Geometriae Dedicata **83:** 1–29, 2000. 1

Flag-Homogeneous Compact Connected Polygons II

Dedicated to Prof. Salzmann on the occasion of his 70th birthday

T. GRUNDHÖFER[1], N. KNARR[2], and L. KRAMER[1]
[1] *Mathematisches Institut, Universität Würzburg, Am Hubland, D-97074 Würzburg, Germany*
[2] *Institut für Analysis, TU Braunschweig, Pockelsstraße 14, D-38106 Braunschweig, Germany*

(Received: 17 August 1998; in final form: 21 July 1999)

Communicated by K. Strambach

Abstract. All flag-homogeneous compact connected polygons are classified explicitly. It turns out that these polygons are precisely the compact connected Moufang polygons.

Mathematics Subject Classifications (2000): 51H10, 51E12, 51A10.

Key words: Moufang polygon, generalized quadrangle.

We determine explicitly all compact connected (generalized) polygons which admit a flag-transitive group of continuous collineations. The result is summarized in the following theorem:

THEOREM A. *A compact connected polygon is flag-homogeneous if and only if it is a Moufang polygon.*

A similar result was obtained by Burns–Spatzier [6] under a stronger homogeneity assumption (viz. transitivity on the ordered ordinary n-gons in a generalized n-gon). In more detail, we prove the following theorem, which completes the results of part I [10].

MAIN THEOREM. *Let $\mathfrak{P}$ be a compact connected n-gon with $n \geqslant 3$, and denote by p (respectively q) the dimension of a point row (of a line pencil) of $\mathfrak{P}$. Let Γ be a flag-transitive group of automorphisms $\mathfrak{P}$ which is closed in the full automorphism group. Then p and q are finite, and we have one of the following cases:*

(a) $n = 3$ *and* $p = q \in \{1, 2, 3, 4\}$. *Then $\mathfrak{P}$ is the projective plane over the real numbers, over the complex numbers, over Hamilton's quaternions $\mathbb{H}$, or over Cayley's octonions, and the possibilities for the group Γ are given in [10] 3.3, 3.7.*

(b$_1$) $n = 4$ *and* $p = q \in \{1, 2\}$. *Then* $\mathfrak{P}$ *is isomorphic or dual to the real symplectic quadrangle or to the complex symplectic quadrangle, and the possibilities for the group* Γ *are given in [10] 3.4, 3.7.*

(b$_2$) $n = 4$ and $\{p, q\} = \{1, m-2\}$ with $m > 3$. *Then* $\mathfrak{P}$ *is isomorphic or dual to the real orthogonal quadrangle defined by the standard quadratic form on* $\mathbb{R}^{m+2}$ *of Witt index 2. Moreover, the connected component* Γ^1 *of* Γ *is one of the groups* $\mathbf{PO}_{m+2}\mathbb{R}(2)^1$ *or* $\mathbf{P}(\mathbf{SO}_m\mathbb{R} \times \mathbf{SO}_2\mathbb{R})$, *or* $\mathbf{G}_2 \times \mathbf{SO}_2\mathbb{R}$ *for* $m = 7$, *or* $\mathbf{Spin}_7\mathbb{R}\times \mathbf{SO}_2\mathbb{R}$ *for* $m = 8$, *and we have* $|\Gamma : \Gamma^1| \leqslant 2$ *if m is odd, and* $|\Gamma : \Gamma^1|$ *divides* 4 *if m is even.*

(b$_3$) $n = 4$ *and* $\{p, q\} = \{2, 2m-3\}$ *with* $m > 2$. *Then* $\mathfrak{P}$ *is isomorphic or dual to the complex Hermitian quadrangle defined by the standard Hermitian form on* $\mathbb{C}^{m+2}$ *of Witt index 2. Moreover,* Γ^1 *is one of the groups* $\mathbf{PU}_{m+2}\mathbb{C}(2)$, $\mathbf{P}(\mathbf{U}_m\mathbb{C} \times \mathbf{U}_2\mathbb{C})$, *or* $\mathbf{P}(\mathbf{SU}_m\mathbb{C} \times \mathbf{SU}_2\mathbb{C})$, *and we have* $|\Gamma : \Gamma^1| \leqslant 2$ *or* $\Gamma^1 = \mathbf{P}(\mathbf{SU}_m\mathbb{C} \times \mathbf{SU}_2\mathbb{C}) \leqslant \Gamma \leqslant \mathbf{P}(\mathbf{U}_m\mathbb{C} \times \mathbf{U}_2\mathbb{C}) \cdot Z_2$, *where* Z_2 *denotes a cyclic group of order* 2.

(b$_4$) $n = 4$ *and* $\{p, q\} = \{4, 4m-5\}$ *with* $m > 1$. *Then* $\mathfrak{P}$ *is isomorphic or dual to the quaternion Hermitian quadrangle defined by the standard Hermitian form on* $\mathbb{H}^{m+2}$ *of Witt index 2. Moreover,* Γ^1 *is one of the groups* $\mathbf{PU}_{m+2}\mathbb{H}(2)$ *or* $\mathbf{P}(\mathbf{U}_m\mathbb{H} \times \mathbf{U}_2\mathbb{H})$, *and we have* $|\Gamma : \Gamma^1| \leqslant 2$ *for* $m = 2$, *and* $\Gamma = \Gamma^1$ *if* $m > 2$.

(b$_5$) $n = 4$ *and* $\{p, q\} = \{4, 5\}$. *Then* $\mathfrak{P}$ *is isomorphic or dual to the quadrangle defined by the* α*-Hermitian form* $\sum_j x_j^\alpha y_j$ *on* $\mathbb{H}^5$, *where the antiautomorphism* α *of* $\mathbb{H}$ *is given by* $x^\alpha = -i\bar{x}i$. *Moreover,* Γ^1 *is one of the groups* $(\mathbf{PU}_5^\alpha\mathbb{H})^1$, $\mathbf{U}_5\mathbb{C}$, $\mathbf{SU}_5\mathbb{C}$, *and we have* $|\Gamma : \Gamma^1| \leqslant 2$ *or* $\Gamma^1 = \mathbf{SU}_5\mathbb{C} \leqslant \Gamma \leqslant \mathbf{U}_5\mathbb{C} \cdot Z_2$.

(b$_6$) $n = 4$ *and* $\{p, q\} = \{6, 9\}$. *Then* $\mathfrak{P}$ *is one of the two mutually dual quadrangles associated to the exceptional simple Lie group* $\mathrm{E}_{6(-14)}/Z$, *where* Z *is the center of the simply connected Lie group* $E_{6(-14)}$. *Moreover,* Γ^1 *is one of the groups* $\mathrm{E}_{6(-14)}/Z$ *or* $\mathbf{Spin}_{10}\mathbb{R}$ *or* $\mathbf{Spin}_{10}\mathbb{R} \cdot \mathbf{SO}_2\mathbb{R}$, *and we have* $|\Gamma : \Gamma^1| \leqslant 2$ *or* $\Gamma^1 = \mathbf{Spin}_{10}\mathbb{R} \leqslant \Gamma \leqslant (\mathbf{Spin}_{10}\mathbb{R} \cdot \mathbf{SO}_2\mathbb{R}) \cdot Z_2$.

(c) $n = 6$ *and* $p = q \in \{1, 2\}$. *Then* $\mathfrak{P}$ *is isomorphic or dual to the split Cayley hexagon over the real numbers or over the complex numbers, and the possibilities for the group* Γ *are given in [10] 3.5, 3.7.*

The cases (b_1) and (b_2) are also covered by Biller in [1]. We remark that the group isomorphisms

$$\mathbf{PSO}_5\mathbb{R}(2)^1 \cong \mathbf{PSp_4}\mathbb{R}, \qquad \mathbf{PSO}_5\mathbb{C} \cong \mathbf{PSp_4}\mathbb{C},$$

$$\mathbf{PSO_6}\mathbb{R}(2)^1 \cong \mathbf{PSU}_4\mathbb{C}(2), \qquad \mathbf{PU}_4^\alpha\mathbb{H} \cong \mathbf{PSO_8}\mathbb{R}(2)^1$$

correspond to dual pairs of quadrangles defined by the pertinent forms. Note that the cases (b_3), (b_5), (b_6) yield examples of flag-transitive automorphism groups which are neither open nor closed in the topological automorphism group: let G be a flag-transitive subgroup with centralizer $\mathbf{SO}_2\mathbb{R}$, such that $G \cap \mathbf{SO}_2\mathbb{R}$ is finite. Then the group $G \cdot \mathbf{SO}_2\mathbb{Q} \subseteq G \cdot \mathbf{SO}_2\mathbb{R}$ is flag-transitive, but not closed or open in the automorphism group.

Every topological polygon which is a Moufang polygon admits a flag-transitive group of automorphisms (because the root collineations are continuous, compare [2] 3.5). Each building associated to a simple real Lie group is a Moufang building, see [30] 5.6. The main theorem of Burns–Spatzier [6] says that each compact connected Moufang building comes from a real simple Lie group. The classification of the real simple Lie groups shows that the polygons appearing in the theorem above are precisely the compact connected Moufang polygons, see Table VI Ch. X in [11]. Thus we obtain Theorem A and the following:

THEOREM B. *A compact connected polygon is a Moufang polygon if and only if it is isomorphic to one of the polygons in the Main Theorem above.*

Inspection of the list of groups Γ in the Main Theorem, together with [10] 3.8, yields the following:

COROLLARY C. *In each case of the Main Theorem, there is a unique conjugacy class of minimal flag-transitive closed groups* Γ. *In fact, the minimal flag-transitive groups* Γ *are precisely the maximal compact connected subgroups of the full automorphism group of* $\mathfrak{P}$, *except in the cases* (b_2) *when* $\{p, q\}$ *is* $\{1, 5\}$ *or* $\{1, 6\}$, (b_3), (b_5), *and* (b_6).

In the exceptional case (b_2) of Corollary C, the conjugacy class is unique by [23] Ex. 9, p. 57 and the remarks before Prop. 8. The compact connected flag-transitive groups have also been determined by Eschenburg–Heintze [7].

COROLLARY D. *Let* $n > k > 1$ *and let* $\mathfrak{P} = (\mathcal{P}, \mathcal{L}, \mathcal{F})$ *be a compact connected n-gon with a group* Γ *of automorphisms which is closed in the full automorphism group and transitive on the pairs* $(x, y) \in \mathcal{P} \times (\mathcal{P} \cup \mathcal{L})$ *of (graph-theoretic) distance k. Then* $\mathfrak{P}$ *is one of the Moufang polygons listed in the Main Theorem, and* Γ *contains the connected component of the full automorphism group of* $\mathfrak{P}$. *In particular,* Γ *contains all root collineations.*

The last corollary follows from the Main Theorem, since such a group Γ is transitive on flags, and not compact, because the set of all these pairs (x, y) is not compact (use [9] 2.8).

1. Transitive Actions of Compact Lie Groups

Homeomorphism of topological spaces is denoted by $\approx$, isomorphism of topological groups G, H is denoted by $G \cong H$, and $G \overset{\text{loc}}{=} H$ means that G and H are locally isomorphic. The rank of a finitely generated Abelian group A is the $\mathbb{Q}$-vector space dimension of $A \otimes \mathbb{Q}$.

We need some results about compact Lie groups and transformation groups, see, e.g. [25] Ch. 9. We will make frequent use of the following fact: if G is a compact connected Lie group, and if $H \subseteq G$ is a normal compact connected subgroup, then there exists a normal compact connected subgroup $K \subseteq G$ such that $H \cap K$ is finite and $G = KH$. In this situation we call G an almost direct product of H and K, and we denote this fact by $G = H \cdot K$. The number of the non-Abelian simple normal subgroups of G will be called the length $len(G)$ of G. Thus, a compact connected Lie group is an almost direct product of $len(G)$ almost simple compact Lie groups and a torus group.

Our notation for groups is the same as in part I [10]. If a Lie group G acts transitively on a space $X = G/H$, then there is a long exact homotopy sequence

$$\ldots \to \pi_i(H) \to \pi_i(G) \to \pi_i(X) \to \pi_{i-1}(H) \to \ldots$$

In particular, if G is connected and X is simply connected, then H is connected, too.

PROPOSITION 1.1 (Compact groups which are transitive on spheres). *Let G be a compact connected transformation group which acts effectively and transitively on a sphere $\mathbb{S}_{m-1}$ with $m \geqslant 2$, and let $H \subseteq G$ be the stabilizer of an element of $\mathbb{S}_{m-1}$. Then there are only the following possibilities for the triple $(G, H, \mathbb{S}_{m-1})$:*

$(\mathbf{SO}_m\mathbb{R}, \mathbf{SO}_{m-1}\mathbb{R}, \mathbb{S}_{m-1})$ *with* $m \geqslant 2$,
$(\mathbf{SU}_d\mathbb{C} \cdot A, \mathbf{SU}_{d-1}\mathbb{C} \cdot A, \mathbb{S}_{2d-1})$ *with* $d \geqslant 2$,
$(\mathbf{U}_d\mathbb{H} \cdot B, \mathbf{U}_{d-1}\mathbb{H} \cdot B, \mathbb{S}_{4d-1})$ *with* $d \geqslant 2$,
$(\mathbf{G}_2, \mathbf{SU}_3\mathbb{C}, \mathbb{S}_6)$,
$(\mathbf{Spin}_7\mathbb{R}, \mathbf{G}_2, \mathbb{S}_7)$,
$(\mathbf{Spin}_9\mathbb{R}, \mathbf{Spin}_7\mathbb{R}, \mathbb{S}_{15})$,

where $A \in \{1, \mathbf{U}_1\mathbb{C}\}$ and $B \in \{1, \mathbf{U}_1\mathbb{C}, \mathbf{U}_1\mathbb{H}\}$ act as scalars. In each case, H is determined uniquely up to conjugation in G, and the action of G on $\mathbb{S}_{m-1}$ is linear, i.e. $G \subseteq \mathbf{O}_m\mathbb{R}$.

Proof. This classification has been achieved by Montgomery–Samelson [21], Borel [3], see also Poncet [24]; for the uniqueness of H (due to Borel and De Siebenthal) see Wolf [32] 8.10.8, 3 and Poncet [24] §2. □

PROPOSITION 1.2. *Let G be a compact connected Lie group which acts transitively and almost effectively on an $(n-1)$-connected compact manifold X, where $\dim X \geqslant n \geqslant 2$. Let $H \subseteq G$ be the stabilizer of some point $x \in X$. We write $G = G_0 \cdot G_1 \cdot G_2 \cdots G_r$ and $H = H_0 \cdot H_1 \cdot H_2 \cdots H_s$, where G_0 and H_0 are tori (or trivial), and the other factors G_i, H_i are almost simple and non-Abelian. Thus, $len(G) = r$ and $len(H) = s$. Then the following holds:*

(0) *If* $n \geqslant 3$, *then* $\dim G_0 = \dim H_0$.
(1) *If* $n \geqslant 5$, *then* $r = len(G) = s = len(H)$.
(2) *If* $n \geqslant 7$, *then we can reorder the almost simple factors of these groups in such a way that the following holds for* $1 \leqslant i \leqslant r = s$:
(2.1) *If* G_i *belongs to the* $\mathbf{U}_k\mathbb{H}$*-family,* $k \geqslant 1$, *then so does* H_i.
(2.2) *If* G_i *belongs to the* $\mathbf{SU}_k\mathbb{C}$*-family,* $k \geqslant 3$, *then so does* H_i.
(2.3) *If* G_i *belongs to the* $\mathbf{SO}_k\mathbb{R}$*-family,* $k \geqslant 7$, *or is exceptional, then* H_i *also belongs to the* $\mathbf{SO}_k\mathbb{R}$*-family or is exceptional.*
(3) *If* $n \geqslant 9$, *then (2.3) can be improved as follows:*
(3.1) *If* G_i *belongs to the* $\mathbf{SO}_k\mathbb{R}$*-family,* $k \geqslant 7$, *then* H_i *also belongs to the* $\mathbf{SO}_k\mathbb{R}$*-family.*
(3.2) *If* G_i *is exceptional, then so is* H_i; *and either both groups* G_i, H_i *are of type* $\mathbf{G}_2$, *or none is.*

Proof. Since X is $(n-1)$-connected, the long exact homotopy sequence of the principal bundle $H \to G \to G/H = X$ breaks down into isomorphisms $\pi_k(H) \cong \pi_k(G)$, for $0 \leqslant k \leqslant n-2$.

The fundamental group of a compact semisimple Lie group is finite. Hence if $n \geqslant 3$, then $\pi_1(G) \cong \pi_1(H)$, and thus G and H have the same number of torus factors. This shows (0). By a result of Bott, $\pi_3(G) \cong \mathbb{Z}^r$, cp. Mimura [19] 3.9, and $\pi_3(H) \cong \mathbb{Z}^s$, hence we have proved (1). In the stable range we have isomorphisms $\pi_k(\mathbf{SO}_n\mathbb{R}) \cong \pi_k(\mathbf{SO}_\infty\mathbb{R})$ for $n \geqslant k+2$, $\pi_k(\mathbf{SU}_n\mathbb{C}) \cong \pi_k(\mathbf{SU}_\infty\mathbb{C})$ for $n \geqslant (k+1)/2$, $\pi_k(\mathbf{U}_n\mathbb{H}) \cong \pi_k(\mathbf{U}_\infty\mathbb{H})$ for $n \geqslant (k-1)/4$. The low-dimensional unstable homotopy groups of all compact almost simple Lie groups are known, see Mimura [19] p. 970. These tables show that the different families of compact almost simple classical Lie groups can be distinguished by their first five homotopy groups. In the range $k \leqslant 5$, the exceptional groups have the same homotopy groups as the orthogonal groups, so one needs to consider also the 6th and 7th homotopy groups. They distinguish between the orthogonal groups and the exceptional groups. □

LEMMA 1.3. *Let* X *be a 1-connected compact homogeneous space of a compact connected Lie group* G. *Then the semisimple commutator group* G' *acts still transitively on* X.

Proof. Replacing G by a finite covering group, if necessary, we may assume that $G = T \times H$ is a direct product of a 1-torus T and a compact connected group H. Proceding inductively, it suffices to show that H acts transitively on X. Let $x \in X$ and consider the group $H_x = G_x \cap H$. Then

$$\dim H - \dim H_x \leqslant \dim X = \dim G - \dim G_x,$$

and, since $\dim G = \dim H + 1$, it follows that $\dim H_x + 1 \geqslant \dim G_x$. If we have equality, then $H/H_x = X$ by invariance of domain. Otherwise we have $\dim H_x = \dim G_x$, since $H_x \subseteq G_x$, and therefore $H_x = G_x$, because X is simply connected and hence G_x is connected. But this would imply that $X \approx G/G_x = (T \times H)/H_x = \mathbb{S}_1 \times (H/H_x)$, a contradiction to $\pi_1(X) = 0$. □

We use this result to prove the following.

LEMMA 1.4. *Let X be a 3-connected compact homogeneous space of a compact connected Lie group G. Assume that G has a normal subgroup of type* $\mathbf{SO}_3\mathbb{R}$ *or* $\mathbf{Spin}_3\mathbb{R}$*, i.e. that* $G \overset{\text{loc}}{\cong} \mathbf{SO}_3\mathbb{R} \times H$*, for some compact connected subgroup* $H \subseteq G$*. Then the subgroup H acts still transitively on X.*

Proof. By Lemma 1.3 we may assume that G has no torus factors, and by passing to a finite covering, that $G = \mathbf{Spin}_3\mathbb{R} \times H$. Fix an element $x \in X$. As in the proof of Lemma 1.3, $H_x \subseteq G_x$ is a subgroup of codimension $\dim G_x - \dim H_x = r \leqslant 3$. If $r = 3$, then $\dim H/H_x = \dim X$, and thus H acts transitively on X.

