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The point and line space of a compact projective plane are homeomorphic

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Abstract. The local Hopf bundles at the points and lines of a compact projective plane with manifold lines are all weakly equivalent. In particular, the point space and the line space of such a projective plane are homeomorphic. This is a consequence of the following topological result.

Theorem. *Let ξ, ξ' be orientable topological $\mathbb{R}^n$-bundles over an n-dimensional CW-complex. If ξ and ξ' are fibre homotopy equivalent and stably equivalent, then ξ and ξ' are equivalent.*

The point space of a compact projective plane with manifold lines can be viewed as the Thom space of a certain fibre bundle η_p. Similarly, the line space is the Thom space of another fibre bundle η_ℓ. We show that the fibre bundles η_p and η_ℓ are weakly equivalent. Unfortunately, there is no obvious geometric reason why such an equivalence should exist. However, one can rather easily see an equivalence between η_p and another bundle ${}^u\eta_\ell$ which is obtained from η_ℓ by turning the fibres of η_ℓ 'upside down'. Essentially, this was already observed by Eisele [5] and Schroth [19]. It follows from the Hirsch-Mazur Theorem that η_ℓ and ${}^u\eta_\ell$ are stably equivalent, and (by definition) η_ℓ and ${}^u\eta_\ell$ are fibre homotopy equivalent. Thus the result announced in the title follows from the topological theorem in the abstract.

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Some results about $\mathbb{R}^n$-bundles

We recall some facts about microbundles and $\mathbb{R}^n$-bundles. Most of the results in this section are well-known to topologists. The necessary background about general fibre bundles can be found in the first chapters of Husemoller [9]. See also Holm [8] for a careful exposition of microbundles and $\mathbb{R}^n$-bundles.

A *space* is a topological space. Unless specified otherwise, all maps are assumed to be continuous.

1 Definition A *bundle* $\phi = (E, B, p)$ over a space B is a surjective map $E \xrightarrow{p} B$. The space $B = B(\phi)$ is called the *base* and $E = E(\phi)$ is called the *total space* of ϕ. The *fibre over* $b \in B$ is $E_b = p^{-1}(b)$, and for $A \subseteq B$ the bundle $E_A = p^{-1}(A) \longrightarrow A$ is called the *restriction* $\phi|_A$. A bundle map $(F, f) : \phi \longrightarrow \phi'$ is a commutative diagram

$$\begin{tikzcd} E(\phi) \arrow[r, "F"] \arrow[d] & E(\phi') \arrow[d] \\ B(\phi) \arrow[r, "f"] & B(\phi'). \end{tikzcd}$$

If f and F are homeomorphisms, then (F, f) is called a *weak equivalence*, $\phi \cong \phi'$, and if in addition $f = \mathrm{id}$, then (F, id) is called an *equivalence* (also denoted by $\phi \cong \phi'$). A *section* $B \xrightarrow{s} E$ is a right inverse to p, i.e. $p \circ s = \mathrm{id}_B$. We put $E_0 = E \setminus s(B)$. For bundles ϕ, ϕ' with sections, we require that bundle maps preserve sections, i.e. that the diagram

$$\begin{tikzcd} B \arrow[r, "s"] \arrow[d, "f"'] & E \arrow[r, "p"] \arrow[d, "F"'] & B \arrow[d, "f"'] \\ B' \arrow[r, "s'"] & E' \arrow[r, "p'"] & B' \end{tikzcd}$$

commutes. The *Whitney sum* $\phi_1 \oplus \phi_2$ of two bundles $\phi_i = (E_i, B, p_i)$ over the same base B, for $i = 1, 2$, has as total space the pull-back

$$E_1 \oplus E_2 = \{(e_1, e_2) \in E_1 \times E_2 |\, p_1(e_1) = p_2(e_2)\}$$

over the diagram

$$\begin{tikzcd} & E_2 \arrow[d, "p_2"] \\ E_1 \arrow[r, "p_1"] & B, \end{tikzcd}$$

with the obvious projection $(e_1, e_2) \longmapsto p_1(e_1)$ (and section $b \longmapsto (s_1(b), s_2(b))$ if ϕ_1 and ϕ_2 have sections).

We are mainly interested in $\mathbb{R}^n$-bundles.

2 Definition An $\mathbb{R}^n$*-bundle* $\xi = (E, B, p, s)$ is a bundle with section, subject to the following additional condition: for every $b \in B$ there exists an open neighborhood V of b and a homeomorphism $h : V \times \mathbb{R}^n \longrightarrow E_V = p^{-1}(V)$ such that the diagram

$$\begin{tikzcd}
 & V \times \mathbb{R}^n \arrow[dr, "\mathrm{pr}_1"] \arrow[dd, "h"] & \\
V \arrow[ur, "(\mathrm{id}_V, 0)"] \arrow[dr, "s|_V"'] & & V \\
 & E_V \arrow[ur, "p|_{E_V}"'] &
\end{tikzcd}$$

commutes. The pair (V, h) is called a *coordinate chart*. The *trivial (product)* $\mathbb{R}^n$*-bundle* $B \times \mathbb{R}^n \xrightarrow{\mathrm{pr}_1} B$ is denoted by $\underline{\mathbb{R}}^n$.

A *locally trivial* $\mathbb{S}^n$*-bundle* is a bundle (E, B, p), subject to the following condition: for every $b \in B$ there is an open neighborhood V of b and a homeomorphism $h : V \times \mathbb{S}^n \longrightarrow E_V$ such that the diagram

$$\begin{tikzcd}
V \times \mathbb{S}^n \arrow[dr, "\mathrm{pr}_1"] \arrow[dd, "h"'] & \\
 & V \\
E_V \arrow[ur, "p|_{E_V}"'] &
\end{tikzcd}$$

commutes. Again, (V, h) is called a *coordinate chart*.

To each such $\mathbb{S}^n$-bundle ϕ one can associate an $\mathbb{R}^{n+1}$-bundle $\mathring{D}(\phi)$ in a canonical way. The total space of $\mathring{D}(\phi)$ is the *open mapping cylinder* $\mathring{C}_p$ of the bundle projection $E(\phi) \xrightarrow{p} B$, i.e. $\mathring{C}_p$ is the quotient space

$$\mathring{C}_p = (E(\phi) \times [0, \infty) \sqcup B)/\sim,$$

where $E(\phi) \times \{0\}$ is identified by the map p with B. There are canonical maps $B \overset{s_0}{\hookrightarrow} \mathring{C}_p \longrightarrow B$ which make $\mathring{D}(\phi)$ into an $\mathbb{R}^{n+1}$-bundle (Fig. 1). If we write $\langle e, t\rangle$ for the equivalence class of $(e, t) \in E(\phi) \times [0, \infty)$, then (e, t) can be thought of as 'cylindrical coordinates' for $\langle e, t\rangle$ in the total space of the $\mathbb{R}^{n+1}$ bundle.

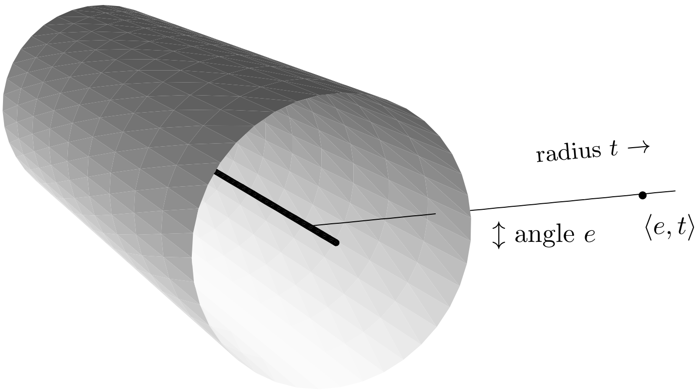

Fig. 1

3 Definition A locally finite covering $\{V_i | \, i \in I\}$ of B by open sets is called *numerable* if there exist maps $f_i : B \longrightarrow [0,1]$ with $f_i^{-1}((0,1]) = V_i$, such that $\sum_{i \in I} f_i = 1$. An $\mathbb{R}^n$-bundle (or $\mathbb{S}^n$-bundle, or n-microbundle, see Definition 7 below) is called *numerable* if there exists a numerable covering of B by coordinate charts. In our setting, most base spaces will be paracompact, so bundles are automatically numerable.

