Geometriae Dedicata **92:** 145–178, 2002. 145

Loop Groups and Twin Buildings

Dedicated to John Stallings on the occasion of his 65th birthday

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(Received: 15 November 2000; in final form: 11 April 2001)

Abstract. In these notes we describe some buildings related to complex Kac–Moody groups. First we describe the spherical building of $\mathrm{SL}_n(\mathbb{C})$ (i.e. the projective geometry $\mathrm{PG}(\mathbb{C}^n)$) and its Veronese representation. Next we recall the construction of the affine building associated to a discrete valuation on the rational function field $\mathbb{C}(z)$. Then we describe the same building in terms of complex Laurent polynomials, and introduce the Veronese representation, which is an equivariant embedding of the building into an affine Kac–Moody algebra. Next, we introduce topological twin buildings. These buildings can be used for a proof which is a variant of the proof by Quillen and Mitchell, of Bott periodicity which uses only topological geometry. At the end we indicate very briefly that the whole process works also for affine real almost split Kac–Moody groups.

Mathematics Subject Classifications (2000). 51E24, 51H15, 22E67, 53C42.

Key words. Bott periodicity, isoparametric submanifolds, loop groups, polar representations, topological buildings, twin buildings.

1. Introduction

We briefly recall the definition of Coxeter complexes and buildings; for more details, we refer to the books by Brown [8], Ronan [35], Scharlau [41], and Tits [48]. For our purposes, a *simplicial complex* is poset $(\Delta, \leqslant)$ whose elements are called simplices, with the following two properties: any two simplices $X, Y \in \Delta$ have a unique infimum $X \sqcap Y$, and for any $X \in \Delta$, the poset $\Delta_{\leqslant X} = \{Y \in \Delta | Y \leqslant X\}$ is order-isomorphic to the power set $(2^F, \subseteq)$ of some finite set F; the cardinality of this set F is called the *rank* of the simplex X. Note that the rank of a simplex differs by one from the dimension of its geometric realization; a k-simplex has rank $k+1$. The rank of a simplicial complex is the maximum of the ranks of its simplices.

COXETER COMPLEXES

Let I be a finite set with r elements, and let (m_{ij}) be a symmetric matrix indexed by $I \times I$, with entries in $\mathbb{N} \cup \{\infty\}$, subject to the following two conditions: $m_{ij} \geqslant 2$ for all $i \neq j$, and $m_{ii} = 1$ for all i. Such a Coxeter matrix is determined by its Coxeter graph;

*Supported by a Heisenberg fellowship by the Deutsche Forschungsgemeinschaft.

the vertices of this graph are the elements of I, and two vertices i, j are joined by $m_{ij} - 2$ edges, or by one edge labeled m_{ij}. The corresponding *Coxeter system* (W, S) is the group W with generating set $S = \{s_i | \ i \in I\}$ and relators $(s_i s_j)^{m_{ij}}$. Associated to such a Coxeter system is a simplicial complex, the *Coxeter complex* $\Sigma = \Sigma(W, S)$ which is defined as follows. For $J \subseteq I$ let W_J denote the subgroup generated by the elements s_j, for $j \in J$. The simplices of Σ are the cosets $wW_J \in W/W_J$, where J runs over all subsets of I; the partial ordering is the reversed inclusion, i.e.

$$gW_J \leqslant hW_{J'} \text{ if and only if } gW_J \supseteq hW_{J'}.$$

The group W acts regularly on the maximal simplices of this simplicial complex (by left translations). The *type* of a coset wW_J is $\mathrm{type}(wW_J) = I \setminus J$; a subset $J \subseteq I$ is called *spherical* if W_J is finite; if I itself is spherical, then Σ is called spherical. The geometric realization of a spherical Coxeter complex of rank k is a triangulated $k-1$-sphere. We define a W-invariant double-coset valued distance function δ on Σ as follows:

$$\delta\colon \Sigma \times \Sigma \longrightarrow \bigcup \{W_J \backslash W / W_K | \ J, K \subseteq I\},$$

$$\delta(uW_J, vW_K) = W_J u^{-1} v W_K.$$

Coxeter groups have nice geometric properties; in particular, the word problem can be solved. Let A be an Abelian group, and let $a\colon S \longrightarrow A$ be a function. We require that $a(s_i) = a(s_j)$ holds whenever m_{ij} is finite and odd. Then there is a well-defined extension $a\colon W \longrightarrow A$, which is defined as $a(s_{i_1} \ldots s_{ir}) = a(s_{i_1}) + \cdots + a(s_{i_r})$, for a *reduced* (minimal) expression $w = s_{i_1} \ldots s_{i_r}$, the *a-length*. In the special case $A = \mathbb{Z}$, with $a(s_i) = 1$ for all i, we obtain the usual *length function* $\ell\colon W \longrightarrow \mathbb{Z}$. In general, the set of generators S is not uniquely determined by the abstract group W, so it is important to consider the pair (W, S); the question to which extent S is determined by the group W is treated in [6] and [30] see also [10] for related results.

BUILDINGS

Let $\Delta \neq \emptyset$ be a simplicial complex, and let $\Sigma = \Sigma(W, S)$ be a Coxeter complex. A simplicial injection $\phi\colon \Sigma \longrightarrow \Delta$ is called a *chart*, and its image $A = \phi(\Sigma)$ is called an *apartment*. The complex Δ is called a *building* (of type (m_{ij}) and rank r) if there exists a collection $\mathcal{A}$ of apartments with the following properties.

Bld$_1$ For any two simplices $X, Y \in \Delta$, there exists an apartment $A \in \mathcal{A}$ with $X, Y \in A$.

Bld$_2$ Given two charts $\phi_i\colon \Sigma \hookrightarrow \Delta$, for $i = 1, 2$, there exists an element $w \in W$ such that $\phi_1 \circ w(X) = \phi_2(X)$ holds for all $X \in \phi_2^{-1}(\phi_1(\Sigma) \cap \phi_2(\Sigma))$.

A simplex of maximal rank r is also called a *chamber*; the set of all chambers is $\mathrm{Cham}(\Delta)$. For any subset $X \subseteq \Delta$, we let $\mathrm{Cham}(X)$ denote the set of all chambers contained in X. The building is *thick* if every simplex of rank $r-1$ is contained in at least three chambers. *All buildings in this paper will be thick.* A *spherical building* is a building with finite apartments.

It follows from the axioms that there is a well-defined double-coset valued *distance function*

$$\delta\colon \Delta\times\Delta \longrightarrow \bigcup\{W_J\backslash W/W_K \mid J, K\subseteq I\}$$

whose restriction to any apartment is given by the function δ defined above in 1.1. The restriction of δ to $\mathrm{Cham}(\Delta)\times\mathrm{Cham}(\Delta)$ is Tits' more familiar W-valued distance function; buildings can also be characterized by properties of the distance function δ, see Ronan's book [35].

Let $\blacktriangle^{r-1}$ denote the standard $r-1$-simplex $(2^I, \subseteq)$. The two axioms yield a simplicial surjection, the *type function* (the 'accordion map') $\mathrm{type}\colon \Delta \longrightarrow \blacktriangle^{r-1}$, whose restriction to any apartment agrees with the type function defined above. The type function is characterized (up to automorphisms of $\blacktriangle^{r-1}$) by the fact that its restriction to every simplex of Δ is injective. A simplex $X\in\Delta$ is called *spherical* if $I\backslash\mathrm{type}(X)$ is spherical.

For nonspherical buildings the apartment system $\mathcal{A}$ is in general not unique (there is always a unique *maximal* apartment system), but the isomorphism type of the apartments and the Coxeter system (W, S) is uniquely determined by the simplicial complex Δ.

AUTOMORPHISMS

The automorphism group $\mathrm{Aut}(\Delta)$ consists of all simplicial automorphisms of Δ; it has a normal subgroup $\mathrm{Spe}(\Delta)$ consisting of all type-preserving automorphisms. An *action* of a group G on Δ is a homomorphism $G\longrightarrow \mathrm{Aut}(\Delta)$. We say that G acts *transitively* on Δ if it acts transitively on the set $\mathrm{Cham}(\Delta)$ of chambers.

Strongly transitive actions and BN-pairs. A group G is said to act *strongly transitively* on Δ (with respect to $\mathcal{A}$) if G acts as a group of special automorphisms on Δ such that

STA$_1$ G acts transitively on the set $\mathcal{A}$ of apartments.

STA$_2$ If $A\in\mathcal{A}$ is an apartment, then the set-wise stabilizer N of A acts transitively on the chambers in A.

Let $C\in A$ be a chamber in an apartment A, let $B=G_C$ denote the stabilizer of C, and N the set-wise stabilizer of A. Then (B, N) is a so-called BN-pair for the group G; the Weyl group $W=N/(N\cap B)$ acts regularly on A and is isomorphic to the Coxeter group of Δ. The stabilizers of the simplices contained in C are called *standard parabolic subgroups* of G.

PANELS AND RESIDUES

Let $X \in \Delta$ be a simplex of type J. The *residue* of X is the poset $\mathrm{Res}(X) = \Delta_{\geqslant X} = \{Y \in \Delta |\ Y \geqslant X\}$. The residue is order-isomorphic to the *link* of X, and thus can be identified with a subcomplex of Δ (although strictly speaking, $\mathrm{Res}(X)$ is not a subcomplex). It is a basic but important fact that residues are again buildings; the corresponding Coxeter complex is obtained from the restricted Coxeter matrix $(m_{i,j})_{J\times J}$. The *type* of the residue $\mathrm{Res}(X)$ is $I\setminus J$, and its rank is card $(I\setminus J)$. A residue $S_i = \mathrm{Res}(X)$ of rank 1 and type $\{i\}$ is called an *i-panel*.

The following observation is very simple, but useful.

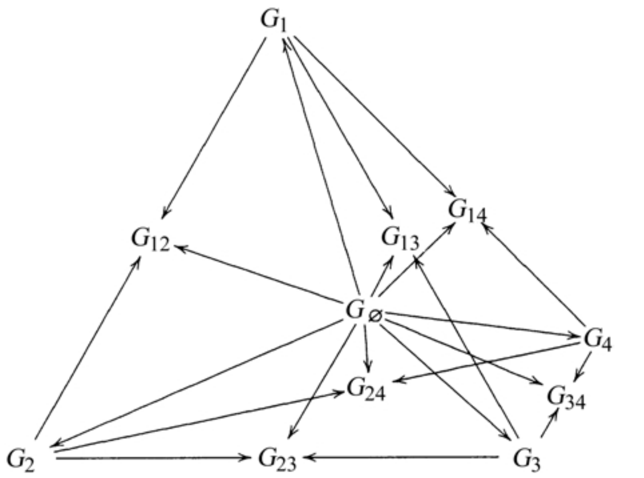
LEMMA 1.5. *Suppose that G acts as a group of special automorphisms on a building Δ of rank $r \geqslant 2$; let $C \in \Delta$ be a chamber. Then G acts transitively on Δ if and only if the following holds for all r simplices $X_1, \dots, X_r \leqslant C$ of rank $r-1$:*

- *the stabilizer G_{X_i} acts transitively on the panel* $\mathrm{Res}(X_i)$.

Tits pointed out that transitive groups acting on buildings of higher rank can be represented as amalgams.

THEOREM 1.6 ([49] 2.3). *Let G be a group acting transitively and type-preservingly on an irreducible building Δ of rank $r \geqslant 3$ (i.e. we assume that the Coxeter diagram is connected and has at least 3 vertices). Let X be a chamber, let $X_1, \dots, X_r$, denote its subsimplices of rank $r-1$, and $X_{\{ij\}}$, $i, j \in I$, $i \neq j$ its $\binom{r}{2}$ subsimplices of rank $r-2$. Then the $\binom{r+1}{2} + 1$ different G-stabilizers of these simplices with their natural inclusions form a diagram (a simple 2-complex of groups* ([7], II.12 and III.C)*— the corresponding poset is the set of all subsets of I with at most two elements) whose limit is G.* □

For example, a building of rank 4 with $I = \{1, 2, 3, 4\}$ yields a complex of groups as follows, if we put $G_{X_J} = G_J$ for short.



In general, the geometric realization of this 2-complex of groups is the cone over the first barycentric subdivision of the 1-skeleton of an $r-1$-simplex.

For group amalgamations related to twin buildings (cf. Section 4), see [2].

2. The Spherical Case: Projective Space and $\mathrm{SL}_n(\mathbb{C})$

In this section we describe the spherical building obtained from $n-1$-dimensional complex projective space, some related groups, and the Veronese representation. Given a ring R, we let $R(k)$ denote the matrix algebra of all $k \times k$-matrices with entries in R.

THE SPHERICAL BUILDING $\Delta(\mathbb{C}^n)$

The *Grassmannian* of k-spaces in $\mathbb{C}^n$ is the complex projective variety

$$\mathrm{Gr_k}(\mathbb{C}^n) = \{U \leqslant \mathbb{C}^n |\ \dim(U) = k\}.$$

A *partial flag* in $\mathbb{C}^n$ is a nested sequence of subspaces $0 < U_{j_1} < U_{j_2} < \cdots < U_{j_r} < \mathbb{C}^n$, with $\dim(U_{j_\nu}) = j_\nu$. Such a partial flag can be viewed as a map

$$\{1, \ldots, n-1\} \supseteq J \stackrel{U}{\longrightarrow} \mathrm{Gr}_1(\mathbb{C}^n) \cup \cdots \cup \mathrm{Gr}_{n-1}(\mathbb{C}^n),$$

with $\dim(U_j) = j$ and $U_j \leqslant U_k$ for $j \leqslant k$. There is a natural order '$\leqslant$' on the collection $\Delta(\mathbb{C}^n)$ of all partial flags: if U and U' are partial flags with $J \subseteq J' \subseteq \{1, \ldots, n-1\}$, then $U \leqslant U'$ if and and only if $U'|_J = U$. The resulting poset is a simplicial complex of rank $n-1$, the spherical building

$$\Delta(\mathbb{C}^n) = (\Delta(\mathbb{C}^n), \leqslant).$$

Of course, $\Delta(\mathbb{C}^n)$ is precisely the same as the $n-1$-dimensional projective geometry $\mathrm{PG}(\mathbb{C}^n)$ in a different guise.

Given an ordered basis $(v_1, \ldots, v_n)$ of $\mathbb{C}^n$, we may consider the maximal flag $U = U^{\mathbb{C}}(v_1, \ldots, v_n)$, where $U_i = \mathrm{span}_{\mathbb{C}}\{v_1, \ldots, v_i\}$, and the apartment $A^{\mathbb{C}}\{v_1, \ldots, v_n\}$ consisting of all flags obtained as partial flags from the $n!$ distinct maximal flags (chambers) $U^{\mathbb{C}}(v_{\pi(1)}, \ldots, v_{\pi(n)})$, where $\pi \in \mathrm{Sym}(n)$. It is not difficult to check that $\Delta(\mathbb{C}^n)$ together with this collection of apartments is a spherical building. As a simplicial complex, $A^{\mathbb{C}}\{v_1, \ldots, v_n\}$ is a triangulation of the sphere $\mathbb{S}^{n-2}$. In fact, let $\blacktriangle^{n-1}$ denote the standard $n-1$-simplex, and $\mathrm{Bd}(\blacktriangle^{n-1})$ its boundary; then $A^{\mathbb{C}}\{v_1, \ldots, v_n\}$ is simplicially isomorphic to the first barycentric subdivision $\Sigma = \mathrm{Sd}(\mathrm{Bd}\ (\blacktriangle^{n-1}))$. The corresponding Coxeter group is the symmetric group $W = \mathrm{Sym}(n)$, given by the presentation

$$W = \langle s_1, \ldots, s_{n-1} |\ (s_i s_j)^{m_{ij}} = 1,\ 1 \leqslant i, j \leqslant n-1\rangle,$$

$$\text{where } m_{ij} = \begin{cases} 1, & \text{if } i = j, \\ 3, & \text{if } |i-j| = 1, \\ 2, & \text{else.} \end{cases}$$

The involution s_i is the transposition $(i, i+1)$; the Coxeter diagram is

A_{n-1} :

$(n-1$ nodes). The type of a flag

$$U\colon J \longrightarrow \mathrm{Gr}_1(\mathbb{C}^n) \cup \cdots \cup \mathrm{Gr}_{n-1}(\mathbb{C}^n)$$

is type $(U) = J$, the set of the dimensions of the subspaces occurring in U. In $\Delta(\mathbb{C}^n)$, a panel S_i is determined by a flag of the form

$$U_1 < U_2 < \cdots < U_{i-1} < U_{i+1} < \cdots U_{n-1},$$

and there is a natural bijection $S_i \longrightarrow \mathrm{Gr}_1(U_{i+1}/U_{i-1}) \cong \mathbb{C}\mathrm{P}^1 = \mathbb{C} \cup \{\infty\}$ onto a projective line.