Otherwise $r = 2, 1, 0$. Since H_x is normal in G_x, there exists an r-dimensional subgroup $K \subseteq G_x$ which is locally isomorphic to the factor group G_x/H_x, and $G_x = (H_x)^1 \cdot K$. Since the dimension of this group K is r, it has to be an r-torus. The exact homotopy sequence implies that $\pi_1(X) \cong \mathbb{Z}^r$, a contradiction for $r = 1, 2$. If $r = 0$, then $H_x = G_x$. This yields $X \approx \mathbb{S}_3 \times H/H_x$, a contradiction to the 3-connectedness of X. □

1.5 DIMENSION OF CERTAIN QUOTIENTS. By a straightforward induction based on spheres as homogeneous spaces as in 1.1, we have the following identities:

$$\dim \mathbf{SO}_m\mathbb{R} - \dim \mathbf{SO}_n\mathbb{R} = n + (n+1) + \ldots + (m-1)$$

$$\dim \mathbf{SU}_m\mathbb{C} - \dim \mathbf{SU}_n\mathbb{C} = (2n+1) + (2n+3) + \ldots + (2m-1)$$

$$\dim \mathbf{U}_m\mathbb{H} - \dim \mathbf{U}_n\mathbb{H} = (4n+3) + (4n+7) + \ldots + (4m-1).$$

LEMMA 1.6. *Let G be a compact connected Lie group which acts effectively on the n-sphere $\mathbb{S}_n$ with $n > 1$ such that each orbit has codimension at most one in $\mathbb{S}_n$. Then G is transitive on $\mathbb{S}_n$.*

Proof. If some orbit has dimension n, then G is transitive, cp. [25] 96.11(a). Thus we have to eliminate the possibility that each orbit has dimension $n-1$. For $n \neq 2, 4$ this is done in Bredon [4] Cor. 3, and for $n = 4$ the assertion is a consequence of a result of Richardson, see [25] 96.34.

For $n = 2$ we follow Nils Rosehr and argue as follows. Each orbit is a singleton or a circle, and each circle decomposes $\mathbb{S}_2$ into two connected components, which are open in $\mathbb{S}_2$. The images of the circles are precisely the cut-points of the orbit space $\mathbb{S}_2/G$. By a theorem of Wallace [31] 1.11 each (non-trivial) compact connected T_1-space has (at least two) non-cut-points, hence singletons do arise as orbits. □

PROPOSITION 1.7. *Let G be a compact Lie group acting on a compact space X. Suppose that all stabilizers are conjugate to one fixed subgroup $H \subseteq G$. The set of all conjugates of H can be identified with $G/\mathrm{Nor}_G(H)$, and becomes in this* *way a compact space. Then the map which assigns to $x \in X$ the stabilizer G_x is a continuous surjection of X onto $G/\mathrm{Nor}_G(H)$.*

Proof. See Bredon [4] II.5.9. □

2. Starting the Proof

For $n \neq 4$ or $p = q$, the Main Theorem has been proved in [10] 3.8. In this paper we consider the situation where $n = 4$ and $p \neq q$. We have to show that this leads to the cases (b_2)–(b_6) of the Main Theorem.

We fix some notation for the rest of this paper, as in [10] Section 3. Let $\mathfrak{P} = (\mathcal{P}, \mathcal{L}, \mathcal{F})$ be a compact connected generalized quadrangle with an automorphism group Γ which is transitive on the flag set $\mathcal{F}$, as in the Main Theorem. The full automorphism group of $\mathfrak{P}$ and its closed subgroups are Lie groups, see [10] 1.5, 1.6, 2.2. The connected component Γ^1 of Γ is still flag-transitive, compare [10] 2.3. We use the letter Δ for compact connected subgroups of Γ^1; if possible, we choose Δ to be flag-transitive, compare 2.3, 1.3, and 1.4. Let $(x, \ell) \in \mathcal{F}$ be a flag. We denote by $\Gamma_{[x]}$ and $\Gamma_{[\ell]}$ the kernels of the actions of the stabilizers Γ_x on the line pencil $\mathcal{L}_x$ and of Γ_ℓ on the point row L of ℓ, respectively. Then $\Gamma/\Gamma_x \approx \mathcal{P}$, $\Gamma/\Gamma_\ell \approx \mathcal{L}$, $\Gamma/\Gamma_{x,\ell} \approx \mathcal{F}$, $\Gamma_x/\Gamma_{x,\ell} \approx \mathcal{L}_x$, and $\Gamma_\ell/\Gamma_{x,\ell} \approx L$ are topological manifolds (of finite dimension). Thus we can apply [12] 2.1 to infer that the point rows of $\mathfrak{P}$ are homeomorphic to spheres $\mathbb{S}_p$, and the line pencils are spheres $\mathbb{S}_q$.

In [10] we considered those cases where the Euler characteristic of $\mathcal{F}$ is positive, or where the topological parameters are one. Our present assumption $p \neq q$ means that $\mathcal{P}, \mathcal{L}$ and $\mathcal{F}$ have Euler characteristic zero or that the fundamental group of $\mathcal{F}$ is infinite, hence [10] 2.4 and [10] 2.3 do not apply. This fact accounts for some of the difficulties of the case $p \neq q$, compared to the case $p = q$.

Up to duality, i.e. up to exchanging p and q, we have one of the following cases: $p = 1 < q$, $p = 2 < q$ and q is odd, $p = 4$ and $q \geqslant 3$ is odd, or $p \geqslant 6$ is even and $q \geqslant 3$ is odd, see [10] 1.7 and [12]. These four cases are considered in Sections 3–6.

LEMMA 2.1. *Let Δ be a compact connected subgroup of Γ. Then the stabilizer Δ_x acts linearly on the pencil $\mathcal{L}_x$, and Δ_ℓ acts linearly on the point row L of ℓ.*

Proof. Δ_x is contained in a maximal compact connected subgroup Φ_x of $(\Gamma_x)^1$, and Γ_x and $(\Gamma_x)^1$ act transitively on $\mathcal{L}_x$. If $\mathcal{L}_x$ is homeomorphic to $\mathbb{S}_1$, then we apply Brouwer's theorem (compare [25] 96.30) to deduce that $(\Gamma_x)^1$ acts on $\mathcal{L}_x$ as $\mathbf{SO}_2\mathbb{R}$ or as a finite covering of $\mathbf{PSL}_2\mathbb{R}$; in particular, Δ_x acts linearly. If $\mathcal{L}_x$ is a sphere of dimension larger than 1, then Φ_x is transitive on $\mathcal{L}_x$, compare [10] 2.3, and the linearity of Φ_x and Δ_x is obtained from 1.1. The dual arguments apply to Δ_ℓ. □

LEMMA 2.2. *For every flag $(x, \ell) \in \mathcal{F}$, we have $\Delta_{[x]} \cap \Delta_{[\ell]} = 1$.*

Proof. By the linearity proved in Lemma 2.1, the group $\Phi = \Delta_{[x]} \cap \Delta_{[\ell]}$ fixes a point distinct from x on each line through x, and a line distinct from ℓ through each point on ℓ. Thus Φ fixes an ordinary quadrangle in $\mathfrak{P}$. The subquadrangle $\mathfrak{Q}$ consisting of all fixed elements of Φ contains the pencil $\mathcal{L}_x$ and the point row of ℓ, hence $\mathfrak{Q} = \mathfrak{P}$ and $\Phi = 1$. □

LEMMA 2.3. *Let Δ be a maximal compact subgroup of Γ^1. If $p > 1$ and $q > 1$, then Δ is flag-transitive. If $p = 1 < q$, then Δ is transitive on the line space $\mathcal{L}$, and the stabilizer Δ_x of any point $x \in \mathcal{P}$ acts transitively on $\mathcal{L}_x$; if Δ is not flag-transitive, then all point orbits of Δ have dimension $q + 1$, and each point orbit intersects each point row in precisely one point.*

Proof. For the first assertion see [10] 3.6. If $p = 1 < q$, then $\mathcal{L}$ is simply connected, see [10] Appendix 4_2, whence Δ is transitive on $\mathcal{L}$ by *loc. cit.* 2.3. It remains to prove the transitivity of Δ_x on $\mathcal{L}_x$ and the assertion on the point orbits. We may assume that Δ is not flag-transitive. This means that for every line $\ell \in \mathcal{L}$, the stabilizer Δ_ℓ is not transitive on the point row $L \approx \mathbb{S}_1$ of l. As Δ_ℓ is compact and connected (note that Δ is connected by the Malcev–Iwasawa theorem), we infer that Δ_ℓ acts trivially on $L \approx \mathbb{S}_1$, i.e. $\Delta_\ell = \Delta_{x,\ell}$ for every point x on ℓ.

We have $\dim \mathcal{P} = q + 2$ and $\dim \mathcal{L} = 2q + 1$, see [10] 1.7, and we obtain

$$\dim \Delta_x + \dim x^\Delta = \dim \Delta = \dim \Delta_\ell + 2q + 1 = \dim \Delta_{x,\ell} + 2q + 1. \qquad (*)$$

In view of $\dim x^\Delta \leqslant q + 2$ this implies $\dim \Delta_x \geqslant \dim \Delta_{x,\ell} + q - 1$, hence the orbits of Δ_x in $\mathcal{L}_x$ have dimension at least $q - 1$. By 1.6 the group Δ_x is transitive on $\mathcal{L}_x$, and $(*)$ shows that $\dim x^\Delta = q + 1$. Each point row intersects each point orbit, as Δ is transitive on $\mathcal{L}$. It remains to show that the intersection contains precisely one point. Let $x, x^\delta \in L$ be points, where $\delta \in \Delta$. By transitivity of Δ_{x^δ} on $\mathcal{L}_{x^\delta}$ we may assume that $L^\delta = L$, whence δ fixes all points on ℓ, and $x = x^\delta$ in particular. □

LEMMA 2.4. *Let $p \geqslant 5$, $q \geqslant 2$ and $(p, q) \neq (5, 2), (7, 4)$. Then $\Delta_{[x]}$ does not contain a subgroup Δ_0 such that the fixed points of Δ_0 acting on $L \approx \mathbb{S}_p$ form a $(p-3)$-sphere $\mathbb{S}_{p-3} \subset L$; in particular, $\Delta_{[x]}$ has no subgroup $\mathbf{SO}_p\mathbb{R}$ acting on L in the standard way (i.e. as the stabilizer of an antipodal pair in $\mathbb{S}_p$).*

Proof. As a consequence of 1.1, the stabilizer $\Delta_{x,\ell}$ acts linearly on the point row L of ℓ, hence $\Delta_{x,\ell}$ fixes a second point x' on ℓ; by duality, $\Delta_{x,\ell}$ fixes also a second line ℓ' through x. Repeating this argument shows that $\Delta_{x,\ell}$ fixes the points and lines of an ordinary quadrangle in $\mathfrak{P}$. The elements of $\mathfrak{P}$ fixed by Δ_0 form a thick subquadrangle with topological parameters $(p-3, q)$. As $p - 3 \geqslant 2$, we infer from [10] 1.7 that $p - 3 = q \in \{2, 4\}$, or both sums $p + q$ and $p - 3 + q$ have to be odd, a contradiction. If $\mathbf{SO}_p\mathbb{R}$ acts as indicated, then $\mathbf{SO}_p\mathbb{R}$ contains a subgroup $\Delta_0 \cong \mathbf{SO}_3\mathbb{R}$ fixing a $(p-3)$-sphere in L. □

We need also the following homotopy-theoretic results which are proved in [13].

PROPOSITION 2.5. *Let $\mathfrak{P}$ be a compact quadrangle with parameters (p, q). Let L be a point row containing the point x. The inclusions $L \subseteq x^\perp \subseteq \mathcal{P}$ are cofibrations; the* *pairs* $(x^\perp, L)$ *and* $(\mathcal{P}, x^\perp)$ *are* $(p-1)$- *and* $(p+q-1)$-*connected, respectively. Consequently, the point space* $\mathcal{P}$ *is* $(p-1)$-*connected; if* $q > 1$, *then the inclusion* $L \subseteq \mathcal{P}$ *induces an isomorphism* $\mathbb{Z} \cong \pi_p(L) \to \pi_p(\mathcal{P})$. *A similar result holds for the line space. Since the flag space fibers over* $\mathcal{P}$ *and* $\mathcal{L}$ *with homotopy* q- *and* p-*spheres as fibers, it is* $\min\{p-1, q-1\}$-*connected.*

Proof. See Kramer [13] Ch.3. □

COROLLARY. *Suppose that* $p, q \geqslant 3$. *Then* $\pi_{p+1}(\mathcal{P})$ *and* $\pi_{q+1}(\mathcal{L})$ *are finite.*

Proof. Let (x, ℓ) be a flag. The point row L is homotopy equivalent to $\mathbb{S}_p$; therefore $\pi_{p+1}(\mathcal{L}_x)$ is finite, cp. Spanier [26] 9.7.7, 9.7.9. The pair $(x^\perp, L)$ is $(p+q-1)$-connected, and, in particular, $(q+1)$-connected. The exact homotopy sequence of the pair shows that $\pi_{q+1}(L) \to \pi_{q+1}(x^\perp)$ is an epimorphism. The pair $(\mathcal{P}, x^\perp)$ is $(2p+q-1)$-connected, hence $\pi_{p+1}(x^\perp) = \pi_{p+1}(\mathcal{P})$. □

LEMMA 2.7. *Let* $\mathfrak{P}$ *be a compact generalized quadrangle with parameters* (p, q), *with* $q \geqslant 2$, *and let* G *be a compact connected line-transitive group of automorphisms. Then the action of* G *on* $\mathcal{L}$ *does not factor in the form* $G = G_1 \cdot G_2$, $G_\ell = (G_1)_\ell \cdot (G_2)_\ell$, *with* $G_1/(G_1)_\ell \approx \mathbb{S}_q$, $G_2/(G_2)_\ell \approx \mathbb{S}_{p+q}$, *except possibly if* G_1 *acts regularly on* $\mathbb{S}_q$ *(in this case* $q = 3$ *and* $G_1 = \mathbf{SU}_2\mathbb{C}$ *acts regularly on* $\mathbb{S}_3$*).*

Proof. Assume that the action factors as indicated. Then $\mathcal{L} \approx \mathbb{S}_q \times \mathbb{S}_{p+q}$ is a product of two spheres. Let $\mathcal{L}_x \subseteq \mathcal{L}$ be a line pencil. Then there exists a homotopy equivalence $g : \mathbb{S}_q \to \mathcal{L}_x$. The composite

$$\mathbb{S}_q \xrightarrow{g} \mathcal{L}_x \xrightarrow{i} \mathcal{L} \approx \mathbb{S}_q \times \mathbb{S}_{p+q} \xrightarrow{p_1} \mathbb{S}_q$$

is a also homotopy equivalence, because $\pi_q(\mathcal{L})$ is generated by the image of $\mathbb{S}_q$, see 2.5; in particular, the map $p_1 i : \mathcal{L}_x \to \mathbb{S}_q$, induced by projecting onto the first factor, is surjective. Hence $\mathcal{L}_x$ meets the set $\{a\} \times \mathbb{S}_{p+q} \subseteq \mathcal{L}$ for every $a \in \mathbb{S}_q$. Note that this holds for every $x \in \mathcal{P}$.

If G_1 does not act regularly on the first factor $\mathbb{S}_q$, then the fixed-point set of $(G_1)_\ell$ (acting on $\mathbb{S}_q$) is a proper subset of $\mathbb{S}_q$ containing $\mathbb{S}_0 \subseteq \mathbb{S}_q$. Thus, every line in $\mathbb{S}_0 \times \mathbb{S}_{p+q}$ is fixed by $(G_1)_\ell$. Every line pencil $\mathcal{L}_y$ meets this set twice. Therefore, y is fixed by $(G_1)_x$, and, consequently, $(G_1)_x$ acts trivially on the point set $\mathcal{P}$, a contradiction. □

Note that it is topologically quite possible that $\mathcal{L} \approx \mathbb{S}_q \times \mathbb{S}_{p+q}$; also, such group factorizations do occur if $G_1 = \mathbf{SU}_2\mathbb{C}$: some of the non-Moufang quadrangles discovered by Ferus–Karcher–Münzner and Thorbergsson have these properties [8], [29].

Now we describe the structure of the proofs in Sections 3–6. We always consider a compact connected quadrangle $\mathfrak{P}$ with a flag-transitive automorphism group Γ. Using 2.3, 2.5, 1.3, 1.4 we find a compact connected subgroup Δ which is still flag-transitive (in Sections 4–6) or at least line-transitive (in Section 3). Then we use the known actions of Δ_x on the pencil $\mathcal{L}_x$ and of Δ_ℓ on the point row L, compare 1.1, to determine the group Δ at least up to local isomorphism. Here the following observation is useful: $\Delta_{[\ell]}$ is a normal subgroup of $\Delta_{\ell,x}/\Delta_{[x]}$ (note that $\Delta_{[x]} \cap \Delta_{[\ell]} = 1$ by 2.2), and dually, $\Delta_{[x]}$ is normal in $\Delta_{\ell,x}/\Delta_{[\ell]}$. Moreover, by 1.1 we know in each case that the embedding of $\Delta_{x,\ell}$ into Δ_x or Δ_ℓ is the standard one.

If Δ is flag-transitive, then $\mathfrak{P}$ is isomorphic to the coset geometry with point set Δ/Δ_x and line set Δ/Δ_ℓ, where two cosets are incident if they are not disjoint. In order to determine $\mathfrak{P}$, we show that the pair (Δ_x, Δ_ℓ) is unique up to automorphisms of Δ, and then we compare with the corresponding Moufang quadrangle. If Δ is only line-transitive, then we use a reconstruction method of Stroppel [27], see 3.2 below.

When we determine the possible groups Γ, we make use of the fact that the maximal compact subgroups of an almost simple Lie group are maximal subgroups, see Helgason [11] Chap. IV, Ex. A3(iv), pp. 276, 567.

3. The Case $p = 1 < q$

Throughout this section $\mathfrak{P}$ is a compact quadrangle with parameters $(1, q)$, $q \geqslant 2$ with a flag-transitive automorphism group Γ. By 2.3, Γ contains a compact connected subgroup Δ which is transitive on the set $\mathcal{L}$ of all lines. In view of 2.5, 1.4, 1.3 we assume that Δ is semisimple, and that Δ has no factor of type $\mathbf{SO}_3\mathbb{R}$, for $q \geqslant 4$. Note that our assumption implies that Δ is *not* transitive on the point space $\mathcal{P}$: the fundamental group of Δ is finite, and thus the fundamental group of a point orbit x^Δ is also finite, whereas $\pi_1(\mathcal{P}) \cong \mathbb{Z}$, see [10] Appendix 4_2. By 2.3 this implies that $\Delta_\ell = \Delta_{[\ell]}$ for every line $\ell \in \mathcal{L}$.