4 Lemma *Let* $(F, \mathrm{id}) : \xi \longrightarrow \eta$ *be a bundle map of* $\mathbb{R}^n$*-bundles. If for each fibre the restriction* $F|_{E(\xi)_b} : E(\xi)_b \longrightarrow E(\eta)_b$ *is a bijection, then* (F, id) *is an equivalence.*

Proof. By domain invariance, F is a homeomorphism on each fibre. We have to show that F^{-1} is continuous. This is a local property, so we can assume that ξ and η are trivial $\mathbb{R}^n$-bundles. Then F is of the form $(b, x) \longmapsto (b, g_b(x))$, where $b \longmapsto g_b$ is a continuous map of the base B into the homeomorphism group $\mathrm{TOP}(n)$ of $(\mathbb{R}^n, 0)$. Thus F^{-1} is given by $(b, y) \longmapsto (b, g_b^{-1}(y))$, and this is continuous, since inversion is continuous in the topological group $\mathrm{TOP}(n)$. $\square$

As a consequence, we have the following result.

5 Lemma *Let* ξ *be a numerable* $\mathbb{R}^n$*-bundle over* $B \times [0,1]$*. Then the restrictions* $\xi|_{B \times \{0\}}$ *and* $\xi|_{B \times \{1\}}$ *are weakly equivalent under a bundle map* $(\Phi, (b,0) \longmapsto (b,1))$.

Proof. The proof given in Husemoller [9], Chapter 3 Corollary 4.6 for vector bundles carries over to $\mathbb{R}^n$-bundles; Husemoller's Theorem 2.5 is replaced by Lemma 4 above, see also Holm [8] Lemma 1.5. $\square$

6 Corollary *Let $\xi = (E, B, p, s)$ be a numerable $\mathbb{R}^n$-bundle. If $s_1 : B \longrightarrow E$ is any section, then the $\mathbb{R}^n$-bundles $B \xrightarrow{s} E \longrightarrow B$ and $B \xrightarrow{s_1} E \longrightarrow B$ are equivalent.*

Proof. The fibres of ξ are contractible, hence there exists a homotopy $\alpha : B \times [0,1] \longrightarrow E$ with $\alpha_0 = s$ and $\alpha_1 = s_1$, such that $p \circ \alpha_t = \mathrm{id}_B$, cp. Dold [4] Corollary 2.8. Define a numerable $\mathbb{R}^n$-bundle η over $B \times [0,1]$ with total space $E \times [0,1]$ and section α. Then $\eta|_{B\times\{0\}} \cong \xi$, and $\eta|_{B\times\{1\}}$ is weakly equivalent to the bundle with section s_1. The result follows from Lemma 5. $\square$

We also need the concept of a microbundle.

7 Definition An *n-microbundle* $\mathfrak{x} = (E, B, p, s)$ is a bundle with section, subject to the following condition: for every $b \in B$ there exists an open neighborhood V of b, an open subset $U \subseteq E_V = p^{-1}(V)$ containing $s(V)$ and a homeomorphism $h : V \times \mathbb{R}^n \longrightarrow U$ such that the diagram

$$\begin{tikzcd}
& V \times \mathbb{R}^n \arrow[dd, "h"] \arrow[dr, "\mathrm{pr}_1"] & \\
V \arrow[ur, "(\mathrm{id},0)"] \arrow[dr, "s|_V"'] & & V \\
& U \arrow[ur, "p|_U"'] &
\end{tikzcd}$$

commutes. We call (V, h) a *coordinate chart* of the microbundle. The difference between an n-microbundle and an $\mathbb{R}^n$-bundle is that the image of h need *not* be all of E_V. Obviously, an $\mathbb{R}^n$-bundle is an n-microbundle. The Kister-Mazur Theorem (see Theorem 9 below) says that conversely, microbundles are essentially the same as $\mathbb{R}^n$-bundles, a fact which is not obvious at all.

If $E' \subseteq E$ is a neighborhood of $s(B)$, then it is not difficult to see that $B \longrightarrow E' \longrightarrow B$ is again a microbundle $\mathfrak{x}'$ *contained in* $\mathfrak{x}$. Two microbundles $\mathfrak{x}_1, \mathfrak{x}_2$ over the same base B are called *micro-equivalent* if they contain microbundles $\mathfrak{x}_1', \mathfrak{x}_2'$ which are equivalent in the sense of Definition 1 (this is also sometimes called a micro-isomorphism or an isomorphism germ). In the case of numerable microbundles one has to be careful: a microbundle contained in a numerable microbundle need *a priori* not be numerable.

8 Lemma *Let $\phi = (E, B, p)$ be a numerable locally trivial sphere bundle, and suppose that $s : B \longrightarrow E$ is a section. Then (E, B, p, s) is a numerable microbundle.*

Proof. The only difficulty is to show that one obtains a numerable microbundle. Assume first that ϕ is a product bundle with a single coordinate chart $h : B \times \mathbb{S}^n \longrightarrow E$. Then $s(b) = h(b, s'(b))$, for some map $s' : B \longrightarrow \mathbb{S}^n$. Fix $b_0 \in B$ and let $x_0 = s'(b_0) \in \mathbb{S}^n$. Consider the two real functions $f_\pm(b) = \frac{1}{2}|s'(b) \pm x_0|$ and let $V_\pm = f_\pm^{-1}((0,1])$. Then $B = V_+ \cup V_-$, and it is not difficult to construct maps $g_\pm : V_\pm \longrightarrow \mathrm{O}(n)$ such that $g_\pm(b)(x_0) = s'(b)$ (rotate with a 1-parameter group along the unique geodesic from $s'(b)$ to $\pm x_0$). Identifying $(\mathbb{R}^n, 0)$ with $(\mathbb{S}^n \setminus \{-x_0\}, x_0)$, we obtain microbundle coordinate charts $(b, x) \longmapsto h(b, g_\pm(b)(x))$ over V_+ and V_-.

Now let ϕ be arbitrary. If $\{V_i |\ i \in I\}$ is a locally finite covering of B, then $\{V_{i,\pm} |\ i \in I\}$ is also locally finite. Similarly as above, we can construct functions $f_{i,\pm}$ from the f_i of the numerable covering such that $V_{i,\pm} = f_{i,\pm}^{-1}((0,1])$. Finally, the resulting set of functions can be normalized by multiplying it with the continuous function

$$b \longmapsto \left(\sum_{i \in I} (f_{i,+}(b) + f_{i,-}(b)) \right)^{-1}$$

. $\square$

9 Theorem (Kister-Mazur) *Let $\mathfrak{x}$ be a numerable n-microbundle. Then there exists a numerable microbundle $\mathfrak{x}'$ contained in $\mathfrak{x}$ which is an $\mathbb{R}^n$-bundle, and $\mathfrak{x}'$ is unique up to equivalence.*

Proof. See Holm [8] Theorem 3.3, and also Siebenmann-Guillou-Hähl [20] 6.4. $\square$

Consider the following construction of a new $\mathbb{R}^n$-bundle from an old one.

10 Definition Let ξ be an $\mathbb{R}^n$-bundle. Compactifying each fibre to a sphere $\mathbb{S}^n$, we obtain a locally trivial $\mathbb{S}^n$-bundle $\overline{\xi}$ and another section s_∞ corresponding to the compactifications of the fibres, i.e. $s_\infty(b) = \infty_b$ is the one point which is added to the fibre E_b over b. We put ${}^uE = E_0 \cup s_\infty(B)$ and call ${}^uE \longrightarrow B$ the *upside-down bundle* ${}^u\xi$ of ξ. Clearly, this is again an $\mathbb{R}^n$-bundle with zero-section s_∞, and $({}^uE)_0 = E_0$ (see Fig. 2). If (V, h) is a coordinate chart for ξ, then $(V, {}^uh)$ is a coordinate chart for ${}^u\xi$, where ${}^uh : V \times \mathbb{R}^n \longrightarrow {}^uE$ is defined by

$${}^uh(b, x) = \begin{cases} h(b, x/|x|^2) & \text{for } x \neq 0 \\ \infty_b & \text{for } x = 0. \end{cases}$$

In particular, ${}^u\xi$ is numerable if and only if ξ is numerable.