PARABOLICS IN $\mathrm{SL}_n(\mathbb{C})$

There are natural actions of the groups $\mathrm{GL}_n(\mathbb{C})$ and $\mathrm{SL}_n(\mathbb{C})$ on the building $\Delta(\mathbb{C}^n)$ (by type preserving automorphisms), and it is not difficult to see that these actions are strongly transitive. The $\mathrm{SL}_n(\mathbb{C})$-stabilizer of a partial flag is thus a parabolic subgroup. The *maximal parabolics* are the stabilizers of the flags of rank 1, i.e. of subspaces $0 < V < \mathbb{C}^n$. Such a maximal parabolic is conjugate to one of the *standard maximal parabolics*

$$P^i = \left\{ \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \middle| A \in \mathbb{C}(i),\ C \in \mathbb{C}(n-i),\ B \in \mathbb{C}^{i \times (n-i)},\ \det(A)\det(C) = 1 \right\},$$

and

$$\mathrm{Gr}_i(\mathbb{C}^n) \cong \mathrm{SL}_n(\mathbb{C})/P^i.$$

The minimal parabolics or *Borel subgroups* are the stabilizers of maximal flags, i.e. chambers in $\Delta(\mathbb{C}^n)$. The Borel subgroups are conjugate to the *standard Borel subgroup*

$$B = P^1 \cap \cdots \cap P^{n-1} = \mathrm{S}\nabla_n(\mathbb{C}) = \left\{ \begin{pmatrix} a_1 & & * \\ & \ddots & \\ 0 & & a_n \end{pmatrix} \middle| a_1 a_2 \cdots a_n = 1 \right\}$$

consisting of all unimodular upper triangular matrices. The corresponding homogeneous space is the *complex flag variety*

$$\begin{aligned} \mathrm{Fl}(\mathbb{C}^n) = \mathrm{Cham}(\Delta(\mathbb{C}^n)) = \{&(U_1, \ldots, U_{n-1}) \\ &\mid 0 < U_1 < \cdots < U_{n-1} < \mathbb{C}^n\} \cong \mathrm{SL}_n(\mathbb{C})/B. \end{aligned}$$

THE ANISOTROPIC REAL STRUCTURE OF $\mathrm{SL}_n(\mathbb{C})$

We define a semi-linear involution $*$ on $\mathbb{C}(n)$ by putting $X^* = \bar{X}^T$ (conjugate transpose). Recall that the Lie algebra $\mathfrak{sl}_n\mathbb{C}$ consists of all traceless $n \times n$-matrices. Let

$$\mathfrak{su}(n) = \{X \in \mathfrak{sl}_n\mathbb{C} | X + X^* = 0\} \quad \text{and} \quad \mathfrak{p}_n = \{X \in \mathfrak{sl}_n\mathbb{C} | X = X^*\}.$$

The decomposition $\mathfrak{sl}_n(\mathbb{C}) = \mathfrak{su}(n) \oplus \mathfrak{p}_n$ is called the *Cartan decomposition* of the Lie algebra $\mathfrak{sl}_n(\mathbb{C})$. On the group level, let

$$\mathrm{SU}(n) = \{g \in \mathrm{SL}_n(\mathbb{C}) |\ g^* = g^{-1}\}$$

denote the group of fixed elements of the involution $(g \mapsto g^{-*}) \in \mathrm{Aut}_{\mathbb{R}}(\mathrm{SL}_n(\mathbb{C}))$. Both the involution $X \mapsto -X^*$ in the Lie algebra and the involution $g \mapsto g^{-*}$ in the group are called *Cartan involutions*. (In terms of algebraic groups, we have defined an *almost simple* $\mathbb{R}$-*group scheme* $\underline{G}$ such that $\underline{G}(\mathbb{R}) = \mathrm{SU}(n)$ and $\underline{G}(\mathbb{C}) = \mathrm{SL}_n(\mathbb{C})$. The Galois group of $\mathbb{C}/\mathbb{R}$ acts on the group $\mathrm{SL}_n(\mathbb{C})$ of $\mathbb{C}$-points of $\underline{G}$, and $\mathrm{SU}(n)$ is the group of fixed elements. The group scheme $\underline{G}$ is $\mathbb{R}$-*anisotropic*: no parabolic of $\underline{G}$ is defined over $\mathbb{R}$. For semi-simple $\mathbb{R}$-algebraic groups, 'anisotropic' is the same as 'compact'.)

More geometrically, the group $\mathrm{SU}(n)$ can be described as follows. Let $\langle x, y \rangle = \bar{x}_1 y_1 + \cdots + \bar{x}_n y_n$ denote the standard hermitian form on $\mathbb{C}^n$. This form induces a map $\mathrm{Gr}_k(\mathbb{C}^n) \longrightarrow \mathrm{Gr}_{n-k}(\mathbb{C}^n)$, $V \mapsto V^\perp$, for all k, which extends in a natural way to an involution $\perp$ on $\Delta(\mathbb{C}^n)$. Then $\mathrm{SU}(n)$ is the centralizer of this involution, $\mathrm{SU}(n) = \mathrm{Cen}_{\mathrm{SL_n}(\mathbb{C})}(\perp)$.

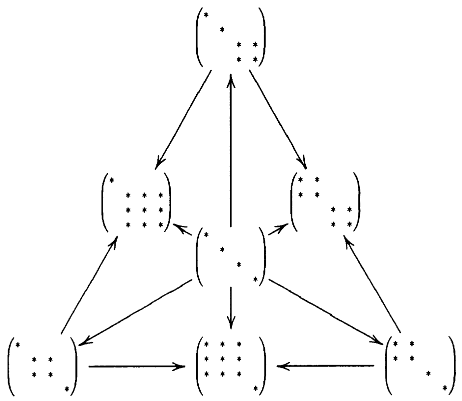
Classically, the involution $\perp$ is called the *standard elliptic polarity* on the complex projective geometry $\mathrm{PG}(\mathbb{C}^n)$; the associated Riemannian symmetric space $X = \mathrm{SL}_n(\mathbb{C})/\mathrm{SU}(n)$ is—as a subset of $\mathrm{Aut}(\Delta)$—precisely the space of all elliptic polarities (i.e. the space of all positive definite hermitian forms on $\mathbb{C}^n$).

Using Gram–Schmidt orthonormalization, one shows that $\mathrm{SU}(n)$ acts transitively on the flag variety $\mathrm{Fl}(\mathbb{C}^n)$. Bernhard Mühlherr pointed out that there is a different proof which uses Lemma 1.5, and which carries over to the Kac–Moody case described later.

LEMMA 2.1. *The group* $\mathrm{SU}(n)$ *acts transitively on* $\mathrm{Fl}(\mathbb{C}^n)$.

Proof. Let $U = U^{\mathbb{C}}(e_1, \ldots, e_n) = (U_1, \ldots, U_{n-1})$ denote the maximal flag arising from the standard basis of $\mathbb{C}^n$. The $\mathrm{SU}(n)$-stabilizer of a panel $S_i = \mathrm{Res}(U_1, \ldots, U_{i-1}, U_{i+1}, \ldots, U_{n-1})$ containing U is isomorphic to $\mathrm{SU}(2) \cdot T^{n-2}$; it induces the transitive group $\mathrm{SO}(3) = \mathrm{SU}(2) \cdot T^{n-2}/T^{n-2}$ on the panel (we let $T^k = \mathrm{U}(1) \times \cdots \times \mathrm{U}(1)$ denote the compact torus of rank k; in terms of algebraic groups, T^k is the group $\underline{T}(\mathbb{R})$ of $\mathbb{R}$-points of an anisotropic $\mathbb{R}$-torus $\underline{T}$ of rank k). The assertion follows with Lemma 1.5. □

For $n = 4$, we have by Theorem 1.6 the following complex of groups which represents $\mathrm{SU}(4)$ as an amalgam (we indicate the nonzero entries of a matrix by $\bullet$, and all matrices are assumed to be unimodular and unitary).



Here

$$\begin{pmatrix} \bullet & \bullet & & \\ \bullet & \bullet & & \\ & & \bullet & \\ & & & \bullet \end{pmatrix} \cong \begin{pmatrix} \bullet & & & \\ & \bullet & \bullet & \\ & \bullet & \bullet & \\ & & & \bullet \end{pmatrix} \cong \begin{pmatrix} \bullet & & & \\ & \bullet & & \\ & & \bullet & \bullet \\ & & \bullet & \bullet \end{pmatrix} \cong \mathrm{SU}(2) \cdot T^2,$$

$$\begin{pmatrix} \bullet & \bullet & \bullet & \\ \bullet & \bullet & \bullet & \\ \bullet & \bullet & \bullet & \\ & & & \bullet \end{pmatrix} \cong \begin{pmatrix} \bullet & & & \\ & \bullet & \bullet & \bullet \\ & \bullet & \bullet & \bullet \\ & \bullet & \bullet & \bullet \end{pmatrix} \cong \mathrm{SU}(3) \cdot T^1,$$

$$\begin{pmatrix} \bullet & \bullet & & \\ \bullet & \bullet & & \\ & & \bullet & \bullet \\ & & \bullet & \bullet \end{pmatrix} \cong \mathrm{SU}(2) \cdot \mathrm{SU}(2) \cdot T^1.$$

In group theoretic terms, we thus have

$$\mathrm{SL}_n(\mathbb{C}) = \mathrm{SU}(n)B, \qquad \mathrm{SU}(n) \cap B = T^{n-1},$$
$$\mathrm{SL}_n(\mathbb{C}) = \mathrm{SU}(n)P^i, \qquad \mathrm{SU}(n) \cap P^i \cong \mathrm{S}(\mathrm{U}(i) \times \mathrm{U}(n-i).$$

In particular,

$$\mathrm{Fl}(\mathbb{C}^n) \cong \mathrm{SU}(n)/T^{n-1} \quad \text{and} \quad \mathrm{Gr}_k(\mathbb{C}^n) \cong \mathrm{SU}(n)/\mathrm{S}(\mathrm{U}(k) \cdot \mathrm{U}(n-k)).$$

KNARR'S CONSTRUCTION

Let $|\Delta(\mathbb{C}^n)|$ denote the *geometric realization* of the simplicial complex $\Delta(\mathbb{C}^n)$. There are various ways to topologize this set. One possibility is the *weak topology* determined by the simplices: by definition, a subset $A \subseteq |\Delta(\mathbb{C}^n)|$ is closed in the weak topology if and only if its intersection with every simplex is closed. We denote the resulting topological space by $|\Delta(\mathbb{C}^n)|_{\mathrm{weak}}$; there are other nice topologies, all of which yield the same weak homotopy type for the space $|\Delta(\mathbb{C}^n)|$ ([7], I.7).

The Solomon–Tits Theorem asserts in our situation that there is a homotopy equivalence

$$|\Delta(\mathbb{C}^n)|_{\mathrm{weak}} \simeq \mathbb{S}^{n-2} \wedge X_+,$$

where X is a discrete set of cardinality $2^{\aleph_0}$ and X_+ its one-point compactification. (In general, the Solomon–Tits Theorem says that for a spherical building Δ of rank r,

$$|\Delta|_{\mathrm{weak}} \simeq \mathbb{S}^{r-1} \wedge X_+ \simeq \bigvee_{\mathrm{card}(X)} \mathbb{S}^{r-1}$$

where X is a discrete space whose cardinality is $\mathrm{card}(X) = \mathrm{card}\{A \in \mathcal{A} | \ C \in A\}$ for some fixed chamber C. The action of the automorphism group $\mathrm{SL}_n(\mathbb{C})$ on the top-dimensional homology group $H_{r-1}(|\Delta|_{\mathrm{weak}})$ of this complex is called the *Steinberg representation*. See [35], App. 4, for more details and further references.)

However, this construction neglects the natural topology of $\Delta(\mathbb{C}^n)$. Consider the following construction. Fix an $(n-2)$-simplex $\blacktriangle^{n-2}$ and label its vertices as $1, 2, \ldots, n-1$. There is a natural surjection

$$\mathrm{Fl}(\mathbb{C}^n) \times |\blacktriangle^{n-2}| \longrightarrow |\Delta(\mathbb{C}^n)|$$

which maps $\{U\} \times |\blacktriangle^{n-2}|$ to the geometric realization of the simplex of $\Delta(\mathbb{C}^n)$ spanned by the vertices $U_1, \ldots, U_{n-1}$ of the given flag U, in such a way that the ith vertex of $\{U\} \times \blacktriangle^{n-2}$ is identified with U_i. There is a natural compact topology on $\mathrm{Fl}(\mathbb{C}^n) \times |\blacktriangle^{n-2}|$, and we endow $|\Delta(\mathbb{C}^n)|$ with the quotient topology. We denote the resulting space by $|\Delta(\mathbb{C}^n)|_{\mathrm{Knarr}}$ (because Knarr—inspired by Mitchell [27]—introduced it first in [21] for compact buildings of rank 2). It can be shown that there is a homeomorphism $|\Delta(\mathbb{C}^n)|_{\mathrm{Knarr}} \cong \mathbb{S}^{n^2-2}$, see [21]. We will prove this in the next section, using the Veronese representation of $\Delta(\mathbb{C}^n)$; a more general result is stated in Section 7.

The Knarr construction works for general topological buildings. If the topology on the chamber set of a spherical building Δ of rank r satisfies certain natural conditions (e.g. the inclusions between its Schubert varieties should be cofibrations), then $|\Delta|_{\mathrm{Knarr}} \simeq \mathbb{S}^{r-1} \wedge O_+$, where O is the set of all chambers opposite to a fixed chamber, and O_+ its one-point compactification. For the special case of a discrete topology on the chamber set, this is precisely the Solomon–Tits Theorem. There is a well-developed theory of compact spherical buildings (the case of rank 2 is worked out in [23], and the results proved there extend immediately to the case of higher rank); the result for the homotopy type of $|\Delta|_{\mathrm{Knarr}}$ can be proved in much greater generality, see [23], Section 3.3. If the building is spherical, irreducible, compact, connected, and of rank at least 3 (see § 7 for definitions), then by the results in [9, 14–16], the space $|\Delta|_{\mathrm{Knarr}}$ can be identified with the *visual boundary* $X(\infty)$, ([7] II.8), of a Riemannian symmetric space X; for $\Delta(\mathbb{C}^n)$, the symmetric space in question is $X = \mathrm{SL}_n(\mathbb{C})/\mathrm{SU}(n)$; the same conclusion holds for buildings of rank 2, provided that the automorphism group acts transitively on the flags ([14–16]).

THE VERONESE REPRESENTATION OF $\Delta(\mathbb{C}^n)$

We endow $\mathbb{C}(n)$ with the positive definite Hermitian form $\langle X, Y\rangle = \mathrm{tr}(X^*Y)$. Consider the subspace $H(n) \leqslant \mathbb{C}(n)$ consisting of all Hermitian matrices in $\mathbb{C}(n)$. Every matrix $X \in H(n)$ has a unique decomposition $X = X^{\mathrm{tls}} + (\mathrm{tr}(X)/n)\mathbf{1}$ into a traceless Hermitian matrix X^{tls} and a real multiple of the identity matrix. The adjoint action of $\mathrm{SU}(n)$ on $H(n)$ is the action by conjugation, $X \mapsto gXg^*$. We have an orthogonal $\mathrm{SU}(n)$-invariant splitting $H(n) = \mathfrak{p}_n \oplus \mathbf{1}\mathbb{R}$ (recall that $\mathfrak{p}_n = \{X \in H(n) | \mathrm{tr}(X) = 0\}$). Suppose that $X \in H(n)$ is a *projector*, i.e. that $X^2 = X$. If $X \neq 0, 1$, then the minimal polynomial of X is $\mu_X(t) = t(t-1)$, and $\mathrm{tr}(X) = k$, for some $k \in \{1, \ldots, n-1\}$. The kernel V of X is then an $n-k$-dimensional subspace of $\mathbb{C}^n$. In this way, we obtain an $\mathrm{SU}(n)$-equivariant one-to-one correspondence between elements of $\mathrm{Gr}_{n-k}(\mathbb{C}^n)$ and self-adjoint projectors with trace k which is given by the map $X \mapsto \ker(X)$. The map $X \mapsto X^{\mathrm{tls}} = X - (\mathrm{tr}(X)/n)\mathbf{1}$ is $\mathrm{SU}(n)$-equivariant; in this way, we obtain an embedding $\Phi : \mathrm{Gr}_k(\mathbb{C}^n) \hookrightarrow \mathfrak{p}_n$ as follows. For $V \in \mathrm{Gr}_k(\mathbb{C}^n)$ let X_V denote the unique self-adjoint projector with $\ker(X_V) = V$, and put $\Phi(V) = (X_V)^{\mathrm{tls}}$. The elliptic polarity $\perp$ is built-in: the other eigenspace of X_V is the image $V^\perp$ of V under the elliptic polarity. Even better, the incidence can be seen in $\mathfrak{p}_n$: two self-adjoint operators $\Phi(V), \Phi(W) \in \mathfrak{p}_n$ representing subspaces $V \in \mathrm{Gr}_i(\mathbb{C}^n)$ and $W \in \mathrm{Gr}_j(\mathbb{C}^n)$ are incident if and only if the Euclidean distance $|\Phi(V) - \Phi(W)|$ attains the minimum possible value $d_{ij} = \mathrm{dist}(\Phi(\mathrm{Gr}_i(\mathbb{C}^n), \ \Phi(\mathrm{Gr}_j(\mathbb{C}^n))$. If U is a flag in $\Delta(\mathbb{C}^n)$, and if $p \in \blacktriangle^{n-2}$ has barycentric coordinates $(p_1, \ldots, p_{n-1})$, then we map $(U, p) \in \mathrm{Fl}(\mathbb{C}^n) \times \blacktriangle^{n-2}$ to the Hermitian operator

$$\Phi(U, p) = \sum_{i=1}^{n-1} p_i \Phi(U_i) \in \mathfrak{p}_n.$$

In this way we obtain an $\mathrm{SU}(n)$-equivariant injection $\Phi \colon |\Delta(\mathbb{C}^n)| \hookrightarrow \mathfrak{p}_n$. We call this the *Veronese representation* of the building $\Delta(\mathbb{C}^n)$. The image of the flag space $\mathrm{Fl}(\mathbb{C}^n)$ in $\mathfrak{p}_n$ is an *isoparametric submanifold* (we identify a chamber with the barycenter of its geometric realization); the images of the partial flag varieties are parallel focal submanifolds in this isoparametric foliation ([22, 32, 46, 47]). (The corresponding construction for the real projective geometry $\mathrm{PG}(\mathbb{R}^3)$ leads to the classical *Veronese embedding* of $\mathbb{R}\mathrm{P}^2 \hookrightarrow \mathbb{S}^4$, whence the name.)