The following idea is due to Biller [1] 4.2 and replaces another, more complicated argument. For every $x \in \mathcal{P}$, the orbit x^Δ is a compact ovoid (i.e. a set of pairwise non-collinear points which meets every line, see, e.g., [16] 1.5) by 2.3. By [16] 3.2, the set x^Δ is homotopy equivalent to $\mathbb{S}_{1+q}$. Being a homogeneous space of Δ, the orbit is in fact a sphere and the action is given by one of the pairs listed in 1.1, see, e.g., [14] 6.6, [23] Chap. 5 §18 Table 10.

PROPOSITION 3.1. *We have* $\Delta \cong \mathbf{SO}_{q+2}\mathbb{R}$, $\Delta_x \cong \mathbf{SO}_{q+1}\mathbb{R}$, $\Delta_\ell = \Delta_{x,\ell} \cong \mathbf{SO}_q\mathbb{R}$, *or we have the case* $q = 5$ *and* $\Delta \cong \mathbf{G}_2$, $\Delta_x \cong \mathbf{SU}_3\mathbb{C}$, $\Delta_\ell = \Delta_{x,\ell} \cong \mathbf{SU}_2\mathbb{C}$, *or we have the case* $q = 6$ *and* $\Delta \cong \mathbf{Spin}_7\mathbb{R}$, $\Delta_x \cong \mathbf{G}_2$, $\Delta_\ell = \Delta_{x,\ell} \cong \mathbf{SU}_3\mathbb{C}$. *In each case, the embeddings* $\Delta_x \subseteq \Delta$ *and* $\Delta_\ell \subseteq \Delta_x$ *are (up to inner automorphisms) the standard embeddings.*

Proof. Since $\Delta_{[x]} \subseteq \Delta_{x,\ell} = \Delta_\ell$, we have $\Delta_{[x]} = 1$ by 2.2. Thus Δ_x acts effectively and transitively on the line pencil $\mathcal{L}_x$ by 2.3. Since $\Delta_{x,\ell} = \Delta_\ell$ is connected, Δ_x is also connected.

If $q = 2$, then $(\Delta_x, \Delta_{x,\ell}) = (\mathbf{SO}_3\mathbb{R}, \mathbf{SO}_2\mathbb{R})$. Therefore $\dim \Delta = \dim \mathcal{L} + \dim \Delta_\ell = 6$, and thus $\Delta \stackrel{\text{loc}}{=} \mathbf{SO}_4\mathbb{R}$ because Δ is semisimple. Since $\Delta/\Delta_x \approx \mathbb{S}_3$, we conclude that $\pi_1(\Delta) = \mathbb{Z}_2$, whence $\Delta \cong \mathbf{SO}_4\mathbb{R}$ or $\Delta \cong \mathbf{SO}_3\mathbb{R} \times \mathbf{SU}_2\mathbb{C}$. In the second case, $\mathcal{L} \approx \mathbf{SO}_3\mathbb{R}/\mathbf{SO}_2\mathbb{R} \times \mathbf{SU}_2\mathbb{C}$ is a product of homogeneous spheres, contradicting 2.7. Thus $(\Delta, \Delta_x) = (\mathbf{SO}_4\mathbb{R}, \mathbf{SO}_3\mathbb{R})$.

Now let $q \geqslant 3$ for the rest of this proof. Then $\Delta_\ell = \Delta_{x,\ell}$ is semisimple by 1.2, and therefore Δ_x is also semisimple, because $\Delta_x/\Delta_{x,\ell} \approx \mathbb{S}_q$ by 2.3.

We have $\dim \mathcal{P} = q + 2$ and $\dim \mathcal{L} = 2q + 1$. Lemma 2.3 implies that $\dim \Delta = \dim \Delta_l + 2q + 1$ and that Δ_x acts (effectively) on the sphere $\mathcal{L}_x \approx \mathbb{S}_q$ as one of the transitive groups listed in 1.1. We consider these groups case by case.

(1) $\Delta_x = \mathbf{SO}_{q+1}\mathbb{R}$. Then $\Delta_{x,\ell} = \mathbf{SO}_q\mathbb{R} = \Delta_\ell$ and Δ is a semi-simple compact Lie group of dimension $2q + 1 + \dim \mathbf{SO}_q\mathbb{R} = 2q + 1 + q(q-1)/2$. We want to show that $\Delta \cong \mathbf{SO}_{q+2}\mathbb{R}$. Note that Δ_x cannot act trivially on the orbit x^Δ, since this would imply that $q + 1 = 3$ by 1.1.

If $q \neq 3$, then Δ_x is almost simple. Thus Δ acts almost effectively on $x^\Delta \approx \mathbb{S}_{q+1}$. From 1.1 we see that the action is effective, and that $(\Delta, \Delta_x) = (\mathbf{SO}_{q+2}\mathbb{R}, \mathbf{SO}_q\mathbb{R})$.

If $q = 3$, then $\Delta \overset{\text{loc}}{=} \mathbf{SO}_5\mathbb{R}$, because there is no other 10-dimensional compact semisimple Lie group. No proper six-dimensional quotient of $\mathbf{SO}_4\mathbb{R}$ appears as a stabilizer in 1.1, hence the action of Δ on x^Δ is effective, and $(\Delta, \Delta_x) = (\mathbf{SO}_5\mathbb{R}, \mathbf{SO}_4\mathbb{R})$.

(2) $\Delta_x = \mathbf{SU}_{(q+1)/2}\mathbb{C}$. Then $\Delta_{x,l} = \mathbf{SU}_{(q-1)/2}\mathbb{C} = \Delta_\ell$. Again, Δ acts effectively on x^Δ, since no proper quotient of Δ_x appears as a stabilizer in 1.1. Thus $(\Delta, \Delta_x) = (\mathbf{SU}_{(q+3)/2}\mathbb{C}, \mathbf{SU}_{(q+1)/2}\mathbb{C})$, or $q = 5$ and $(\Delta, \Delta_x) = (\mathbf{G}_2, \mathbf{SU}_3\mathbb{C})$. By 1.5, the first possibility leads to the contradiction $\dim \mathcal{L} = 2q + 1 = q + (q + 2)$.

(3) $\Delta_x = \mathbf{U}_{(q+1)/4}\mathbb{H} \cdot \mathbf{U}_1\mathbb{H}$, for $q \geqslant 7$. Then $\Delta_{x,\ell} = \mathbf{U}_{(q-3)/4}\mathbb{H} \cdot \mathbf{U}_1\mathbb{H} = \Delta_\ell$. Moreover, $len(\Delta) = len(\Delta_\ell) = len(\Delta_x) = 2$. As above we conclude that $(\Delta, \Delta_x) = (\mathbf{U}_{(q+5)/4}\mathbb{H} \cdot \mathbf{U}_1\mathbb{H}, \mathbf{U}_{(q+1)/4}\mathbb{H} \cdot \mathbf{U}_1\mathbb{H})$, and this leads to $\dim \mathcal{L} = 2q + 1 = q + (q + 4)$, a contradiction.

(4) $\Delta_x = \mathbf{U}_{(q+1)/4}\mathbb{H}$, for $q \geqslant 7$. As in case (3), this leads to $2q + 1 = q + (q + 4)$, a contradiction.

(5) $\Delta_x = \mathbf{G}_2$ and $q = 6$. Then $\Delta_{x,\ell} = \mathbf{SU}_3\mathbb{C} = \Delta_\ell$. Using 1.1 as before, we obtain that $(\Delta, \Delta_x) = (\mathbf{Spin}_7\mathbb{R}, \mathbf{G}_2)$.

(6) $\Delta_x = \mathbf{Spin}_7\mathbb{R}$ and $q = 7$. Then $\Delta_{x,\ell} = \mathbf{G}_2 = \Delta_\ell$, and hence Δ is almost simple of dimension $\dim \Delta = 15 + 14 = 29$. Such a group does not exist.

(7) $\Delta_x = \mathbf{Spin}_9\mathbb{R}$ and $q = 15$. Then $\Delta_{x,\ell} = \mathbf{Spin}_7\mathbb{R} = \Delta_\ell$ and Δ is simple of dimension $31 + 21 = 52$. By 1.2 Δ is an orthogonal group. Such an orthogonal group does not exist.

In each case, the uniqueness of the embeddings follows from 1.1. □

PROPOSITION 3.2. *If $p = 1 < q$ then we have case (b_2) of the Main Theorem.*

Proof. By 3.1, the triple $(\Delta, \Delta_x, \Delta_\ell = \Delta_{x,\ell})$ is determined uniquely up to conjugation in Δ. Now the method of Stroppel [27] for the reconstruction of geometries from groups which are not necessarily flag-transitive shows that $\mathfrak{P}$ is determined uniquely by the triple $(\Delta, \Delta_\ell, \{\Delta_y \mid y \in L\})$. We show that $\{\Delta_y \mid y \in L\}$ is already determined by $(\Delta, \Delta_x, \Delta_\ell)$.

The set $\{\Delta_y \mid y \in L\}$ is topologized as in 1.7. Note that the map $y \mapsto \Delta_y$ is injective on L; therefore $\{\Delta_y \mid y \in L\} \approx \mathbb{S}_1$ is a 1-sphere. On the other hand, the collection of all conjugates of Δ_x which contain Δ_ℓ is also homeomorphic to $\mathbb{S}_1$: the fixed point set of such a conjugate of $\mathbf{SO}_{q+1}\mathbb{R}$, acting on $\mathbb{R}^{q+2}$, is one-dimensional and has to be contained in the fixed point set of $\Delta_\ell = \mathbf{SO}_q\mathbb{R}$, which is two-dimensional. The two exceptional cases $\Delta \cong \mathbf{Spin}_7\mathbb{R}$ and $\Delta \cong \mathbf{G}_2$ are similar.

By domain invariance we conclude that $\{\Delta_y \mid y \in L\}$ is precisely the set of all Δ-conjugates of Δ_x which are contained in Δ_ℓ. This is a purely group-theoretic description of the triple $(\Delta, \Delta_\ell, \{\Delta_y \mid y \in L\})$. Thus $\mathfrak{P}$ is isomorphic to the incidence structure $\mathfrak{P}'$ with point set $\bigcup_{y \in L} \Delta/\Delta_y$ and line set Δ/Δ_ℓ, where cosets of the form $\delta\Delta_y$ and $\delta\Delta_\ell$ with $\delta \in \Delta$ are incident; an isomorphism of $\mathfrak{P}' \to \mathfrak{P}$ is given by evaluation, see Stroppel [27].

Let $\mathfrak{Q}$ be the real orthogonal quadrangle defined by a quadratic form on $\mathbb{R}^{q+4}$ of Witt index 2. Then Δ acts effectively on $\mathfrak{Q}$ as a collineation group which is transitive on the lines of $\mathfrak{Q}$, and the stabilizer of any point of $\mathfrak{Q}$ is transitive on the pencil through that point; the transitivity properties required in the cases $\Delta = \mathbf{G}_2$ and $\Delta = \mathbf{Spin}_7\mathbb{R}$ are expressed by the isomorphisms $\mathbf{SO}_7\mathbb{R}/\mathbf{SO}_5\mathbb{R} = \mathbf{G}_2/\mathbf{SU}_2\mathbb{C}$, $\mathbf{SO}_6\mathbb{R}/\mathbf{SO}_5\mathbb{R} = \mathbf{SU}_3\mathbb{C}/\mathbf{SU}_2\mathbb{C}$, $\mathbf{SO}_8\mathbb{R}/\mathbf{SO}_6\mathbb{R} = \mathbf{Spin}_7\mathbb{R}/\mathbf{SU}_3\mathbb{C}$, $\mathbf{SO}_7\mathbb{R}/\mathbf{SO}_6\mathbb{R} = \mathbf{G}_2/\mathbf{SU}_3\mathbb{C}$ of homogeneous spaces, see e.g. Onishchik [23] Chap. I, §5, 3, p. 90f, Kramer [13]. By the uniqueness obtained above, we conclude that $\mathfrak{P}$ is isomorphic to $\mathfrak{Q}$.

Finally we determine the possible flag-transitive groups Γ. The group $\Sigma := \mathbf{PO}_{q+4}\mathbb{R}(2)$ is the full automorphism group of $\mathfrak{Q} \cong \mathfrak{P}$, and the maximal compact subgroups $\mathbf{P}(\mathbf{O}_{q+2}\mathbb{R} \times \mathbf{O}_2\mathbb{R})$ of Σ are maximal subgroups of Σ. Let Γ_0 be a maximal compact subgroup of Γ^1. By definition of Δ, we have $\Delta \leqslant \Gamma_0'$, and $\Gamma_0' \leqslant \mathbf{SO}_{q+2}\mathbb{R}$. Now $\mathbf{G}_2$ is maximal in $\mathbf{SO}_7\mathbb{R}$, and $\mathbf{Spin}_7\mathbb{R}$ is maximal in $\mathbf{SO}_8\mathbb{R}$, see, e.g., [25] 95.12. Hence we obtain that $\Gamma_0' = \mathbf{SO}_{q+2}\mathbb{R}$ or $\Gamma_0' = \Delta$ for $q = 5, 6$.

In order to obtain the groups containing Γ_0', we look at the Cartan decomposition of the corresponding simple Lie algebras. The adjoint action of $\mathbf{SO}_{q+2}\mathbb{R}$ on the Lie algebra Lie(Σ) of Σ is an action by conjugation; by considering skew-symmetric matrices we see that Lie(Σ) decomposes under this action as $\mathrm{Lie}(\mathbf{SO}_{q+2}\mathbb{R}) + \mathbb{R}^{q+2} + \mathbb{R}^{q+2} + \mathbb{R}$, where $\mathbf{SO}_{q+2}\mathbb{R}$ acts naturally on $\mathbb{R}^{q+2}$. In the case where $\Gamma_0' = \mathbf{SO}_{q+2}\mathbb{R}$, we conclude that $\Gamma^1 = \Sigma^1$ or $\Gamma^1 = \mathbf{P}(\mathbf{SO}_{q+2}\mathbb{R} \times A)$ with $A \in \{1, \mathbf{SO}_2\mathbb{R}\}$ or $\Gamma^1 = \mathbf{SO}_{q+3}\mathbb{R}(1)$, a stabilizer in Σ^1. The flag-transitivity of Γ^1 eliminates the last possibility as well as the case $A = 1$.

For $q = 5, 6$ and $\Gamma_0' = \mathbf{G}_2, \mathbf{Spin}_7\mathbb{R}$, the adjoint action of Γ_0' decomposes Lie(Σ) as $\mathrm{Lie}(\Gamma_0') + \mathbb{R}^7 + \mathbb{R}^{q+2} + \mathbb{R}^{q+2} + \mathbb{R}$. One can check that each Lie subalgebra of Lie(Σ) which contains $\mathrm{Lie}(\Gamma_0') + \mathbb{R}$ also contains $\mathrm{Lie}(\mathbf{SO}_q\mathbb{R}) + \mathbb{R} = \mathrm{Lie}(\Gamma_0') + \mathbb{R}^7 + \mathbb{R}$, where $\mathbb{R} = \mathrm{Lie}(\mathbf{SO}_2\mathbb{R})$. This proves the assertions on Γ^1 in part (b_2) of the Main Theorem. The assertions on Γ follow by computing the normalizer of Γ^1. □

4. The Case where $p = 2 < q$ and q is Odd

Throughout this section $\mathfrak{P}$ is a compact connected quadrangle with parameters $(2, q)$, $q \geqslant 3$ odd with a flag-transitive automorphism group Γ. By 2.3 we find a compact connected subgroup Δ which acts transitively on the flags. *By 2.5 and 1.3 we may (and will) assume that Δ is semisimple*. As a consequence, Δ_ℓ is semisimple, see 1.2 and 2.5. Since the point rows are 2-spheres, the group Δ_ℓ is locally isomorphic to $\mathbf{SU}_2\mathbb{C} \times \Delta_{[\ell]}$ by 1.1.

We consider first the case that $q \geqslant 9$. Then $\pi_i(\Delta) = \pi_i(\Delta_\ell)$ for $i \leqslant 7$ by 2.5. In particular, Δ and Δ_ℓ have the same number of almost simple factors and no torus factors, and we can compare their respective types as in 1.2.

PROPOSITION 4.1. *If $q \geqslant 9$, then $\Delta \overset{\text{loc}}{=} \mathbf{SU}_2\mathbb{C} \times \mathbf{SU}_{(q+3)/2}\mathbb{C}$, $\Delta_x \overset{\text{loc}}{=} \mathbf{SU}_{(q+1)/2}\mathbb{C} \times \mathbf{U}_1\mathbb{C}$, $\Delta_\ell \overset{\text{loc}}{=} \mathbf{SU}_2\mathbb{C} \times \Delta_{[\ell]}$ and $\Delta_{x,\ell} \overset{\text{loc}}{=} \mathbf{U}_1\mathbb{C} \times \Delta_{[\ell]}$, where $\Delta_{[\ell]} \overset{\text{loc}}{=} \mathbf{SU}_{(q-1)/2}\mathbb{C}$.*

Proof. We have $\Delta_\ell \overset{\text{loc}}{=} \Delta_{[\ell]} \times \mathbf{SU}_2\mathbb{C}$, and $\pi_6(\mathbf{SU}_2\mathbb{C}) = \mathbb{Z}_{12}$. No other compact group has $\mathbb{Z}_{12}$ as its 6th homotopy group, see Mimura [19] p. 970, hence Δ splits off a normal factor of type $\mathbf{SU}_2\mathbb{C}$. The possible pairs $(\Delta_x/\Delta_{[x]}, \Delta_{x,\ell}/\Delta_{[x]})$ are listed in 1.1. Since $\Delta_{[\ell]}$ is a normal subgroup of $\Delta_{x,\ell}/\Delta_{[x]}$, see 2.2, and has no torus factor, it has at most two almost simple factors. Since $len(\Delta) = len(\Delta_\ell) = len(\Delta_{[\ell]}) + 1$, the group Δ has at most three almost simple factors. Moreover, $len(\Delta_x) = len(\Delta_{x,\ell}) = len(\Delta_\ell) - 1 = len(\Delta_{[\ell]})$.

(1) If $\Delta_{[\ell]}$ is discrete, then Δ is almost simple and hence locally isomorphic to $\mathbf{SU}_2\mathbb{C}$, which is absurd.

(2) If $len(\Delta_{[\ell]}) = 2$, then $\Delta_{[\ell]} \overset{\text{loc}}{=} \mathbf{U}_1\mathbb{H} \times \mathbf{U}_m\mathbb{H}$, where $q = 4m + 3$. By 1.2, Δ has three almost simple factors which belong to the $\mathbf{U}_k\mathbb{H}$-family. Moreover, the Abelian group $\pi_7(\Delta_\ell) \cong \pi_7(\Delta)$ has rank 1, cp. [19], and so two of the three factors are of type $\mathbf{U}_1\mathbb{H}$, since $\pi_7\mathbf{U}_k\mathbb{H}$ has rank 1 for $k \geqslant 2$. Thus $\Delta \overset{\text{loc}}{=} \mathbf{SU}_2\mathbb{C} \times \mathbf{SU}_2\mathbb{C} \times \mathbf{U}_n\mathbb{H}$. We have $\dim \mathcal{L} = 2q + 2 = 8m + 8 = \dim \mathbf{U}_n\mathbb{H} - \dim \mathbf{U}_m\mathbb{H} = (4m + 3) + (4m + 7) + \ldots$, a contradiction.

(3) If $len(\Delta_{[\ell]}) = 1 = len(\Delta_x)$, then $\Delta_{[\ell]} \subseteq \Delta_x$ is one of the almost simple stabilizers in table 1.1, and we go through the whole list.