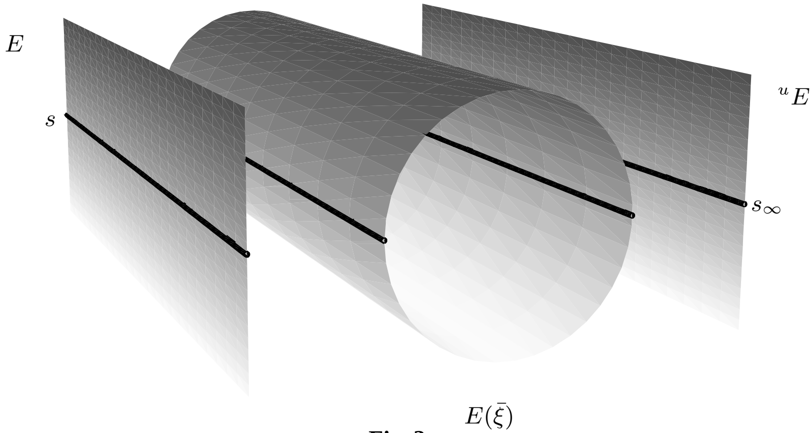

Fig. 2

If ξ happens to be a vector bundle with a Riemannian metric, then ξ and ${}^u\xi$ are equivalent under the bundle map $E \longmapsto {}^uE$, $v \longmapsto v/|v|^2$ (with the obvious extension to the zero-vectors). I do not know if such an isomorphism exists for arbitrary $\mathbb{R}^n$-bundles. For orientable $\mathbb{R}^n$-bundles over CW-complexes of dimension at most n, this is true by the results in Sect. 2. A first step in this direction is the following version of the Hirsch-Mazur Theorem (which is essentially due to Browder [2] Proposition 2.2, see also Holm [8] Theorem 4.5 for a proof which avoids microbundles) and its corollary which shows that ξ and ${}^u\xi$ are *stably equivalent*.

11 Proposition *Let ϕ be a locally trivial numerable $\mathbb{S}^n$-bundle with a section $B \xrightarrow{s} E$. Then (B, E, p, s) is a numerable n-microbundle $\mathfrak{x}$ by Lemma 8. Let ξ be a numerable $\mathbb{R}^n$-bundle contained in $\mathfrak{x}$. Then there is an equivalence*

$$\xi \oplus \underline{\mathbb{R}} \cong \overset{\circ}{D}(\phi).$$

Proof. Let $\eta = \overset{\circ}{D}(\phi) = (\overset{\circ}{C}, B, p_0, s_0)$ for short. The equivalence class of $(x,t) \in \overset{\circ}{C}_p$ is denoted $\langle x, t\rangle$, cp. Definition 2. Consider the section $s_1 : b \longmapsto \langle s(b), 1\rangle$. Then both $B \xrightarrow{s_1} \overset{\circ}{C}_p$ and $B \xhookrightarrow{s_0} \overset{\circ}{C}_p$ make $\overset{\circ}{C}_p \xrightarrow{p_0} B$ into an $\mathbb{R}^{n+1}$-bundle. Let $\eta_1 = (\overset{\circ}{C}_p, B, p_0, s_1)$. By Corollary 6 there is an equivalence $\eta \cong \eta_1$.

By Lemma 8, $\mathfrak{x} = (E, B, p, s)$ is a numerable n-microbundle. Let ξ be a numerable $\mathbb{R}^n$-bundle contained in $\mathfrak{x}$, let $\lambda : \mathbb{R} \longrightarrow (-1, 1)$ be a

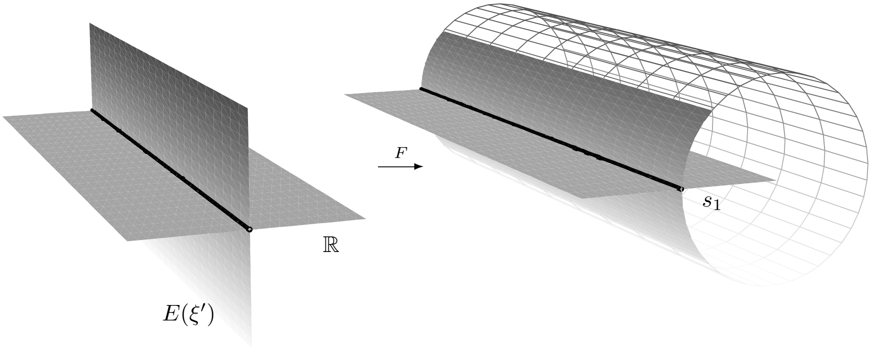

Fig. 3

homeomorphism with $\lambda(0) = 0$ and put

$$F : E(\xi) \times \mathbb{R} \longrightarrow \mathring{C}_p, \qquad (x,t) \longmapsto \langle x, \lambda(t) \rangle$$

(see Fig. 3). Then $F(E(\xi) \times \mathbb{R}) \xrightarrow{p_0} B$ is a numerable $\mathbb{R}^{n+1}$-bundle contained in the numerable $\mathbb{R}^{n+1}$-bundle η_1. By the Kister-Mazur Theorem 9, we have $\xi \oplus \underline{\mathbb{R}} \cong \eta_1$, and $\eta_1 \cong \eta$. The result follows. $\square$

12 Corollary *Let ξ be a numerable $\mathbb{R}^n$-bundle. Then there is an equivalence*

$$\xi \oplus \underline{\mathbb{R}} \cong {}^u\xi \oplus \underline{\mathbb{R}}.$$

Proof. Note that by definition $\overline{\xi} = \overline{{}^u\xi}$. The $\mathbb{R}^n$-bundle ξ is contained in the n-microbundle $(E(\overline{\xi}), B, p, s)$, and thus $\mathring{D}(\overline{\xi}) \cong \xi \oplus \underline{\mathbb{R}}$ by Proposition 11. Similarly, ${}^u\xi$ is an $\mathbb{R}^n$-bundle contained in the n-microbundle $(E(\overline{\xi}), B, p, s_\infty)$, whence ${}^u\xi \oplus \underline{\mathbb{R}} \cong \mathring{D}(\overline{\xi})$. $\square$

Our next aim is to show that in certain situations there is an equivalence $\xi \cong {}^u\xi$. Now we need some results about classifying spaces.

The necessary classifying spaces

Recall that a *fibration* is a bundle which has the homotopy lifting property, see Spanier [21] Ch. 2.2. Let $E_1 \xrightarrow{p_1} B$ and $E_2 \xrightarrow{p_2} B$ be fibrations. A commutative diagram of bundle maps between fibrations

$$\begin{tikzcd} E_1 \arrow[r, "F"] \arrow[dr, "p_1"'] & E_2 \arrow[r, "G"] \arrow[d, "p_2"] & E_1 \arrow[dl, "p_1"] \\ & B & \end{tikzcd}$$

is called a *fibre homotopy equivalence* if there are homotopies $G \circ F \simeq \mathrm{id}_{E_1}$ and $F \circ G \simeq \mathrm{id}_{E_2}$ which are constant under p_1 and p_2 (a homotopy $f : E_1 \times [0,1] \longrightarrow E_1$ is constant under p_1 if $p_1 \circ f_t = p_1 \circ f_0$ for all $t \in [0,1]$). An *n-spherical fibration* is a fibration whose fibres are homotopy equivalent to an n-sphere. A spherical fibration is called *orientable* if the fundamental groupoid of B acts trivially on the homotopy groups of the fibre; in particular, a spherical fibration over a 1-connected base is always orientable. Orientability is preserved under fibre homotopy equivalence.