The following variation of the map Φ is also useful. For a nonzero matrix $X \in \mathbb{C}(n)$, put $\hat{X} = |X|^{-1}X$, where $|.|$ denotes the Euclidean norm, and consider the map $\hat{\Phi} \colon (U, p) \mapsto \widehat{\Phi(U, p)}$. Then is is not difficult to see that

$$\hat{\Phi}(|\Delta(\mathbb{C}^n)|) = \mathbb{S}^{n^2-2} \subset \mathfrak{p}_n \cong \mathbb{R}^{n^2-1};$$

the map is injective, since we can recover a flag $(U_{i_1}, \ldots, U_{i_r})$ from its image $X \in \mathfrak{p}_n$ as follows: the Hermitian matrix X has eigenvalues $\lambda_1 < \lambda_2 < \cdots < \lambda_r$, and $U_{i_k} = \ker(X - \lambda_1\mathbf{1}) \oplus \cdots \oplus \ker(X - \lambda_k\mathbf{1})$. The surjectivity follows from the fact that every Hermitian matrix can be diagonalized under the $\mathrm{SU}(n)$-action, because the image of the Veronese representation contains certainly all diagonal traceless matrices of norm 1; these are precisely the images of the simplices in the apartment $A^{\mathbb{C}}\{e_1, \ldots, e_n\}$. Since Φ is continuous on $|\Delta(\mathbb{C}^n)|_{\mathrm{Knarr}}$, we obtain in particular the claimed homeomorphism $|\Delta(\mathbb{C}^n)|_{\mathrm{Knarr}} \cong \mathbb{S}^{n^2-2}$.

3. The Affine Building of $\mathbb{C}(z)$

In this section we describe the affine building associated to the discrete valuation on the rational function field $\mathbb{C}(z)$. This building is discussed in considerably more detail in the books by Brown [8] and Ronan [35].

LATTICES IN $\mathbb{C}(z)^n$

We let $\mathbb{L} = \mathbb{C}(z)$ denote the field of fractions of the polynomial ring $\mathbb{C}[z]$. Thus, $\mathbb{L}$ is the field of rational functions on the complex projective line $\mathbb{C}\mathrm{P}^1 = \mathbb{C} \cup \{\infty\}$. For $c \in \mathbb{C}\mathrm{P}^1$ we let $\mathcal{O}_c \leqslant \mathbb{L}$ denote the subring of all rational functions which don't have a pole in c, and $m_c = \{f \in \mathcal{O}_c | \, f(c) = 0\}$ the maximal ideal of $\mathcal{O}_c$. Evaluation at c yields a map $ev_c\colon \mathcal{O}_c \longrightarrow \mathbb{C}$, $f \mapsto f(c)$ with kernel m_c, and we obtain exact sequences

$$0 \longrightarrow m_c \longrightarrow \mathcal{O}_c \stackrel{ev_c}{\rightarrow} \mathbb{C} \longrightarrow 0$$

and

$$1 \longrightarrow \mathrm{SL}_n(m_c) \longrightarrow \mathrm{SL}_n(\mathcal{O}_c) \stackrel{ev_c}{\rightarrow} \mathrm{SL}_n(\mathbb{C}) \longrightarrow 1$$

(the group $\mathrm{SL}_n(m_c)$ is defined to be the kernel of the evaluation map ev_c). We may view the elements of $\mathrm{SL}_n(\mathbb{L})$ as *rational maps* from $\mathbb{C}\mathrm{P}^1$ into $\mathrm{SL}_n(\mathbb{C})$. Note also that every element $q \neq 0$ of $\mathbb{L}$ can be expressed in the form $q = (z-c)^k \frac{f}{g}$, with $k \in \mathbb{Z}$, $f, g \in \mathbb{C}[z]$, and $f(c) \neq 0 \neq g(c)$. We put $v_c(q) = k$, and $v_c(0) = \infty$. The map $v_c\colon \mathbb{L} \longrightarrow \mathbb{Z} \cup \{\infty\}$ is a *discrete valuation* on $\mathbb{L}$ (with some modifications for $c = \infty$). Note that $\mathcal{O}_c^\times = \{q \in \mathcal{O}_c | \, v_c(q) = 0\}$. There is nothing special about the choice of $c \in \mathbb{C}\mathrm{P}^1$, and we put $c = 0$ for the remainder of this section.

The group $\mathrm{SL}_n(\mathbb{L})$ acts on the projective geometry $\mathrm{PG}(\mathbb{L}^n)$ in very much the same way as $\mathrm{SL}_n(\mathbb{C})$ on $\mathrm{PG}(\mathbb{C}^n)$, and we could consider the spherical building $\Delta(\mathbb{L}^n)$. But now we introduce a different geometry for this group, the *affine building* $\Delta(\mathbb{L}^n, \mathcal{O}_0)$. Given an $\mathbb{L}$-basis $v_1, \ldots, v_n$ of $\mathbb{L}^n$, we have the free $\mathcal{O}_0$-module

$$M = \mathrm{span}_{\mathcal{O}_0}\{v_1, \ldots, v_n\} = v_1\mathcal{O}_0 + \cdots + v_n\mathcal{O}_0$$

of rank n generated by these basis vectors. We call M an $\mathcal{O}_0$-*lattice*, and we let $\mathrm{Lat}_n(\mathbb{L}, \mathcal{O}_0)$ denote the collection of all such lattices. (The following simple observation is useful. If $M \subseteq \mathbb{L}^n$ is a free $\mathcal{O}_0$-module of rank k, with $\mathcal{O}_0$-basis $\{v_1, \ldots, v_k\}$, then $\{v_1, \ldots, v_k\}$ is linearly independent over $\mathbb{L}$, because $\mathbb{L}$ is the field of fractions of $\mathcal{O}_0$. Thus, the $\mathcal{O}_0$-lattices are precisely the free $\mathcal{O}_0$-modules of rank n in $\mathbb{L}^n$.) Evidently, the group $\mathrm{GL}_n(\mathbb{L})$ acts transitively on $\mathrm{Lat}_n(\mathbb{L}, \mathcal{O}_0)$; the $\mathrm{GL}_n(\mathbb{L})$-stabilizer of the $\mathcal{O}_0$-module M_0 spanned by the canonical basis $e_1, \ldots, e_n$ of $\mathbb{L}^n$ is the group $\mathrm{GL}_n(\mathcal{O}_0)$, and therefore

$$\mathrm{Lat}_n(\mathbb{L}, \mathcal{O}_0) \cong \mathrm{GL}_n(\mathbb{L})/\mathrm{GL}_n(\mathcal{O}_0).$$

We call two lattices $M, M' \in \mathrm{Lat}_n(\mathbb{L}, \mathcal{O}_0)$ *projectively equivalent* if $M = qM'$ for some $q \in \mathbb{L}^\times$. In view of the factorization of q given above, this is clearly equivalent with the condition that $M = z^k M'$ holds for some $k \in \mathbb{Z}$. The projective equivalence class of $M \in \mathrm{Lat}_n(\mathbb{L}, \mathcal{O}_0)$ is denoted by

$$[M] = \{z^k M | \ k \in \mathbb{Z}\}$$

Thus we have obtained an action of the projective groups $\mathrm{PGL}_n(\mathbb{L})$ and $\mathrm{PSL}_n(\mathbb{L})$ on the set

$$\{[M] | \ M \in \mathrm{Lat}_n(\mathbb{L}, \mathcal{O}_0)\}$$

of projective equivalence classes of $\mathcal{O}_0$-lattices.

THE ACTION OF $\mathrm{SL}_n(\mathbb{L})$ AND THE TYPE FUNCTION

The group $\mathrm{SL}_n(\mathbb{L})$ is *not* transitive on set of projective equivalence classes of $\mathcal{O}_0$-lattices. Let $M_0 = \mathrm{span}_{\mathcal{O}_0}\{e_1, \ldots, e_n\}$ denote the $\mathcal{O}_0$-module spanned by the canonical basis $e_1, \ldots, e_n$ of $\mathbb{L}^n$. Suppose that $g(M_0) = M'$, for some $g \in \mathrm{GL}_n(\mathbb{L})$. Since $v_0(\det(h)) = 0$ for all $h \in \mathrm{GL}_n(\mathcal{O}_0)$ (because $v_0(q) = 0$ if $q \in \mathcal{O}_0$ is a unit), the number $v_0(\det(g))$ depends only on the module M'. The determinant of the map $\lambda_{z^k} : v \mapsto z^k v$ is $\det(\lambda_{z^k}) = z^{kn}$. Thus we have a well-defined map

$$\mathrm{type}([M']) = v(\det(g)) + n\mathbb{Z} \in \mathbb{Z}/n$$

which is $\mathrm{SL}_n(\mathbb{L})$-invariant. Note also that the stabilizers agree, $\mathrm{SL}_n(\mathbb{L})_M = \mathrm{SL}_n(\mathbb{L})_{[M]}$, since $\det(\lambda_{z^k}) \neq 1$ for $k \neq 0$. We put

$$\mathcal{V}_i = \{[M] | \ M \in \mathrm{Lat}_n(\mathbb{L}, \mathcal{O}_0), \mathrm{type}([M]) = i\}.$$

Let

$$M_i = \mathrm{span}_{\mathcal{O}_0}\{ze_1, \ldots, ze_i, e_{i+1}, \ldots, e_n\}$$

denote the $\mathcal{O}_0$-module spanned by the vectors $ze_1, \ldots, ze_i, e_{i+1}, \ldots, e_n$. Then $[M_i] \in \mathcal{V}_i$.

LEMMA 3.1. *The action of* $\mathrm{SL}_n(\mathbb{L})$ *on* $\mathcal{V}_i$ *is transitive.*

Proof. If $[M] = [g(M_i)] \in \mathcal{V}_i$ for some $g \in \mathrm{GL}_n(\mathbb{L})$, then $v_0(\det(g)) \equiv 0 \pmod n$, whence $v_0(\det(z^k g)) = 0$ for a suitable $k \in \mathbb{Z}$. Put $g' = z^k g$, then $[M] = [g'(M_i)]$, and $v_0(\det(g')) = 0$, whence $\det(g') \in \mathcal{O}_0$. Finally, put $h = \mathrm{diag}(\det(g')^{-1}, 1, \ldots, 1)$. Then h fixes M_i, and thus $g'h \in \mathrm{SL}_n(\mathbb{L})$ maps $[M_i]$ to $[M]$. □

These n different $\mathcal{O}_0$-modules $M_0, \ldots, M_{n-1}$ thus form a cross-section for the action of $\mathrm{SL}_n(\mathbb{L})$ on the set of projective equivalence classes of $\mathcal{O}_0$-lattices, and we put

$$P^0_{\mathrm{rat}} = \mathrm{SL}_n(\mathbb{L})_{M_0} = \mathrm{SL}_n(\mathcal{O}_0),$$
$$P^i_{\mathrm{rat}} = \mathrm{SL}_n(\mathbb{L})_{M_i} = \left\{ \begin{pmatrix} A & z^{-1}B \\ zC & D \end{pmatrix} \middle| A \in \mathcal{O}_0(i), \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{SL}_n(\mathcal{O}_0) \right\}$$

for $i = 1, \dots, n-1$. The sets $\mathcal{V}_0, \dots, \mathcal{V}_{n-1}$ play the same rôle as the Grassmannians $\mathrm{Gr}_j(\mathbb{C}^n)$ in Section 1.1, and the stabilizers $P^0_{\mathrm{rat}}, \dots, P^{n-1}_{\mathrm{rat}}$ play the same rôle as the standard maximal parabolics $P^1, \dots, P^{n-1}$ in $\mathrm{SL}_n(\mathbb{C})$. However, there is one fundamental difference: the stabilizers P^i_{rat}, $i = 0, \dots, n-1$, are conjugate to P^0_{rat} in $\mathrm{GL}_n(\mathbb{L})$,

$$P^i_{\mathrm{rat}} = g_i \mathrm{SL}_n(\mathcal{O}_0) g_i^{-1},$$

where

$$g_i = \mathrm{diag}(\underbrace{z, \dots, z}_{i}, \underbrace{1, \dots, 1}_{n-i}) \in \mathrm{GL}_n(\mathbb{L}).$$

INCIDENCE, PERIODIC FLAGS, AND APARTMENTS

We define an *incidence relation* $\mathtt{I}$ on the set $\{[M] \mid M \in \mathrm{Lat}_n(\mathbb{L}, \mathcal{O}_0)\}$ as follows:

$$[M]\mathtt{I}[M'] \quad \overset{\mathrm{def}}{\Longleftrightarrow} \quad zM \leqslant z^k M' \leqslant M \quad \text{for some } k \in \mathbb{Z}.$$

This relation is symmetric, since $zM \leqslant z^k M' \leqslant M$ implies that $zM' \leqslant z^{1-k} M \leqslant M'$. Clearly, $\mathrm{SL}_n(\mathbb{L})$ acts by incidence preserving automorphisms on this incidence geometry. Similarly as before, we can use the incidence relation to construct a poset $\Delta(\mathbb{L}^n, \mathcal{O}_0)$, the *affine building* of $(\mathbb{L}^n, \mathcal{O}_0)$; the elements of $\Delta(\mathbb{L}^n, \mathcal{O}_0)$ are the sets of pairwise incident elements of $\mathcal{V}_0 \cup \dots \cup \mathcal{V}_{n-1}$.

Note also that $[M_i]\mathtt{I}[M_j]$ holds for all i, j. The chambers are of the following form. Let $\mathfrak{B} = (v_1, \dots, v_n)$ be an ordered basis of $\mathbb{L}^n$, and put $M_i^{\mathfrak{B}} = \mathrm{span}_{\mathcal{O}_0}\{zv_1, \dots, zv_i, v_{i+1}, \dots, v_n\}$. Then $M(\mathfrak{B}) = \{[M_0^{\mathfrak{B}}], \dots, [M_{n-1}^{\mathfrak{B}}]\}$ is a maximal flag, which can be viewed as an infinite sequence of free $\mathcal{O}_0$-modules

$$\dots > z^{-1} M_{n-1}^{\mathfrak{B}} > M_0^{\mathfrak{B}} > M_1^{\mathfrak{B}} > \dots > M_{n-1}^{\mathfrak{B}} > zM_0^{\mathfrak{B}} > zM_1^{\mathfrak{B}} > \dots$$

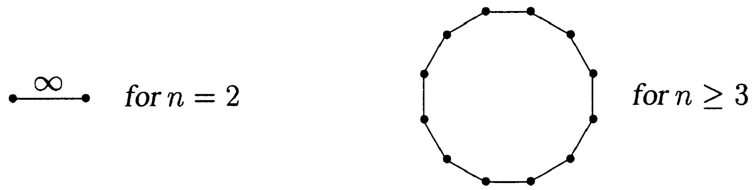
of rank n. The quotient of two consecutive modules in this chain is a one-dimensional complex vector space. Note that $\Delta(\mathbb{L}^n, \mathcal{O}_0)$ has rank n.