(3.1) $\Delta_{[\ell]} = \mathbf{SO}_q\mathbb{R}$. By 1.2 $\Delta \overset{\text{loc}}{=} \mathbf{SU}_2\mathbb{C} \times \mathbf{SO}_m\mathbb{R}$. Then $\dim \mathcal{L} = \dim \mathbf{SO}_m\mathbb{R} - \dim \mathbf{SO}_q\mathbb{R} = 2q + 2 = q + (q + 1) + \cdots + (m - 1)$ by 1.5, a contradiction.

(3.2) $\Delta_{[\ell]} = \mathbf{U}_{(q-3)/4}\mathbb{H}$. Then by 1.2 $\Delta \overset{\text{loc}}{=} \mathbf{SU}_2\mathbb{C} \times \mathbf{U}_m\mathbb{H}$. As in (2), the equation $\dim \mathcal{L} = 2q + 2 = q + (q + 4) + \ldots$ leads to a contradiction.

(3.3) $\Delta_{[\ell]} = \mathbf{Spin}_7\mathbb{R}$ and $q = 15$. By 1.2 $\Delta \overset{\text{loc}}{=} \mathbf{SU}_2\mathbb{C} \times \mathbf{SO}_m\mathbb{R}$. Moreover, $\dim \mathcal{L} = 2 + 30$, whence $\dim \Delta = 56$, and thus $\dim \mathbf{SO}_m\mathbb{R} = 53$, which is impossible.

(3.4) $\Delta_{[\ell]} = \mathbf{SU}_{(q-1)/2}\mathbb{C}$. By 1.2 $\Delta \overset{\text{loc}}{=} \mathbf{SU}_2\mathbb{C} \times \mathbf{SU}_m\mathbb{C}$. Then $\dim \mathcal{L} = 2q + 2 = q + (q + 2) + \ldots$, hence $m = (q + 3)/2$, and $\Delta \overset{\text{loc}}{=} \mathbf{SU}_2\mathbb{C} \times \mathbf{SU}_{(q+3)/2}\mathbb{C}$. Thus

$\Delta_\ell \overset{\text{loc}}{=} \mathbf{SU}_2\mathbb{C} \times \mathbf{SU}_{(q-1)/2}\mathbb{C}$ and $\Delta_{x,\ell} \overset{\text{loc}}{=} \mathbf{U}_1\mathbb{C} \times \mathbf{SU}_{(q-1)/2}\mathbb{C}$; whence $\Delta_x \overset{\text{loc}}{=} \mathbf{U}_1\mathbb{C}\times$ $\mathbf{SU}_{(q+1)/2}\mathbb{C}$. □

We consider the remaining cases $q = 3, 5, 7$ separately. If $q = 7$, then we still have that the first 5 homotopy groups of Δ and Δ_ℓ agree. Moreover, $len(\Delta) = len(\Delta_\ell) = len(\Delta_{[\ell]}) + 1 = len(\Delta_{x,\ell}) + 1 = len(\Delta_x)$ for $q \geqslant 5$.

PROPOSITION 4.2. *If* $q = 7$, *then* $\Delta \overset{\text{loc}}{=} \mathbf{SU}_2\mathbb{C} \times \mathbf{SU}_5\mathbb{C}$, $\Delta_x \overset{\text{loc}}{=} \mathbf{SU}_4\mathbb{C} \times \mathbf{U}_1\mathbb{C}$, $\Delta_\ell \overset{\text{loc}}{=} \mathbf{SU}_2\mathbb{C} \times \Delta_{[\ell]}$ *and* $\Delta_{x,\ell} \overset{\text{loc}}{=} \mathbf{U}_1\mathbb{C} \times \Delta_{[\ell]}$, *where* $\Delta_{[\ell]} \overset{\text{loc}}{=} \mathbf{SU}_3\mathbb{C}$.

Proof. We have $\dim \mathcal{L} = 16$ and $\dim \mathcal{P} = 11$. The possible pairs $(\Delta_x/\Delta_{[x]}, \Delta_{x,\ell}/\Delta_{[x]})$ are listed in 1.1. By 2.2, the group $\Delta_{[\ell]}$ is a normal subgroup of one of the following stabilizers: $\mathbf{SO}_7\mathbb{R}$, $\mathbf{SU}_3\mathbb{C}$, $\mathbf{U}_1\mathbb{H} \cdot \mathbf{U}_1\mathbb{H}$, $\mathbf{G}_2$, hence we have one of the following cases.

(1) $\Delta_{[\ell]}$ is discrete. Then Δ is almost simple and of dimension $\dim \mathcal{L} + \dim \mathbf{SU}_2\mathbb{C} = 19$; but there is no almost simple group of that dimension.
(2) $\Delta_{[\ell]} = \mathbf{SO}_7\mathbb{R}$. Then $len(\Delta) = 2$ and $\dim \Delta = 40$. Again, such a group does not exist.
(3) $\Delta_{[\ell]} = \mathbf{U}_1\mathbb{H}$. Then $len(\Delta) = 2$ and $\dim \Delta = 22$. There is only one such group, namely $\mathbf{SU}_3\mathbb{C} \times \mathbf{G}_2$. But $0 = \pi_4(\Delta) \neq \pi_4(\Delta_\ell) = \mathbb{Z}_2 + \mathbb{Z}_2$, a contradiction.
(4) $\Delta_{[\ell]} = \mathbf{U}_1\mathbb{H} \cdot \mathbf{U}_1\mathbb{H}$. Then $len(\Delta) = 3$ and $\dim \Delta = 25$. The only possibility is $\mathbf{SU}_3\mathbb{C} \times \mathbf{SU}_2\mathbb{C} \times \mathbf{G}_2$. But $\mathbb{Z} + \mathbb{Z}_2 = \pi_5(\Delta) \neq \pi_5(\Delta_\ell) = \mathbb{Z}_2 + \mathbb{Z}_2 + \mathbb{Z}_2$, a contradiction.
(5) $\Delta_{[\ell]} = \mathbf{G}_2$. Then $len(\Delta) = 2$ and $\dim \Delta = 33$, which is impossible.
(6) $\Delta_{[\ell]} = \mathbf{SU}_3\mathbb{C}$. Then $len(\Delta) = 2$ and $\dim \Delta = 27$. The only possibility is $\Delta \overset{\text{loc}}{=} \mathbf{SU}_5\mathbb{C} \times \mathbf{SU}_2\mathbb{C}$. Then $\Delta_\ell \overset{\text{loc}}{=} \mathbf{SU}_3\mathbb{C} \times \mathbf{SU}_2\mathbb{C}$, $\Delta_{x,\ell} \overset{\text{loc}}{=} \mathbf{SU}_3\mathbb{C} \times \mathbf{U}_1\mathbb{C}$ and $\Delta_x \overset{\text{loc}}{=} \mathbf{SU}_4\mathbb{C} \times \mathbf{U}_1\mathbb{C}$. □

The next case is $q = 5$. Now we know only that the first three homotopy groups of Δ and Δ_ℓ agree.

PROPOSITION 4.3. *If* $q = 5$ *then* $\Delta \overset{\text{loc}}{=} \mathbf{SU}_2\mathbb{C} \times \mathbf{SU}_4\mathbb{C}$, $\Delta_x \overset{\text{loc}}{=} \mathbf{SU}_3\mathbb{C} \times \mathbf{U}_1\mathbb{C}$, $\Delta_\ell \overset{\text{loc}}{=} \mathbf{SU}_2\mathbb{C} \times \Delta_{[\ell]}$ *and* $\Delta_{x,\ell} \overset{\text{loc}}{=} \mathbf{U}_1\mathbb{C} \times \Delta_{[\ell]}$, *where* $\Delta_{[\ell]} \overset{\text{loc}}{=} \mathbf{SU}_2\mathbb{C}$.

Proof. We have $\dim \mathcal{L} = 12$ and $\dim \mathcal{P} = 9$. The group $\Delta_{[\ell]}$ is normal in either $\mathbf{SO}_5\mathbb{R}$ or $\mathbf{SU}_2\mathbb{C}$.

(1) $\Delta_{[\ell]}$ is discrete. Then $\Delta_\ell \overset{\text{loc}}{=} \mathbf{SU}_2\mathbb{C}$ and $\dim \Delta = 12 + 3 = 15$, hence $\dim \Delta_x = 6$. Since $\pi_2(\mathcal{P}) = \mathbb{Z}$ and $\pi_2(\Delta) = 0$, cp. [25] 94.36, the center of Δ_x has to be one-dimensional. But there is no five-dimensional semisimple compact group.
(2) $\Delta_{[\ell]} = \mathbf{SO}_5\mathbb{R}$. Then $\Delta_\ell \overset{\text{loc}}{=} \mathbf{SU}_2\mathbb{C} \times \mathbf{SO}_5\mathbb{R}$, and $\pi_4(\Delta_\ell) = \mathbb{Z}_2 + \mathbb{Z}_2$. On the other hand $\pi_5(\Delta)$ has to be finite, and $\pi_5(\mathcal{L}) = \mathbb{Z}$. These homotopy groups do not fit together to an exact sequence.
(3) $\Delta_{[\ell]} = \mathbf{SU}_2\mathbb{C}$. Then $\dim \Delta = 12 + 6 = 18$ and therefore $\Delta \overset{\text{loc}}{=} \mathbf{SO}_5\mathbb{R} \times \mathbf{SU}_3\mathbb{C}$ or $\Delta \overset{\text{loc}}{=} \mathbf{SU}_4\mathbb{C} \times \mathbf{SU}_2\mathbb{C}$. In case $\Delta \overset{\text{loc}}{=} \mathbf{SO}_5\mathbb{R} \times \mathbf{SU}_3\mathbb{C}$ we find that $\pi_4(\Delta) = \mathbb{Z}_2$ is

a finite group. As in the case (2), this is not possible. Hence $\Delta \stackrel{\text{loc}}{=} \mathbf{SU}_4\mathbb{C} \times \mathbf{SU}_2\mathbb{C}$, whence $\Delta_\ell \stackrel{\text{loc}}{=} \mathbf{SU}_2\mathbb{C} \times \mathbf{SU}_2\mathbb{C}$, $\Delta_{x,\ell} \stackrel{\text{loc}}{=} \mathbf{SU}_2\mathbb{C} \times \mathbf{U}_1\mathbb{C}$ and $\Delta_x \stackrel{\text{loc}}{=} \mathbf{SU}_3\mathbb{C} \times \mathbf{U}_2\mathbb{C}$. □

The last case is $q = 3$. The fundamental group tells us that Δ_ℓ is semisimple. By 2.5 we have $\pi_3(\mathcal{L}) = \mathbb{Z}$, and by Corollary 2.6 the group $\pi_4(\mathcal{L})$ is finite, hence we obtain a short exact sequence

$$0 \to \pi_3(\Delta_\ell) \to \pi_3(\Delta) \to \pi_3(\mathcal{L}) \to 0.$$

This implies $len(\Delta) = len(\Delta_\ell) + 1 = len(\Delta_{[\ell]}) + 2 = len(\Delta_{x,\ell}) + 2 = len(\Delta_x) + 1$, since $\Delta_{x,\ell} \stackrel{\text{loc}}{=} \mathbf{U}_1\mathbb{C} \times \Delta_{[\ell]}$, and because $\pi_4(\mathbb{S}^3) = \pi_4(\mathcal{L}_x)$ is finite.

PROPOSITION 4.4. *If* $q = 3$ *then* $\Delta \stackrel{\text{loc}}{=} \mathbf{SU}_2\mathbb{C} \times \mathbf{SU}_3\mathbb{C}$, $\Delta_x \stackrel{\text{loc}}{=} \mathbf{SU}_2\mathbb{C} \times \mathbf{U}_1\mathbb{C}$, $\Delta_\ell \stackrel{\text{loc}}{=} \mathbf{SU}_2\mathbb{C}$ *and* $\Delta_{x,\ell} \stackrel{\text{loc}}{=} \mathbf{U}_1\mathbb{C}$.

Proof. Here, $\Delta_{[\ell]}$ has to be a normal subgroup of $\mathbf{SU}_2\mathbb{C}$ or $\mathbf{U}_2\mathbb{C}$.

(1) If $\Delta_{[\ell]}$ is not discrete, then $\Delta_{[\ell]}$ is semisimple because Δ_ℓ is semisimple, hence $\Delta_{[\ell]} \stackrel{\text{loc}}{=} \mathbf{SU}_2\mathbb{C}$, and Δ is a 14-dimensional group with 3 almost simple factors. The only possibility is $\Delta \stackrel{\text{loc}}{=} \mathbf{SU}_3\mathbb{C} \times \mathbf{SU}_2\mathbb{C} \times \mathbf{SU}_2\mathbb{C}$, and $\Delta_\ell \stackrel{\text{loc}}{=} \mathbf{SU}_2\mathbb{C} \times \mathbf{SU}_2\mathbb{C}$. Replacing Δ by its universal covering group $\tilde{\Delta}$, we have $\tilde{\Delta} = \mathbf{SU}_3\mathbb{C} \times \mathbf{SU}_2\mathbb{C} \times \mathbf{SU}_2\mathbb{C}$; furthermore, $\mathcal{L}$ is 2-connected, see 2.5, whence $(\tilde{\Delta})_\ell = \mathbf{SU}_2\mathbb{C} \times \mathbf{SU}_2\mathbb{C}$.

We argue that up to automorphisms there is only one imbedding $(\tilde{\Delta})_\ell \subseteq \tilde{\Delta}$ which yields an effective action, namely the one where one factor of $\mathbf{SU}_2\mathbb{C} \times \mathbf{SU}_2\mathbb{C}$ is imbedded into $\mathbf{SU}_3\mathbb{C}$ and the other factor $\mathbf{SU}_2\mathbb{C}$ is imbedded diagonally into $\mathbf{SU}_2\mathbb{C} \times \mathbf{SU}_2\mathbb{C}$. This is true because neither $\mathbf{SU}_3\mathbb{C}$ nor $\mathbf{SU}_2\mathbb{C} \times \mathbf{SU}_2\mathbb{C}$ contain $\mathbf{SU}_2\mathbb{C} \times \mathbf{SU}_2\mathbb{C}$ as a non-normal subgroup.

But now $\mathcal{L}$ is – as a homogeneous space – a product of two spheres

$$((\mathbf{SU}_2\mathbb{C} \times \mathbf{SU}_2\mathbb{C})/diag(\mathbf{SU}_2\mathbb{C})) \times (\mathbf{SU}_3\mathbb{C}/\mathbf{SU}_2\mathbb{C}) \approx \mathbb{S}_3 \times \mathbb{S}_5,$$

contradicting 2.7.

(2) If $\Delta_{[\ell]}$ is discrete, then $\dim \Delta = 11$ and hence $\Delta \stackrel{\text{loc}}{=} \mathbf{SU}_3\mathbb{C} \times \mathbf{SU}_2\mathbb{C}$. Therefore $\Delta_\ell \stackrel{\text{loc}}{=} \mathbf{SU}_2\mathbb{C}$, $\Delta_{x,\ell} \stackrel{\text{loc}}{=} \mathbf{U}_1\mathbb{C}$, and $\Delta_x \stackrel{\text{loc}}{=} \mathbf{SU}_2\mathbb{C} \times \mathbf{U}_1\mathbb{C}$. □

PROPOSITION 4.5. *If* $p = 2$ *and* $q \geqslant 5$ *is odd, then we have case* (b_3) *of the Main Theorem.*

Proof. Let $m = (q + 3)/2$. Thus $m \geqslant 4$. By the results of this section, the universal covering group $\tilde{\Delta} = \mathbf{SU}_2\mathbb{C} \times \mathbf{SU}_m\mathbb{C}$ acts almost effectively on $\mathfrak{P}$, with connected stabilizers $(\tilde{\Delta})_x \stackrel{\text{loc}}{=} \mathbf{U}_1\mathbb{C} \times \mathbf{SU}_{m-1}\mathbb{C}$, $(\tilde{\Delta})_\ell \stackrel{\text{loc}}{=} \mathbf{SU}_2\mathbb{C} \times \Delta_{[\ell]}$ and $(\tilde{\Delta})_{x,\ell} \stackrel{\text{loc}}{=} \mathbf{U}_1\mathbb{C} \times \Delta_{[\ell]}$, where $\Delta_{[\ell]} \stackrel{\text{loc}}{=} \mathbf{SU}_{m-2}\mathbb{C}$.

We claim that $\tilde{\Delta} = \mathbf{SU}_2\mathbb{C} \times \mathbf{SU}_m\mathbb{C}$ contains only one conjugacy class of subgroups which are locally isomorphic to $\mathbf{SU}_{m-1}\mathbb{C}$ (viz. the stabilizers of non-zero vectors). Indeed, each inclusion of such a subgroup into $\mathbf{SU}_m\mathbb{C}$ gives an almost effective representation of $\mathbf{SU}_{m-1}\mathbb{C}$ on $\mathbb{C}^m$, and our claim follows from [23] Prop. 5, p. 53 and Prop. 8, p. 56, compare also [22] p. 42, case I, and [25] 95.12 for $n \leqslant 8$. The embeddings of $(\tilde{\Delta})_{x,\ell}$ into $(\tilde{\Delta})_x$ or into $(\tilde{\Delta})_\ell$ are the standard ones (as stabilizers of vectors, compare 1.1). We conclude that the pair $((\tilde{\Delta})'_x, (\tilde{\Delta})_{[\ell]} = (\tilde{\Delta})_{x,\ell}{}')$ is unique in $\tilde{\Delta}$ up to conjugation, and $(\tilde{\Delta})'_x, (\tilde{\Delta})_{[\ell]}$ are contained in the factor $1 \times \mathbf{SU}_m\mathbb{C}$ of $\tilde{\Delta}$. We have $(\tilde{\Delta})_\ell = S \times (\tilde{\Delta})_{[\ell]}$ with a subgroup $S \cong \mathbf{SU}_2\mathbb{C}$, and $S = S' \subseteq Cen_{\tilde{\Delta}}((\tilde{\Delta})'_{[\ell]}) = (\mathbf{SU}_2\mathbb{C} \times \mathbf{U}_2\mathbb{C})' = \mathbf{SU}_2\mathbb{C} \times \mathbf{SU}_2\mathbb{C}$. The projections of S into the two factors of $\mathbf{SU}_2\mathbb{C} \times \mathbf{SU}_2\mathbb{C}$ are both not trivial, because S is distinct from the normal factor $\mathbf{SU}_2\mathbb{C} \times 1$ of $\tilde{\Delta}$, and $(\tilde{\Delta})_\ell$ is not contained in the normal factor $1 \times \mathbf{SU}_m\mathbb{C}$, since $\tilde{\Delta}$ is generated by $(\tilde{\Delta})_x$ and $(\tilde{\Delta})_\ell$, cp. [10] p.104. Hence S is a diagonal subgroup of $\mathbf{SU}_2\mathbb{C} \times \mathbf{SU}_2\mathbb{C}$, and therefore unique up to conjugation by elements of $\mathbf{SU}_2\mathbb{C} \times 1$. We conclude that the quadruple $((\tilde{\Delta})_x{}', (\tilde{\Delta})_{[\ell]}, S, (\tilde{\Delta})_\ell = S \times (\tilde{\Delta})_{[\ell]})$ is unique up to conjugation in $\tilde{\Delta}$. Since S contains only one conjugacy class of subgroups isomorphic to $\mathbf{U}_1\mathbb{C}$, the triple $((\tilde{\Delta})_x = \mathbf{U}_1\mathbb{C} \times (\tilde{\Delta})_x{}', (\tilde{\Delta})_\ell, (\tilde{\Delta})_{x,\ell} = \mathbf{U}_1\mathbb{C} \times \Delta_{[\ell]})$ is determined uniquely up to conjugation in $\tilde{\Delta}$.