The *spherical fibration of a numerable $\mathbb{R}^n$-bundle* is the fibration $E_0(\xi) \longrightarrow B$ (this is indeed a fibration, see Spanier [21] Theorem 2.7.12) and two numerable $\mathbb{R}^n$-bundles ξ, ξ' are called *fibre homotopy equivalent, $\xi \simeq \xi'$*, if their respective spherical fibrations are fibre homotopy equivalent. Note that an equivalence of $\mathbb{R}^n$-bundles induces a fibre homotopy equivalence, and that $\xi \simeq {}^u\xi$, since $E_0(\xi) = E_0({}^u\xi)$. We call a numerable $\mathbb{R}^n$-bundle *orientable* if its spherical fibration is orientable.

The aim of this section is to prove the following theorem.

13 Theorem *Let ξ, ξ' be orientable numerable $\mathbb{R}^n$-bundles over a space B with the homotopy type of a CW-complex of dimension at most n. If ξ and ξ' are stably isomorphic, i.e. if $\xi \oplus \underline{\mathbb{R}}^k \cong \xi' \oplus \underline{\mathbb{R}}^k$ for some $k \geq 0$, and if there is a fibre homotopy equivalence $\xi \simeq \xi'$, then there is an equivalence $\xi \cong \xi'$.*

This implies in particular the following result.

14 Corollary *Let ξ be an orientable numerable $\mathbb{R}^n$-bundle over a space B which has the homotopy type of a CW-complex of dimension at most n. Then there is an equivalence*

$$\xi \cong {}^u\xi.$$

Proof. By Corollary 12 we have $\xi \oplus \underline{\mathbb{R}} \cong {}^u\xi \oplus \underline{\mathbb{R}}$, and $\xi \simeq {}^u\xi$. The result follows from Theorem 13. $\square$

In order to prove Theorem 13 we need some facts about classifying spaces and vector bundles. More precisely, we will prove that the kernels of the maps

$$\begin{array}{ccc} \pi_i(\mathrm{BSTOP}(n)) & \longrightarrow & \pi_i(\mathrm{BSTOP}(n+k)) \\ \downarrow & & \\ \pi_i(\mathrm{BSG}(n)) & & \end{array}$$

intersect trivially for $i \leq n$ (see below for definitions). First we prove this result for *vector bundles*. (An n-vector bundle is an $\mathbb{R}^n$-bundle with extra structure, the vector space structure on the fibres. Husemoller [9] gives a comprehensive introduction.) Let $\mathrm{BSO}(n)$ denote the classifying space for oriented n-vector bundles. For any space X, the set of free homotopy classes $[X; \mathrm{BSO}(n)]$ is in a natural one-to-one correspondence with the collection of the equivalence classes of oriented numerable n-vector bundles over X, see eg. Husemoller [9] Ch. 4. Since $\mathrm{BSO}(n)$ is 1-connected, this set coincides also with the set $[X; \mathrm{BSO}(n)]_0$ of *based homotopy classes*, see eg. Whitehead [25] III.1.11.

15 A model for $\mathrm{BSO}(n)$ Let $V_m(\mathbb{R}^k)$ denote the Stiefel manifold of orthonormal m-frames in $\mathbb{R}^k$, and let

$$V_m(\mathbb{R}^\infty) = \bigcup_{k \geq 0} V_m(\mathbb{R}^k),$$

cp. Husemoller [9] Ch. 8. For $n \leq m$ there is a natural free action of the matrix group $\mathrm{SO}(n)$ on $V_m(\mathbb{R}^\infty)$, and the orbit space

$$\mathrm{BSO}(n) = V_m(\mathbb{R}^\infty)/\mathrm{SO}(n)$$

is a classifying space for $\mathrm{SO}(n)$. Note that $\mathrm{SO}(n)$ can be identified with a subset of $V_n(\mathbb{R}^n) \subseteq V_m(\mathbb{R}^\infty)$.

16 The tangent bundle of $\mathbb{S}^n$ The tangent bundle of $\mathbb{S}^n$ can be described by the Borel-Hirzebruch method. The n-sphere is a homogeneous space

$$\mathbb{S}^n = \mathrm{SO}(n+1)/\mathrm{SO}(n),$$

and the tangent vector bundle τ^n of $\mathbb{S}^n$ is the vector bundle associated to the principal bundle $\mathrm{SO}(n) \hookrightarrow \mathrm{SO}(n+1) \longrightarrow \mathbb{S}^n$ via the standard representation of $\mathrm{SO}(n)$ on $\mathbb{R}^n$. Consider the diagram

$$\begin{tikzcd}
\mathrm{SO}(n) \arrow[r, equal] \arrow[d, hook] & \mathrm{SO}(n) \arrow[r, hook] \arrow[d] & \mathrm{SO}(n+1) \arrow[d] \\
\mathrm{SO}(n+1) \arrow[r] \arrow[d] & V_{n+1}(\mathbb{R}^\infty) \arrow[r, equal] \arrow[d] & V_{n+1}(\mathbb{R}^\infty) \arrow[d] \\
\mathrm{SO}(n+1)/\mathrm{SO}(n) \arrow[r, "c"] & V_{n+1}(\mathbb{R}^\infty)/\mathrm{SO}(n) \arrow[r] & V_{n+1}(\mathbb{R}^\infty)/\mathrm{SO}(n+1).
\end{tikzcd}$$

The bottom line is a fibration

$$\mathrm{SO}(n+1)/\mathrm{SO}(n) \xrightarrow{c} \mathrm{BSO}(n) \longrightarrow \mathrm{BSO}(n+1).$$

The left column, which is the pull-back fibration of the middle column, i.e. the principal bundle over $\mathrm{BSO}(n)$, is the principal bundle associated to the tangent bundle of $\mathbb{S}^n$. Thus c is a classifying map for the tangent bundle of $\mathbb{S}^n$. Note that the map $\mathrm{BSO}(n) \longrightarrow \mathrm{BSO}(n+1)$ corresponds to the process of stabilizing vector bundles: if an oriented n-vector bundle ξ is classified by $f : B(\xi) \longrightarrow \mathrm{BSO}(n)$, then the composite $B(\xi) \longrightarrow \mathrm{BSO}(n) \longrightarrow \mathrm{BSO}(n+1)$ classifies $\xi \oplus \underline{\mathbb{R}}$.

Let T_n denote the kernel of the map $\pi_n(\mathrm{BSO}(n)) \longrightarrow \pi_n(\mathrm{BSO}(n+1))$. As before, $\mathbb{S}^n \xrightarrow{c} \mathrm{BSO}(n)$ is a classifying map for the tangent bundle of $\mathbb{S}^n$. As we remarked before, $\mathrm{BSO}(n)$ is 1-connected, so the set of free homotopy classes $[\mathbb{S}^k; \mathrm{BSO}(n)]$ coincides with the homotopy group $\pi_k(\mathrm{BSO}(n)) = [\mathbb{S}^k; \mathrm{BSO}(n)]_0$, see Whitehead [25] III.1.11.

17 Proposition *The group T_n is generated by the image of $\pi_n(c)$.*
If n is even, then $\pi_n(c)$ is an isomorphism onto T_n.
If $n = 1, 3, 7$, then $T_n = 0$ and c is homotopic to a constant map.
If $n \neq 1, 3, 7$ is odd, then T_n is cyclic of order two.

Proof. The long exact homotopy sequence shows that $\pi_n(c)(\mathrm{id}_{\mathbb{S}^n}) = c \in T_n$ (this is also clear from the geometric point of view: $\tau^n \oplus \underline{\mathbb{R}} \cong \underline{\mathbb{R}}^{n+1}$ is a trivial bundle). Moreover, T_n is a cyclic group generated by c, because $\pi_n(\mathbb{S}^n) \cong \mathbb{Z}$.

If n is even, then $\pi_{n+1}(\mathrm{BSO}(n+1))$ is isomorphic to a subgroup of $\mathbb{Z}/2 \oplus \mathbb{Z}/2$, see eg. Mimura and Toda [15] IV.6.14. Since $\pi_n(\mathbb{S}^n) \cong \mathbb{Z}$, the long exact homotopy sequence shows that the map $\pi_n(c)$ is an injection if n is even.