The collection of all chambers is called the *periodic flag variety* $\mathrm{Fl}(\mathbb{L}^n, \mathcal{O}_0)$. The stabilizer of the chamber $\{[M_0], \dots, [M_{n-1}]\}$ is the Borel group

$$B_{\mathrm{rat}} = P^0_{\mathrm{rat}} \cap P^1_{\mathrm{rat}} \cap \dots \cap P^{n-1}_{\mathrm{rat}}$$
$$= \left\{ \begin{pmatrix} \mathcal{O}_0 & & \mathcal{O}_0 \\ & \ddots & \\ z\mathcal{O}_0 & & \mathcal{O}_0 \end{pmatrix} \in \mathrm{SL}_n(\mathcal{O}_0) \right\} = ev_0^{-1}(\mathrm{S}\nabla_n(\mathbb{C})).$$

Given a basis $v_1, \dots, v_n$ of $\mathbb{L}^n$, we define the standard apartment $A_0^{\mathbb{L}}\{v_1, \dots, v_n\}$ as the collection of all partial flags obtained from the maximal flags $M(\mathfrak{B})$, where $\mathfrak{B}$ runs through the collection of all bases of the form $\mathfrak{B} = (z^{k_1}v_{\pi(1)}, \ldots, z^{k_n}v_{\pi(n)})$, for $\pi \in \mathrm{Sym}(n)$ and $k_1, \ldots, k_n \in \mathbb{Z}$. The set-wise $\mathrm{SL}_n(\mathbb{L})$-stabilizer N_{rat} of $A_0^{\mathbb{L}} = A_0^{\mathbb{L}}\{e_1, \ldots, e_n\}$ is the collection of all unimodular permutation matrices with entries in $\mathbb{L}^\times$, and the element-wise stabilizer T_{rat} of $A_0^{\mathbb{L}}$ is the collection of all diagonal unimodular matrices with entries in $\mathcal{O}_0^\times$. The quotient is the *affine Weyl group* $W = N_{\mathrm{rat}}/T_{\mathrm{rat}} \cong \tilde{A}_{n-1}$ of $\mathrm{SL}_n(\mathbb{L})$. As a simplicial complex, $A_0^{\mathbb{L}}$ is a triangulation of $\mathbb{R}^{n-1}$.

THEOREM 3.2. *The simplicial complex* $\Delta(\mathbb{L}^n, \mathcal{O}_0)$ *is a building of rank n and type* $\tilde{A}_{n-1}$*; as an apartment system, we may choose the set* $\mathcal{A}_0^{\mathbb{L}} = \{A_0^{\mathbb{L}}\{v_1, \ldots, v_n\} \mid v_1, \ldots, v_n$ a basis for $\mathbb{L}^n\}$. *The Coxeter diagram is* $\tilde{A}_{n-1}$*;*



(*n nodes*).

Proof. Probably the easiest way to see that this is a building is to verify that $(B_{\mathrm{rat}}, N_{\mathrm{rat}})$ is a *BN*-pair for the group $\mathrm{SL}_n(\mathbb{L})$; cf [8], Ch. V.8, and [35], Ch. 9.2. □

It is not difficult to see that the residue of $[M_i]$ is isomorphic to $\Delta(\mathbb{C}^n)$, for $i = 0, \ldots, n-1$. Thus the panels in $\Delta(\mathbb{L}^n, \mathcal{O}_0)$ are again isomorphic to the complex projective line $\mathbb{C}\mathrm{P}^1$.

One final remark. We have constructed the affine building $\Delta(\mathbb{L}^n, \mathcal{O}_0)$ related to the discrete valuation v_0. There is nothing special about the point $0 \in \mathbb{C}\mathrm{P}^1$; if we choose a different point $c \in \mathbb{C}\mathrm{P}^1$, then we obtain a different affine building $\Delta(\mathbb{L}^n, \mathcal{O}_c)$. These buildings are pairwise isomorphic; in fact they are permuted by the group $\mathrm{PSL}_2(\mathbb{C})$. Each of these building has a distinct collection of parabolics, so $\mathrm{SL}_n(\mathbb{L})$ contains an uncountable set of *BN*-pairs.

4. The Twinning over $\mathbb{C}[z, 1/z]$

Now we describe the buildings $\Delta(\mathbb{L}^n, \mathcal{O}_0)$ and $\Delta(\mathbb{L}^n, \mathcal{O}_\infty)$ in a slightly different way, replacing the field $\mathbb{C}(z)$ by the ring $\mathbb{C}[z, 1/z]$. This section owes much to the paper [4] by Abramenko and Van Maldeghem. So let

$$\mathbb{A} = \mathbb{C}[z, 1/z] = \bigcap\{\mathcal{O}_x \mid x \in \mathbb{C}^\times\}$$

denote the ring of all rational functions which are holomorphic on $\mathbb{C}^\times \subseteq \mathbb{C}\mathrm{P}^1$. This is a subring of $\mathbb{L}$, the ring of *Laurent polynomials*. Note that $\mathbb{A} \cap \mathcal{O}_0 = \mathbb{C}[z]$ and $\mathbb{A} \cap \mathcal{O}_\infty = \mathbb{C}[1/z]$. Similarly as before, we let $\mathrm{Lat}_n(\mathbb{A}, \mathbb{C}[z])$ denote the collection of all free $\mathbb{C}[z]$-modules in $\mathbb{A}^n$ which are spanned by an $\mathbb{A}$-basis, and $\mathrm{Lat}_n(\mathbb{A}, \mathbb{C}[1/z])$ the collection of all free $\mathbb{C}[1/z]$-modules spanned by $\mathbb{A}$-bases. Thus

$$\mathrm{Lat}_n(\mathbb{A}, \mathbb{C}[z]) = \{g(E_0^+) \mid g \in \mathrm{GL}_n(\mathbb{A})\}$$

and

$$\mathrm{Lat}_n(\mathbb{A}, \mathbb{C}[1/z]) = \{g(E_0^-) | \ g \in \mathrm{GL}_n(\mathbb{A})\},$$

where

$$E_0^+ = \mathrm{span}_{\mathbb{C}[z]}\{e_1, \dots, e_n\} \quad \text{and} \quad E_0^- = \mathrm{span}_{\mathbb{C}[1/z]}\{e_1, \dots, e_n\}.$$

For

$$E \in \mathrm{Lat}_n(\mathbb{A}, \mathbb{C}[z]) \cup \mathrm{Lat}_n(\mathbb{A}, \mathbb{C}[1/z])$$

we put, as before, $[E] = \{z^k E | \ k \in \mathbb{Z}\}$,
and

$$E_i^+ = \mathrm{span}_{\mathbb{C}[z]}\{ze_1, \dots, ze_i, e_{i+1}, \dots, e_n\},$$

$$E_i^- = \mathrm{span}_{\mathbb{C}[1/z]}\{ze_1, \dots, ze_i, e_{i+1}, \dots, e_n\},$$

and

$$\mathcal{V}_i^\pm = \{[gE_i^\pm] | \ g \in \mathrm{SL}_n(\mathbb{A})\}.$$

The incidence is also defined as before. If $[E] \in \mathcal{V}_i^\pm$ and $[E'] \in \mathcal{V}_j^\pm$, then

$$[E]\mathrm{I}[E'] \quad \overset{\text{def}}{\iff} \quad z^{\pm 1}E \leqslant z^k E' \leqslant E \quad \text{for some } k \in \mathbb{Z}$$

Thus we obtain two simplicial complexes $\Delta^+(\mathbb{A}^n)$ and $\Delta^-(\mathbb{A}^n)$ which are isomorphic under the map induced by the ring automorphism $z \mapsto 1/z$. The set of all maximal simplices is denoted by $\mathrm{Fl}(\Delta^\pm(\mathbb{A}^n))$.

Now there is a canonical map $\mathrm{Lat}_n(\mathbb{A}, \mathbb{C}[z]) \longrightarrow \mathrm{Lat}_n(\mathbb{L}, \mathcal{O}_0)$ which maps E to $\mathrm{span}_{\mathcal{O}_0}(E)$, and a similar map $\mathrm{Lat}_n(\mathbb{A}, \mathbb{C}[1/z]) \longrightarrow \mathrm{Lat}_n(\mathbb{L}, \mathcal{O}_\infty)$; these maps induce canonical $\mathrm{SL}_n(\mathbb{A})$-equivariant simplicial maps $\Delta^+(\mathbb{A}^n) \longrightarrow \Delta(\mathbb{L}^n, \mathcal{O}_0)$ and $\Delta^-(\mathbb{A}^n) \longrightarrow \Delta(\mathbb{L}^n, \mathcal{O}_\infty)$.

PROPOSITION 4.1. *The two maps*

$$\Delta^+(\mathbb{A}^n) \longrightarrow \Delta(\mathbb{L}^n, \mathcal{O}_0) \quad \textit{and} \quad \Delta^-(\mathbb{A}^n) \longrightarrow \Delta(\mathbb{L}^n, \mathcal{O}_\infty)$$

are isomorphisms and thus $\Delta^+(\mathbb{A}^n)$, $\Delta^-(\mathbb{A}^n)$ *are buildings; the group* $\mathrm{SL}_n(\mathbb{A})$ *acts transitively on both buildings.*

Proof. For the proof we note that the $\mathrm{SL}_n(\mathbb{A})$-stabilizer of the $n-2$-simplex $B_i = \{[M_0], \dots, [M_{i-1}], [M_{i+1}], \dots, [M_{n-1}]\} \in \Delta(\mathbb{L}^n, \mathcal{O}_0)$ induces the transitive group $\mathrm{PSL}_2\mathbb{C}$ on the corresponding panel. Thus $\mathrm{SL}_n(\mathbb{A})$ acts transitively on $\Delta(\mathbb{L}^n, \mathcal{O}_0)$ by Lemma 1.5. Since $\mathrm{SL}_n(\mathbb{A})$ has the same stabilizers both in $\Delta^+(\mathbb{A}^n)$ and in $\Delta(\mathbb{L}^n, \mathcal{O}_0)$ (see below), we obtain the claimed isomorphism $\Delta^+(\mathbb{A}^n) \cong \Delta(\mathbb{L}^n, \mathcal{O}_0)$. The involution $z \mapsto 1/z$ on the ring $\mathbb{A}$ and the projective line $\mathbb{C}\mathrm{P}^1 = \mathbb{C} \cup \{\infty\}$ normalizes $\mathrm{SL}_n(\mathbb{A})$, and we obtain $\Delta^-(\mathbb{A}^n) \cong \Delta(\mathbb{L}^n, \mathcal{O}_\infty)$. □

If $\mathfrak{B} = (v_1, \dots, v_n)$ is an $\mathbb{A}$-basis for $\mathbb{A}^n$, then we have similarly as before the chamber $E^+(\mathfrak{B}) = \{[E_0^{+,\mathfrak{B}}], \dots, [E_{n-1}^{+,\mathfrak{B}}]\}$ corresponding to the modules $E_i^{+,\mathfrak{B}} =$

$\mathrm{span}_{\mathbb{C}[z]}\{zv_1, \ldots, zv_i, v_{i+1}, \ldots, v_n\}$, and the apartment $A^{+,\mathbb{A}}\{v_1, \ldots, v_n\}$ obtained from the $\mathbb{A}$-bases $(z^{k_1}e_{\pi(1)}, \ldots, z^{k_n}e_{\pi(n)})$; similarly, we obtain an apartment $A^{-,\mathbb{A}}\{v_1, \ldots, v_n\}$ for the building $\Delta^-(\mathbb{A}^n)$. It can be checked that $\mathrm{SL}_n(\mathbb{A})$ acts strongly transitively on $\Delta^+(\mathbb{A}^n)$ and $\Delta^-(\mathbb{A}^n)$ with respect to these apartment systems

$$\mathcal{A}^\pm = \{A^{\pm,\mathbb{A}}\{v_1, \ldots, v_n\} \mid v_1, \ldots, v_n \text{ an } \mathbb{A} - \text{basis for } \mathbb{A}^n\}.$$

Note that the apartment system $\mathcal{A}^{+,\mathbb{A}}$ is strictly smaller than $\mathcal{A}_0^{\mathbb{L}}$.

PROPOSITION 4.2. *The group* $\mathrm{SL}_n(\mathbb{A})$ *acts strongly transitively both on* $\Delta^+(\mathbb{A}^n)$ *and on* $\Delta^-(\mathbb{A}^n)$.

Proof. This follows from the fact that $\mathrm{SL}_n(\mathbb{A})$ acts transitively on pairs of *opposite* chambers in $\Delta^+(\mathbb{A}^n) \times \Delta^-(\mathbb{A}^n)$, ([4]); the opposition relation is defined in the next section. □

We put

$$P^{i,+}_{\mathrm{alg}} = P^i_{\mathrm{rat}} \cap \mathrm{SL}_n(\mathbb{A}) \quad \text{and} \quad B^+_{\mathrm{alg}} = B_{\mathrm{rat}} \cap \mathrm{SL}_n(\mathbb{A}).$$

Thus

$$P^{0,+}_{\mathrm{alg}} = \mathrm{SL}_n(\mathbb{C}[z]) \quad \text{and} \quad P^{i,+}_{\mathrm{alg}} = g_i(P^{0,+}_{\mathrm{alg}})g_i^{-1},$$

where $g_i = \mathrm{diag}(z, \ldots, z, 1, \ldots, 1)$ as before. These are the parabolics corresponding to $\Delta^+(\mathbb{A}^n)$; there are similar parabolics for $\Delta^-(\mathbb{A}^n)$.

TWIN BUILDINGS

The notion of a twin building was developed by Ronan and Tits in order to supply geometries for Kac–Moody groups; we refer to [1, 3, 4, 28, 29, 31, 36, 37, 50, 51] for more details about twin buildings.

Let (W, S) be a Coxeter system, and let (Δ^+, Δ^-) be a pair of buildings with this given Coxeter system. The W-valued distance in $\Delta^\pm$ is denoted $\delta_{\Delta^\pm}$. A *twinning* of Δ^+ with Δ^- is a W-valued *codistance* function

$$\delta^*\colon \mathrm{Cham}(\Delta^\pm) \times \mathrm{Cham}(\Delta^\mp) \longrightarrow W,$$

subject to the following axioms. *The intuitive idea is that objects with a small codistance are far apart*.

Tw$_1$ The relation $\delta^*(C^\pm, D^\mp) = \delta^*(D^\mp, C^\pm)^{-1}$ holds for all chambers $C^\pm \in \Delta^\pm$, $D^\mp \in \Delta^\mp$ (the codistance is 'symmetric').

Tw$_2$ Let $w \in W$ and $s \in S$, and suppose that $\ell(ws) = \ell(w) - 1$ (i.e. that w has a reduced expression with s as the last letter). If $C^\pm \in \Delta^\pm$, $D^\mp, E^\mp \in \Delta^\mp$ are chambers with $\delta^*(C^\pm, D^\mp) = w$ and $\delta_{\Delta^\mp}(D^\mp, E^\mp) = s$, then $\delta^*(C^\pm, E^\mp) = ws$ (all chambers $E^\mp$ in the s-panel through $D^\mp$ are 'further away' from $C^\pm$).

Tw$_3$ If $C^\pm \in \Delta^\pm$, $D^\mp \in \Delta^\mp$ are chambers with codistance $\delta^*(C^\pm, D^\mp) = w$, and if $s \in S$, then there exists a chamber $E^\mp \in \Delta^\mp$, with $\delta_{\Delta^\mp}(D^\mp, E^\mp) = s$, and with $\delta^*(C^\pm, E^\mp) = ws$ (the codistance leads to galleries).

Note that there is a symmetry in the axioms if we exchange the signs '+' and '−'. Sometimes we will state a result for a specific choice of the signs; the corresponding result for the opposite choice of signs follows in the same way.

It is clear how to extend the codistance to a double coset valued codistance

$$\delta^*\colon \Delta^\pm \times \Delta^\mp \longrightarrow \bigcup\{W_J\backslash W/W_K \mid J, K \subseteq I\}.$$

The *opposition relation* op $\subseteq \Delta^+ \times \Delta^-$ is defined as follows: two chambers are opposite if their codistance is 1; this relation extends in a natural way to the simplices. The codistance can be used to sync the type functions in both buildings in such a way that opposite vertices have the same type, and we will assume this to be done.

A (special) *automorphism* of a twin building is a pair (g^+, g^-) of automorphisms $g^\pm \in \mathrm{Spe}(\Delta^\pm)$ which preserves the codistance, $\delta^*(C^+, C^-) = \delta^*(g^+(C^+), g^-(C^-))$.

EXAMPLE 4.3. Let Δ be a spherical building, let w_0 denote the unique longest element in the Coxeter group W of Δ, put $\Delta^+ = \Delta^- = \Delta$, and $\delta^*(C, D) = \delta(C, D)w_0$. The resulting geometry is a twin building.

All twin buildings with spherical halves arise in this way, see [50]. Twin buildings are natural generalizations of spherical buildings; they share many of the particular geometric properties of spherical buildings. The twinning is in general not determined by the pair (Δ^+, Δ^-); a pair of buildings (e.g. a pair of trees) can admit many nonequivalent twinnings.