Let $\mathfrak{Q}$ be the complex Hermitian quadrangle defined by a Hermitian form on $\mathbb{C}^{m+2}$ of Witt index 2. Then $\tilde{\Delta}$ acts almost effectively on $\mathfrak{Q}$ as a flag-transitive automorphism group. By the uniqueness obtained above, we conclude that $\mathfrak{P}$ is isomorphic to $\mathfrak{Q}$. As a consequence, the semisimple group Δ is isomorphic to $\mathbf{P}(\mathbf{SU}_m\mathbb{C} \times \mathbf{SU}_2\mathbb{C})$.

It remains to determine the possible flag-transitive groups Γ. We have shown that the connected components Γ^1 of these groups contain the subgroup $\mathbf{P}(\mathbf{SU}_m\mathbb{C} \times \mathbf{SU}_2\mathbb{C})$ of $\mathbf{PU}_{m+2}\mathbb{C}(2)$. The overgroups of this group (in the full automorphism group of $\mathfrak{Q}$) can be found as in 3.2. Thus Γ satisfies the conclusions in case (b_3) of the Main Theorem. □

Finally, we have to consider the case $q = 3$.

PROPOSITION 4.6. *If $p = 2$ and $q = 3$, then we have case (b_3) of the Main Theorem.*

Proof. By 4.4 the universal covering group $\tilde{\Delta} = \mathbf{SU}_2\mathbb{C} \times \mathbf{SU}_3\mathbb{C}$ acts almost effectively on $\mathfrak{P}$, with connected stabilizers $(\tilde{\Delta})_x \stackrel{\text{loc}}{=} \mathbf{U}_1\mathbb{C} \times \mathbf{SU}_2\mathbb{C}$, $(\tilde{\Delta})_\ell \stackrel{\text{loc}}{=} \mathbf{SU}_2\mathbb{C}$ and $(\tilde{\Delta})_{x,\ell} \stackrel{\text{loc}}{=} \mathbf{U}_1\mathbb{C}$. We apply the results of our Appendix to $\mathcal{P} = \Delta/\Delta_x$. From [10] Appendix 4_3 we infer that $H^i(\mathcal{P}) \cong \mathbb{Z} \cong H_i(\mathcal{P})$ for $i = 0, 2, 5, 7$, and that $H^i(\mathcal{P}) = 0 = H_i(\mathcal{P})$ else. From 7.1 we see that the only possibility which is left is case 7.1 (2), with $\mathcal{P} \approx M_{k,l}$, and $l = 1$.

Next we note that $(\tilde{\Delta})_\ell$ cannot be contained in the normal factor $\mathbf{SU}_3\mathbb{C}$ of $\tilde{\Delta}$, because Δ_x and Δ_ℓ generate the whole group Δ. It is also clear that $(\tilde{\Delta})_\ell$ cannot be contained in the other normal factor $\mathbf{SU}_2\mathbb{C}$, since the action on $\mathcal{L}$ is effective. Thus Δ_ℓ is embedded 'diagonally', and $\mathbf{SU}_3\mathbb{C}$ acts regularly on $\mathcal{L}$; in particular, there is a homeomorphism $\mathcal{L} \approx \mathbf{SU}_3\mathbb{C}$. By 7.3, $\mathcal{P} \approx M_{k,1}$ is not a spin manifold, and thus k is odd by 7.1 (2). From this and the description of $(\tilde{\Delta})_x$ in 7.1 we see that the pair of matrices $(-1, 1) \in \mathbf{SU}_2\mathbb{C} \times \mathbf{SU}_3\mathbb{C}$ is not contained in Δ_x, and thus the action of $\tilde{\Delta}$ on $\mathcal{P}$ is effective, i.e. $\Delta = \tilde{\Delta} = \mathbf{SU}_2\mathbb{C} \times \mathbf{SU}_3\mathbb{C}$. Thus the embedding of

$\Delta_\ell \cong \mathbf{SU}_2\mathbb{C}$ is given by $\mathrm{id} \times \iota$ (same terminology as in 7.1, i.e. $\iota(A) = \begin{pmatrix} A & \\ & 1 \end{pmatrix}$), since $\mathrm{id} \times \kappa$ would yield a non-effective action on $\mathcal{L}$.

Therefore we may assume that

$$\Delta_\ell = \{(A, \iota(A)) |\ A \in \mathbf{SU}_2\mathbb{C}\} \tag{$*$}$$

and

$$\Delta_{x,\ell} = \left\{(C, \iota(C)) \middle|\ C = \mathrm{diag}(c, c^{-1}),\ c \in \mathbf{U}_1\mathbb{C}\right\}. \tag{$**$}$$

In the natural action of Δ on $\mathbb{C}^3$ (via the second factor), $\Delta_{x,\ell}$ fixes a one-dimensional complex subspace elementwise.

We now show that Δ_x is uniquely determined (up to automorphisms of Δ) by the groups given in ($*$) and ($**$). By 7.1 (2), the point stabilizer Δ_x is conjugate to the subgroup consisting of pairs of matrices of the form

$$\left(\begin{pmatrix} c^2 & \\ & c^{-2} \end{pmatrix}, \begin{pmatrix} Ac^k & \\ & c^{-2k} \end{pmatrix}\right), \quad A \in \mathbf{SU}_2\mathbb{C},\ c \in \mathbf{U}_1\mathbb{C},$$

for some fixed odd integer k. A 1-parameter subgroup of Δ_x which fixes a 1-dimensional subspace in $\mathbb{C}^3$ elementwise and which is not contained in $\mathbf{SU}_3\mathbb{C}$ (such as $\Delta_{x,\ell}$) is then clearly conjugate to the subgroup

$$\left\{\left(\begin{pmatrix} c^{-2} & \\ & c^2 \end{pmatrix}, \begin{pmatrix} c^{2k} & & \\ & 1 & \\ & & c^{-2k} \end{pmatrix}\right) \middle|\ c \in \mathbf{U}_1\mathbb{C}\right\}.$$

Suppose that this group is conjugate to $\Delta_{x,\ell}$ (this obviously implies $k = \pm 1$, but we do not need this fact here). The group $\Delta_{x,\ell}$ given in ($**$) has (in its action on $\mathbb{C}^3$) two unique eigenspaces on which the group acts non-trivially (viz. the subspaces spanned by the canonical basis vectors e_1, e_2 of $\mathbb{C}^3$). We can interchange these two eigenspaces by an element of Δ_ℓ which normalizes $\Delta_{x,\ell}$. Thus we can assume that the semisimple part $\mathbf{SU}_2\mathbb{C}$ of Δ_x fixes e_2 and hence coincides with

$$\left\{\left(1, \begin{pmatrix} a_{11} & 0 & a_{12} \\ 0 & 1 & 0 \\ a_{21} & 0 & a_{22} \end{pmatrix}\right) \middle|\ A \in \mathbf{SU}_2\mathbb{C}\right\}.$$

The group generated by these matrices and $\Delta_{x,\ell}$ is

$$\Delta_x = \left\{\left(\begin{pmatrix} c^{-2} & \\ & c^2 \end{pmatrix}, \begin{pmatrix} a_{11}c & 0 & a_{12}c \\ 0 & c^{-2} & 0 \\ a_{21}c & 0 & a_{22}c \end{pmatrix}\right) \middle|\ A \in \mathbf{SU}_2\mathbb{C},\ c \in \mathbf{U}_1\mathbb{C}\right\}.$$

Thus the triple $(\Delta_x, \Delta_\ell, \Delta_{x,\ell})$ of subgroups of Δ is determined uniquely up to automorphisms of Δ, and the proof can be completed as in the last two paragraphs of 4.5. □

5. The Case where $p = 4$ and $q \geqslant 3$ is Odd

Throughout this section $\mathfrak{P}$ is a compact connected quadrangle with parameters $(4, q)$, for $q \geqslant 3$ odd, with a flag-transitive automorphism group Γ. By 2.3 we find a compact connected subgroup Δ which acts transitively on the flags. *By 2.5, 1.3 we may (and will) assume that Δ is semisimple.* Since $p = 4$, we have $\Delta_\ell \overset{\text{loc}}{=} \mathbf{SO}_5\mathbb{R} \times \Delta_{[\ell]}$, compare 1.1.

If $q \geqslant 5$, then $len(\Delta) = len(\Delta_\ell)$. *By 1.3, 1.2 and 1.4 we may and will assume that Δ, Δ_ℓ, and Δ_x have no torus factors for $q \geqslant 3$, and that Δ has no normal subgroups of type $\mathbf{SU}_2\mathbb{C} = \mathbf{U}_1\mathbb{H}$ for $q \geqslant 5$.*

LEMMA 5.1. *If $q \geqslant 9$, then $\Delta_{[\ell]}$ is not of type $\mathbf{U}_1\mathbb{H}$.*

Proof. Assume otherwise. Then $\Delta_\ell \overset{\text{loc}}{=} \mathbf{U}_2\mathbb{H} \times \mathbf{U}_1\mathbb{H}$, and thus $\Delta \overset{\text{loc}}{=} \mathbf{U}_n\mathbb{H} \times \mathbf{U}_m\mathbb{H}$ by 1.2. Moreover, $\pi_7(\Delta_\ell)$ has rank 1, see [19], and so $\pi_7(\Delta)$ also has rank 1. It follows that $n = 1$ or $m = 1$, contradicting our assumptions. □

The tenth homotopy group of $\mathbf{SO}_5\mathbb{R}$ is $\mathbb{Z}_{120}$, see [19], and (up to local isomorphy) no other compact almost simple Lie group has this group as its tenth homotopy group (note that $\mathbf{U}_2\mathbb{H} \overset{\text{loc}}{=} \mathbf{SO}_5\mathbb{R}$). Therefore we consider first the case $q \geqslant 13$.

PROPOSITION 5.2. *If $q \geqslant 13$ then $\Delta \overset{\text{loc}}{=} \mathbf{U}_2\mathbb{H} \times \mathbf{U}_{(q+5)/4}\mathbb{H}$, $\Delta_x \overset{\text{loc}}{=} \mathbf{U}_1\mathbb{H} \times \mathbf{U}_1\mathbb{H} \times \mathbf{U}_{(q+1)/4}\mathbb{H}$, $\Delta_\ell \overset{\text{loc}}{=} \mathbf{SO}_5\mathbb{R} \times \Delta_{[\ell]}$ and $\Delta_{x,\ell} \overset{\text{loc}}{=} \mathbf{SO}_4\mathbb{R} \times \Delta_{[\ell]}$, where $\Delta_{[\ell]} \overset{\text{loc}}{=} \mathbf{U}_{(q-3)/4}\mathbb{H}$.*

Proof. The group $\Delta_{[\ell]}$ is a normal semisimple subgroup of $\Delta_{x,\ell}/\Delta_{[x]}$, see 2.2, hence by 1.1 the group $\Delta_{[\ell]}$ is a normal subgroup of $\mathbf{SO}_q\mathbb{R}$, $\mathbf{SU}_{(q-1)/2}\mathbb{C}$, $\mathbf{U}_1\mathbb{H} \cdot \mathbf{U}_{(q+1)/4}\mathbb{H}$, or $\mathbf{Spin}_7\mathbb{R}$ (for $q = 15$), and $\Delta_{[\ell]}$ is not of type $\mathbf{U}_1\mathbb{H}$ by 5.1.

Since $\pi_{10}(\Delta) = \pi_{10}(\Delta_\ell)$, the group Δ splits off a normal subgroup of type $\mathbf{SO}_5\mathbb{R}$, $\mathbf{SO}_6\mathbb{R}$, $\mathbf{SU}_4\mathbb{C}$, or $\mathbf{SU}_5\mathbb{C}$. In the case of $\mathbf{SU}_4\mathbb{C}$ or $\mathbf{SO}_6\mathbb{R}$ we get a factor $\mathbb{Z}_2$ in the 10th homotopy group, hence Δ_ℓ itself has to split off $\mathbf{SU}_4\mathbb{C}$, $\mathbf{SO}_6\mathbb{R}$, or $\mathbf{SO}_{11}\mathbb{R}$.

(1) If $\Delta_{[\ell]}$ is discrete, then $len(\Delta) = 1$ and $\Delta_\ell \overset{\text{loc}}{=} \mathbf{SO}_5\mathbb{R}$. But no compact almost simple Lie group apart from $\mathbf{SO}_5\mathbb{R}$ has $\mathbb{Z}_{120}$ as its tenth homotopy group, cp. [19] p. 970, hence this case is impossible.
(2) $\Delta_{[\ell]} = \mathbf{SO}_q\mathbb{R}$. This case is excluded by the dual of 2.4.
(3) $\Delta_{[\ell]} = \mathbf{SU}_{(q-1)/2}\mathbb{C}$. By 1.2, $\Delta \overset{\text{loc}}{=} \mathbf{SU}_m\mathbb{C} \times \mathbf{U}_n\mathbb{H}$, for $m \geqslant 3$. Thus $\dim \mathcal{L} = 2q + 4 = \dim \mathbf{U}_n\mathbb{H} - 10 + q + (q + 2) + (q + 4) + \ldots + (m - 1)/2$. The only possibility is $2q + 14 = \dim \mathbf{U}_n\mathbb{H} + q + (q + 2)$, hence $\dim \mathbf{U}_n\mathbb{H} = 12$, which is absurd.
(4) $\Delta_{[\ell]} = \mathbf{Spin}_7\mathbb{R}$ and $q = 15$. By 1.2 $\Delta \overset{\text{loc}}{=} \mathbf{U}_m\mathbb{H} \times \mathbf{SO}_n\mathbb{R}$. Moreover, $\pi_9(\Delta_\ell) = \mathbb{Z}_2 + \mathbb{Z}_2 = \pi_9(\Delta)$, whence $n = 7$ or $n = 9$. If $n = 7$, then $\Delta \overset{\text{loc}}{=} \mathbf{U}_m\mathbb{H} \times \mathbf{SO}_7\mathbb{R}$ and $\dim \mathcal{L} = 34 = \dim \mathbf{U}_m\mathbb{H} - \dim \mathbf{SO}_5\mathbb{R} = \dim \mathbf{U}_m\mathbb{H} - 10$, which is impossible. If $n = 9$, then $\Delta \overset{\text{loc}}{=} \mathbf{U}_m\mathbb{H} \times \mathbf{SO}_9\mathbb{R}$ and $\dim \mathcal{L} = 34 = 15 + \dim \mathbf{U}_m\mathbb{H} - 10$, again a contradiction.

(5) $\Delta_{[\ell]} = \mathbf{U}_1\mathbb{H} \cdot \mathbf{U}_{(q-3)/4}\mathbb{H}$. By 1.2 $\Delta \overset{\text{loc}}{=} \mathbf{U}_r\mathbb{H} \times \mathbf{U}_m\mathbb{H} \times \mathbf{U}_n\mathbb{H}$. Since $\pi_{10}(\Delta_\ell) = \mathbb{Z}_{120} + \mathbb{Z}_{15}$, the group Δ splits off a factor $\mathbf{U}_1\mathbb{H}$, contrary to our assumptions on Δ.

(6) $\Delta_{[\ell]} = \mathbf{U}_{(q-3)/4}\mathbb{H}$. By 1.2 $\Delta \overset{\text{loc}}{=} \mathbf{U}_m\mathbb{H} \times \mathbf{U}_n\mathbb{H}$. Since $\pi_{10}(\Delta_\ell) = \mathbb{Z}_{120}$ we may assume that $m = 2$, and thus $n = (q+1)/4 + 1$. Hence $\Delta_\ell \overset{\text{loc}}{=} \mathbf{SO}_5\mathbb{R} \times \mathbf{U}_{(q-3)/4}\mathbb{H}$, $\Delta_{x,\ell} \overset{\text{loc}}{=} \mathbf{SO}_4\mathbb{R} \times \mathbf{U}_{(q-3)/4}\mathbb{H}$, and $\Delta_x \overset{\text{loc}}{=} \mathbf{SO}_4\mathbb{R} \times \mathbf{U}_{(q+1)/4}\mathbb{H}$. □

The remaining cases are $q = 11, 9, 7, 5, 3$, and we consider them separately.

PROPOSITION 5.3. *If* $q = 11$, *then* $\Delta \overset{\text{loc}}{=} \mathbf{U}_2\mathbb{H} \times \mathbf{U}_4\mathbb{H}$, $\Delta_x \overset{\text{loc}}{=} \mathbf{U}_1\mathbb{H} \times \mathbf{U}_1\mathbb{H} \times \mathbf{U}_3\mathbb{H}$, $\Delta_\ell \overset{\text{loc}}{=} \mathbf{SO}_5\mathbb{R} \times \mathbf{U}_2\mathbb{H}$ *and* $\Delta_{x,\ell} \overset{\text{loc}}{=} \mathbf{SO}_4\mathbb{R} \times \mathbf{U}_2\mathbb{H}$.

Proof. Here $\dim \mathcal{L} = 26$ and $\dim \mathcal{P} = 19$. The almost simple group $\Delta_{[\ell]}$ is normal in one of the following stabilizers $\mathbf{SO}_{11}\mathbb{R}$, $\mathbf{SU}_5\mathbb{C}$, or $\mathbf{U}_1\mathbb{H} \cdot \mathbf{U}_2\mathbb{H}$, and not locally isomorphic to $\mathbf{U}_1\mathbb{H}$ by 5.1.

(1) $\Delta_{[\ell]}$ is discrete. Then Δ is almost simple and of dimension $26 + 10 = 36$, hence $\dim \Delta_x = 36 - 19 = 17$. But no 17-dimensional compact Lie group can act transitively on $\mathbb{S}_{11}$ by 1.1, a contradiction.

(2) $\Delta_{[\ell]} = \mathbf{SO}_{11}\mathbb{R}$. This case is excluded by the dual of 2.4.

(3) $\Delta_{[\ell]} = \mathbf{SU}_5\mathbb{C}$. By 1.2 $\Delta \overset{\text{loc}}{=} \mathbf{SU}_m\mathbb{C} \times \mathbf{U}_n\mathbb{H}$, and $m = 5$, $n = 4$ since $\dim \Delta = 26 + 10 + 24 = 60$. The projection of $\mathbf{SU}_5\mathbb{C} \subseteq \Delta_\ell$ into the factor $\mathbf{U}_4\mathbb{H} \subseteq \Delta$ has to be trivial (since $\mathbf{SU}_5\mathbb{C}$ has to fix many vectors of $\mathbb{H}^4$, cp. [25] 95.10, p. 626), thus $\mathbf{SU}_5\mathbb{C}$ is normal in Δ and acts trivially on $\mathcal{L}$, a contradiction.

(4) $\Delta_{[\ell]} = \mathbf{U}_1\mathbb{H} \cdot \mathbf{U}_2\mathbb{H}$. Then $\dim \Delta = 26 + 23 = 49$ and Δ has three almost simple factors. By 1.2 we see that they are all quaternion unitary; whence $\Delta \overset{\text{loc}}{=} \mathbf{U}_1\mathbb{H} \times \mathbf{U}_2\mathbb{H} \times \mathbf{U}_4\mathbb{H}$. But we assumed that Δ has no normal factor of type $\mathbf{U}_1\mathbb{H}$.

