm{FI}_s(\mathcal{G})$ such that $A_h \subseteq B \bmod D$.*
(b) *For each $B \subseteq \omega$, there exists a countable subset $\mathcal{G}_0 \subseteq \mathcal{G}$ such that B is supported by $\mathrm{FI}_s(\mathcal{G}_0)$ mod D.*
(c) *Suppose that $\mathcal{G} = \mathcal{G}_1 \sqcup \mathcal{G}_2$ and that $A \subseteq \omega$ is supported by $\mathrm{FI}_s(\mathcal{G}_1)$ mod D. If $h \in FI(\mathcal{G})$ and $A_h \subseteq A \bmod D$, then $A_{h_1} \subseteq A \bmod D$, where $h_1 = h \restriction \mathcal{G}_1$.*

Proof. We have already discussed clause (b). Clauses (a) and (c) are the statements of Claims 3.17(1) and 3.17(4) from Shelah [27, Chapter VI]. □

The following lemma will ensure that our construction concentrates on the initial segments of $(\omega^\omega/\mathcal{D})^*$.

Lemma 4.6. *Suppose that $\mathcal{G} \subseteq \omega^\omega$ is a family of surjective functions and that D is a maximal filter over ω modulo which $\mathcal{G}$ is independent. Suppose also that $g \in \omega^\omega$ is a function such that $\ell < g/D$ for every $\ell \in \omega$. Then $f/D < g/D$ for every $f \in \mathcal{G}$.*

Proof. This is Claim 3.19(1) from Shelah [27, Chapter VI]. □

Finally, the next lemma is the key to our construction of the required set of ultrafilters.

Lemma 4.7. *Suppose that $\mathcal{G} \sqcup \mathcal{G}^* \subseteq \omega^\omega$ is a family of surjective functions and that D is a maximal filter over ω modulo which $\mathcal{G} \sqcup \mathcal{G}^*$ is independent. Suppose that I is a linear order and that $\mathcal{G} = \{f_t \mid t \in I\}$ is indexed by the elements of I. Then there exists a filter $D^+ \supseteq D$ over ω which satisfies the following conditions*:

(a) *If $s \in I$ and $\ell \in \omega$, then $\ell < f_s/D^+$.*

(b) *If* $s<t\in I$, *then* $f_s/D^+<f_t/D^+$.

(c) *Suppose that* (I_1, I_2) *is a* (λ, θ)*-cut of* I *such that* λ, $\theta>\omega$. *Then for every ultrafilter* $\mathcal{U}\supseteq D^+$ *over* ω, *there does* not *exist a function* $g\in\omega^\omega$ *such that*

$$f_s/\mathcal{U}<g/\mathcal{U}<f_t/\mathcal{U}$$

for all $s\in I_1$, $t\in I_2$.

(d) D^+ *is a maximal filter over* ω *modulo which* $\mathcal{G}^*$ *is independent*.

Proof. For each $t\in I$ and $\ell\in\omega$, let

$$A_{\ell,t}=\{n\in\omega \mid \ell<f_t(n)\}.$$

For each pair $s<t\in I$, let

$$B_{s,t}=\{n\in\omega \mid f_s(n)<f_t(n)\}.$$

Finally for each pair $r<s\in I$ and each function $g\in\omega^\omega$ such that $g^{-1}(\ell)$ is supported by $\mathrm{FI}_s(\mathcal{G}^*\sqcup\{f_t \mid t\in I\backslash[r,s]\})$ mod D for all $\ell\in\omega$, let

$$C_{g,r,s}=\{n\in\omega \mid g(n)<f_r(n) \quad \text{or} \quad f_s(n)<g(n)\}.$$

Let E be the filter on ω generated by D, together with all of the sets $A_{\ell,t}$, $B_{s,t}$, $C_{g,r,s}$ defined above. (Note that the next claim implies that E is a nontrivial filter.)

Claim 4.8. *If* $h\in FI(\mathcal{G}^*)$, *then* $A_h\neq\emptyset$ mod E.

Assuming Claim 4.8, we shall now complete the proof of Lemma 4.7. Applying Lemma 4.4(a), $\mathcal{G}^*$ is independent mod E. Let $D^+\supseteq E$ be a maximal such filter.