If n is odd, then $\pi_n(\mathrm{BSO}(n))$ is isomorphic to a subgroup of $\mathbb{Z}/2 \oplus \mathbb{Z}/2$ by *loc.cit.*, so the image of $\pi_n(c)$ in $\pi_n(\mathrm{BSO}(n))$ is either trivial or isomorphic to $\mathbb{Z}/2$. It is well-known that the tangent bundle of $\mathbb{S}^n$ is not fibre homotopy equivalent to the trivial bundle, provided that $n \neq 1,3,7$, see James-Whitehead [10] Theorem 1.12, combined with Adams' result on elements of Hopf invariant 1 (Husemoller [9] Ch. 15). In particular, $\tau^n \ncong \underline{\mathbb{R}}^n$, and thus $c \in \pi_n(\mathrm{BSO}(n))$ has order two for $n \neq 1,3,7$. Finally, the tangent bundle of $\mathbb{S}^n$ is known to be trivial for $n = 1,3,7$. $\square$

Let $\mathrm{SG}(n+1)$ denote the topological semigroup of all maps $\mathbb{S}^n \longrightarrow \mathbb{S}^n$ of degree 1, and let $\mathrm{SF}(n) = \Omega^n\mathbb{S}^n \cap \mathrm{SG}(n+1)$ denote the subset of all based degree 1 self maps of $\mathbb{S}^n$, cp. Milnor [14] Sect. 2. There is a fibration $\mathrm{SF}(n-1) \longrightarrow \mathrm{SG}(n) \longrightarrow \mathbb{S}^{n-1}$ and an associated classifying space $\mathrm{BSG}(n)$ which classifies oriented $(n-1)$-spherical fibrations over sufficiently nice spaces (eg. over CW-complexes) up to (oriented) fibre homotopy equivalence, see Stasheff [22], Madsen-Milgram [12] Ch. 1. Corresponding to the process of stabilizing bundles and to the forgetful maps from n-vector bundle to $\mathbb{R}^n$-bundles to spherical fibrations, there is a commutative diagram of fibrations and bundle maps

$$\begin{tikzcd}
\mathrm{SO}(n+1)/\mathrm{SO}(n) \arrow[r] \arrow[d] & \mathrm{BSO}(n) \arrow[r, "stab"] \arrow[d, "f^{\mathrm{SO}}_{\mathrm{SG}}"'] & \mathrm{BSO}(n+1) \arrow[d, "f^{\mathrm{SO}}_{\mathrm{SG}}"'] \\
\mathrm{SG}(n+1)/\mathrm{SG}(n) \arrow[r] & \mathrm{BSG}(n) \arrow[r, "stab"] & \mathrm{BSG}(n+1).
\end{tikzcd}$$

The spaces on the left are the homotopy fibres of the stabilization maps. (Up to homotopy, every map can be converted into a fibration by a standard process, see Spanier [21] Theorem 2.8.9. The resulting fibre is called the homotopy fibre of the map.)

18 Lemma *The forgetful map $f^{\mathrm{SO}}_{\mathrm{SG}}$ induces isomorphisms*

$$\pi_k(\mathbb{S}^n) = \pi_k(\mathrm{SO}(n+1)/\mathrm{SO}(n)) \cong \pi_k(\mathrm{SG}(n+1)/\mathrm{SG}(n))$$

for $k \leq 2n-3$ and maps the group T_n isomorphically onto the kernel of the stabilization map

$$\pi_n(\mathrm{BSG}(n)) \longrightarrow \pi_n(\mathrm{BSG}(n+1))$$

for all $n \geq 1$.

Proof. Note that $\mathrm{SG}(1) \cong \{pt\}$, and $\mathrm{SF}(1) \cong \{pt\}$ is the connected component of $\Omega\mathbb{S}^1 \simeq \mathbb{Z}$, hence $\mathrm{SG}(2) \simeq \mathbb{S}^1$. Also, $\pi_1(\mathrm{BSO}(n)) = 0 = \pi_1(\mathrm{BSG}(n)) = 0$ for all n. Easy diagram chasing shows that $\pi_1(\mathrm{SO}(2)/\mathrm{SO}(1)) \longrightarrow \pi_1(\mathrm{SG}(2)/\mathrm{SG}(1))$ is an isomorphism and that the kernels of both stabilization maps in dimension 1 are trivial, so the claim follows for $n = 1$.

For $n = 2$ we use the fact that $\pi_i(\mathrm{SG}(3)) = \pi_i(\mathrm{SF}(3))$ for $i \leq 2$, see Milnor [14] Sect. 2. Thus $\pi_2(\mathrm{BSG}(3)) \cong \mathbb{Z}/2$. Again, easy diagram chasing shows that $\pi_2(\mathrm{BSO}(3)) \longrightarrow \pi_2(\mathrm{BSG}(3))$ is an isomorphism, and so $T_2 \subseteq \pi_2(\mathrm{BSO}(2))$ maps isomorphically onto the kernel of $\pi_2(\mathrm{BSG}(2)) \longrightarrow \pi_2(\mathrm{BSG}(3))$ (note however that $\pi_2(\mathbb{S}^2) \not\cong \pi_2(\mathrm{SG}(3)/\mathrm{SG}(2))$).

For $n \geq 3$ we have isomorphisms $\pi_i(\mathrm{SO}(n+1)/\mathrm{SO}(n)) \longrightarrow \pi_i(\mathrm{SG}(n+1)/\mathrm{SG}(n))$ for all $i \leq 2n-3$, see Burghelea-Lashof [3] p. 38 no. (5), Milgram [13] Theorem A.

It remains to show that T_n maps isomorphically onto the kernel of the stabilization map for spherical fibrations. If $n \neq 1, 3, 7$ is odd, then τ^n is not fibre homotopically trivial, hence $f^{\mathrm{SO}}_{\mathrm{SG}} \circ c \neq 0$ in $\pi_n(\mathrm{BSG}(n))$, cp. the remarks in the proof of Proposition 17. Assume that n is even, and let $g : \mathbb{S}^n \longrightarrow \mathbb{S}^n$ be a map of degree k. Then the Euler class of the bundle corresponding to $c \circ g = kc$ is k times the Euler class of τ^n; in particular, it is not zero. The Euler class is invariant under fibre homotopy equivalence (since it can be defined for spherical fibrations, see eg. Milnor [14], Spanier [21] Ch. 9.5; he denotes the Euler class by Ω), hence $f^{\mathrm{SO}}_{\mathrm{SG}} c \circ g = k \cdot f^{\mathrm{SO}}_{\mathrm{SG}} c \neq 0$ in $\pi_n(\mathrm{BSG}(n))$. Thus $f^{\mathrm{SO}}_{\mathrm{SG}}$ maps T_n isomorphically onto the kernel of the stabilization map $\pi_n(\mathrm{BSG}(n)) \longrightarrow \pi_n(\mathrm{BSG}(n+1))$.

Now we get back to $\mathbb{R}^n$ bundles. Let $\mathrm{STOP}(n)$ denote the topological group of all orientation preserving based homeomorphisms of $\mathbb{R}^n$. There is a corresponding classifying space $\mathrm{BSTOP}(n)$ which classifies oriented numerable $\mathbb{R}^n$-bundles up to (oriented) equivalence. Let $\mathrm{STOP}(n+1)/\mathrm{STOP}(n)$ denote the homotopy fibre of the stabilization map $\mathrm{BSTOP}(n) \longrightarrow \mathrm{BSTOP}(n+1)$. The forgetful map $f^{\mathrm{SO}}_{\mathrm{SG}}$ factors as

$$\mathrm{BSO}(n) \xrightarrow{f^{\mathrm{SO}}_{\mathrm{STOP}}} \mathrm{BSTOP}(n) \xrightarrow{f^{\mathrm{STOP}}_{\mathrm{SG}}} \mathrm{BSG}(n)$$

19 Proposition (cp. Varadarajan [24] Sect. 1.) *The map*

$$\pi_i(\mathrm{SO}(n+1)/\mathrm{SO}(n)) \longrightarrow \pi_i(\mathrm{STOP}(n+1)/\mathrm{STOP}(n))$$

is an isomorphism for all $i \leq n-1$, and also for $i = n$, provided that $n \neq 3$. In dimension 3, we have $\pi_3(\mathrm{STOP}(4)/\mathrm{SO}(4)) \cong \mathbb{Z} \oplus \mathbb{Z}/2$ instead.