(5) $\Delta_{[\ell]} = \mathbf{U}_2\mathbb{H}$. Then $\dim \Delta = 26 + 23 = 46$ and Δ has two almost simple factors. By 1.2 they are quaternion unitary, whence $\Delta \overset{\text{loc}}{=} \mathbf{U}_2\mathbb{H} \times \mathbf{U}_4\mathbb{H}$, and $\Delta_{x,\ell} \overset{\text{loc}}{=} \mathbf{SO}_4\mathbb{R} \times \mathbf{U}_2\mathbb{H}$. Thus Δ_x is a semisimple 27-dimensional group with three almost simple factors. Using 1.1 we find that $\Delta_x \overset{\text{loc}}{=} \mathbf{SO}_4\mathbb{R} \times \mathbf{U}_3\mathbb{H}$ and $\Delta_\ell \overset{\text{loc}}{=} \mathbf{SO}_5\mathbb{R} \times \mathbf{U}_2\mathbb{H}$. □

PROPOSITION 5.4. *The case* $q = 9$ *is not possible.*

Proof. Here $\dim \mathcal{L} = 22$, and $\Delta_{[\ell]}$ is normal in $\mathbf{SO}_9\mathbb{R}$ or $\mathbf{SU}_4\mathbb{C}$. If $\Delta_{[\ell]}$ is discrete, then Δ is almost simple of dimension $22 + 10 = 32$, which is absurd. The case $\Delta_{[\ell]} = \mathbf{SO}_9\mathbb{R}$ is excluded by the dual of 2.4. Thus $\Delta_{[\ell]} = \mathbf{SU}_4\mathbb{C}$, and $\dim \Delta = 22 + 25 = 37$. By 1.2 we have $\Delta \overset{\text{loc}}{=} \mathbf{SU}_m\mathbb{C} \times \mathbf{U}_n\mathbb{H}$, but such a group does not exist. □

PROPOSITION 5.5. *If* $q = 7$ *then* $\Delta \overset{\text{loc}}{=} \mathbf{U}_2\mathbb{H} \times \mathbf{U}_3\mathbb{H}$, $\Delta_x \overset{\text{loc}}{=} \mathbf{U}_1\mathbb{H} \times \mathbf{U}_1\mathbb{H} \times \mathbf{U}_2\mathbb{H}$, $\Delta_\ell \overset{\text{loc}}{=} \mathbf{SO}_5\mathbb{R} \times \mathbf{U}_1\mathbb{H}$ *and* $\Delta_{x,\ell} \overset{\text{loc}}{=} \mathbf{SO}_4\mathbb{R} \times \mathbf{U}_1\mathbb{H}$.

Proof. Here $\dim \mathcal{L} = 18$ and $\Delta_{[\ell]}$ is normal in $\mathbf{SO}_7\mathbb{R}$, $\mathbf{SU}_3\mathbb{C}$, $\mathbf{U}_1\mathbb{H} \cdot \mathbf{U}_1\mathbb{H}$, or $\mathbf{G}_2$.

(1) $\Delta_{[\ell]}$ is discrete. Then $\dim\Delta = 18 + 10 = 28$ and Δ is almost simple. Therefore $\Delta \stackrel{\text{loc}}{=} \mathbf{SO}_8\mathbb{R}$; but $0 = \pi_5(\Delta) \neq \pi_5(\Delta_\ell) = \mathbb{Z}_2$, a contradiction.

(2) $\Delta_{[\ell]} = \mathbf{SO}_7\mathbb{R}$. Then $\dim\Delta = 18 + 31 = 49$, and by 1.2 $\Delta \stackrel{\text{loc}}{=} \mathbf{U}_m\mathbb{H} \times E$, where E is orthogonal or exceptional. The only possibility is $\Delta \stackrel{\text{loc}}{=} \mathbf{U}_3\mathbb{H} \times \mathbf{SO}_8\mathbb{R}$. The point stabilizer has dimension $\dim\Delta_x = 49 - 15 = 34$ and three almost simple factors. The only possibility is $\Delta_x \stackrel{\text{loc}}{=} \mathbf{SO}_8\mathbb{R} \times \mathbf{SO}_4\mathbb{R}$. The projection $\mathbf{SO}_8\mathbb{R} \subseteq \Delta_x$ into $\mathbf{U}_3\mathbb{H} \subseteq \Delta$ has to be trivial (otherwise $\mathbf{SO}_8\mathbb{R}$ would fix a non-zero vector in $\mathbb{H}^3$, by [25] 95.10, p. 626), hence $\mathbf{SO}_8\mathbb{R} \subseteq \Delta_x$ is normal in Δ, a contradiction.

(3) $\Delta_{[\ell]} = \mathbf{SU}_3\mathbb{C}$. Then $\dim\Delta = 18 + 18 = 36$ and $\pi_5(\Delta) = \mathbb{Z} + \mathbb{Z}_2$. The only possibility is $\Delta \stackrel{\text{loc}}{=} \mathbf{SU}_4\mathbb{C} \times \mathbf{U}_3\mathbb{H}$. The point stabilizer Δ_x has dimension $36 - 15 = 21$ and three almost simple factors, hence $\Delta_x \stackrel{\text{loc}}{=} \mathbf{SU}_4\mathbb{C} \times \mathbf{SO}_4\mathbb{R}$. The projection of $\mathbf{SU}_4\mathbb{C} \subseteq \Delta_x$ into $\mathbf{U}_3\mathbb{H} \subseteq \Delta$ has to be trivial (otherwise $\mathbf{SU}_4\mathbb{C}$ would fix vectors in $\mathbb{H}^3$, by [25] 95.10, p. 624), hence this case is excluded.

(4) $\Delta_{[\ell]} = \mathbf{G}_2$. Then $\dim\Delta = 18 + 24 = 42$ and $\pi_5(\Delta_\ell) = \mathbb{Z}_2$. Thus $\Delta \stackrel{\text{loc}}{=} \mathbf{U}_3\mathbb{H} \times \mathbf{SO}_7\mathbb{R}$. On the other hand, $\dim\Delta_x = 42 - 15 = 27$, and Δ_x has three almost simple factors. Hence $\Delta_x \stackrel{\text{loc}}{=} \mathbf{SO}_7\mathbb{R} \times \mathbf{SO}_4\mathbb{R}$. Again, the projection of $\mathbf{SO}_7\mathbb{R} \subseteq \Delta_x$ into $\mathbf{U}_3\mathbb{H} \subseteq \Delta$ has to be trivial (otherwise $\mathbf{SO}_7\mathbb{R}$ would fix vectors in $\mathbb{H}^3$, by [25] 95.10, p. 625), a contradiction.

(5) $\Delta_{[\ell]} = \mathbf{U}_1\mathbb{H} \cdot \mathbf{U}_1\mathbb{H}$. Then $\dim\Delta = 18 + 16 = 34$ and $\pi_5(\Delta) = \mathbb{Z}_2 + \mathbb{Z}_2 + \mathbb{Z}_2$. Hence $\Delta \stackrel{\text{loc}}{=} \mathbf{U}_1\mathbb{H} \times \mathbf{U}_2\mathbb{H} \times \mathbf{U}_3\mathbb{H}$. This contradicts our assumption that Δ has no factor of type $\mathbf{SU}_2\mathbb{C}$.

(6) $\Delta_{[\ell]} = \mathbf{U}_1\mathbb{H}$. Then $\dim\Delta = 18 + 13 = 31$ and $\pi_5(\Delta) = \mathbb{Z}_2 + \mathbb{Z}_2$. Hence $\Delta \stackrel{\text{loc}}{=} \mathbf{U}_2\mathbb{H} \times \mathbf{U}_3\mathbb{H}$ and $\Delta_\ell \stackrel{\text{loc}}{=} \mathbf{SO}_5\mathbb{R} \times \mathbf{U}_1\mathbb{H}$ and $\Delta_{x,\ell} \stackrel{\text{loc}}{=} \mathbf{SO}_4 \times \mathbf{U}_1\mathbb{H}$. Then Δ_x is a semisimple 16-dimensional group with three almost simple factors, hence $\Delta_x \stackrel{\text{loc}}{=} \mathbf{SO}_4\mathbb{R} \times \mathbf{U}_2\mathbb{H}$. □

PROPOSITION 5.6. *If* $q = 5$ *then* $\Delta \stackrel{\text{loc}}{=} \mathbf{SU}_5\mathbb{C}$, $\Delta_x \stackrel{\text{loc}}{=} \Delta_{[x]} \times \mathbf{SU}_3\mathbb{C}$, $\Delta_\ell \stackrel{\text{loc}}{=} \mathbf{U}_2\mathbb{H}$ *and* $\Delta_{x,\ell} \stackrel{\text{loc}}{=} \Delta_{[x]} \times \mathbf{SU}_2\mathbb{C}$, *where* $\Delta_{[x]} \stackrel{\text{loc}}{=} \mathbf{SU}_2\mathbb{C}$.

Proof. Here $\dim\mathcal{L} = 14$ and $\Delta_{[\ell]}$ is normal in one of the groups $\mathbf{SO}_5\mathbb{R}$ or $\mathbf{SU}_2\mathbb{C}$. The case where $\Delta_{[\ell]} = \mathbf{SO}_5\mathbb{R}$ is excluded by the dual of 2.4.

Assume that $\Delta_{[\ell]} = \mathbf{SU}_2\mathbb{C}$. Then Δ is a 27-dimensional group with two almost simple factors. The only possibility is $\tilde{\Delta} = \mathbf{SU}_5\mathbb{C} \times \mathbf{SU}_2\mathbb{C}$, and $(\tilde{\Delta})_\ell = \mathbf{U}_2\mathbb{H} \times \mathbf{SU}_2\mathbb{C}$. The image of $\mathbf{U}_2\mathbb{H}$ under the projection $p_2 : \mathbf{SU}_5\mathbb{C} \times \mathbf{SU}_2\mathbb{C} \to \mathbf{SU}_2\mathbb{C}$ onto the second factor has to be trivial, hence we have $\mathbf{U}_2\mathbb{H} \subseteq \mathbf{SU}_5\mathbb{C}$, and the representation has to be the obvious inclusion $\mathbf{U}_2\mathbb{H} \subseteq \mathbf{SU}_4\mathbb{C} \subseteq \mathbf{SU}_5\mathbb{C}$. The centralizer of $\mathbf{U}_2\mathbb{H}$ in $\mathbf{SU}_5\mathbb{C}$ is a torus; therefore the image of the factor $\mathbf{SU}_2\mathbb{C}$ in $(\tilde{\Delta})_\ell$ under the projection p_1 has to be trivial. But then $\mathbf{SU}_2\mathbb{C}$ is contained in the second factor, $\mathbf{SU}_2\mathbb{C}$, of $\tilde{\Delta}$, a contradiction to the fact that the action of $\tilde{\Delta}$ on $\mathcal{L}$ is almost effective.

Therefore $\Delta_{[\ell]}$ is discrete, whence $\Delta_\ell \stackrel{\text{loc}}{=} \mathbf{U}_2\mathbb{H}$. By 2.5 and 1.2 (1) we have $len(\Delta) = len(\Delta_\ell) = 1$, and furthermore, $\dim\Delta = 2 \cdot 5 + 4 + \dim\Delta_\ell = 24$, hence $\Delta \stackrel{\text{loc}}{=} \mathbf{SU}_5\mathbb{C}$. The group Δ_x has length 2 and dimension $\dim\Delta - \dim\mathcal{P} =$

$24 - (5 + 2 \cdot 4) = 11$, thus $\Delta_x \overset{\text{loc}}{=} \mathbf{SU}_3\mathbb{C} \times \mathbf{SU}_2\mathbb{C}$. The transitive action of Δ_x on the pencil $\mathbb{S}_5$ shows that $\Delta_{x,\ell} \overset{\text{loc}}{=} \mathbf{SU}_2\mathbb{C} \times \Delta_{[x]}$, where $\Delta_{[x]} \overset{\text{loc}}{=} \mathbf{SU}_2\mathbb{C}$. □

PROPOSITION 5.7. *If* $q = 3$ *then* $\Delta \overset{\text{loc}}{=} \mathbf{U}_2\mathbb{H} \times \mathbf{U}_2\mathbb{H}$, $\Delta_\ell \overset{\text{loc}}{=} \mathbf{U}_2\mathbb{H}$, $\Delta_x \overset{\text{loc}}{=} (\mathbf{U}_1\mathbb{H})^3$, *and* $\Delta_{x,\ell} \overset{\text{loc}}{=} \mathbf{U}_1\mathbb{H} \times \mathbf{U}_1\mathbb{H}$.

Proof. Here $\dim \mathcal{L} = 10$. The fundamental group tells us that Δ_ℓ is semisimple. By Corollary 2.6 the group $\pi_4(\mathcal{L})$ is finite and $\pi_3(\mathcal{L}) \cong \mathbb{Z}$ by 2.5, hence we obtain a short exact sequence

$$0 \to \pi_3(\Delta_\ell) \to \pi_3(\Delta) \to \pi_3(\mathcal{L}) \to 0,$$

that is, $len(\Delta) = len(\Delta_\ell) + 1$. The group $\Delta_{[\ell]}$ is discrete, or locally isomorphic to $\mathbf{U}_1\mathbb{H}$.

Assume that $\Delta_{[\ell]} \overset{\text{loc}}{=} \mathbf{U}_1\mathbb{H}$. Then Δ is 23-dimensional with three almost simple factors. The only possibility is $\tilde{\Delta} = \mathbf{U}_2\mathbb{H} \times \mathbf{U}_2\mathbb{H} \times \mathbf{U}_1\mathbb{H}$, and $(\tilde{\Delta})_\ell = \mathbf{U}_2\mathbb{H} \times \mathbf{U}_1\mathbb{H}$. The factor $\mathbf{U}_2\mathbb{H}$ of $(\tilde{\Delta})_\ell$ is embedded diagonally into the subgroup $\mathbf{U}_2\mathbb{H} \times \mathbf{U}_2\mathbb{H}$ of $\tilde{\Delta}$ and has centralizer $\{1\} \times \{1\} \times \mathbf{U}_1\mathbb{H}$ in $\tilde{\Delta}$. This centralizer is normal in $\tilde{\Delta}$ and coincides with the factor $\mathbf{U}_1\mathbb{H}$ of $(\tilde{\Delta})_\ell$, a contradiction to the effective action of Δ_ℓ on $\mathcal{L}$.

Thus $\Delta_{[\ell]}$ is discrete. Then Δ is 20-dimensional with two almost simple factors. The only possibility is $\tilde{\Delta} = \mathbf{U}_2\mathbb{H} \times \mathbf{U}_2\mathbb{H}$ and $(\tilde{\Delta})_\ell = \mathbf{U}_2\mathbb{H}$. Then $(\tilde{\Delta})_{x,\ell} = \mathbf{U}_1\mathbb{H} \times \mathbf{U}_1\mathbb{H}$. The stabilizer Δ_x has three almost simple factors and dimension 9, hence $(\tilde{\Delta})_x = (\mathbf{U}_1\mathbb{H})^3$. □

PROPOSITION 5.8. *If* $p = 4$ *and* $3 \leqslant q \neq 5$ *is odd, then we have case* (b_4) *of the Main Theorem.*

Proof. Let $m = (q + 5)/4$. By the results of this section, the universal covering group $\tilde{\Delta} = \mathbf{U}_2\mathbb{H} \times \mathbf{U}_m\mathbb{H}$ acts almost effectively on $\mathfrak{P}$, with stabilizers $(\tilde{\Delta})_x \overset{\text{loc}}{=} \mathbf{U}_1\mathbb{H} \times \mathbf{U}_1\mathbb{H} \times \mathbf{U}_{m-1}\mathbb{H}$, $(\tilde{\Delta})_\ell \overset{\text{loc}}{=} \mathbf{SO}_5\mathbb{R} \times \Delta_{[\ell]}$ and $(\tilde{\Delta})_{x,\ell} \overset{\text{loc}}{=} \mathbf{SO}_4\mathbb{R} \times \Delta_{[\ell]}$, where $\Delta_{[\ell]} \overset{\text{loc}}{=} \mathbf{U}_{m-2}\mathbb{H}$.

First we determine the embedding $(\tilde{\Delta})_x \subseteq \tilde{\Delta}$, considering separately the cases $m \geqslant 3$ and $m = 2$.

If $m \geqslant 3$, then $\mathbf{U}_m\mathbb{H}$ contains only one conjugacy class of subgroups which are locally isomorphic to $\mathbf{U}_{m-1}\mathbb{H}$. Indeed, each inclusion of this type gives an almost effective representation of $\mathbf{U}_{m-1}\mathbb{H}$ on $\mathbb{H}^m$ (and the centralizer of this representation contains $\mathbb{H}$); this representation is unique up to equivalence, which yields the uniqueness claimed above, compare [22] p. 43, case IV, [23] Prop. 5, p. 53 and Prop. 8, p. 56 (or [25] 95.12 for $m = 3, 4, 5$).

As a consequence, $\mathbf{U}_{m-1}\mathbb{H}$ is embedded into $\tilde{\Delta}$ in the standard way (as a stabilizer of a vector); for $m = 3$ we observe that the subgroup $\mathbf{U}_2\mathbb{H}$ of $(\tilde{\Delta})_x$ cannot be embedded diagonally into $\tilde{\Delta} = \mathbf{U}_2\mathbb{H} \times \mathbf{U}_3\mathbb{H}$, because the centralizers of these diagonal subgroups are too small to accommodate the remaining factors of Δ_x.

Now we consider the case $m = 2$. We have

$$(\tilde{\Delta})_\ell = \{(A, \phi(A)) | A \in \mathbf{U}_2\mathbb{H}\},$$

where $\phi : \mathbf{U}_2\mathbb{H} \to \mathbf{U}_2\mathbb{H}$ is a homomorphism, which is not trivial since $(\tilde{\Delta})_\ell$ is not normal in $\tilde{\Delta}$. As $\mathbf{U}_2\mathbb{H}$ is simple and ϕ is bijective, we may assume that $(\tilde{\Delta})_\ell = \{(A, A) | A \in \mathbf{U}_2\mathbb{H}\}$. The embedding of $(\tilde{\Delta})_{x,\ell}$ into $(\tilde{\Delta})_\ell$ is the standard one, as a stabilizer of a 1-dimensional subspace in $\mathbb{H}^2$, compare 1.1. Thus

$$(\tilde{\Delta})_{x,\ell} = \left\{ \left(\begin{pmatrix} a & \\ & b \end{pmatrix}, \begin{pmatrix} a & \\ & b \end{pmatrix} \right) \middle| \, a, b \in \mathbf{U}_1\mathbb{H} \right\} \cong \mathbf{U}_1\mathbb{H} \times \mathbf{U}_1\mathbb{H}.$$

We may assume that one of the two factors $\mathbf{U}_1\mathbb{H}$, say

$$\left\{ \left(\begin{pmatrix} a & \\ & 1 \end{pmatrix}, \begin{pmatrix} a & \\ & 1 \end{pmatrix} \right) \middle| \, a \in \mathbf{U}_1\mathbb{H} \right\}$$

of $(\tilde{\Delta})_{x,\ell}$ is normal in $(\tilde{\Delta})_x$. The normalizer N of that factor in $\tilde{\Delta}$ has the connected component

$$N^1 = \left\{ \left(\begin{pmatrix} a & \\ & b \end{pmatrix}, \begin{pmatrix} a & \\ & d \end{pmatrix} \right) \middle| \, a, b, d \in \mathbf{U}_1\mathbb{H} \right\} \overset{\text{loc}}{\cong} (\mathbf{U}_1\mathbb{H})^3$$

This shows that $(\tilde{\Delta})_x = N^1$ is unique up to automorphisms of $\tilde{\Delta}$, since $(\tilde{\Delta})_x$ is connected.