Since $A_{\ell,t}\in D^+$ for each $t\in I$ and $\ell\in\omega$, it follows that $\ell<f_s/D^+$ and so clause 4.7(a) holds. Similarly, since $B_{s,t}\in D^+$ for each $s<t\in I$, it follows that clause 4.7(b) holds. Finally suppose that (I_1, I_2) is a (λ, θ)-cut of I such that λ, $\theta>\omega$ and that $g\in\omega^\omega$. By Lemma 4.5(b), for each $\ell\in\omega$, there exists a countable subset $\mathcal{G}_\ell\subseteq\mathcal{G}\sqcup\mathcal{G}^*$ such that $g^{-1}(\ell)$ is supported by $\mathrm{FI}_s(\mathcal{G}_\ell)$ mod D. Since λ, $\theta>\omega$, it follows that there exist $r\in I_1$ and $s\in I_2$ such that $g^{-1}(\ell)$ is supported by $\mathrm{FI}_s(\mathcal{G}^*\sqcup\{f_t \mid t\in I\backslash[r,s]\})$ mod D for all $\ell\in\omega$ and hence $C_{g,r,s}\in D^+$. This implies that if $\mathcal{U}\supseteq D^+$ is an ultrafilter over ω, then either $g/\mathcal{U}<f_r/\mathcal{U}$ or $f_s/\mathcal{U}<g/\mathcal{U}$. Thus clause 4.7(c) also holds.

Thus it only remains to prove Claim 4.8. Suppose that $h\in\mathrm{FI}(\mathcal{G}^*)$. Then it is enough to prove that

$$A_h\cap\bigcap_{i\leqslant a}A_{\ell_i,t_i}\cap\bigcap_{i<j\leqslant a}B_{t_i,t_j}\cap\bigcap_{k\leqslant b}C_{g_k,r_k,s_k}\neq\emptyset \bmod D$$

in the case when the following conditions are satisfied:

- $t_0<t_1<\cdots<t_a$.
- If $k\leqslant b$, then r_k, $s_k\in\{t_i \mid i\leqslant a\}$.

Let $\mathcal{T} = \{f_{t_i} \mid i \leqslant a\}$. We shall define a sequence of functions $h_m \in FI(\mathcal{G} \sqcup \mathcal{G}^*)$ inductively for $m \in \omega$ so that the following conditions are satisfied:

(1) $h_0 = h$ and $h_m \subseteq h_{m+1}$.
(2) $\mathrm{dom}\, h_m \cap \mathcal{T} = \emptyset$.
(3) If $h^* \in \mathrm{FI}(\mathcal{T})$ and $k \leqslant b$, then one of the following occurs for almost all m,
 (i) there exists $\ell \in \omega$ such that $A_{h_m \cup h^*} \subseteq g_k^{-1}(\ell) \bmod D$; or
 (ii) $A_{h_m \cup h^*} \cap g_k^{-1}(\ell) = \emptyset \bmod D$ for all $\ell \in \omega$.

(Clearly if (i) occurs, then there exists a *fixed* ℓ such that $A_{h_m \cup h^*} \subseteq g_k^{-1}(\ell) \bmod D$ for almost all m.) To see that the induction can be carried out, first fix an enumeration of the countably many pairs h^*, k that must be dealt with. Now suppose that h_m has been defined and that we must next deal with the pair h^*, k. There are two cases to consider. First suppose that there exists $\ell \in \omega$ such that $A_{h_m \cup h^*} \cap g_k^{-1}(\ell) \neq \emptyset \bmod D$. By Lemma 4.5(a), there exists $\tilde{h} \in \mathrm{FI}(\mathcal{G} \sqcup \mathcal{G}^*)$ such that

$$A_{\tilde{h}} \subseteq A_{h_m \cup h^*} \cap g_k^{-1}(\ell) \bmod D.$$

Clearly we must have that $h_m \cup h^* \subseteq \tilde{h}$; and in this case, we set

$$h_{m+1} = \tilde{h} \restriction ((\mathcal{G} \sqcup \mathcal{G}^*) \setminus \mathcal{T}).$$

Otherwise, we must have that $A_{h_m \cup h^*} \cap g_k^{-1}(\ell) = \emptyset \bmod D$ for all $\ell \in \omega$; and in this case, we set $h_{m+1} = h_m$.