The group $\mathrm{SL}_n(\mathbb{A})$ induces a twinning on the pair $(\Delta^+(\mathbb{A}^n), \Delta^-(\mathbb{A}^n))$; the codistance can be defined as follows. The group $\mathrm{SL}_n(\mathbb{A})$ has a *Birkhoff decomposition* (a Bruhat twin decomposition) as

$$\mathrm{SL}_n(\mathbb{A}) = B^-_{\mathrm{alg}} N B^+_{\mathrm{alg}} = \dot{\bigcup}\{B^-_{\mathrm{alg}} w B^+_{\mathrm{alg}} \mid w \in N/(B^+_{\mathrm{alg}} \cap B^-_{\mathrm{alg}})\}$$

where N is the set-wise stabilizer of the twin apartment $(A^{+,\mathbb{A}}\{e_1, \ldots, e_n\}, A^{-,\mathbb{A}}\{e_1, \ldots, e_n\})$. Since

$$\mathrm{Cham}(\Delta^\pm(\mathbb{A}^n)) = \mathrm{SL}_n(\mathbb{A})/B^\pm_{\mathrm{alg}} \quad \text{and} \quad N/(B^+_{\mathrm{alg}} \cap B^-_{\mathrm{alg}}) \cong W,$$

we may use the Birkhoff decomposition to define the codistance as

$$\delta^*(gB^-_{\mathrm{alg}}, hB^+_{\mathrm{alg}}) = w \quad \text{if and only if} \quad B^-_{\mathrm{alg}} g^{-1} h B^+_{\mathrm{alg}} = B^-_{\mathrm{alg}} w B^+_{\mathrm{alg}}.$$

The following Theorem is 'folklore'. A nice proof is given in Abramenko and Van Maldeghem [4].

THEOREM 4.4. *The triple* $(\Delta^+(\mathbb{A}^n), \Delta^-(\mathbb{A}^n), \delta^*)$ *is a twin building, and the group* $\mathrm{SL}_n(\mathbb{A})$ *acts as a strongly transitive group of automorphisms. Two chambers* $(C^+, C^-) \in \Delta^+(\mathbb{A}^n) \times \Delta^-(\mathbb{A}^n)$ *are opposite if and only if they arise from lattices which are 'back to back', i.e. if there exists an ordered* $\mathbb{A}$*-basis* $(v_1, \ldots, v_n)$ *such that* $C^+ = E^+(v_1, \ldots, v_n)$ *and* $C^- = E^-(v_1, \ldots, v_n)$.

5. Loop Groups

There is one big difference between $\mathrm{SL}_n(\mathbb{L})$ and $\mathrm{SL}_n(\mathbb{A})$ which is due to the fact that $\mathbb{A}$ is a ring, while $\mathbb{L}$ is a field: the group $\mathrm{SL}_n(\mathbb{L})$ is almost simple, whereas the group $\mathrm{SL}_n(\mathbb{A})$ is a semi-direct product.

BASED LOOPS

Fix $c \in \mathbb{C}^\times$ and consider the evaluation map $ev_c\colon \mathrm{SL}_n(\mathbb{A}) \longrightarrow \mathrm{SL}_n(\mathbb{C})$ given by $ev_c : f(z) \mapsto f(c)$. We denote the kernel of this map by $\Omega_{\mathrm{alg}}(\mathrm{SL}_n(\mathbb{C}), c)$ and we put $\Omega_{\mathrm{alg}}\mathrm{SL}_n(\mathbb{C}) = \Omega_{\mathrm{alg}}(\mathrm{SL}_n(\mathbb{C}), 1)$ for short. The injection $\mathrm{SL}_n(\mathbb{C}) \subseteq \mathrm{SL}_n(\mathbb{A})$ leads to a split exact sequence

$$1 \longrightarrow \Omega_{\mathrm{alg}}(\mathrm{SL}_n(\mathbb{C}), c) \longrightarrow \mathrm{SL}_n(\mathbb{A}) \underset{\leftarrow}{\overset{ev_c}{\longrightarrow}} \mathrm{SL}_n(\mathbb{C}) \longrightarrow 1,$$

hence $\mathrm{SL}_n(\mathbb{A})$ is a semi-direct product. (There is an obvious $\mathbb{C}^\times$-action on $\mathrm{SL}_n(\mathbb{A})$ given by $f(z) \mapsto f(az)$; under this action, the collection of normal subgroups $\{\Omega_{\mathrm{alg}}(\mathrm{SL}_n(\mathbb{C}), c) |\ c \in \mathbb{C}^\times\}$ is permuted transitively. Thus one is lead to the semi-direct product $\mathrm{SL}_n(\mathbb{A}) \rtimes \mathbb{C}^\times$.)

LEMMA 5.1. *For every $c \in \mathbb{C}^\times$, the group $\Omega_{\mathrm{alg}}(\mathrm{SL}_n(\mathbb{C}), c)$ acts transitively on each of the sets $\mathcal{V}_0^+, \mathcal{V}_0^-, \ldots, \mathcal{V}_{n-1}^+, \mathcal{V}_{n-1}^-$.*

Proof. We clearly have $\mathrm{SL}_n(\mathbb{C}) \subseteq \mathrm{SL}_n(\mathbb{C}[z]) = P_{\mathrm{alg}}^{0,+}$, and thus

$$P_{\mathrm{alg}}^{0,+}\,\Omega_{\mathrm{alg}}(\mathrm{SL}_n(\mathbb{C}), c) \supseteq \mathrm{SL}_n(\mathbb{C})\,\Omega_{\mathrm{alg}}(\mathrm{SL}_n(\mathbb{C}), c) = \mathrm{SL}_n(\mathbb{A}).$$

Now $P_{\mathrm{alg}}^{i,+} = g_i P_{\mathrm{alg}}^{0,+} g_i^{-1}$ and $\Omega_{\mathrm{alg}}(\mathrm{SL}_n(\mathbb{C}), c)$ is g_i-invariant, whence

$$P_{\mathrm{alg}}^{i,+}\Omega_{\mathrm{alg}}(\mathrm{SL}_n(\mathbb{C}), c) = g_i P_{\mathrm{alg}}^{0,+} g_i^{-1}\Omega_{\mathrm{alg}}(\mathrm{SL}_n(\mathbb{C}), c) = \mathrm{SL}_n(\mathbb{A}). \qquad \square$$

There is a natural map $\mathrm{SL}_n(\mathbb{A}) \longrightarrow C^\infty(\mathbb{S}^1, \mathrm{GL}_n(\mathbb{C})) = L_{\mathrm{diff}}\mathrm{SL}_n(\mathbb{C})$ into the set $L_{\mathrm{diff}}\mathrm{SL}_n(\mathbb{C})$ of smooth maps from the unit circle into $\mathbb{SL}_n(\mathbb{C})$; this map is obtained by viewing the elements of $\mathrm{SL}_n(\mathbb{A})$ as maps from $\mathbb{S}^1 \subseteq \mathbb{C}^\times$ into $\mathrm{SL}_n(\mathbb{C})$. If $f = \sum_{\mathrm{fin}} f_k z^k \in \mathrm{SL}_n(\mathbb{A})$, then

$$f_k = \frac{1}{2\pi i}\oint_{|z|=1} \frac{1}{z^{k+1}} f(z)\mathrm{d}z.$$

Therefore, the map into $L_{\mathrm{diff}}\mathrm{SL}_n(\mathbb{C})$ is an injection. From now on, we denote the group $\mathrm{SL}_n(\mathbb{A})$ also by $L_{\mathrm{alg}}\mathrm{SL}_n(\mathbb{C})$. The subgroup $\Omega_{\mathrm{alg}}(\mathrm{SL}_n(\mathbb{C}), c)$ is thus the subgroup $L_{\mathrm{alg}}\mathrm{SL}_n(\mathbb{C}) \cap \Omega_{\mathrm{diff}}(\mathrm{SL}_n(\mathbb{C}), c)$ of c-based algebraic loops (by c-based we mean that the base-point of $\mathbb{S}^1$ is c—the base point of the group is always the identity element).

THE CARTAN INVOLUTION

Consider the semi-linear involution # on the complex vector space $\mathbb{A}(n)$ which is given by

$$f = \sum_{\text{fin}} f_k z^k \mapsto f^{\#} = \sum_{\text{fin}} f^*_{-k} z^k.$$

The subgroup of all elements $g \in \mathrm{SL}_n(\mathbb{A})$ with $g^{-\#} = g$ is denoted by $L_{\text{alg}}\mathrm{SU}(n)$. This terminology is motivated by the fact that

$$L_{\text{alg}}\mathrm{SU}(n) = L_{\text{diff}}\mathrm{SU}(n) \cap L_{\text{alg}}\mathrm{SL}_n(\mathbb{C}).$$

To see this, note that for $z \in \mathbb{S}^1$ we have $z^{-1} = \bar{z}$, hence $f(z)f(z)^* = 1$ holds for all $z \in \mathbb{S}^1$ if and only if $f \in L_{\text{alg}}\mathrm{SU}(n)$. For $c \in \mathbb{S}^1$ we have a semi-direct decomposition

$$L_{\text{alg}}\mathrm{SU}(n) = \mathrm{SU}(n)\Omega_{\text{alg}}(\mathrm{SU}(n), c)$$

as before, and an $\mathbb{S}^1$-action which is given by $f(z) \mapsto f(az)$.

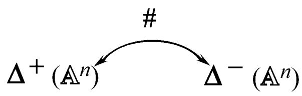
LEMMA 5.2. *We have* $L_{\text{alg}}\mathrm{SU}(n) \cap \mathrm{SL}_n(\mathbb{C}[z]) = \mathrm{SU}(n)$.

Proof. Let $f = f_0 + f_1 z + \cdots + f_k z^k \in \mathrm{SL}_n(\mathbb{C}[z])$. Then $f^{\#} = f^*_k z^{-k} + \cdots + f^*_1 z^{-1} + f_0{}^*$. If $f^{\#} = f^{-1} \in \mathrm{SL}_n(\mathbb{C}[z])$, then $f_i = 0$ for $i \geqslant 1$. □

In particular,

$$B^+_{\text{alg}} \cap L_{\text{alg}}\mathrm{SU}(n) \cong T^{n-1} \quad \text{and} \quad P^{i,+}_{\text{alg}} \cap L_{\text{alg}}\mathrm{SU}(n) \cong \mathrm{S}(\mathrm{U}(i) \cdot \mathrm{U}(n-i))$$

(as in Section 2, T^k denotes a compact torus of rank k). There is a more geometric description of the group $L_{\text{alg}}\mathrm{SU}(n)$. The map $\mathbb{A}^n \xrightarrow{\#} \mathbb{A}^n$ induces isomorphisms $\Delta^+(\mathbb{A}^n) \xrightarrow{\#} \Delta^-(\mathbb{A}^n)$ and $\Delta^-(\mathbb{A}^n) \xrightarrow{\#} \Delta^+(\mathbb{A}^n)$,



and $L_{\text{alg}}\mathrm{SU}(n) = \mathrm{Cen}_{\mathrm{SL}_n(\mathbb{A})}(\#)$. There is a corresponding Cartan decomposition of the loop algebra $\mathfrak{sl}_n(\mathbb{A})$ into eigenspaces of the involution #, $\mathfrak{sl}_n(\mathbb{A}) = L_{\text{alg}}\mathfrak{su}(n) \oplus \mathfrak{X}$, where $\mathfrak{X}$ denotes the traceless hermitian matrices in $\mathbb{A}(n)$.

THEOREM 5.3. *The group* $L_{\text{alg}}\mathrm{SU}(n)$ *acts transitively on the periodic flags,*

$$L_{\text{alg}}\mathrm{SL}_n(\mathbb{C}) = B^+_{\text{alg}} L_{\text{alg}}\mathrm{SU}(n), \qquad \mathrm{Fl}(\Delta^+(\mathbb{A}^n)) \cong L_{\text{alg}}\mathrm{SU}(n)/T^{n-1}.$$

Proof. The proof is exactly the same as in Lemma 2.1. The $L_{\text{alg}}\mathrm{SU}(n)$-stabilizer of a panel is isomorphic to $\mathrm{SU}(2) \cdot T^{n-2}$ and induces the transitive group SO(3) on the panel. □

In particular, $L_{\text{alg}}\mathrm{SU}(n)$ acts transitively on $\mathcal{V}^+_0$. Since $L_{\text{alg}}\mathrm{SU}(n)_{[M_0]} = \mathrm{SU}(n)$, the group $\Omega_{\text{alg}}\mathrm{SU}(n)$ acts *regularly* on $\mathcal{V}^+_0$,

$$\mathcal{V}_0^+ \cong L_{\mathrm{alg}}\mathrm{SU}(n)/\mathrm{SU}(n) \cong \Omega_{\mathrm{alg}}\mathrm{SU}(n),$$

and

$$\mathrm{Fl}(\Delta^+(\mathbb{A}^n)) \cong \Omega_{\mathrm{alg}}\mathrm{SU}(n) \times (\mathrm{SU}(n)/T^{n-1}).$$

Note that the proof above (which is due to Bernhard Mühlherr) is much simpler than the classical proof given, e.g., in [34], Theorem 8.3.2.

6. The Affine Veronese Representation of $\Delta^+(\mathbb{A}^n)$

In this section we construct an equivariant embedding $\Delta^+(\mathbb{A}^n) \hookrightarrow \mathfrak{sl}_n(\mathbb{A})$ which is very similar to the finite-dimensional Veronese representation $\Delta(\mathbb{C}^n) \hookrightarrow \mathfrak{p}_n \subseteq \mathfrak{sl}_n(\mathbb{C})$. Recall that we presented the flags in $\Delta(\mathbb{C}^n)$ as certain Hermitian operators. Similarly, we want to associate an operator to the $\mathbb{C}[z]$-module E_0. To this end we consider the first order linear differential operator

$$\mathbb{A} \longrightarrow \mathbb{A}, \qquad f \mapsto z\partial_z f = z\frac{\partial f}{\partial z}.$$

If $f = \sum_{\mathrm{fin}} f_k z^k$, then $z\partial_z f = \sum_{\mathrm{fin}} k f_k z^k$. Thus

$$\ker(z\partial_z - \lambda) = \begin{cases} z^\lambda \mathbb{C}, & \text{for } \lambda \in \mathbb{Z}, \\ 0, & \text{for } \lambda \in \mathbb{C} \setminus \mathbb{Z} \end{cases}$$

In particular,

$$\mathbb{C}[z] = \bigoplus_{k \geqslant 0} \ker(z\partial_z - k) \quad \text{and} \quad \mathbb{C}[1/z] = \bigoplus_{k \geqslant 0} \ker(z\partial_z + k).$$

The operator $z\partial_z$ extends in an obvious way to $\mathbb{A}^n$ and to the matrix algebra $\mathbb{A}(n)$. For $f \in \mathbb{A}^n$ we put $Df = z\partial_z f$. Let

$$\Pi_k = \mathrm{diag}(\underbrace{1, \ldots, 1}_{k}, 0, \ldots, 0) \in H(n)$$

and let $\Pi_k^{\mathrm{tls}} = \Pi_k - (k/n)\,\mathbf{1}$ denote its traceless image in $\mathfrak{p}_n$.
Thus

$$\begin{aligned} E_i^+ &= \mathrm{span}_{\mathbb{C}[z]}\{ze_1, \ldots, ze_i, e_{i+1}, \ldots, e_n\} \\ &= \bigoplus_{k \geqslant 0} \ker(D - \Pi_i - k\mathbf{1}) \\ &= \bigoplus_{k \geqslant 0} \ker\Big((D - \Pi_i^{\mathrm{tls}}) - \Big(k + \frac{i}{n}\Big)\mathbf{1}\Big); \end{aligned}$$

the elements of the flag varieties $\mathcal{V}_i^+$ correspond bijectively to the $L_{\mathrm{alg}}\mathrm{SU}(n)$-conjugates of $D - \Pi_i^{\mathrm{tls}}$. Let $g \in L_{\mathrm{alg}}\mathrm{SU}(n)$. Then

$$0 = z\partial_z(gg^\#) = (z\partial_z g)g^\# + gz\partial_z(g^\#).$$

Therefore

$$gDg^{\#}f = g(z\partial_z(g^{\#}))f + gg^{\#}Df = Df - (z\partial_z g)g^{\#}f,$$

whence $gDg^{\#} = D - (z\partial_z g)g^{\#}$, and

$$g(D - \Pi_i^{\mathrm{tls}})g^{\#} = D - (z\partial_z g)g^{\#} - g\Pi_i^{\mathrm{tls}}g^{\#}.$$

We put $\mathfrak{X} = \{X \in \mathfrak{sl}_n(\mathbb{A}) \mid X^{\#} = X\}$. This is an infinite-dimensional real vector space, and $(z\partial_z g)g^{\#} \in \mathfrak{X}$ for all $g \in L_{\mathrm{alg}}\mathrm{SU}(n)$. We construct an $L_{\mathrm{alg}}\mathrm{SU}(n)$-equivariant embedding of $\Delta^+(\mathbb{A}^n)$ into the infinite-dimensional real vector space $\mathfrak{X} \oplus \mathbb{R}D$ as follows.

$$\begin{aligned} &\mathcal{V}_k^+ \hookrightarrow \mathfrak{X} \oplus \mathbb{R}D, \\ &[gE_k^+] \mapsto g(D - \Pi_k^{\mathrm{tls}})g^{\#} = D - (z\partial_z g)g^{\#} - g\Pi_k^{\mathrm{tls}}g^{\#}. \end{aligned}$$

We extend this mapping to the geometric realization $|\Delta^+(\mathbb{A}^n)|$ of $\Delta^+(\mathbb{A}^n)$ in the canonical way, and we call the resulting $L_{\mathrm{alg}}\mathrm{SU}(n)$-equivariant map

$$|\Delta^+(\mathbb{A}^n)| \hookrightarrow \mathfrak{X} \oplus \mathbb{R}D \xrightarrow{-pr_1} \mathfrak{X}$$

the *affine Veronese representation* of $\Delta^+(\mathbb{A}^n)$. Explicitly, the Veronese representation of the vertex $[gE_k]$ is

$$\Phi([gE_k]) = g\Pi_k^{\mathrm{tls}}g^{\#} + (z\partial_z g)g^{\#}.$$

Note that the group $L_{\mathrm{alg}}\mathrm{SU}(n)$ acts through *gauge transformations*

$$X \mapsto gXg^{\#} + (z\partial_z g)g^{\#}$$

on $\mathfrak{X}$. Similarly as in the spherical case, it is not difficult to prove that Φ injects the geometric realization $|\Delta^+(\mathbb{A}^n)|$ into $\mathfrak{X}$. The partial flags in the building $\Delta^+(\mathbb{A}^n)$ correspond thus to certain operators $D - X \in \mathfrak{X} \oplus \mathbb{R}D$ with finite-dimensional nontrivial eigenspaces. Note also that # swaps $\Delta^+(\mathbb{A}^n)$ and $\Delta^-(\mathbb{A}^n)$, so Φ is at the same time a Veronese representation for $\Delta^-(\mathbb{A}^n)$ with exactly the same image.