Now we know the embedding of $(\tilde{\Delta})_x$. Let $m \geqslant 2$.

The embeddings of $(\tilde{\Delta})_{x,\ell}$ into $(\tilde{\Delta})_x$ or into $(\tilde{\Delta})_\ell$ are the standard ones (as stabilizers of vectors, compare 1.1). In particular, $(\tilde{\Delta})_{[\ell]} \cong \mathbf{U}_{m-2}\mathbb{H}$ is embedded into $\tilde{\Delta}$ in the standard way, and $(\tilde{\Delta})_x = R \times U_{m-1}$, where $R \cong \mathbf{U}_1\mathbb{H} \times \mathbf{U}_1\mathbb{H} \overset{\text{loc}}{\cong} \mathbf{SO}_4\mathbb{R}$ and $U_{m-1} \cong \mathbf{U}_{m-1}\mathbb{H}$ is a subgroup of the factor $1 \times \mathbf{U}_m\mathbb{H}$ of $\tilde{\Delta}$, and U_{m-1} is unique up to automorphisms of $\tilde{\Delta}$ (even for $m = 2$). We have $Cs_{\tilde{\Delta}}(\Delta_{[\ell]}) \cong \mathbf{U}_m\mathbb{H} \times \mathbf{U}_2\mathbb{H}$, and $(\tilde{\Delta})_\ell = V_2 \times (\tilde{\Delta})_{[\ell]}$ with a subgroup $V_2 \cong \mathbf{U}_2\mathbb{H}$. Since $(\tilde{\Delta})_x$ is not contained in the normal factor $1 \times \mathbf{U}_m\mathbb{H}$ of $\tilde{\Delta}$, one of the two factors of $R \subseteq (\tilde{\Delta})_{x,\ell}$ projects nontrivially into the factor $\mathbf{U}_2\mathbb{H} \times 1$ of $\tilde{\Delta}$, and this factor lies in V_2 (as it centralizes $(\tilde{\Delta})_{[\ell]}$). We conclude that the projection of V_2 into the factor $\mathbf{U}_2\mathbb{H} \times 1$ of $\tilde{\Delta}$ is not trivial, hence surjective. Thus V_2 is a diagonal subgroup of $Cs_{\tilde{\Delta}}(\Delta_{[\ell]}) \cong \mathbf{U}_2\mathbb{H} \times \mathbf{U}_2\mathbb{H}$. The group $Aut(\mathbf{U}_2\mathbb{H} \times 1)$ permutes these diagonal subgroups transitively, and leaves U_{m-1} and $\Delta_{[\ell]}$ unchanged. Therefore the triple $(U_{m-1}, (\tilde{\Delta})_{[\ell]}, V_2)$ is determined uniquely up to automorphisms of $\tilde{\Delta}$, and the same is true for the triple $(U_{m-1}, (\tilde{\Delta})_{[\ell]}, (\tilde{\Delta})_\ell = V_2 \times (\tilde{\Delta})_{[\ell]})$. The group V_2 contains only one conjugacy class of subgroups which are candidates for R, see 1.1. In view of $(\tilde{\Delta})_x = R \times U_{m-1}, (\tilde{\Delta})_{x,\ell} = R \times (\tilde{\Delta})_{[\ell]}$, the triple $((\tilde{\Delta})_x, (\tilde{\Delta})_\ell, (\tilde{\Delta})_{x,\ell})$ is determined uniquely up to automorphisms of $\tilde{\Delta}$.

Let $\mathfrak{Q}$ be the quaternion Hermitian quadrangle defined by the standard Hermitian form on $\mathbb{H}^{m+2}$ of Witt index 2. Then $\tilde{\Delta}$ acts almost effectively on $\mathfrak{Q}$ as a flag-transitive automorphism group. By the uniqueness obtained above, we conclude that $\mathfrak{P}$ is isomorphic to $\mathfrak{Q}$. As a consequence, Δ is isomorphic to $\mathbf{P}(\mathbf{U}_2\mathbb{H} \times \mathbf{U}_m\mathbb{H})$.

Finally we determine the possible flag-transitive groups Γ. We have shown that the connected components Γ^1 of these groups contain the maximal (compact) subgroup $\mathbf{P}(\mathbf{U}_2\mathbb{H} \times \mathbf{U}_m\mathbb{H})$ of $\mathbf{PU}_{m+2}\mathbb{H}(2)$, which is the full automorphism group of $\mathfrak{Q} \cong \mathfrak{P}$ unless $q = 3$, where it has index 2 in the full automorphism group, cp. Tits [30] 8.6 or Takeuchi [28]. Thus Γ satisfies the conclusions in case (b_4) of the Main Theorem. □

PROPOSITION 5.9. *If $p = 4$ and $q = 5$, then we have case (b_5) of the Main Theorem.*

Proof. By 5.5 the universal covering group $\tilde{\Delta} = \mathbf{SU}_5\mathbb{C}$ acts almost effectively on $\mathfrak{P}$, with stabilizers $(\tilde{\Delta})_\ell = \mathbf{U}_2\mathbb{H}$, $(\tilde{\Delta})_x = (\tilde{\Delta})_{[x]} \times \mathbf{SU}_3\mathbb{C}$, $(\tilde{\Delta})_{x,\ell} = (\tilde{\Delta})_{[x]} \times \mathbf{SU}_2\mathbb{C}$, where $(\tilde{\Delta})_{[x]} = \mathbf{SU}_2\mathbb{C}$. In fact, Δ_ℓ is globally isomorphic to $\mathbf{U}_2\mathbb{H}$ and Δ is globally isomorphic to $\mathbf{SU}_5\mathbb{C}$, as we infer from the exact homotopy sequence

$$\pi_2(\mathcal{L}) = 0 \to \pi_1(\Delta_\ell) \to \pi_1(\Delta) \to \pi_1(\mathcal{L}) = 0,$$

which shows that $\pi_1(\Delta_\ell) \cong \pi_1(\Delta)$ is a subgroup of the cyclic groups Z_2 and Z_5, hence trivial. Thus the effective representation of $(\tilde{\Delta})_\ell \subset \mathbf{SU}_5\mathbb{C}$ on $\mathbb{C}^5 = \mathbb{R}^{10}$ corresponds to a decomposition $\mathbb{R}^{10} = \mathbb{H}^2 + \mathbb{R} + \mathbb{R}$, see [25] 95.10, p. 624. Using [23] Prop. 8, p. 56, we conclude that $(\tilde{\Delta})_\ell = \mathbf{U}_2\mathbb{H}$ is unique up to conjugation in $\tilde{\Delta}$. The embeddings of $(\tilde{\Delta})_{x,\ell}$ into $(\tilde{\Delta})_x$ or into $(\tilde{\Delta})_\ell$ are the standard ones (as stabilizers of vectors, compare 1.1). Furthermore $(\tilde{\Delta})_{[x]} = \mathbf{SU}_2\mathbb{C}$ is unique in $(\tilde{\Delta})_{x,l} = \mathbf{SU}_2\mathbb{C} \times \mathbf{SU}_2\mathbb{C}$ up to automorphisms which are induced by inner automorphisms of $(\tilde{\Delta})_\ell$. Since $(\tilde{\Delta})_x = (\tilde{\Delta})_{[x]} \times Cs_{\tilde{\Delta}}((\tilde{\Delta})_{[x]})$, we conclude that the triple $((\tilde{\Delta})_x, (\tilde{\Delta})_\ell, (\tilde{\Delta})_{x,\ell})$ is determined uniquely up to conjugation in $\tilde{\Delta}$.

Let $\mathfrak{Q}$ be the quadrangle defined by the α-Hermitian form $\sum_j x_j x_j^\alpha$ on $\mathbb{H}^5$, as in case (b_5) of the Main Theorem. Then $\tilde{\Delta}$ acts almost effectively on $\mathfrak{Q}$ as a flag-transitive automorphism group, because $\mathbf{U}_5\mathbb{C}$ is a (maximal compact) subgroup of $(\mathbf{PU}_5^\alpha\mathbb{H})^1$, compare Helgason [11] Ch. 10 Table V, and this subgroup is flag-transitive by 2.3 and 1.3. By the uniqueness obtained above, we conclude that $\mathfrak{P}$ is isomorphic to $\mathfrak{Q}$.

Finally we determine the possible flag-transitive groups Γ. We have shown that the connected components Γ^1 of these groups contain the subgroup $\mathbf{SU}_5\mathbb{C}$. The overgroups of this group can be determined by using the Cartan decomposition of the Lie algebra Lie($\mathbf{PU}_5^\alpha\mathbb{H}$). Under the adjoint action of $\mathbf{SU}_5\mathbb{C}$, this Lie algebra decomposes as Lie($\mathbf{SU}_5\mathbb{C}$) $+ \mathbb{R} + \mathbb{R}^{20}$. This gives the assertions as in case (b_5). □

6. The Case where $p \geqslant 6$ is Even and $q \geqslant 3$ is Odd

Throughout this section $\mathfrak{P}$ is a compact connected quadrangle with parameters (p, q), $p \geqslant 6$ even, $q \geqslant 3$ odd, with a flag-transitive automorphism group Γ. By 2.3 we find a compact connected subgroup Δ which acts transitively on the flags. *By 2.5, 1.3 we may (and will) assume that Δ is semisimple.*

PROPOSITION 6.1. *The case $p \geqslant 8$ even, $q \geqslant 3$ odd is impossible.*

Proof. Since $p \geqslant 8$ is even, we have $\Delta_\ell \overset{\text{loc}}{=} \mathbf{SO}_{p+1}\mathbb{R} \times \Delta_{[\ell]}$ by 1.1. The group $\Delta_{[x]}$ is normal in $\mathbf{SO}_p\mathbb{R}$, and the possibility $\Delta_{[x]} = \mathbf{SO}_p\mathbb{R}$ is excluded by 2.4. Thus $\Delta_{[x]}$ is discrete. Then Δ_x acts almost effectively on the sphere $\mathbb{S}_q$. Since $\Delta_\ell \overset{\text{loc}}{=} \mathbf{SO}_{p+1}\mathbb{R} \times \Delta_{[\ell]}$, we have $\Delta_{x,\ell} \overset{\text{loc}}{=} \mathbf{SO}_p\mathbb{R} \times \Delta_{[\ell]}$. But by 1.1 no group which acts transitively on an odd-dimensional sphere $\mathbb{S}_q$ has a stabilizer of this form. □

It remains to consider the case $p = 6$. In this case, we have either (a) $\Delta_\ell \overset{\text{loc}}{=} \mathbf{G}_2 \times \Delta_{[\ell]}$, or (b) $\Delta_\ell \overset{\text{loc}}{=} \mathbf{SO}_7\mathbb{R} \times \Delta_{[\ell]}$. The sixth homotopy group of a compact Lie group is always finite, cp. [19] p. 970, and since $\pi_6(\mathcal{P}) = \mathbb{Z}$, the exact sequence

$$\to \pi_6(\Delta) \to \pi_6(\mathcal{P}) \to \pi_5(\Delta_x) \to$$

implies that the rank of $\pi_5(\Delta_x)$ is at least 1. Therefore Δ_x has an almost simple factor of type $\mathbf{SU}_n\mathbb{C}$, for some $n \geqslant 3$.

PROPOSITION 6.2. *If $p = 6$, then $\Delta_{[x]}$ is discrete.*

Proof. The group $\Delta_{[x]}$ is normal in the stabilizer of the action of Δ_ℓ on the point row, hence (a) $\Delta_{[x]}$ is normal in $\mathbf{SU}_3\mathbb{C}$, or (b) $\Delta_{[x]}$ is normal in $\mathbf{SO}_6\mathbb{R}$. The case $\Delta_{[x]} = \mathbf{SO}_6\mathbb{R}$ is excluded by 2.4. We assume now that $\Delta_{[x]} = \mathbf{SU}_3\mathbb{C}$, and we aim for a contradiction (which proves the proposition). We have $\Delta_\ell \overset{\text{loc}}{=} \mathbf{G}_2 \times \Delta_{[\ell]}$, and we consider several cases.

(1) $\Delta_{[\ell]}$ is discrete. For $q \geqslant 5$ the group Δ is almost simple. If $q \geqslant 9$, then $\pi_6(\Delta) = \pi_6(\Delta_\ell) = \mathbb{Z}_3$ yields $\Delta \overset{\text{loc}}{=} \Delta_\ell$, which is absurd. If $q = 7$ then Δ is almost simple and of dimension $\dim \Delta = 20 + 14 = 34$, which is impossible. If $q = 5$ then Δ is almost simple of dimension $\dim \Delta = 16 + 14 = 30$, which again is impossible. If $q = 3$ then $\dim \Delta = 12 + 14 = 26$ and $len(\Delta) = len(\Delta_\ell) + 1 = 2$, which is again impossible.

(2) $\Delta_{[\ell]}$ is nondiscrete and $q \geqslant 9$. By 2.2, $\Delta_{[\ell]}$ is a normal semisimple subgroup of a stabilizer of a transitive group on $\mathbb{S}_q$ as in 1.1. By 1.2, $len(\Delta) = len(\Delta_\ell)$, and $\Delta \overset{\text{loc}}{=} \mathbf{G}_2 \times E$ for some semisimple group E, and again by 1.2, $(E, \Delta_{[\ell]})$ is (locally) one of the following pairs.

$(\mathbf{SO}_m\mathbb{R}, \mathbf{SO}_q\mathbb{R})$	$\dim \mathcal{L} = q + (q+1) + (q+2) + \cdots$
$(\mathbf{SU}_m\mathbb{C}, \mathbf{SU}_{(q+1)/2}\mathbb{C})$	$\dim \mathcal{L} = q + (q+2) + (q+4) + \cdots$
$(\mathbf{U}_m\mathbb{H}, \mathbf{U}_{(q+1)/4}\mathbb{H})$	$\dim \mathcal{L} = q + (q+4) + (q+8) + \cdots$
$(\mathbf{U}_1\mathbb{H} \cdot \mathbf{U}_m\mathbb{H}, \mathbf{U}_1\mathbb{H} \cdot \mathbf{U}_{(q+1)/4}\mathbb{H})$	$\dim \mathcal{L} = q + (q+4) + (q+8) + \cdots$
$(\mathbf{Spin}_m\mathbb{R}, \mathbf{Spin}_7\mathbb{R})$	$\dim \mathcal{L} = 36$

Moreover, $\dim \Delta - \dim \Delta_\ell = \dim E - \dim \Delta_{[\ell]} = \dim \mathcal{L} = 2q + 6$. This is impossible in the first four cases. In the last case we obtain $\dim \mathbf{Spin}_m\mathbb{R} = 36 + 14 + 21 = 71$, which is also impossible.

(3) $\Delta_{[\ell]}$ is nondiscrete and $q = 7$. Then $\Delta_{[\ell]}$ is one of the groups $\mathbf{SO}_7\mathbb{R}$, $\mathbf{G}_2$, $\mathbf{SU}_3\mathbb{C}$, $\mathbf{U}_1\mathbb{H} \cdot \mathbf{U}_1\mathbb{H}$, or $\mathbf{U}_1\mathbb{H}$. We exclude each case.

The case $\Delta_{[\ell]} = \mathbf{SO}_7\mathbb{R}$ is excluded by the dual of 2.4. If $\Delta_{[\ell]} = \mathbf{G}_2$, then $\dim \Delta = 20 + 28 = 48$, and by 1.2 Δ is a product of two orthogonal or exceptional groups, which is impossible. If $\Delta_{[\ell]} = \mathbf{SU}_3\mathbb{C}$, then $\dim \Delta = 20 + 22 = 42$, and by 1.2 Δ is a 42-dimensional product of $\mathbf{SU}_n\mathbb{C}$ with an orthogonal or exceptional group, which is impossible. If $\Delta_{[\ell]} = \mathbf{U}_1\mathbb{H} \cdot \mathbf{U}_1\mathbb{H}$, then $\dim \Delta = 20 + 20 = 40$, and by 1.2 $\Delta \stackrel{\text{loc}}{=} \mathbf{U}_m\mathbb{H} \times \mathbf{U}_n\mathbb{H} \times E$, where E is orthogonal or exceptional, but such a 40-dimensional group does not exist. Finally, if $\Delta_{[\ell]} = \mathbf{U}_1\mathbb{H} \cdot \mathbf{U}_1\mathbb{H}$, then $\dim \Delta = 20 + 17 = 37$, and by 1.2 $\Delta \stackrel{\text{loc}}{=} \mathbf{U}_m\mathbb{H} \times E$, where E is orthogonal or exceptional, but such a 37-dimensional group does not exist.

(4) $\Delta_{[\ell]}$ is nondiscrete and $q = 5$. Then $\Delta_{[\ell]}$ is one of the groups $\mathbf{SO}_5\mathbb{R}$ or $\mathbf{SU}_2\mathbb{C}$, and $len(\Delta) = 2$. If $\Delta_{[\ell]} = \mathbf{SO}_5\mathbb{R}$, then $\dim \Delta = 16 + 24 = 40$, which is impossible. If $\Delta_{[\ell]} = \mathbf{SU}_2\mathbb{C}$, then $\dim \Delta = 16 + 17 = 33$, again impossible.

(5) $\Delta_{[\ell]}$ is nondiscrete and $q = 3$. Then $\Delta_{[\ell]} = \mathbf{SO}_3\mathbb{R}$ and $\dim \Delta = 12 + 17 = 29$, hence $\dim \Delta_x = 14$. But $len(\Delta_x) = 2$; such a group does not exist. □

PROPOSITION 6.3. *If* $p = 6$*, then* $q = 9$ *and* $\Delta \stackrel{\text{loc}}{=} \mathbf{SO}_{10}\mathbb{R}$*,* $\Delta_x \stackrel{\text{loc}}{=} \mathbf{SU}_5\mathbb{C}$*,* $\Delta_\ell \stackrel{\text{loc}}{=} \mathbf{SO}_7\mathbb{R}$*, and* $\Delta_{x,\ell} \stackrel{\text{loc}}{=} \mathbf{SU}_4\mathbb{C}$.