For all $n \geq 1$, the forgetful map $f^{\mathrm{SO}}_{\mathrm{STOP}}$ maps T_n isomorphically onto the kernel of the stabilization map $\pi_n(\mathrm{BSTOP}(n)) \longrightarrow \pi_n(\mathrm{BSTOP}(n+1))$.

Proof. We divide the proof in several steps.

Step (i). The claim holds for $n \leq 3$.

If $n \leq 3$, then $\mathrm{SO}(n) \hookrightarrow \mathrm{STOP}(n)$ is a homotopy equivalence by the results of Kneser [16] and Hatcher [7], see also Kirby-Siebenmann [11] Essay V Sect. 5, so the same is true for $f^{\mathrm{SO}}_{\mathrm{STOP}}$ in these dimensions, and the claim follows easily for $n \leq 2$. Also, $\pi_3(\mathrm{BSTOP}(3)) \cong \pi_3(\mathrm{BSO}(3)) = 0$, so $0 = T_3$ maps isomorphically onto the kernel of the stabilization map in dimension 3.

Let $\mathrm{STOP} = \bigcup_{n \geq 0} \mathrm{STOP}(n)$ and $\mathrm{SO} = \bigcup_{n \geq 0} \mathrm{SO}(n)$. Quinn proved that

$$\pi_i(\mathrm{STOP}/\mathrm{SO}, \mathrm{STOP}(4)/\mathrm{SO}(4)) = 0 \tag{1}$$

for $i \leq 5$, cp. Freedman-Quinn [6] 8.7A., so $\pi_i(\mathrm{STOP}(4)/\mathrm{SO}(4))$ is isomorphic to $0, 0, 0, \mathbb{Z}/2, 0$ for $i = 0, 1, 2, 3, 4$, cp. Kirby-Siebenman [11] Essay V Sect. 5, 5.0(5). The exact sequence

$$\begin{aligned} 0 &\longrightarrow \pi_3(\mathrm{SO}(4)/\mathrm{SO}(3)) \\ &\longrightarrow \pi_3(\mathrm{STOP}(4)/\mathrm{SO}(3)) \to \pi_3(\mathrm{STOP}(4)/\mathrm{SO}(4)) \\ &\longrightarrow 0 \end{aligned}$$

shows that $\pi_3(\mathrm{STOP}(4)/\mathrm{SO}(3)) \cong \pi_3(\mathrm{STOP}(4)/\mathrm{STOP}(3)) \in \{\mathbb{Z}, \mathbb{Z} \oplus \mathbb{Z}/2\}$, and $\mathbb{Z}$ is excluded by the diagram

$$\begin{tikzcd} 0 \arrow[r] & \pi_3(\mathrm{SO}(4)/\mathrm{SO}(3)) \arrow[r] \arrow[dr, "\cong"'] & \pi_3 \mathrm{STOP}(4)/\mathrm{STOP}(3)) \arrow[r] \arrow[d] & \pi_3(\mathrm{STOP}(4)/\mathrm{SO}(4)) \arrow[r] & 0 \\ & & \pi_3(\mathrm{SG}(4)/\mathrm{SG}(3)). & & \end{tikzcd}$$

Thus $\pi_3(\mathrm{STOP}(4)/\mathrm{STOP}(3)) \cong \mathbb{Z} \oplus \mathbb{Z}/2$, and the case $n = 3$ is finished.

Step (ii). Some stability results for $\mathrm{STOP}(m)/\mathrm{SO}(m)$.

If $m \geq 5$, then

$$\pi_i(\mathrm{STOP}/\mathrm{SO}, \mathrm{STOP}(m)/\mathrm{SO}(m)) = 0 \tag{2}$$

for all $i \leq m + 2$, see Kirby-Siebenmann [11] Essay V Sect. 5, 5.0(4) and Burghelea-Lashof [3] 5.1 (note that this is one dimension better than Quinn's result (1) for $m = 4$ mentioned above). There are two long exact sequences

$$\longrightarrow \pi_i(\mathrm{STOP}(m), \mathrm{SO}(m)) \longrightarrow \pi_i(\mathrm{STOP}, \mathrm{SO}) \longrightarrow \pi_i(\Box) \longrightarrow$$

and

$$\longrightarrow \pi_i(\mathrm{SO}, \mathrm{SO}(m)) \longrightarrow \pi_i(\mathrm{STOP}, \mathrm{STOP}(m)) \longrightarrow \pi_i(\Box) \longrightarrow$$

where $\pi_i(\Box)$ is the ith homotopy group of the diagram

$$\begin{tikzcd} \mathrm{SO}(m) \arrow[r, hook] \arrow[d, hook] & \mathrm{SO} \arrow[d, hook] \\ \mathrm{STOP}(m) \arrow[r, hook] & \mathrm{STOP}, \end{tikzcd}$$

see Stern [23] (these homotopy groups are defined for $i \geq 3$, and $\pi_2(\square)$ is a pointed set). Note also that $\pi_i(G, H) = \pi_i(G/H)$ for $G, H \in \{\mathrm{SO}(m), \mathrm{STOP}(m), \mathrm{SO}, \mathrm{STOP}\}$ whenever this makes sense (properly speaking, one has to replace the topological groups by certain semi-simplicial groups to justify these identifications, see Stern [23], Rourke-Sanderson [17]). Combining the two long exact sequences with the stability results (1), (2) above, we conclude that

$$\pi_i(\mathrm{STOP}/\mathrm{STOP}(m), \mathrm{SO}/\mathrm{SO}(m)) = 0$$

for $m \geq 5$ and $i \leq m+2$, and for $m = 4$ and $i \leq 5$. In the stable range $i \leq m-1$ we have $\pi_i(\mathrm{SO}/\mathrm{SO}(m)) = 0$, so $\pi_i(\mathrm{STOP}/\mathrm{STOP}(m)) = 0$ as well.

Step (iii). The claim holds for $n \geq 4$.

Suppose now that $n \geq 4$. Then $\pi_i(\mathrm{STOP}(n+1)) = \pi_i(\mathrm{STOP})$ and $\pi_i(\mathrm{SO}) = \pi_i(\mathrm{SO}(n+1))$ for $i \leq n$. For $i \geq 3$, this follows from Step (ii); for $i = 0, 1, 2$ one has to modify some arguments slightly. In any case, we obtain a diagram

$$\begin{tikzcd}
\pi_i(\mathrm{SO}(n+1)/\mathrm{SO}(n)) \arrow[r] \arrow[d, "\cong"'] & \pi_i(\mathrm{STOP}(n+1)/\mathrm{STOP}(n)) \arrow[d, "\cong"'] \\
\pi_i(\mathrm{SO}/\mathrm{SO}(n)) \arrow[r, "\cong"] & \pi_i(\mathrm{STOP}/\mathrm{STOP}(n)).
\end{tikzcd}$$

for $i \leq n$ and $n \geq 4$. This implies that $\pi_n(\mathrm{STOP}(n+1)/\mathrm{STOP}(n))$ is infinite cyclic for $n \geq 4$ and thus T_n maps onto the kernel of the stabilization map for $\mathrm{BSTOP}(n)$ in dimension n. However, we know already that T_n maps isomorphically onto the kernel of the n-dimensional stabilization map for $\mathrm{BSG}(n)$ by Lemma 18,

$$\begin{tikzcd}[column sep=small]
\pi_n\left(\frac{\mathrm{SO}(n+1)}{\mathrm{SO}(n)}\right) \arrow[rr] \arrow[dr] \arrow[dd] & & T_n \arrow[rr] \arrow[dr] \arrow[dd] & & \pi_n(\mathrm{BSO}(n)) \arrow[dr] \arrow[dd] & \\
& \pi_n\left(\frac{\mathrm{STOP}(n+1)}{\mathrm{STOP}(n)}\right) \arrow[rr] \arrow[dl] & & \pi_n(f^{\mathrm{SO}}_{\mathrm{STOP}})T_n \arrow[rr] \arrow[dl] & & \pi_n(\mathrm{BSTOP}(n)) \arrow[dl] \\
\pi_n\left(\frac{\mathrm{SG}(n+1)}{\mathrm{SG}(n)}\right) \arrow[rr] & & \pi_n(f^{\mathrm{SO}}_{\mathrm{SG}})T_n \arrow[rr] & & \pi_n(\mathrm{BSG}(n)). &
\end{tikzcd}$$

The result follows. $\square$

Repeated application of this stability result yields the following corollary.