Now fix some $k \leqslant b$. Let $r_k = t_{\imath(k)}$ and $s_k = t_{\tau(k)}$. Let

$$\mathcal{T}_k = \{f_{t_i} \mid i \notin [\imath(k), \tau(k)]\}.$$

Suppose that $h^* \in \mathrm{FI}(\mathcal{T})$. Then for all sufficiently large m, either:

(i) there exists $\ell \in \omega$ such that $A_{h_m \cup h^*} \subseteq g_k^{-1}(\ell) \bmod D$; or
(ii) $A_{h_m \cup h^*} \cap g_k^{-1}(\ell) = \emptyset \bmod D$ for all $\ell \in \omega$.

First suppose that (i) holds. Since $g_k^{-1}(\ell)$ is supported by

$$\mathrm{FI}_s(\mathcal{G}^* \sqcup \{f_t \mid t \in I \setminus [r_k, s_k]\}),$$

Lemma 4.5(c) implies that:

(i)′ there exists $\ell \in \omega$ such that $A_{h_m \cup (h^* \restriction \mathcal{T}_k)} \subseteq g_k^{-1}(\ell)$ mod D for almost all m.

Similarly, if (ii) holds, then Lemma 4.5(c) implies that:

(ii)′ for almost all m, $A_{h_m \cup (h^* \restriction \mathcal{T}_k)} \cap g_k^{-1}(\ell) = \emptyset$ mod D for all $\ell \in \omega$.

Let $\psi_k : \omega^{\mathcal{T}_k} \to \omega \cup \{\infty\}$ be the function defined by

$$\psi_k(h^* \restriction \mathcal{T}_k) = \begin{cases} \ell & \text{if (i)' holds,} \\ \infty & \text{if (ii)' holds.} \end{cases}$$

For each infinite set $W \subseteq \omega$, let $W^{(\mathcal{T})}$ be the set of functions $h^* : \mathcal{T} \to W$ such that $h^*(f_{t_i}) < h^*(f_{t_j})$ for all $i < j \leqslant a$; and for each $k \leqslant b$, let $\varphi_k : \omega^{(\mathcal{T})} \to 3$ be the function defined by

$$\varphi_k(h^*) = \begin{cases} 0 & \text{if } \psi_k(h^* \restriction \mathcal{T}_k) < h^*(f_{t_{\imath(k)}}), \\ 1 & \text{if } \psi_k(h^* \restriction \mathcal{T}_k) > h^*(f_{t_{\tau(k)}}), \\ 2 & \text{otherwise.} \end{cases}$$

By Ramsey's Theorem, there exists an infinite set $W \subseteq \omega$ such that $\varphi_k \restriction W^{(\mathcal{T})}$ is a constant function for all $k \leqslant b$.

Claim 4.9. *For each $k \leqslant b$, either $\varphi_k \restriction W^{(\mathcal{T})} \equiv 0$ or $\varphi_k \restriction W^{(\mathcal{T})} \equiv 1$.*

Proof. Suppose that $\varphi_k \restriction W^{(\mathcal{T})} \equiv 2$, so that

$$h^*(f_{t_{\imath(k)}}) \leqslant \psi_k(h^* \restriction \mathcal{T}_k) \leqslant h^*(f_{t_{\tau(k)}})$$

for all $h^* \in W^{(\mathcal{T})}$. Let $|\{j \mid \imath(k) \leqslant j \leqslant \tau(k)\}| = p$ and let $h' : \mathcal{T}_k \to W$ be a strictly increasing function such that

$$|\{w \in W \mid h'(f_{t_{\imath(k)-1}}) < w < h'(f_{t_{\tau(k)+1}})\}| = 2p.$$

Then we can extend h' to a function $h^* \in W^{(\mathcal{T})}$ such that either

$$\psi_k(h^* \restriction \mathcal{T}_k) = \psi_k(h') < h^*(f_{t_{\imath(k)}})$$

or

$$\psi_k(h^* \restriction \mathcal{T}_k) = \psi_k(h') > h^*(f_{t_{\tau(k)}}),$$

which is a contradiction. □

Choose an increasing sequence $j_0 < j_1 < \cdots < j_a$ of elements of W such that $j_i > \ell_i$ for each $i \leqslant a$; and let $h^* \in FI(\mathcal{T})$ be the function defined by $h^*(f_{t_i}) = j_i$ for each $i \leqslant a$. To complete the proof of Claim 4.8, it is enough to show that for almost all m,

$$A_{h_m \cup h^*} \subseteq A_h \cap \bigcap_{i \leqslant a} A_{\ell_i, t_i} \cap \bigcap_{i < j \leqslant a} B_{t_i, t_j} \cap \bigcap_{k \leqslant b} C_{g_k, r_k, s_k} \bmod D.$$