CAVEAT

The affine Veronese representation is *not* surjective. Let $a(z) = z + 1/z$ and $r \in \mathbb{R}^\times$. The differential equation

$$(z\partial_z - ra)f = \lambda f$$

has the solution $f(z) = \mathrm{e}^{r(z-1/z)}z^{\lambda} \cdot \mathrm{const}$. This function is holomorphic on $\mathbb{C}^\times$ if and only if λ is an integer, but it is not meromorphic on $\mathbb{C} \cup \{\infty\}$, so $f \notin \mathbb{A}$. Thus, the traceless diagonal matrix $X = \mathrm{diag}(a, \ldots, a, (1-n)a) \in \mathfrak{X}$ does *not* represent a flag of the building because $D - X$ has no nontrivial eigenspaces.

The space $\mathfrak{X}$ is a subspace of the *loop algebra* $\mathfrak{sl}_n(\mathbb{A})$, which in turn is contained in the semi-direct product $\mathfrak{sl}_n(\mathbb{A}) \oplus \mathbb{C}D$. So $\mathfrak{X} \oplus \mathbb{R}D$ plays a very similar rôle as $\mathfrak{p}_n \subseteq \mathfrak{sl}_n(\mathbb{C})$. Let Q denote the collection of all barycenters of images of chambers in $\mathfrak{X}$. The closure M of Q in the Hilbert space completion of the real pre-Hilbert space $\mathfrak{X}$ is an *infinite-dimensional isoparametric submanifold.* See [19, 33] for more information. The set Q coincides with the subset $Q(p) \subseteq M$ introduced in [18] by Heintze and Liu. In [12], we prove that all known isoparametric submanifolds of rank at least 3 in Hilbert spaces arise in a uniform way from Veronese representations of twin buildings.

7. Topological Geometry and Bott Periodicity

In this section we propose a definition of topological twin buildings. Since spherical buildings are twin buildings, this is at the same time a definition of topological spherical buildings. Definitions of spherical topological buildings have been proposed by Burns and Spatzier [9], Jäger [20], Kühne [25], and myself; Mitchell [27] proposes an *ad hoc* definition of topological *BN*-pairs. For spherical buildings of rank 2 there is a well-established theory, see [13, 14, 17, 23, 24, 39, 40, 42]. The starting point is always a topology on the set of vertices of the building (i.e. on the 0-simplices). Using the type function, the simplices of higher rank can be interpreted as ordered tuples of vertices, and thus one obtains a topology on Δ; the question then is which maps should be continuous. Burns and Spatzier [9] require only that $\mathrm{Cham}(\Delta)$ should be closed, i.e. that every net of chambers, viewed as a net of r-tuples of vertices, converges to some chamber. Moreover, they claim that for the classical geometries (projective spaces) this agrees with the traditional notion of a topological geometry, see *loc.cit*. p. 1. This is definitely not true, and it is not difficult to construct perverse topologies on nice geometries which satisfy their condition nevertheless. However, in the *compact* spherical case, their definition is the correct one (and that's the only instance where they need it in their work [9]), see Proposition 7.5 below.

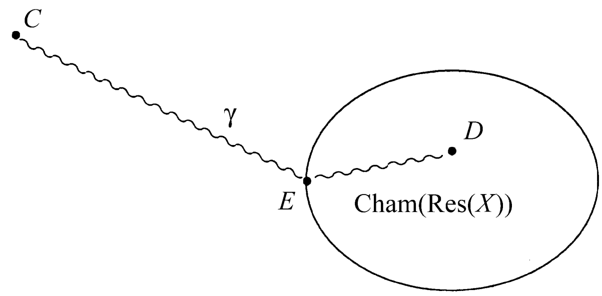
Compactness or local compactness in the non-spherical case leads to locally finite buildings (finite panels), and this in turn is related to locally compact CAT(0)-spaces; however, these matter are not within the scope of the present article.

Similarly as in [20], our notion of a topological building asks for the continuity of certain *projections*. The fact that this definition makes sense for twin buildings was pointed out by Bernhard Mühlherr during a meeting in Oberwolfach back in 1992; then, we planned to write a joint paper with Martina Jäger on projections and topologies in buildings which, however, never came to existence. This section is a first approximation of what we had in mind.

The fact that topological buildings can be used to prove Bott periodicity is particularly appealing, since topological *K*-theory is an important ingredient in topological geometry; many classification results in [23, 40] depend in an essential way on Bott periodicity.

PROJECTIONS IN (TWIN) BUILDINGS

Tits defined projections in (spherical) buildings in [48]; a modern account based on metric properties of buildings is given in [11]. Let C, D be chambers in a building Δ, let $\delta(C, D) = w$, and let $w = s_{i_1} \cdots s_{i_r}$ be a reduced (minimal) expression for w in terms of the generating set S. Then there exists a unique *minimal gallery* $\gamma = (C_0 = C, C_1, C_2, \ldots, C_r = D)$ *of type* $(s_{i_1}, \ldots, s_{i_r})$, consisting of chambers $C_0, \ldots, C_r$, such that $\delta(C_{k-1}, C_k) = s_{i_k}$ holds for $k = 1, \ldots, r$. Now let $X \in \Delta$ be a simplex and let C be a chamber. Then there exists a unique chamber E in $\mathrm{Res}(X)$ which we denote $E = \mathrm{proj}_X C$, the *projection of C onto X*, with the following property: for every chamber $D \in \mathrm{Res}(X)$, and for every minimal gallery γ starting at C and ending at D, the first chamber in γ which is contained in $\mathrm{Res}(X)$ is E, the *gate* of $\mathrm{Res}(X)$ with respect to C.

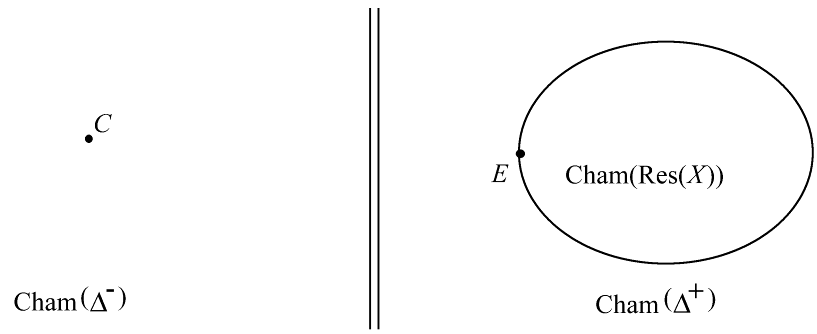


Note that $\mathrm{proj}_X C = C$ if $C \in \mathrm{Cham}(\mathrm{Res}(X))$. If $Y \in \Delta$ is an arbitrary simplex, then there exists a unique simplex Z which is contained in some chamber in $\mathrm{Res}(X)$, such that

$$\mathrm{Cham}(\mathrm{Res}(Z)) = \mathrm{proj}_X \mathrm{Cham}(\mathrm{Res}(Y)),$$

and we put $Z = \mathrm{proj}_X Y$.

Now suppose that $(\Delta^+, \Delta^-, \delta^*)$ is a twin building, that $X \in \Delta^+$ is a *spherical* simplex (recall from Section 1 that this means that $\mathrm{Res}(X)$ is spherical), and that $C \in \Delta^-$ is a chamber. Then there exists a unique chamber $E \in \mathrm{Res}(X)$ which maximizes the numerical codistance function $D \mapsto \ell(\delta^*(C, D))$ on $\mathrm{Cham}(\mathrm{Res}(X))$, see Ronan [36] (4.1). Intuitively, a 'small' codistance corresponds to a 'big' distance, so E is the chamber 'closest' to C; note that $D \mapsto \ell(\delta^*(C, D))$ is bounded above because X is spherical.



The chamber E is again denoted $E = \mathrm{proj}_X C$, and the projection $\mathrm{proj}_X Y$ of an arbitrary simplex $Y \in \Delta^-$ onto X is defined exactly as before (but only for spherical X!).

7.1 *Schubert Cells*

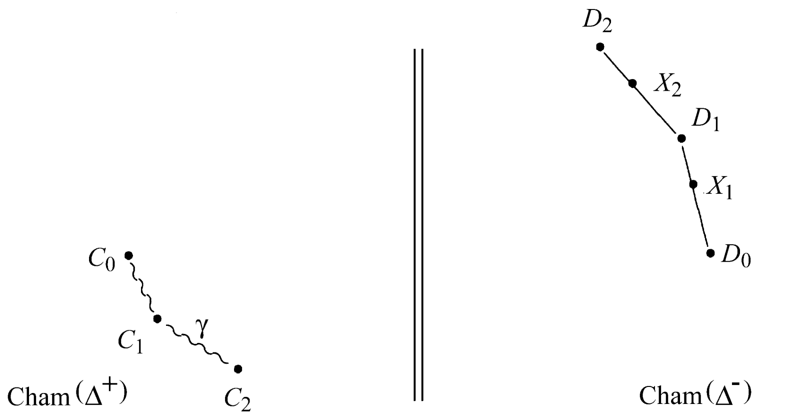
Let C_0 be a chamber in a building Δ, let $J \subseteq I$, and let $wW_J \in W/W_J$. The set

$$\mathcal{C}_{wW_J}(C_0) = \{X \in \Delta |\ \delta(C_0, X) = wW_J\}$$

is called a *Schubert cell* in Δ. Schubert cells in general buildings have no special structure (e.g., if Δ is a tree), but Schubert cells in halves of twin buildings (and in particular Schubert cells in spherical buildings) have a nice product structure, i.e. they admit coordinates (labels).

7.2. *Coordinatizing a Half Twin*

Let $C_0 \in \Delta^+$ and $D_0 \in \Delta^-$ be a pair of opposite chambers in a twin building $(\Delta^+, \Delta^-, \delta^*)$. These two chambers determine a unique apartment $A \subseteq \Delta^-$ (half of the *twin apartment* spanned by (C_0, D_0), see, e.g., [36], 2.8). Let $w \in W$, and let $w = s_{i_1} \cdots s_{i_r}$ be a reduced expression. Let $E \in \mathcal{C}_w(C_0)$, and let $C_0, \ldots, C_r = E$ be a (necessarily minimal) gallery of type $(s_{i_1}, \ldots, s_{i_r})$. Let $D_0, \ldots, D_r$ be the unique minimal gallery of the same type in the apartment $A \subseteq \Delta^-$. Then $C_k \,\mathrm{op}\, D_k$ holds for $k = 0, \ldots, r$. We define *r coordinates* $(X_1, \ldots, X_r)$ by $X_k = \mathrm{proj}_{D_k \sqcap D_{k-1}} C_k$.



Note that $X_k \in \mathrm{Res}(D_k \sqcap D_{k-1}) \setminus \{D_k\}$. Now the point is that step by step, the whole gallery γ can be recovered from these coordinates;

$$C_1 = \mathrm{proj}_{C_0 \sqcap C_1} X_1, \qquad C_2 = \mathrm{proj}_{C_1 \sqcap C_2} X_2, \qquad C_3 = \mathrm{proj}_{C_2 \sqcap C_3} X_2, \quad \text{etc.},$$

since C_{k-1} and s_{i_k} determine $C_{k-1} \sqcap C_k$, the information needed is only C_0, the coordinates $(X_1, \ldots, X_r)$, and the reduced expression $(s_{i_1}, \ldots, s_{i_r})$. Note also that different reduced expressions for a Schubert cell lead to different coordinates; our coordinatization process depends on a choice of a reduced expression for every element $w \in W$. In the case of spherical buildings of rank two, the different expressions of the longest element in the Coxeter group and the resulting ‘changes of coordinates’ lead to Van Maldeghem’s coordinatizing rings [52].

The process above yields coordinates for the Schubert cells $\mathcal{C}_w(C_0) \subseteq \mathrm{Cham}(\Delta^+)$. For a coset $wW_J \in W/W_J$ we may assume that w is the unique *shortest coset representative*. The canonical ‘forgetful’ map $\mathcal{C}_w(C_0) \longrightarrow \mathcal{C}_{wW_J}(C_0)$ which sends a chamber C to the unique subsimplex $C|_{I\setminus J}$ of type $I \setminus J$ contained in C is a bijection; in fact, $C = \mathrm{proj}_{C|_{I\setminus J}} C_0$. The Schubert cell $\mathcal{C}_{wW_J}(C_0)$ can also directly be coordinatized, by the same method as described above. In any case we see that each Schubert cell is in a natural correspondence with a finite product of punctured panels; for $\Delta^+(\mathbb{A}^n)$ we see that the Schubert cell $\mathcal{C}_{wW_J}(C_0)$ bijects onto $\mathbb{C}^m$, where m is the length of the shortest coset representative of wW_J.

TOPOLOGICAL TWIN BUILDINGS

Let $(\Delta^+, \Delta^-, \delta^*)$ be a twin building. Suppose that there is a Hausdorff topology on the set of vertices (the 0-simplices) of both buildings. The simplices of type J can be regarded as J-tuples of vertices; in this way, the topology on the vertices determines a topology on both buildings. For $J, K \subseteq I$ and $w \in W$ we put

$$\mathcal{D}^{J,K}_{W_J w W_K} = \{(X, Y) \in \Delta^+ \times \Delta^- \mid \mathrm{type}(X) = I \setminus J, \mathrm{type}(Y) = I \setminus K, \delta^*(X, Y) = W_J w W_K\}.$$

DEFINITION 7.3. A twin building is called a *topological twin building* if the following condition is satisfied:

TTB If $J \subseteq I$ is spherical (i.e. if W_J is finite) and $K \subseteq I$ is arbitrary, then $(X, Y) \longmapsto \mathrm{proj}_X Y$ is continuous on the set $\mathcal{D}^{J,K}_{W_J W_K}$.

The condition $\delta^*(X, Y) = W_J W_K$ means that X and Y are *almost opposite*, i.e. that there exist chambers $C \geqslant X$ and $D \geqslant Y$ with $C \,\mathrm{op}\, D$.

Remark 7.4. For spherical buildings of rank 2 and for projective spaces, this *is* the common notion of a topological building, ([23, 25, 26]).

In the compact spherical case, there is a nice criterion due to Grundhöfer and Van Maldeghem: □

PROPOSITION 7.5. *If Δ is spherical, and if the topology on the vertex set is compact, then Δ is a topological building if and only if the chamber set is compact.*

Proof. This follows from the closed graph theorem for maps into compact spaces, see [17]. □

We just mention the following result.

PROPOSITION 7.6. *Let $(\Delta^+, \Delta^-, \delta^*)$ be a topological twin building, and let $X \in \Delta^+$ be spherical. Then* $\mathrm{Res}(X)$ *is in a natural way a topological building. If Z is another* *simplex of the same type as* X (*in either half of the twin building*), *then* $\mathrm{Res}(Z)$ *is continuously isomorphic to* $\mathrm{Res}(X)$.