20 Corollary *For $i \le n$ and $1 \le n \le n+k$, the kernels of the maps*

$$\begin{tikzcd} \pi_i(\mathrm{BSTOP}(n)) \arrow[r] \arrow[d] & \pi_i(\mathrm{BSTOP}(n+k)) \\ \pi_i(\mathrm{BSG}(n)) & \end{tikzcd}$$

intersect trivially. $\square$

Proof of Theorem 13. It suffices to prove the theorem for B, a CW-complex of dimension $m \le n$. We proceed by induction on m. For $m = 0$, there is nothing to show. Let ξ, ξ' be $\mathbb{R}^n$-bundles over an $(m+1)$-dimensional CW-complex B as in the theorem, for $m+1 \le n$, with classifying maps $c, c' : B \longrightarrow \mathrm{BSTOP}(n)$. We orient both bundles in such a way that $f_{\mathrm{SG}}^{\mathrm{STOP}} \circ c \simeq f_{\mathrm{SG}}^{\mathrm{STOP}} \circ c'$. Then $stab \circ c \simeq stab \circ c'$, where $stab : \mathrm{BSTOP}(n) \longrightarrow \mathrm{BSTOP}(n+k)$ is the stabilization map.

Let $I = [0,1]$. By our induction hypothesis, we may assume that the theorem holds for m-dimensional CW-complexes, and in particular for the m-skeleton $B^{(m)}$. We want to prove it for B, an $(m+1)$-dimensional CW-complex. There exists a map $C : B^{(m)} \times I \cup B \times \{0,1\} \longrightarrow \mathrm{BSTOP}(n)$ with $C|_{B\times\{0\}} = c$ and $C|_{B\times\{1\}} = c'$, and we are dealing with the *extension problem*

$$\begin{tikzcd} B^{(m)} \times I \cup B \times \{0,1\} \arrow[r, "C"] \arrow[d, hook] & \mathrm{BSTOP}(n) \\ B \times I. \arrow[ur, dotted] & \end{tikzcd}$$

Note that $B \times I$ is an $(m+2)$-dimensional CW-complex, and that $(B\times I)^{(m+1)} = B^{(m)} \times I \cup B \times \{0,1\}$. Let $e^{m+1} \longrightarrow B$ be an $(m+1)$-cell. Then $e^{m+1} \times I \longrightarrow B \times I$ is an $(m+2)$-cell, and we have to show that the map C can be extended over this cell. Let $\dot{\chi} : \partial(e^{m+1} \times I) \longrightarrow B \times I$ denote the attaching map. We obtain a diagram

$$\begin{tikzcd} \partial(e^{m+1} \times I) \arrow[r, "\dot{\chi}"] \arrow[dd, hook] & B^{(m)} \times I \cup B \times \{0,1\} \arrow[r, "C"] \arrow[dd, hook] & \mathrm{BSTOP}(n) \arrow[dl, "f_{\mathrm{SG}}^{\mathrm{STOP}}"'] \arrow[dd, "stab"] \\ & \mathrm{BSG}(n) & \\ e^{m+1} \times I \arrow[r] & B \times I \arrow[r] \arrow[u] & \mathrm{BSTOP}(n+k) \end{tikzcd}$$

which commutes up to homotopy. The three spaces on the right are 1-connected, so we can view $C \circ \dot{\chi}$ as an element of $\pi_{m+1}(\mathrm{BSTOP}(n))$. The diagram shows that both $stab \circ C \circ \dot{\chi}$ and $f_{\mathrm{SG}}^{\mathrm{STOP}} \circ C \circ \dot{\chi}$ are homotopic to a constant map, since they factor through the contractible space $e^{m+1} \times I$. By Corollary 20, the map $C \circ \dot{\chi}$ is also homotopic to a constant map, so we can extend $C \circ \dot{\chi}$ over $e^{m+1} \times I$. Every $(m+2)$-cell of $B \times I$ is of this form, so the map C can be extended to a map $B \times I \longrightarrow \mathrm{BSTOP}(n)$ (see Whitehead [25] V.5.2) and c and c' are homotopic. $\square$

Hopf bundles of compact projective planes

A *projective plane* is an incidence geometry $\mathfrak{P} = (\mathcal{P}, \mathcal{L}, \mathcal{F})$ consisting of a *point set* $\mathcal{P}$, a *line set* $\mathcal{L}$ and a *flag set* $\mathcal{F} \subseteq \mathcal{P} \times \mathcal{L}$ describing the incidence relation: a point $p \in \mathcal{P}$ and a line $\ell \in \mathcal{L}$ are incident if and only if $(p, \ell) \in \mathcal{F}$. For distinct points $p, q \in \mathcal{P}$ we denote the unique line joining them by $pq \in \mathcal{L}$; dually, the intersection point of two distinct lines $h, \ell \in \mathcal{L}$ is $h\ell \in \mathcal{P}$. The set $\mathcal{L}_p = \{h \in \mathcal{L} | \ (p, h) \in \mathcal{F}\}$ is called a *line pencil* and $\mathcal{P}_\ell = \{q \in \mathcal{P} | \ (q, \ell) \in \mathcal{F}\}$ is called a *point row*.

If $\mathcal{P}$ and $\mathcal{L}$ are compact Hausdorff spaces, and if the maps $(p, q) \longmapsto pq$ and $(h, \ell) \longmapsto h\ell$ are continuous on their respective domains $\mathcal{P} \times \mathcal{P} \setminus \mathrm{id}_{\mathcal{P}}$ and $\mathcal{L} \times \mathcal{L} \setminus \mathrm{id}_{\mathcal{L}}$, then $\mathfrak{P}$ is called a *compact projective plane*; see Salzmann *et al.* [18] for a comprehensive introduction. The continuity condition is equivalent with $\mathcal{F} \subseteq \mathcal{P} \times \mathcal{L}$ being closed, see [18] 41.5. The classical Moufang planes $PG_2(\mathbb{F})$, for $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ are examples for such planes, but there exists a continuum of non-classical compact projective planes.

21 Definition Let $\mathfrak{P} = (\mathcal{P}, \mathcal{L}, \mathcal{F})$ be a compact projective plane with manifold lines. Then every point row is homeomorphic to an n-sphere, for some $n \in \{1, 2, 4, 8\}$, cp. Salzmann *et al.* [18] 52.3, Breitsprecher [1] 2.1 and 2.3.1. Let $q \in \mathcal{P}$ be a point and define a map (the central projection from q) $\mathcal{P} \setminus \{q\} \longrightarrow \mathcal{L}_q$ by $p \longmapsto pq$. It is easy to see that this map is a locally trivial fibre bundle which we denote by η_q, the *local Hopf bundle at* p [18] 51.23. The one-point compactification of the total space of η_q is clearly homeomorphic to $\mathcal{P}$. If m is a line which is not incident with p, then we can define a section s by $s(\ell) = m\ell$. Since the lines of $\mathfrak{P}$ are n-spheres, η_q is an $\mathbb{R}^n$-bundle, and $\mathcal{P} \cong M(\eta_q)$ is the Thom space of this bundle. The map $\mathcal{F} \longrightarrow \mathcal{P}$ is a locally trivial $\mathbb{S}^n$-bundle, see [18] 51.23. For $X \subseteq \mathcal{P}$ we denote the restriction by $\mathcal{F}_X \longrightarrow X$; similarly for $\mathcal{F} \longrightarrow \mathcal{L}$.