It is clear that $A_{h_m} \subseteq A_h$ for all m and that

$$A_{h^*} \subseteq \bigcap_{i \leqslant a} A_{\ell_i, t_i} \cap \bigcap_{i<j \leqslant a} B_{t_i, t_j}.$$

Finally let $k \leqslant b$. If $\varphi_k(h^*) = 0$, then $\psi_k(h^* \upharpoonright \mathcal{T}_k) < h^*(f_{t_{\iota(k)}}) < \infty$ and so for almost all m,

$$A_{h_m \cup h^*} \subseteq \{n \in \omega \mid g_k(n) = \psi_k(h^* \upharpoonright \mathcal{T}_k) < h^*(f_{t_{\iota(k)}}) = f_{t_{\iota(k)}}(n)\} \bmod D.$$

Similarly, if $\varphi_k(h^*) = 1$ and $\psi_k(h^* \upharpoonright \mathcal{T}_k) < \infty$, then for almost all m,

$$A_{h_m \cup h^*} \subseteq \{n \in \omega \mid f_{t_{\tau(k)}}(n) = h^*(f_{t_{\tau(k)}}) < \psi_k(h^* \upharpoonright \mathcal{T}_k) = g_k(n)\} \bmod D.$$

On the other hand, if $\varphi_k(h^*) = 1$ and $\psi_k(h^* \upharpoonright \mathcal{T}_k) = \infty$, then for almost all m,

$$A_{h_m \cup h^*} \cap \bigcup_{\ell \leqslant h^*(f_{t_{\tau(k)}})} g_k^{-1}(\ell) = \emptyset \bmod D$$

and so

$$A_{h_m \cup h^*} \subseteq \{n \in \omega \mid f_{t_{\tau(k)}}(n) = h^*(f_{t_{\tau(k)}}) < g_k(n)\} \bmod D.$$

Hence, in every case, we have that for almost all m,

$$A_{h_m \cup h^*} \subseteq C_{g_k, r_k, s_k} \bmod D.$$

This completes the proof of Lemma 4.7. □

It is now straightforward to construct a set of ultrafilters satisfying the conclusion of Theorem 3.3.

Proof of Theorem 3.3. Suppose that $2^\omega = \kappa > \omega_1$. Let $\{J_\alpha \mid \alpha < 2^\kappa\}$ be the set of linear orders given by Theorem 3.8; and for each $\alpha < \kappa$, let $I_\alpha = \omega_1 + J_\alpha$. Fix some $\alpha < \kappa$. To simplify notation, let $I_\alpha = I$. Then the corresponding ultrafilter $\mathcal{D}_\alpha = \mathcal{D}$ is constructed as follows.

Let $F_0 = \{X \subseteq \omega \mid |\omega \backslash X| < \omega\}$ be the Fréchet filter over ω. By Theorem 1.5(1) of Shelah [27, Appendix], there exists a family $\mathcal{G} \subseteq \omega^\omega$ of surjective functions of cardinality $\kappa = 2^\omega$ such that $\mathcal{G}$ is independent mod F_0. Let $\mathcal{P}(\omega) = \{X_\mu \mid \mu < \kappa\}$ be an enumeration of the powerset of ω and "enumerate" $\mathcal{G}$ as $\{f_\xi^\mu \mid \mu, \xi < \kappa\}$. We shall define by induction on $\mu < \kappa$

- a decreasing sequence of subsets $\mathcal{G}_\mu \subseteq \{f_\xi^\nu \mid \xi < \kappa \text{ and } \mu \leqslant \nu < \kappa\}$; and
- an increasing sequence of filters D_μ over ω

such that the following conditions are satisfied:

(a) $\mathcal{G}_0 = \mathcal{G}$.