Proof. Pick $Y \in \Delta^-$ opposite X. Then proj induces an isomorphism $\mathrm{Res}(X) \cong \mathrm{Res}(Y)$. Let $U, V \in \mathrm{Res}(X)$ be simplices with maximal W_J-distance. We need to show that $\mathrm{proj}_V U$ is continuous. But $\bar{U} = \mathrm{proj}_Y U$ depends continuously on U, and $\mathrm{proj}_V U = \mathrm{proj}_V \bar{U}$. For the second claim one uses the following fact which is not difficult to prove (see the proof by Tits [48], 3.30, in the spherical case): given simplices X, X' (in the same half of the building and of the same type), there exists a simplex X'' in the other half which is opposite both to X and to X'. □

EXAMPLE 7.7. The building $\Delta(\mathbb{C}^n)$, with the natural topology on its vertex set $\mathrm{Gr}_1(\mathbb{C}^n) \cup \mathrm{Gr}_2(\mathbb{C}^n) \cup \cdots \cup \mathrm{Gr}_{n-1}(\mathbb{C}^n)$ is a topological building by Proposition 7.5, since $\mathrm{Fl}(\mathbb{C}^n)$ is compact.

There is a natural topology on the group $\mathrm{SL}_n(\mathbb{A})$; the evaluation map $\mathrm{SL}_n(\mathbb{A}) \longrightarrow \mathcal{V}_i$ induces a topology on the set $\mathcal{V}_0 \cup \cdots \cup \mathcal{V}_{n-1}$ of vertices of $\Delta^+(\mathbb{A}^n)$ and similarly on the vertices of $\Delta^-(\mathbb{A}^n)$.

THEOREM 7.8. *The twin building* $(\Delta^+(\mathbb{A}^n), \Delta^-(\mathbb{A}^n), \delta^*)$ *is a topological twin building*.

Sketch of proof. Let $(C, D) \in \Delta^+(\mathbb{A}^n) \times \Delta^-(\mathbb{A}^n)$ be a pair of opposite chambers, let $X \leqslant C$ be spherical of type $I \setminus J$ and $Y \leqslant D$ of type $I \setminus K$. Since $\mathrm{SL}_n(\mathbb{A})$ acts strongly transitively on the twin building, it acts transitively on the set $\mathcal{D}^{J,K}_{W_J W_K}$, and $\mathcal{D}^{J,K}_{W_J W_k} \cong \mathrm{SL}_n(\mathbb{A})/\mathrm{SL}_n(\mathbb{A})_{X,Y}$. Now $\mathrm{SL}_n(\mathbb{A})_{X,Y}$ fixes $Z = \mathrm{proj}_X Y$, and we have a continuous map $\mathrm{SL}_n(\mathbb{A})/\mathrm{SL}_n(\mathbb{A})_{X,Y} \longrightarrow \mathrm{SL}_n(\mathbb{A})/\mathrm{SL}_n(\mathbb{A})_Z$. It follows that the map $(X', Y') \longmapsto \mathrm{proj}_{X'} Y'$ is continuous on $\mathcal{D}^{J,K}_{W_J W_K}$. □

Let $\preceq$ denote the *Bruhat order* on W/W_J, for all $J \subseteq I$, and put

$$\mathcal{C}_{\preceq w W_J}(C_0) = \bigcup \{\mathcal{C}_{vW_J}(C_0) |\; vW_J \preceq wW_J\};$$

this set is called the *Schubert variety* corresponding to wW_J. Let $m(wW_J)$ denote the ℓ-length of the shortest coset representative of wW_J.

PROPOSITION 7.9. *The Schubert varieties* $\mathcal{C}_{wW_J}(C_0)$ *in* $\Delta^+(\mathbb{A}^n)$ *are CW-complexes, with Poincaré series* $\sum_{vW_J \preceq wW_J} t^{2m(vW_J)}$.

Sketch of proof. Let w be a shortest coset representative for wW_J, and let $\mathcal{G}_{s_1,\ldots,s_r}(C_0)$ denote the collection of all (possibly stammering) galleries of type $(s_1, \ldots, s_r)$ (for some fixed reduced expression $s_1 \cdots s_r$ for w), starting with the chamber C_0. It is not difficult that to show that $\mathcal{G}_{s_1,\ldots,s_r}(C_0)$ is an iterated $\mathbb{C}\mathrm{P}^1$-bundle (sometimes called a *Bott-Samelson cycle*); the total space is a smooth manifold of dimension $2m(wW_J) = 2\ell(w)$. These gallery spaces are also known as *Bott-Samelson desingularizations* of Schubert varieties. Consider the endpoint map

$$\mathcal{G}_{s_1,\ldots,s_r}(C_0) \xrightarrow{\rho} \mathcal{C}_{\preceq wW_J}(C_0).$$

The non-stammering galleries in $\mathcal{G}_{s_1,\ldots,s_r}(C_0)$ are mapped bijectively onto the Schubert cell $\mathcal{C}_{wW_J}(C_0)$. There is a canonical injection

$$\mathcal{G}_{s_1,\ldots,s_{r-1}}(C_0) \longrightarrow \mathcal{G}_{s_1,\ldots,s_r}(C_0)$$

(by stammering at the end). The stammering galleries in $\mathcal{G}_{s_1,\ldots,s_r}(C_0)$ are either of this type, or galleries which don't stammer at the end, but somewhere before the end. Using this and the fact that

$$\mathcal{G}_{s_1,\ldots,s_r}(C_0) \longrightarrow \mathcal{G}_{s_1,\ldots,s_{r-1}}(C_0)$$

is a $\mathbb{C}\mathrm{P}^1$-bundle, one can use induction on $\ell(w)$ to define a cellular map

$$\mathrm{e}^2 \times \mathrm{e}^{2\ell(w)-2} \xrightarrow{\phi} \mathcal{G}_{s_1,\ldots,s_r}(C_0)$$

which maps the boundary of the cell $\mathrm{e}^2 \times \mathrm{e}^{2\ell(w)-2}$ onto the stammering galleries, see [27], Thm. 2.22, and [23], Section 4.1. Now $\rho \circ \phi$ is an attaching map for a $2\ell(w)$-cell. □

It follows that the based loop group $\Omega_{\mathrm{alg}}\mathrm{SU}(n)$ has a CW decomposition with Poincaré series

$$\sum_{wA_{n-1} \in \tilde{A}_{n-1}/A_{n-1}} t^{2m(wA_{n-1})}$$

where W is the affine Weyl group of type $\tilde{A}_{n-1}$ generated by $s_1, \ldots, s_n$, and A_{n-1} the subgroup generated by $s_1, \ldots, s_{n-1}$.

KNARR'S CONSTRUCTION FOR $\Delta^+(\mathbb{A}^n)$

We apply Knarr's construction to the the halves of the topological twin building $(\Delta^+(\mathbb{A}^n), \Delta^-(\mathbb{A}^n), \delta^*)$. The main ideas can be found in Mitchell's paper [27], although our approach (which is the same as Knarr's approach [21]) is more based on geometric properties (i.e. the coordinatization of twin buildings), whereas Mitchell makes strong use of the BN-pair. Let $|\Delta^+(\mathbb{A}^n)|$ denote the geometric realization of $\Delta^+(\mathbb{A}^n)$. By Theorem 7.8, there is a canonical topology on the flag space $\mathrm{Fl}(\Delta^+(\mathbb{A}^n))$ and we endow $|\Delta^+(\mathbb{A}^n)|$ with the quotient topology induced by the map $\mathrm{Fl}(\Delta^+(\mathbb{A}^n)) \times |\blacktriangle^{n-1}| \longrightarrow |\Delta^+(\mathbb{A}^n)|$. The resulting space is denoted $|\Delta^+(\mathbb{A}^n)|_{\mathrm{Knarr}}$.

More generally, assume that $(\Delta^+, \Delta^-, \delta^*)$ is a topological twin building, and that the panels are topological spheres. For example, in $\Delta^+(\mathbb{A}^n)$ the panels are homeomorphic to $\mathbb{C}\mathrm{P}^1 \cong \mathbb{S}^2$. Moreover, panels of the same type are homeomorphic by Proposition 7.6. Let $m(s_i)$ denote the topological dimension of the i-panels in Δ^+. It is proved in Kramer [23] Prop. 2.0.2 that $m(s_i) = m(s_j)$ holds whenever m_{ij} is odd. Thus we obtain a $\mathbb{Z}$-length $m : W \longrightarrow \mathbb{Z}$. For a Schubert cell $\mathcal{C}_w(C_0)$, we have

$$\mathcal{C}_w(C_0) \cong \mathbb{R}^{m(w)}.$$

PROPOSITION 7.10 ([21] and [27] 2.16). *Let* $(\Delta^+, \Delta^-, \delta^*)$ *be a topological twin building. Assume that the panels are topological spheres. For* $w \in W$; *let* X_w *denote the image of* $\mathcal{C}_w(C_0) \times |\blacktriangle^{r-1}|$ *in* $|\Delta^+|_{\mathrm{Knarr}}$. *Then for each* $w \in W$, *the set* $\bigcup\{X_v|\ v \prec w\}$ *is contractible.*

Sketch of proof. The proof is by induction on the length $\ell(w)$. Note that $X_1 = \{C_0\} \times |\blacktriangle^{r-1}|$ is contractible. Moreover,

$$\mathcal{C}_{\preceq w}(C_0)\Big/\bigcup\{\mathcal{C}_u(C_0)|\ u \prec w\} \cong \mathbb{S}^{m(w)},$$

because this quotient is the one-point compactification of the Schubert cell $\mathcal{C}_w(C_0)$.

The inductive step is accomplished as follows. First of all, $\bigcup\{X_v|\ v \prec u\}$ is contractible for all $u \prec w$. Then it is not hard to see that $\bigcup\{X_v|\ v \preceq u\}/\bigcup\{X_v|\ v \prec u\}$ is also contractible. This implies that $\bigcup\{X_v|\ v \preceq u\}$ is contractible. Finally, $\bigcup\{X_v|\ v \prec w\}$ is homotopy equivalent to a wedge of such contractible spaces and hence itself contractible. □

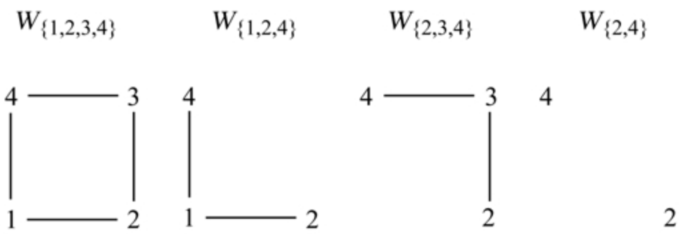
COROLLARY 7.11. *If* Δ^+ *is spherical, then* $|\Delta^+|_{\mathrm{Knarr}}$ *is homeomorphic to a sphere of dimension* $m(w_0) + r - 1$, *where* w_0 *is the unique longest element in the Coxeter group W, and r is the rank of the building. In the non-spherical case,* $|\Delta^+|_{\mathrm{Knarr}}$ *is contractible.*

Sketch of proof. In the spherical case, let $w_0 \in W$ denote the longest element; this is at the same time the unique maximal element in the Bruhat order. Then $X_{w_0}\setminus\bigcup\{X_v|\ v \prec w_0\} \cong \mathbb{R}^{m(w_0)} \times \mathbb{R}^{r-1}$, so $X_{w_0}/\bigcup\{X_v|\ v \prec w_0\} \cong \mathbb{S}^{m(w_0)} \wedge \mathbb{S}^{r-1}$, and this space is homotopy equivalent to $|\Delta^+|_{\mathrm{Knarr}}$, since $\bigcup\{X_v|\ v \prec w_0\}$ is contractible. Finally, it is not difficult to see that X_{w_0} is a manifold, cf. [23], Prop. 4.2.1, hence $|\Delta^+|_{\mathrm{Knarr}}$ is a compact manifold (of dimension at least 5) homotopy equivalent to a sphere, and thus to homeomorphic to $\mathbb{S}^{m(w_0)+r-1}$ by the proof of the generalized Poincaré conjecture due to Smale [43], Stallings [44], and Zeemann [53]. In the nonspherical case, $|\Delta^+|_{\mathrm{Knarr}}$ is a limit of contractible spaces and hence itself contractible. □

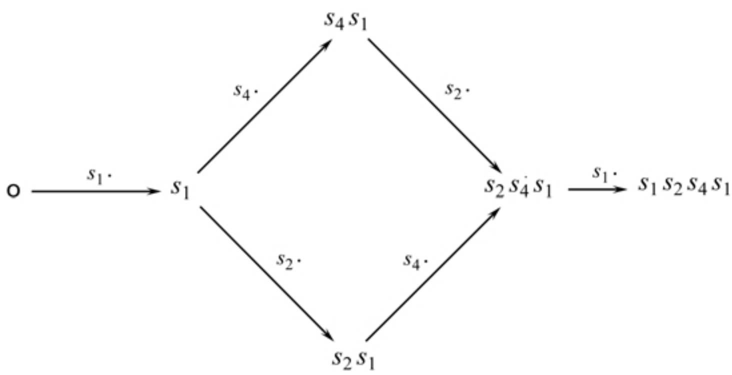
The theorem above can be proved in much greater generality; it suffices to assume that the panels are compact, connected, and of finite covering dimension. Under these assumptions, the corollary holds up to homotopy equivalence, see [23] Section 3.3. On the other hand, if the vertices of the building are endowed with the discrete topology, then the theorem leads to a quick proof of the Solomon–Tits Theorem.

BOTT PERIODICITY

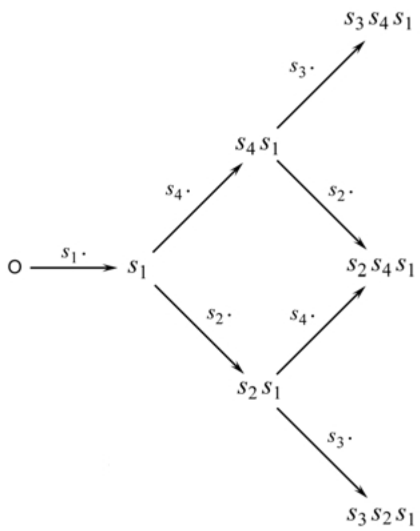
The crucial step in the proof of Bott periodicity is the following observation. We have $\mathrm{Res}([E_0]) \cong \Delta(\mathbb{C}^n)$, and thus a natural map $\mathrm{Gr}_k(\mathbb{C}^n) \hookrightarrow \mathcal{V}_k$. But the first terms in the Poincaré series for these two spaces agree; the first few shortest coset representatives for W/W_J and $W_K/W_{J\cap K}$ are the same, where $J = \{2, \ldots, n\}$ and $K = \{2, \ldots, k-1, k+1, \ldots, n\}$. For example, we have the following cell structure for $n = 4$. Put $I = \{1, 2, 3, 4\}$ and let $J = \{2, 3, 4\}$ and $K = \{1, 2, 4\}$; thus we have the following Coxeter groups.



For $W_{\{1,2,4\}}/W_{\{4,2\}}$ we have the following Bruhat order for the shortest coset representatives ([41], 2.5).



Thus we see that $Gr_2(\mathbb{C}^4)$ has a cell decomposition as $e^0 \cup e^2 \cup e^4 \cup e^4 \cup e^6 \cup e^8$. The Bruhat order for the shortest coset representatives of $W_{\{1,2,3,4\}}/W_{\{2,3,4\}}$ starts as



(and continues infinitely to the right). Accordingly, the cell structure of $\mathcal{V}_2 \cong \Omega_{\mathrm{alg}}\mathrm{SU}(4)$ is $\mathrm{e}^0 \cup \mathrm{e}^2 \cup \mathrm{e}^4 \cup \mathrm{e}^4 \cup \mathrm{e}^6 \cup \mathrm{e}^6 \cup \mathrm{e}^6 \cup \cdots$; in particular, the 5-skeleton of $\mathcal{V}_2$ is the same as for $Gr_2(\mathbb{C}^4)$. The general result is as follows.