Eisele's homeomorphism criterion [5] is a complicated bundle-free version of the bundle map (G, g) below. See [18] 52.15 for a streamlined version of his result. A similar construction is used by Schroth in [19] who proved that for any two points q, q' the bundles η_q and $\eta_{q'}$ are weakly equivalent.

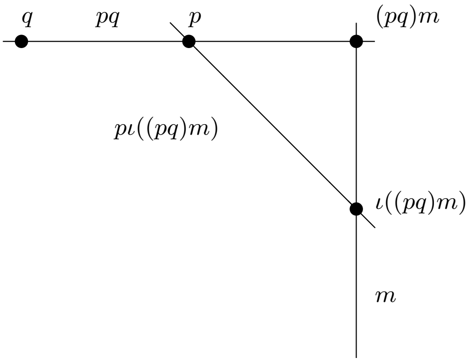

Fig. 4

(And dually for any two lines. Schroth did not consider the case of projective planes, but his arguments are valid nevertheless. Conversely, his result follows directly from Proposition 22 below, since $\eta_q \cong {}^u\eta_m \cong \eta_{q'}$.)

22 Proposition *Let (q, m) be a non-incident point-line pair in a compact projective plane with manifold lines. Then there is a weak equivalence*

$$\eta_q \cong {}^u\eta_m$$

between the local Hopf bundle at q and the upside-down local Hopf bundle at m.

Proof. Let $\bar{E} = \{(p, \ell) \in \mathcal{F} |\ (p, m) \in \mathcal{F}\}$ denote the set of all flags whose points lie on the line m. Then there is an injection

$$E(\eta_m) \hookrightarrow \bar{E}, \qquad \ell \longmapsto (m\ell, \ell)$$

whose image consists of all flags in $\bar{E}$ whose line is different from m. Clearly, $\bar{E}$ can be identified with the total space of the $\mathbb{S}^n$-bundle $\overline{\eta}_\ell$ obtained by compactifying the fibres: the element which is added to the fibre $E(\eta_m)_p$ is the flag (p, m). The zero-section $\mathcal{P}_\ell \longrightarrow E(\eta_m)$ of η_m is given by $p \longmapsto pq$; the corresponding subset in $\bar{E}$ consists of the flags whose line passes through q.

Now consider the bundle η_q. Its zero-section is given by $\ell \longmapsto \ell m$. We define an injection $G : E(\eta_q) \longrightarrow \bar{E}$ as follows. Let $\iota : \mathcal{P}_m \longrightarrow \mathcal{P}_m$ be a fixed-point free homeomorphism (recall that $\mathcal{P}_m \cong \mathbb{S}^n$). Put

$$G(p) = (\iota((pq)m), p\iota((pq)m))$$

(see Fig. 4). The image of G consists of all flags in $\bar{E}$ whose line does not pass through q, and the section of η_q is mapped onto the set of flags whose line is m. Thus, if we put $g(\ell) = \iota(m\ell)$, then $(G, g) : \eta_q \longrightarrow {}^u\eta_m$ is a weak equivalence of $\mathbb{R}^n$-bundles. $\square$

23 Theorem *Let (p, ℓ) be a point-line pair in a compact projective plane with manifold lines. Then the $\mathbb{R}^n$-bundles η_p and η_ℓ are weakly equivalent.*

Proof. It clearly suffices to prove the theorem for non-incident point-line pairs. If $n = 1$, then η_p corresponds to one of the two elements of $\pi_1(\mathrm{BTOP}(1)) \cong \pi_0(\mathrm{TOP}(1)) \cong \mathbb{Z}/2$, so either η_p is the trivial line bundle over $\mathbb{S}^1$, or η_p is the Möbius strip. The Thom space of the trivial bundle is not a manifold, so η_p has to be the Möbius strip. The same argument applies to η_ℓ.

Assume now that $n \geq 2$. Then $n \in \{2, 4, 8\}$, and there is a weak equivalence $\eta_p \cong {}^u\eta_\ell$ by Proposition 22. By Corollary 14, there is an equivalence $\eta_\ell \cong {}^u\eta_\ell$. $\square$

24 Corollary *The point space $\mathcal{P}$ and the line space $\mathcal{L}$ of a compact projective plane with manifold lines are homeomorphic.*

Proof. The point space is the Thom space of η_p, and the line space is the Thom space of η_ℓ. $\square$

References

1. S. Breitsprecher, Projektive Ebenen, die Mannigfaltigkeiten sind, Math. Z. **121** (1971) 157–174.
2. W. Browder, Open and closed disk bundles, Ann. Math. **83** (1966) 218–230.
3. D. Burghelea and R. Lashof, The homotopy type of the space of diffeomeorphisms I,II, Trans. Amer. Math. Soc. **196** (1974) 1–50.
4. A. Dold, Partitions of unity in the theory of fibrations, Ann. Math. **78** (1963) 223–255.
5. E. Eisele, On homeomorphisms and incidence relations of compact projective planes, Arch. Math. **58** (1992) 615–620.
6. M.H. Freedman and F. Quinn, Topology of 4-manifolds, Princeton University Press, Princeton, New Jersey (1990).
7. A.E. Hatcher, A proof of the Smale conjecture, Diff$(S^3) \simeq O(4)$. Ann. Math. **117** (1983) 553–607.
8. P. Holm, The microbundle representation theorem, Acta Math.. **117** (1967) 191–213.
9. D. Husemoller, Fibre bundles, 3rd ed., Springer Verlag, New York Berlin (1994).
10. I.M. James and J.H.C. Whitehead, The homotopy theory of sphere bundles over spheres. I., Proc. London Math. Soc. **4** (1954), 196–218.
11. R.C. Kirby and L.C. Siebenmann, Foundational essays on topological manifolds, smoothings, and triangulations, Princeton University Press, Princeton, New Jersey (1977).
12. I. Madsen and J.R. Milgram, The classifying spaces for surgery and cobordism of manifolds, Princeton University Press, Princeton, New Jersey (1979).
13. J.R. Milgram, On the Haefliger knot groups. Bull. Amer. Math. Soc. **78** (1972) 861–865.

14. J. Milnor, On characteristic classes for spherical fibre spaces, Comment. Math. Helv. **43** (1968) 51–77.
15. M. Mimura and H. Toda, Topology of Lie groups, I and II, Translations of Math. Monographs Vol. 91, Amer. Math. Soc., Providence, Rhode Island (1991).
16. H. Kneser, Die Deformationssätze der einfach zusammenhängenden Flächen, Math. Z. **25** (1926) 362–372.
17. C.P. Rourke and B.J. Sanderson, Δ-sets I,II, Quarterly J. Math. Oxford **22** (1971) 321–338 and 465–485.
18. H. Salzmann, D. Betten, T. Grundhöfer, H. Hähl, R. Löwen, and M. Stroppel, Compact projective planes, de Gruyter Expositions in Mathematics 21, Berlin (1995).
19. A.E. Schroth, On the topological structure of generalized polygons, Simon Stevin **67** (1993) 259–266.
20. L. Siebenmann, L. Guillou and H. Hähl, Les voisinages ouverts réguliers, Annales Ecole Normale Sup. **6** (1973) 253–293.
21. E.H. Spanier, Algebraic topology, Springer Verlag, New York Berlin (1966).
22. J. Stasheff, A classification theorem for fibre spaces. Topology **2** (1963) 239–246.
23. R.J. Stern, On topological and piecewise linear vector fields, Topology **14** (1975) 257–269.
24. K. Varadarajan, Span and stably trivial bundles, Pacific J. Math. **60** (1975) 277–287.
25. G.W. Whitehead, Elements of homotopy theory, Springer Verlag, New York Berlin (1978).