(b) $|\{f_\xi^\nu \mid \xi<\kappa \text{ and } \mu\leqslant\nu<\kappa\}\setminus\mathcal{G}_\mu|\leqslant|\mu|+\omega$.
(c) D_μ is a maximal filter modulo which $\mathcal{G}_\mu$ is independent.
(d) Either $X_\mu\in D_{\mu+1}$ or $\omega\setminus X_\mu\in D_{\mu+1}$.

When $\mu=0$, we let $D_0\supseteq F_0$ be a maximal filter modulo which $\mathcal{G}_0=\mathcal{G}$ is independent. If μ is a limit ordinal, then we define $\mathcal{G}_\mu=\cap_{\nu<\mu}\mathcal{G}_\nu$ and let $D_\mu\supseteq\cup_{\nu<\mu}D_\nu$ be a maximal filter modulo which $\mathcal{G}_\mu$ is independent. Finally suppose that $\mu=\nu+1$. By Lemma 4.4, there exists a finite subset $\mathcal{F}_\nu\subseteq\mathcal{G}_\nu$ such that $\mathcal{G}_\nu\setminus\mathcal{F}_\nu$ is independent modulo either the filter generated by $D_\nu\cup\{X_\nu\}$ or the filter generated by $D_\nu\cup\{\omega\setminus X_\nu\}$. Without loss of generality, suppose that $\mathcal{G}_\nu\setminus\mathcal{F}_\nu$ is independent modulo the filter $D_\nu{}'$ generated by $D_\nu\cup\{X_\nu\}$; and let $E_\nu\supseteq D_\nu{}'$ be a maximal filter modulo which $\mathcal{G}_\nu\setminus\mathcal{F}_\nu$ is independent. Let

$$\mathcal{G}_\mu=\{f_\xi^\tau\in\mathcal{G}_\nu\setminus\mathcal{F}_\nu \mid \mu\leqslant\tau<\kappa\}.$$

Note that

$$\mathcal{H}_\nu=\{f_\xi^\tau\in\mathcal{G}_\nu\setminus\mathcal{F}_\nu \mid \tau=\nu\}$$

has cardinality κ. Hence we can re-index $\mathcal{H}_\nu$ as $\mathcal{H}_\nu=\{f_t^\nu \mid t\in I\}$. By Lemma 4.7, there exists a filter $D_\mu\supseteq E_\nu$ which satisfies the following conditions:

(1) If $s\in I$ and $\ell\in\omega$, then $\ell<f_s^\nu/D_\mu$.
(2) If $s<t\in I$, then $f_s^\nu/D_\mu<f_t^\nu/D_\mu$.
(3) Suppose that (I_1,I_2) is a (λ,θ)-cut of I such that $\lambda,\ \theta>\omega$. Then for every ultrafilter $\mathcal{U}\supseteq D_\mu$ over ω, there does *not* exist a function $g\in\omega^\omega$ such that

$$f_s^\nu/\mathcal{U}<g/\mathcal{U}<f_t^\nu/\mathcal{U}$$

for all $s\in I_1,\ t\in I_2$.
(4) D_μ is a maximal filter over ω modulo which $\mathcal{G}_\mu$ is independent.

Finally let $\mathcal{D}=\cup_{\mu<\kappa}D_\mu$. By clause (d), $\mathcal{D}$ is an ultrafilter.

Claim 4.10. *If L is an initial segment of $(\omega^\omega/\mathcal{D})^*$, then there exists $\mu<\kappa$ such that $\{f_t^\mu/\mathcal{D} \mid t\in I\}\subseteq L$.*

Proof of Claim 4.10. Let $g/\mathcal{D}\in L$. Then there exists $\mu<\kappa$ such that

$$A_\ell=\{n\in\omega \mid \ell<g(n)\}\in D_\mu$$

for all $\ell\in\omega$. Since D_μ is a maximal filter modulo which $\mathcal{G}_\mu$ is independent, Lemma 4.6 implies that $f/D_\mu<g/D_\mu$ for all $f\in\mathcal{G}_\mu$. Hence $\{f_t^\mu/\mathcal{D} \mid t\in I\}\subseteq L$. □

From now on, it is necessary to write $\mathcal{D}_\alpha$, I_α, etc.