Proof. By the previous proposition, $\Delta_{[x]}$ is discrete. Since Δ_x has an almost simple factor of type $\mathbf{SU}_n\mathbb{C}$, $n \geqslant 3$, and acts transitively and almost effectively on $\mathbb{S}_q$, we have $\Delta_x \stackrel{\text{loc}}{=} \mathbf{SU}_n\mathbb{C}$, and $q = 2n - 1 \geqslant 5$. Thus Δ and Δ_ℓ are almost simple; in particular, $\Delta_{[\ell]}$ is discrete as well, and thus (a) $\Delta_\ell \stackrel{\text{loc}}{=} \mathbf{G}_2$ or (b) $\Delta_\ell \stackrel{\text{loc}}{=} \mathbf{SO}_7\mathbb{R}$. Since $\Delta_x \stackrel{\text{loc}}{=} \mathbf{SU}_n\mathbb{C}$, the flag stabilizer is $\Delta_{x,\ell} \stackrel{\text{loc}}{=} \mathbf{SU}_{n-1}\mathbb{C}$. In case (a) the flag stabilizer is locally isomorphic to $\mathbf{SU}_3\mathbb{C}$, and in case (b) it is locally isomorphic to $\mathbf{SO}_6\mathbb{R} \stackrel{\text{loc}}{=} \mathbf{SU}_4\mathbb{C}$; thus $n = 4$ in case (a) and $n = 5$ in case (b). If $n = 4$, then $q = 7$, and Δ is an almost simple compact group of dimension $\dim \Delta = \dim \mathcal{P} + \dim \Delta_x = 19 + 15 = 34$, which does not exist. Thus $n = 5$ and $q = 9$, and Δ is an almost simple compact group of dimension $\dim \Delta = \dim \mathcal{P} + \dim \Delta_x = 21 + 24 = 45$. The only possibility is $\Delta \stackrel{\text{loc}}{=} \mathbf{SO}_{10}\mathbb{R}$. □

PROPOSITION 6.4. *If* $p = 6$ *and* $q = 9$*, then we have case* (b_6) *of the Main Theorem.*

Proof. By 6.3, the universal covering group $\tilde{\Delta} = \mathbf{Spin}_{10}\mathbb{R}$ acts almost effectively on $\mathfrak{P}$, with stabilizers $(\tilde{\Delta})_x \stackrel{\text{loc}}{=} \mathbf{SU}_5\mathbb{C}$, $(\tilde{\Delta})_\ell \stackrel{\text{loc}}{=} \mathbf{Spin}_7\mathbb{R}$ and $(\tilde{\Delta})_{x,\ell} \stackrel{\text{loc}}{=} \mathbf{SU}_4\mathbb{C} \stackrel{\text{loc}}{=} \mathbf{SO}_6\mathbb{R}$. Up to equivalence, the group $\mathbf{SU}_5\mathbb{C}$ admits only one non-trivial representation on $\mathbb{R}^{10}$, which is in fact irreducible, see [25] 95.10, p. 626. The embeddings of $(\tilde{\Delta})_{x,\ell}$ into $(\tilde{\Delta})_x$ or into $(\tilde{\Delta})_\ell$ are the standard ones (as stabilizers of vectors, compare 1.1). In particular, $(\tilde{\Delta})_{x,\ell}$ decomposes the natural representation space $\mathbb{R}^{10}$ of $\tilde{\Delta}$ into irreducible subspaces of dimensions 8,1,1. In view of [25] 95.10, p. 625 we conclude that $(\tilde{\Delta})_\ell \stackrel{\text{loc}}{=} \mathbf{Spin}_7\mathbb{R}$ acts with the same irreducible subspaces; thus the representation of $(\tilde{\Delta})_\ell$ on $\mathbb{R}^{10}$ is uniquely determined. By [23] Prop. 8, p. 56, the image of the representation of $(\tilde{\Delta})_\ell$ on $\mathbb{R}^{10}$ is uniquely determined within $\mathbf{SO}_{10}\mathbb{R}$ up to automorphisms of $\mathbf{SO}_{10}\mathbb{R}$, and these automorphisms are induced by elements of

$\mathbf{O}_{10}\mathbb{R}$, cp. [23] Ex. 6, p. 49. Since $\tilde{\Delta}$ is simply connected, we conclude that $(\tilde{\Delta})_\ell$ is unique in $\tilde{\Delta}$ up to automorphisms of $\tilde{\Delta}$. In view of the standard embedding of $(\tilde{\Delta})_{x,\ell}$ into $(\tilde{\Delta})_\ell$ we conclude that the pair $((\tilde{\Delta})_\ell, (\tilde{\Delta})_{x,\ell})$ of subgroups of $\tilde{\Delta}$ is uniquely determined up to automorphisms of $\tilde{\Delta}$. The image in $\mathbf{SO}_{10}\mathbb{R}$ of the centralizer $Cs_{\tilde{\Delta}}((\tilde{\Delta})_{x,\ell})$ is $\mathbf{SO}_2\mathbb{R} \times \mathbf{SO}_2\mathbb{R} \cong \mathbf{U}_1\mathbb{C} \times \mathbf{U}_1\mathbb{C}$. This centralizer contains precisely two cyclic subgroups of order 4 which are generated by elements without real eigenvalues on $\mathbb{R}^{10}$; these two complex structures on $\mathbb{R}^{10}$ are the two groups of order 4 which are generated by the two diagonal matrices $diag(i, i, i, i, \pm i)$. The image $\mathbf{SU}_5\mathbb{C}$ of $(\tilde{\Delta})_x$ in $\mathbf{SO}_{10}\mathbb{R}$ is the commutator group of the centralizer of such a subgroup (complex structure). The two diagonal matrices mentioned above are conjugate in $1 \times \mathbf{O}_2\mathbb{R}$, hence conjugate under automorphisms of $\mathbf{SO}_{10}\mathbb{R}$ which act trivially on the image of $(\tilde{\Delta})_\ell$ (and its subgroups). Therefore the triple $((\tilde{\Delta})_x, (\tilde{\Delta})_\ell, (\tilde{\Delta})_{x,\ell})$ is determined uniquely up to automorphisms of $\tilde{\Delta}$.

Now $\tilde{\Delta}$ acts on the two (mutually dual) Moufang quadrangles associated to the simple Lie group $\Sigma := \mathrm{E}_{6(-14)}/Z$ as the commutator group of the maximal compact subgroup $\mathbf{Spin}_{10}\mathbb{R} \cdot \mathbf{SO}_2\mathbb{R}$ of Σ, cp. Helgason [11] Ch. 10 Table V, and this action is flag-transitive (e.g. by 2.3 and 1.3). The uniqueness statement of the previous paragraph implies that $\mathfrak{P}$ is isomorphic to one of these two Moufang quadrangles. As a consequence, Δ is isomorphic to $\mathbf{Spin}_{10}\mathbb{R}$.

Finally we determine the possible flag-transitive groups Γ. We have shown that the connected component Γ^1 of such a group satisfies $\mathbf{Spin}_{10}\mathbb{R} \leqslant \Gamma^1 \leqslant \Sigma$. Now we argue as in the last paragraph of the proof of 5.9, replacing $\mathbb{R}^{20}$ by the $\mathbf{Spin}_{10}\mathbb{R}$-module $\mathbb{R}^{32}$, which is irreducible. Therefore either $\Gamma^1 = \mathbf{Spin}_{10}\mathbb{R}$ and Γ is as specified in (b_6), or Γ^1 contains the maximal compact subgroup $\mathbf{Spin}_{10}\mathbb{R} \cdot \mathbf{SO}_2\mathbb{R}$ of Σ, which is maximal in Σ. Since Σ has index 2 in the full automorphism group of the quadrangle, cp. Takeuchi [28], we infer in the second case that $\Gamma^1 \in \{\Sigma, \mathbf{Spin}_{10}\mathbb{R} \cdot \mathbf{SO}_2\mathbb{R}\}$ has index at most 2 in Γ, hence Γ satisfies the conclusions in case (b_6) of the Main Theorem. □

7. Appendix

Here we collect some topological results about 7-manifolds which are needed for the case of quadrangles with parameters (2, 3) in 4.6. Lemma 7.2 was kindly pointed out to us by Stephan Stolz.

7.1. CERTAIN HOMOGENEOUS 7-MANIFOLDS. Let $G = \mathbf{SU}_2\mathbb{C} \times \mathbf{SU}_3\mathbb{C}$, and let $M = G/H$ be a 1-connected 7-dimensional homogeneous space of G. We assume that the kernel of the action is finite. The stabilizer H is a four-dimensional compact connected subgroup of G, and thus $H \overset{\text{loc}}{\cong} \mathbf{U}_1\mathbb{C} \times \mathbf{SU}_2\mathbb{C}$. Let $K = H' \overset{\text{loc}}{\cong} \mathbf{SU}_2\mathbb{C}$ denote the semisimple part of H. Note that the centralizer of K in G has positive dimension. According to the embedding $K \subseteq G$, we distinguish several cases.

There is a standard embedding $\iota : \mathbf{SU}_2\mathbb{C} \subseteq \mathbf{SU}_3\mathbb{C}$ as matrices of the form $\begin{pmatrix} A & \\ & 1 \end{pmatrix}$, for $A \in \mathbf{SU}_2\mathbb{C}$. Let κ denote the composition of the standard maps

$\mathbf{SU}_2\mathbb{C} \to \mathbf{SO}_3\mathbb{R} \subseteq \mathbf{SU}_3\mathbb{C}$. Up to conjugation by automorphisms of $\mathbf{SU}_3\mathbb{C}$, the maps ι and κ are the only non-trivial homomorphisms from $\mathbf{SU}_2\mathbb{C}$ to $\mathbf{SU}_3\mathbb{C}$. From this fact, one easily deduces that the following five cases exhaust all conjugacy classes of subgroups K of G which are locally isomorphic to $\mathbf{SU}_2\mathbb{C}$.

(1) $K \subseteq \mathbf{SU}_3\mathbb{C}$ is isomorphic to $\mathbf{SO}_3\mathbb{R}$ in the standard embedding. The connected centralizer of this subgroup is the normal factor $\mathbf{SU}_2\mathbb{C}$ of G, and thus

$$M = \mathbf{SU}_2\mathbb{C}/\mathbf{U}_1\mathbb{C} \times \mathbf{SU}_3\mathbb{C}/\mathbf{SO}_3\mathbb{R}.$$

In particular, $H_2(G/H) = \pi_2(G/H) \cong \mathbb{Z} \oplus \mathbb{Z}/2$.

(2) $K \subseteq \mathbf{SU}_3\mathbb{C}$ is the standard embedding ι. The connected centralizer of K consists of pairs of matrices of the form $(A, \mathrm{diag}(z, z, z^{-2}))$, for $A \in \mathbf{SU}_2\mathbb{C}$ and $z \in \mathbf{U}_1\mathbb{C}$. Thus H consists of pairs of matrices of the form $\left(\begin{pmatrix} z^{2l} & \\ & z^{-2l} \end{pmatrix}, \begin{pmatrix} Az^k & \\ & z^{-2k} \end{pmatrix}\right)$, for a fixed pair of integers k, l. We may assume that k, l are relatively prime. The resulting homogeneous space is denoted by $M_{k,l}$. These spaces are certain Einstein manifolds which have been studied by Kreck and Stolz in [17]. From [17] Sec. 4 we infer that

$$H^2(M_{k,l}) \cong \mathbb{Z}, \qquad H^3(M_{k,l}) = 0, \qquad H^4(M_{k,l}) \cong \mathbb{Z}/l^2.$$

Also, $M_{k,l}$ is a spin manifold if and only if k is even. Note that $H^3(M_{k,l}; \mathbb{Z}/2) \cong \mathbb{Z}/2$ if l is even (this is incorrectly stated in [17], see [18] 2.6).

(3) K consists of matrices of the form $\left(A, \begin{pmatrix} A & \\ & 1 \end{pmatrix}\right)$, for $A \in \mathbf{SU}_2\mathbb{C}$. The connected centralizer of this subgroup consists of matrices of the form $\left(1, \begin{pmatrix} z & & \\ & z & \\ & & z^{-2} \end{pmatrix}\right)$, for $z \in \mathbf{U}_1\mathbb{C}$. Thus $\mathbf{SU}_3\mathbb{C} \subseteq \mathbf{G}$ acts transitively on M, and M is in fact a Wallach space. In the terminology of Kreck-Stolz [18], $M = M_{1,1}$ and thus

$$H^2(M) \cong \mathbb{Z}, \qquad H^3(M) = 0, \qquad H^4(M) \cong \mathbb{Z}/3$$

by [18] p. 474.

(4) K is normal in G. Then the action of G is not almost effective, contradicting our assumptions.

(5) K is the image of $\mathrm{id} \times \kappa$. The connected centralizer of this group is trivial, contrary to our assumption that K is the semisimple part of a 4-dimensional subgroup H.

LEMMA 7.2. *A closed connected orientable n-manifold M is a spin manifold if and only if* $\mathrm{Sq}^2 : H^{n-2}(M; \mathbb{Z}/2) \to H^n(M; \mathbb{Z}/2)$ *is trivial.*

Proof. (cf. Bredon [5] p. 423.) A manifold is spin if and only if its 2nd Stiefel–Whitney class w_2 vanishes (note that $w_1 = 0$, since we assume that M is orientable). From the Wu formula $w_i = \sum_\nu \mathrm{Sq}^{i-\nu} \upsilon_\nu$ we see that $w_1 = \upsilon_1$ and $w_2 = \upsilon_2 + \upsilon_1^2$, whence $\upsilon_2 = w_2$. Now the relation $\langle \mathrm{Sq}^i x, [M] \rangle = \langle \upsilon_i x, [M] \rangle$ holds for all $x \in H^{n-i}(M; \mathbb{Z}/2)$ by the definition of the Wu classes υ_i. Thus $w_2 = \upsilon_2 = 0$ if and only if Sq^2 is trival on the $n-2$-dimensional cohomology. □

LEMMA 7.3. *Let $\mathfrak{P}$ be a compact connected quadrangle. If the line space $\mathcal{L}$ is homeomorphic to $\mathbf{SU}_3\mathbb{C}$, then Sq^2 is non-trivial on $H^5(\mathcal{P}; \mathbb{Z}/2)$. In particular, if the point space $\mathcal{P}$ is a manifold, then $\mathcal{P}$ is not a spin manifold by 7.2.*

Proof. From the cohomology of $\mathcal{L}$ we see that the quadrangle has parameters $(2, 3)$. We choose additive generators $1, x_2, x_5, x_7 = x_2x_5$ of the $\mathbb{Z}/2$-cohomology of $\mathcal{P}$, and $1, y_3, y_5, y_8 = y_3y_5$ for $\mathcal{L}$ (the subscripts indicate the degrees, see [10] Appendix 4_3). In the cohomology ring of $\mathcal{L} \approx \mathbf{SU}(3)$ the relation $\mathrm{Sq}^2 y_3 = y_5$ holds, see eg. [20] p. 424. In the $\mathbb{Z}^2$-cohomology of $\mathcal{F}$ we have the relations $x_5 + y_5 = x_2y_3$ and $x_7 = x_2y_5$, see [10] Appendix 4_3. Thus

$$\mathrm{Sq}^2 x_5 = \mathrm{Sq}^2(y_5 + x_2y_3) = \mathrm{Sq}^2(x_2y_3) = x_2\mathrm{Sq}^2 y_3 = x_2y_5 = x_7.$$

This implies that $\mathcal{P}$ is not spin by 7.2. □

Acknowledgements

We would like to thank the referee for requiring a number of clarifications and corrections, and Stephan Stolz for some remarks on homogeneous 7-manifolds.

References

1. Biller, H.: Actions of compact groups on spheres and on generalized quadrangles, PhD Thesis, Stuttgart (1999).
2. Bödi, R. and Kramer, L.: On homomorphisms between generalized polygons, *Geom. Dedicata* **58** (1995), 1–14.
3. Borel, A.: Le plan projectif des octaves et les sphéres comme espaces homogénes, *C. R. Acad. Sci. Paris* **230** (1950), 1378–1380.
4. Bredon, G. E.: Transformation groups with orbits of uniform dimension, *Michigan Math. J.* **8** (1961), 139–147.
5. Bredon, G. E.: *Topology and Geometry*, Springer-Verlag, New York, 1993.
6. Burns, K. and Spatzier, R.: On topological Tits buildings and their classification, *Publ. Math. I.H.E.S.* **65** (1987), 5–34.
7. Eschenburg, J.-H. and Heintze, E.: On the classification of polar representations, *Math. Z.* **232** (1999), 391–398.
8. Ferus, D., Karcher, H. and Münzner, H.-F.: Cliffordalgebren und neue isoparametrische Hyperflächen, *Math. Z.* **177** (1981), 479–502.
9. Grundhöfer, T. and Knarr, N.: Topology in generalized quadrangles, *Topology Appl.* **34** (1990), 139–152.
10. Grundhöfer, T., Knarr, N. and Kramer, L.: Flag-Homogeneous compact connected polygons, *Geom. Dedicata* **55** (1995), 95–114.
11. Helgason, S.: *Differential Geometry, Lie Groups, and Symmetric Spaces*, Academic Press, San Diego, 1978.
12. Knarr, N.: The non-existence of certain topological polygons, *Forum Math.* **2** (1990), 603–612.
13. Kramer, L.: Compact polygons, PhD Thesis, Tübingen, 1994.
14. Kramer, L.: Compact Homogeneous Quadrangles and Focal Manifolds, Habilitationsschrift, Würzburg, 1998.

15. Kramer, L.: *Compact Polygons and Isoparametric Hypersurfaces,* Birkhäuser-Verlag, Basel, to appear.
16. Kramer, L. and Van Maldeghem, H.: Compact ovoids in quadrangles I: Geometric constructions, *Geom. Dedicata.* **78** (1999), 279–300.
17. Kreck, M. and Stolz, S.: A diffeomorphism classification of 7-dimensional homogeneous Einstein manifolds with $SU(3) \times SU(2) \times U(1)$-symmetry, *Ann. of Math.* **127** (1988), 373–388.
18. Kreck, M. and Stolz, S.: Some nondiffeomorphic homeomorphic homogeneous 7-manifolds with positive sectional curvature, *J. Differential Geom.* **33** (1991), 465–486. A correction on: 'Some nondiffeomorphic homeomorphic homogeneous 7-manifolds with positive sectional curvature', *J. Differential Geom.* **49** (1998), 203–204.
19. Mimura, M.: Homotopy theory of Lie groups, In: I. M. James (ed.), *Handbook of Algebraic Topology*, Elsevier, Amsterdam, 1995.
20. Mimura, M. and Toda, H.: *Topology of Lie groups. I, II*, Transl. Math. Monogr. 91, Amer. Math. Soc., Providence, RI, 1991.
21. Montgomery, D. and Samelson, H.: Transformation groups on spheres, *Ann. of Math.* **44** (1943), 454–470.
22. Nagano, T.: Homogeneous sphere bundles and the isotropic Riemann manifolds, *Nagoya Math. J.* **15** (1959), 29–55.
23. Onishchik, A. L.: *Topology of Transitive Transformation Groups,* J. A. Barth, Leipzig, 1994.
24. Poncet, J.: Groupes de Lie compacts de transformations de l'espace euclidien et les sphéres comme espaces homogenes, *Comment. Math. Helv.* **33** (1959), 109–130.
25. Salzmann, H., Betten, D., Grundhöfer, T., Hähl, H., Löwen, R. and Stroppel, M.: *Compact Projective Planes*, De Gruyter, Berlin, 1995.
26. Spanier, E.H.: *Algebraic Topology,* McGraw-Hill, New York, 1966.
27. Stroppel, M.: Reconstruction of incidence geometries from groups of automorphisms, *Arch. Math.* **58** (1992), 621–624.
28. Takeuchi, M.: On the fundamental group and the group of isometries of a symmetric space, *J. Fac. Sci. Univ. Tokyo* **10** (1964), 88–123.
29. Thorbergsson, G.: Clifford algebras and polar planes, *Duke Math J.* **67** (1992), 627–632.
30. Tits, J.: *Buildings of Spherical Type and Finite BN-Pairs*, Lecture Notes in Math. 386, Springer, Berlin, 1986.
31. Wallace, A. D.: Monotone transformations, *Duke Math. J.* **9** (1942), 487–506.
32. Wolf, J.A.: *Spaces of Constant Curvature*, Publish or Perish, Wilmington, 1974.