PROPOSITION 7.12. *Suppose that $n = 2k$ is even. Then the inclusion $Gr_k(\mathbb{C}^{2k}) \hookrightarrow \mathcal{V}_k$ is a 2k-equivalence.* □

The proof for Bott periodicity is now as follows. If k is large, then $Gr_k(\mathbb{C}^{2k})$ is a good approximation for the classifying space BU; so in the limit, we obtain a homotopy equivalence $\mathrm{BU} \simeq \Omega_{\mathrm{alg}}\mathrm{SU}$. There is one problem, though; we have only considered the space $\Omega_{\mathrm{alg}}\mathrm{SU}(n)$ consisting of all based loops which can be expressed as *Laurent polynomials*, whereas topologists consider the space $\Omega_{\mathrm{cts}}\mathrm{SU}(n)$ of all *continuous* (or smooth) based loops. So the proof is not yet finished. At this point it is convenient to introduce Quillen's *space of special paths*

$$\mathcal{S} = \{(t \mapsto g(\mathrm{e}^{2\pi i t})\mathrm{e}^{2\pi i t X} g^{\#}(1)) \mid X \in \blacktriangleleft,\ g \in L_{\mathrm{alg}}\mathrm{SU}(n)\} \subseteq (\mathrm{SU}(n), 1)^{([0,1],0)}$$

where $\blacktriangleleft \subseteq \mathfrak{p}_n$ is a Weyl chamber. The group $L_{\mathrm{alg}}\mathrm{SU}(n)$ acts in a natural way on $\mathcal{S}$. But $\blacktriangleleft \subseteq \mathfrak{p}_n \subseteq \mathfrak{X}$ can also be identified with the image of the chamber $\{[E_0], \ldots, [E_{n-1}]\}$ under the Veronese representation. This identification extends in a natural way to an $L_{\mathrm{alg}}\mathrm{SU}(n)$-equivariant homeomorphism

$$\mathcal{S} \xrightarrow{\cong} |\Delta^+(\mathbb{A}^n)|_{\mathrm{Knarr}} \subseteq \mathfrak{X}$$

as follows: a path $\gamma \in \mathcal{S}$ is mapped to its 'logarithmic derivative' $1/2\pi\mathtt{i}(\mathrm{d}/\mathrm{d}t\gamma(t))\gamma(t)^{-1}$ which is a path in the tangent space $T_1\mathrm{SU}(n)$. (Conversely, a smooth path in the tangent space can be read as a differential equation whose solution starting at 1 is a path in the Lie group.) If we put $z = \mathrm{e}^{2\pi\mathtt{i}t}$, then

$$\frac{\mathrm{d}}{\mathrm{d}t}(g(z)\mathrm{e}^{2\pi\mathtt{i}tX}g^{\#}(1)) = 2\pi\mathtt{i}((z\partial_z g(z))\mathrm{e}^{2\pi\mathtt{i}tX}g^{\#}(1) + g(z)X\mathrm{e}^{2\pi\mathtt{i}tX}g^{\#}(1)),$$
$$\left(\frac{\mathrm{d}}{\mathrm{d}t}(g(z)\mathrm{e}^{2\pi\mathtt{i}tX}g^{\#}(1))\right)(g(z)\mathrm{e}^{2\pi\mathtt{i}tX}g^{\#}(1))^{-1} = 2\pi\mathtt{i}((z\partial_z g(z))g^{\#}(z) + g(z)Xg^{\#}(z)).$$

LEMMA 7.13. *The map*

$$\gamma \longmapsto \frac{1}{2\pi\mathtt{i}}\left(\frac{\mathrm{d}}{\mathrm{d}t}\gamma(t)\right)\gamma(t)^{-1}$$

is an $L_{\mathrm{alg}}\mathrm{SU}(n)$-equivariant homeomorphism $\mathcal{S} \longrightarrow |\Delta^+(\mathbb{A}^n)|_{\mathrm{Knarr}} \subseteq \mathfrak{X}$, where the action on $\Delta^+(\mathbb{A}^n)$ is the standard one, and the action on Quillen's space of special paths is by $\gamma(t) \longmapsto g(\mathrm{e}^{2\pi\mathtt{i}t})\gamma(t)g^{-1}(1)$. □

In particular, $\mathcal{S}$ is contractible. The endpoint map $(\mathrm{SU}(n), \mathbf{1})^{([0,1],0)} \longrightarrow \mathrm{SU}(n)$ yields the universal bundle

$$\Omega_{\mathrm{cts}}\mathrm{SU}(n) \longrightarrow (\mathrm{SU}(n), \mathbf{1})^{([0,1],0)} \longrightarrow \mathrm{SU}(n),$$

(where cts refers to the group of continuous loops) and as a subbundle we have

$$\Omega_{\mathrm{alg}}\mathrm{SU}(n) \longrightarrow \mathcal{S} \longrightarrow \mathrm{SU}(n).$$

Since the total spaces of both bundles are contractible, the inclusion $\Omega_{\mathrm{alg}}\mathrm{SU}(n) \hookrightarrow \Omega_{\mathrm{cts}}\mathrm{SU}(n)$ is a weak (and therefore also a strong) homotopy equivalence. We have proved Quillen's following result:

THEOREM 7.14 (Quillen). *The orbit space* $|\Delta^+(\mathbb{A}^n)|_{\mathrm{Knarr}}/\Omega_{\mathrm{alg}}\mathrm{SU}(n)$ *is homeomorphic to* $\mathrm{SU}(n)$. □

Recall that $\mathrm{Gr}_k(\mathbb{C}^n)$ is a good approximation of the classifying space BU in small dimensions.

COROLLARY 7.15 (Unitary Bott Periodicity). *The inclusion*

$$\mathrm{Gr}_k(\mathbb{C}^{2k}) \longrightarrow \Omega_{\mathrm{alg}}\mathrm{SU}(k) \overset{\simeq}{\hookrightarrow} \Omega_{\mathrm{cts}}\mathrm{SU}(k)$$

is a $2k$*-equivalence. In the limit, the natural map*

$$\mathrm{BU} \longrightarrow \Omega_{\mathrm{cts}}\mathrm{SU}$$

is a homotopy equivalence. □

This implies in particular the famous Bott isomorphisms $\pi_{2k}(\mathrm{U}) = 0$ and $\pi_{2k+1}(\mathrm{U}) \cong \mathbb{Z}$, for all $k \geqslant 0$. Note that we have also proved that the $\Omega_{\mathrm{alg}}\mathrm{SU}(n)$-orbit space map

$$|\Delta^+(\mathbb{A}^n)|_{\mathrm{Knarr}} \longrightarrow \mathrm{SU}(n)$$

is a universal classifying bundle for $\Omega_{\mathrm{alg}}\mathrm{SU}(n)$.

8. Real Forms and Compact Symmetric Spaces

So far, we have discussed the group $\mathrm{SL}_n(\mathbb{A})$ which is the proper generalization of the complex group $\mathrm{SL}_n(\mathbb{C})$. In this last section we consider briefly how real groups fit into the picture; more details can be found in [12] and [27]. Consider the involution ι given by

$$\sum_{\mathrm{fin}} X_\nu z^\nu \longmapsto \sum_{\mathrm{fin}} \bar{X}_\nu z^\nu.$$

The group of ι-fixed elements in $\mathrm{SL}_n(\mathbb{A})$ is $\mathrm{SL}_n(\mathbb{R}[z, 1/z])$, and there is a corresponding twin building over $\mathbb{R}[z, 1/z]$ which is defined exactly in the same way as the one over $\mathbb{C}[z, 1/z]$ considered so far. Note however that we cannot interpret the elements of $\mathrm{SL}_n(\mathbb{R}[z, 1/z])$ as loops in $\mathrm{SL}_n(\mathbb{R})$. Instead, we view the elements of $\mathrm{SL}_n(\mathbb{R}[z, 1/z])$ as paths

$$\begin{aligned}[0, 1] &\longrightarrow L_{\mathrm{alg}}\mathrm{SL}_n(\mathbb{C})\\ t &\longmapsto g(\mathrm{e}^{\mathrm{i}\pi t}).\end{aligned}$$

These paths have the special property that they start and end in $\mathrm{SL}_n(\mathbb{R})$. If we intersect $\mathrm{SL}_n(\mathbb{R}[z, z^{-1}])$ with $L_{\mathrm{alg}}\mathrm{SU}(n)$, then we obtain the group

$$\mathrm{SL}_n(\mathbb{R}[z, 1/z]) \cap L_{\mathrm{alg}}\mathrm{SU}(n)$$

consisting of paths in $\mathrm{SU}(n)$ which start and end in $\mathrm{SO}(n)$. Similarly as before, this group is homotopy equivalent with the based loop space

$$\Omega_{\mathrm{cts}}(\mathrm{SU}(n)/\mathrm{SO}(n)).$$

These loop spaces of compact Riemannian symmetric spaces play an important rôle in topology. They can be used to prove the other versions of Bott periodicity (real and quaternionic), see Mitchell [27].

Acknowledgement

I am indebted to Peter Abramenko for sharing some of his insights and to Theo Grundhöfer for some remarks on the paper. I would particularly like to thank Bernhard Mühlherr, who introduced me to twin buildings in Oberwolfach eight years ago and convinced me of their usefulness and beauty; many ideas in this paper stem from discussions with him during the last years.

References

1. Abramenko, P.: *Twin Buildings and Applications to S-Arithmetic Groups*, Lecture Notes in Math. 1641, Springer, New York, 1996.
2. Abramenko, P. and Mühlherr, B.: Présentations de certaines *BN*-paires jumelées comme sommes amalgamées, *C.R. Acad. Sci. Paris* **325** (1997), 701–706.
3. Abramenko, P. and Ronan, M.: A characterization of twin buildings by twin apartments, *Geom. Dedicata* **73** (1998), 1–9.
4. Abramenko, P. and Van Maldeghem, H.: 1-twinnings of buildings, in *Math. Z.* **238** (2001), 187–203.
5. Back-Valente, V., Bardy-Panse, N., Ben Massaoud, H. and Rousseau, G.: Formes presque-déployées des algébres de Kac-Moody: Classification et raciness relatives, *J. Algebra* **171** (1995), 43–96.
6. Brady, N., McCammond, J. P., Mühlherr, B. and Neumann, W. D.: Non-rigid Coxeter and Artin groups, Preprint, Dortmund (2000).
7. Bridson, M. R. and Haefliger, A.: *Metric Spaces of Non-positive Curvature*, Grundlehren Math. Wiss. 319, Springer-Verlag, Berlin, 1999.
8. Brown, K.: *Buildings*, Springer-Verlag, Berlin, 1989.
9. Burns, K. and Spatzier, R.: On topological Tits buildings and their classification, *Publ. Math. I.H.E.S.* **65** (1987), 5–34.
10. Charney, R. and Davis, M.: When is a Coxeter system determined by its Coxeter group?, *J. London Math. Soc.* **61** (2000), 441–461.
11. Dress, A. and Scharlau, R.: Gated sets in metric spaces, *Aequationes Math.* **34** (1987), 112–120.
12. Groß, C., Kramer, L. and Mühlherr, B.: manuscript in preparation.
13. Grundhöfer, T. and Knarr, N.: Topology in generalized quadrangles, *Topology Appl.* **34** (1990), 139–152.

14. Grundhöfer, T., Knarr, N. and Kramer, L.: Flag-homogeneous compact connected polygons, *Geom. Dedicata* **55** (1995), 95–114.
15. Grundhöfer, T., Knarr, N. and Kramer, L.: Flag-homogeneous compact connected polygons II, *Geom. Dedicata* **83** (2000), 1–29.
16. Grundhöfer, T., Knarr, N. and Kramer, L.: The classification of compact homogeneous Tits buildings, manuscript in preparation.
17. Grundhöfer, T. and Van Maldeghem, H.: Topological polygons and affine buildings of rank three, *Atti Sem. Mat. Fis. Univ. Modena* **38** (1990), 459–479.
18. Heintze, E. and Liu, X.: Homogeneity of infinite-dimensional isoparametric submanifolds, *Ann. Math.* **149** (1999), 149–181.
19. Heintze, E., Palais, R., Terng, C.-l. and Thorbergsson, G.: Hyperpolar actions on symmetric spaces, In: S.-T. Yau (ed.), *Geometry, Topology, and Physics, for Raoul Bott,* (*Cambridge, MA, 1993*) International Press, Somerville, MA, 1995, pp. 214–245.
20. Jäger, M.: Topologische Gebäude, Dissertation, Univ. Kiel (1994).
21. Knarr, N.: The nonexistence of certain topological polygons, *Forum Math.* **2** (1990), 603–612.
22. Knarr, N. and Kramer, L.: Projective planes and isoparametric hypersurfaces, *Geom. Dedicata* **58** (1995), 193–202.
23. Kramer, L.: Compact polygons, Dissertation, Univ. Tübingen (1994). available as math.DG/0104064 in the Mathematics ArXiv, http://front.math.ucdavis.edu/math.DG/0104064
24. Kramer, L.: Homogeneous spaces, Tits buildings, and isoparametric hypersurfaces, to appear in *Mem. Amer. Math. Soc.* (2002).
25. Kühne, R.: Topologische sphärische Tits-Gebäude, Dissertation, Univ. Braunschweig (1994) Mitt. Math. Seminar Giessen 219 (1994).
26. Kühne, R. and Löwen, R.: Topological projective spaces, *Abh. Math. Seminar Univ. Hamburg* **62** (1992), 1–9.
27. Mitchell, S. A.: Quillen's theorem on buildings and the loops on a symmetric space, *Enseign. Math.* **34** (1988), 123–166.
28. Mühlherr, B.: A rank 2 characterization of twinnings, *European J. Combin.* **10** (1998), 603–612.
29. Mühlherr, B.: On the existence of 2-spherical twin buildings, Habilitationsschrift, Univ. Dortmund (1999).
30. Mühlherr, B.: On isomorphisms between Coxeter groups, *Des. Codes Cryptogr.* **21** (2000), 188–189.
31. Mühlherr, B. and Ronan, M.: Local to global structure in twin buildings, *Invent. Math.* **122** (1995), 71–81.
32. Palais, R. and Terng, C.-l.: *Critical Point Theory and Submanifold Geometry*, Lecture Notes in Math. 1353, Springer, New York, 1988.
33. Pinkall, U. and Thorbergsson, G.: Examples of infinite dimensional isoparametric submanifolds, *Math. Z.* **205** (1990), 279–286.
34. Pressley, A. and Segal, G.: *Loop Groups*, Oxford Univ. Press, 1986, corr. reprint, 1988.
35. Ronan, M.: *Lectures on Buildings*, Perspect. Math. 7, Academic Press, Boston, MA, 1989.
36. Ronan, M.: Local isometries of twin buildings, *Math. Z.* **234** (2000), 435–455.
37. Ronan, M. and Tits, J.: Twin trees I, *Invent. Math.* **116** (1994), 463–479.
38. Rousseau, G.: On forms of Kac–Moody algebras, In: W. Haboush and B. Parshall (eds), *Algebraic Groups and their Generalizations: Quantum and Infinite-Dimensional Methods,* (*University Park, PA, 1991*), Proc. Sympos. Pure Math. 56, Amer. Math. Soc., Providence, 1994, pp. 393–399.

39. Salzmann, H.: Topological planes, *Adv. Math.* **2** (1967), 1–60.
40. Salzmann, H., Betten, D., Grundhöfer, T., Hähl, H., Löwen, R. and Stroppel, M.: *Compact Projective Planes*, Expos. in Math. 21, De Gruyter, Berlin, 1995.
41. Scharlau, R.: Buildings, In: F. Buekenhout (ed.), *Handbook of Incidence Geometry*, North Holland, Amsterdam, 1995, pp. 477–645.
42. Schroth, A.: *Topological Circle Planes and Topological Quadrangles*, Pitman Res. Notes Math. 337 Longman, Harlow, 1995.
43. Smale, S.: Generalized Poincaré's conjecture in dimensions greater than four, *Ann. Math.* **74** (1961), 391–406.
44. Stallings, J.: Polyhedral homotopy-spheres, *Bull Amer. Math. Soc.* **66** (1960), 485–488.
45. Terng, C.-l.: Recent progress in submanifold geometry, In: R. Green and S.-T. Yau (ed.), *AMS Summer Research Institute on Differential Geometry,* (*Los Angeles, CA, 1990*), *Proc. Sympos. Pure Math.* **54** (1993), 439–484.
46. Thorbergsson, G.: Isoparametric foliations and their buildings, *Ann. Math.* **133** (1991), 429–446.
47. Thorbergsson, G.: A survey on isoparametric hypersurfaces and their generalizations, In: F. Dillen and L. Verstraelen (eds), *Handbook of Differential Geometry*, North-Holland, Amsterdam, 2000, pp. 963–995.
48. Tits, J.: *Buildings of Spherical Type and Finite BN-pairs*, Lecture Notes in Math. 386, Springer, New York, 1974.
49. Tits, J.: Buildings and group amalgamations, In: E. F. Robertson and C. M. Campbell, (eds), *Proceedings of Groups – St Andrews 1985*, Cambridge Univ. Press, Cambridge, 1986, pp. 110–127.
50. Tits, J.: *Immeubles jumelés*, Résumé de cours. Annuaire du Collège de France, 89^e année (1988–1989) 81–95. Annuaire du Collège de France, 90^e année (1989–1990) 87–103. Annuaire du Collège de France, 98^e année (1997–1998) 97–112.
51. Tits, J.: Twin buildings and groups of Kac–Moody type, In: M. Liebeck and J. Saxl (eds), *Groups, Combinatorics and Geometry* (Durham, 1990), London Math. Soc. Lecture Notes Ser., Cambridge Univ. Press, 1992, pp. 249–286.
52. Van Maldeghem, H.: *Generalized Polygons*, Monogr. Math. 93, Birkhäuser, Basel, 1998.
53. Zeeman, E. C.: The generalized Poincaré conjecture, *Bull Amer. Math. Soc.* **67** (1961), 270.