Claim 4.11. *Fix some $\alpha < 2^\kappa$. Then the set*

$$E_\alpha = \{\beta < 2^\kappa \mid (\omega^\omega/\mathcal{D}_\alpha)^* \approx_i (\omega^\omega/\mathcal{D}_\beta)^*\}$$

has cardinality at most κ.

Proof of Claim 4.11. Suppose that $|E_\alpha| \geqslant \kappa^+$. For each $\beta \in E_\alpha$, let L_β, M_β be initial segments of $(\omega^\omega/\mathcal{D}_\beta)^*$, $(\omega^\omega/\mathcal{D}_\alpha)^*$, respectively, such that there exists an isomorphism $\varphi_\beta : L_\beta \to M_\beta$. By Claim 4.10, for each $\beta \in E_\alpha$, there exists $\mu_\beta < \kappa$ such that

$$R_\beta = \{f_t^{\mu_\beta}/\mathcal{D}_\beta \mid t \in I_\beta\} \subseteq L_\beta.$$

Recall that $I_\beta = \omega_1 + J_\beta$. Let $S_\beta = \{f_t^{\mu_\beta}/\mathcal{D}_\beta \mid t \in \omega_1\}$ and $T_\beta = \{f_t^{\mu_\beta}/\mathcal{D}_\beta \mid t \in J_\beta\}$. Let $\theta = \mathrm{cf}(\kappa) + \omega_2$. Then $(\varphi_\beta[S_\beta], \varphi_\beta[T_\beta])$ determines the (ω_1, θ)-cut (A_β, B_β) of $(\omega^\omega/\mathcal{D}_\alpha)^*$ defined by

$$A_\beta = \{g/\mathcal{D}_\alpha \in (\omega^\omega/\mathcal{D}_\alpha)^* \mid g/\mathcal{D}_\alpha < \varphi_\beta(s) \quad \text{for some } s \in S_\beta\}$$

and

$$B_\beta = \{g/\mathcal{D}_\alpha \in (\omega^\omega/\mathcal{D}_\alpha)^* \mid g/\mathcal{D}_\alpha > \varphi_\beta(t) \quad \text{for some } t \in T_\beta\}.$$

By Theorem 3.9, there exist $\beta \neq \gamma \in E_\alpha$ such that $(A_\beta, B_\beta) = (A_\gamma, B_\gamma)$. But this is impossible, since we can define invariant coinitial embeddings of J_β, J_γ into $B_\beta = B_\gamma$ by $c \mapsto \varphi_\beta(f_c^{\mu_\beta})$ and $d \mapsto \varphi_\gamma(f_d^{\mu_\gamma})$, respectively, which contradicts Theorem 3.8. □

Clearly Claim 4.11 implies that there exists a subset $W \subseteq 2^\kappa$ of cardinality 2^κ such that

$$(\omega^\omega/\mathcal{D}_\alpha)^* \not\approx_i (\omega^\omega/\mathcal{D}_\beta)^*$$

for all $\alpha \neq \beta \in W$. This completes the proof of Theorem 3.3. □

5. Asymptotic cones under CH

In this section, we shall prove Theorem 1.10. Let Γ be an infinite finitely generated group and let d be the word metric with respect to some finite generating set. Then the main point is that each asymptotic cone $\mathrm{Con}_{\mathcal{D}}(\Gamma)$ can be uniformly constructed from an associated ultraproduct

$$\prod \mathcal{M}_n/\mathcal{D},$$

where each $\mathcal{M}_n$ is a suitable countable structure for a fixed countable first-order language $\mathcal{L}$. If CH holds, then $\prod \mathcal{M}_n/\mathcal{D}$ is a saturated structure of cardinality ω_1 and hence is determined up to isomorphism by its complete first-order theory $T_{\mathcal{D}}$. Consequently, if CH holds, then since there are at most 2^{ω} possibilities for $T_{\mathcal{D}}$, there are also at most 2^{ω} possibilities for $\prod \mathcal{M}_n/\mathcal{D}$ and hence also for $\mathrm{Con}_{\mathcal{D}}(\Gamma)$.

Definition 5.1. Let $\mathcal{L}$ be the first-order language consisting of the following symbols:

(a) the binary relation symbol R_q for each $0<q\in\mathbb{Q}$; and
(b) the constant symbol e.

Definition 5.2. For each $n\geqslant 1$, $\mathcal{M}_n$ is the $\mathcal{L}$-structure with universe Γ such that:

(a) $\mathcal{M}_n \models R_q(x,y)$ if and only if $d(x,y)\leqslant qn$; and
(b) e is the identity element of Γ.

Definition 5.3. For each nonprincipal ultrafilter $\mathcal{D}$ over ω, let $T_{\mathcal{D}}$ be the complete first-order theory of $\prod \mathcal{M}_n/\mathcal{D}$.

Theorem 5.4. *If $\mathcal{D}$, $\mathcal{D}'$ are nonprincipal ultrafilters over ω, then the following are equivalent*:

(a) $\prod \mathcal{M}_n/\mathcal{D} \cong \prod \mathcal{M}_n/\mathcal{D}'$.
(b) $\mathrm{Con}_{\mathcal{D}}(\Gamma)$ *is isometric to* $\mathrm{Con}_{\mathcal{D}'}(\Gamma)$.

Proof. We shall begin by describing how the asymptotic cone $\mathrm{Con}_{\mathcal{D}}(\Gamma)$ can be uniformly constructed from the ultraproduct

$$\prod \mathcal{M}_n/\mathcal{D} = \langle X; R_q, e\rangle.$$

First define $\mathcal{M}^0_{\mathcal{D}}$ to be the set of those $x\in X$ such that

$$\prod \mathcal{M}_n/\mathcal{D} \models R_q(x,e)$$

for some $q>0$. Next define an equivalence relation $\approx$ on $\mathcal{M}^0_{\mathcal{D}}$ by

$$x\approx y \quad \text{if and only if} \quad \prod \mathcal{M}_n/\mathcal{D} \models R_q(x,y) \quad \text{for all } q>0.$$

For each $x\in\mathcal{M}^0_{\mathcal{D}}$, let $\langle x\rangle$ denote the corresponding $\approx$-class and let

$$C_{\mathcal{D}} = \{\langle x\rangle \mid x\in\mathcal{M}^0_{\mathcal{D}}\}.$$

Then we can define a metric $d_{\mathcal{D}}$ on $C_{\mathcal{D}}$ by

$$d_{\mathcal{D}}(\langle x\rangle, \langle y\rangle) = \inf\{q \mid \prod \mathcal{M}_n/\mathcal{D} \models R_q(x,y)\}.$$

It is easily checked that $\langle C_{\mathcal{D}}, d_{\mathcal{D}}\rangle$ is isometric to the asymptotic cone $\mathrm{Con}_{\mathcal{D}}(\Gamma)$. Consequently, if $\prod \mathcal{M}_n/\mathcal{D} \cong \prod \mathcal{M}_n/\mathcal{D}'$, then $\mathrm{Con}_{\mathcal{D}}(\Gamma)$ is isometric to $\mathrm{Con}_{\mathcal{D}'}(\Gamma)$.

It is also easily checked that $\prod \mathcal{M}_n/\mathcal{D}$ consists of 2^ω disjoint isomorphic copies of $\mathcal{M}^0_\mathcal{D}$ and that each $\approx$-class has cardinality 2^ω. It follows that any isometry between $\mathrm{Con}_\mathcal{D}(\Gamma)$ and $\mathrm{Con}_{\mathcal{D}'}(\Gamma)$ can be lifted to a corresponding isomorphism between $\prod \mathcal{M}_n/\mathcal{D}$ and $\prod \mathcal{M}_n/\mathcal{D}'$. □

Corollary 5.5. *Assume CH. If* $\mathcal{D}$, $\mathcal{D}'$ *are nonprincipal ultrafilters over* ω, *then the following are equivalent.*

(a) $T_\mathcal{D} = T_{\mathcal{D}'}$.
(b) $\mathrm{Con}_\mathcal{D}(\Gamma)$ *is isometric to* $\mathrm{Con}_{\mathcal{D}'}(\Gamma)$.

Proof. As we mentioned earlier, if CH holds, then $\prod \mathcal{M}_n/\mathcal{D}$ is a saturated structure of cardinality ω_1 and hence is determined up to isomorphism by its complete first-order theory $T_\mathcal{D}$. (For example, see [10].) □

This completes the proof of Theorem 1.10.

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