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Compact Ovoids in Quadrangles III: Clifford Algebras and Isoparametric Hypersurfaces

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Abstract. In this third part, we consider those compact quadrangles which arise from isoparametric hypersurfaces of Clifford type and their focal manifolds. Sections 9–11 give a comprehensive introduction to these quadrangles from the incidence-geometric point of view. Section 10 contains also a new (algebraic) proof that these geometries are quadrangles.

We determine which of these quadrangles have ovoids or spreads and also whether the normal sphere bundles of the focal manifolds admit sections, or whether they are topologically trivial. We give explicit geometric constructions for spreads, ovoids, and sections.

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9. Clifford Systems and Clifford Algebras

We consider the quadrangles discovered by Ferus–Karcher–Münzner [5] and Thorbergsson [20]. First we review the Veronese imbedding of the standard Hermitian quadrangles $H_n(\mathbb{F})$, where $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$. Put $d = \dim_{\mathbb{R}} \mathbb{F}$.

9.1. THE VERONESE REPRESENTATION

Let $H_n(\mathbb{F}) = (\mathcal{P}, \mathcal{L}, \mathcal{F})$ be a standard Hermitian quadrangle. We have seen in Part II [15] 7.4 that the line space $\mathcal{L}$ is homeomorphic to the Stiefel manifold $V_2(\mathbb{F}^{n-1})$, and we can push it into the unit sphere by mapping $(u, v) \in V_2(\mathbb{F}^{n-1})$ to the unit vector $(1/\sqrt{2})(u, v) \in \mathbb{S}^{2d(n-1)-1}$; let $\widetilde{\mathcal{L}} \subseteq \mathbb{S}^{2(n-1)d-1}$ denote the image of this map. The points of the standard Hermitian quadrangles are of the form $p = (c, s, w)\mathbb{F}$, where $c, s \in \mathbb{F}$ are scalars, $w \in \mathbb{F}^{n-1}$, and $|c|^2 + |s|^2 = |w|^2$. We can assume that $c \in \mathbb{R}$ is real, and that $|w|^2 = 1$. The triple (c, s, w) is not well-defined, but the unit vector $(wc, w\bar{s}) \in \mathbb{S}^{2d(n-1)-1}$ is, and the map $p = (c, s, w)\mathbb{F} \mapsto (wc, w\bar{s})$ is an injection. Let $\widetilde{\mathcal{P}} \subseteq \mathbb{S}^{2d(n-1)-1}$ denote its image. The incidence relation becomes very simple in terms of $\widetilde{\mathcal{P}}, \widetilde{\mathcal{L}} \subseteq \mathbb{S}^{2(n-1)d-1}$. Let $(wc, w\bar{s}) \in \widetilde{\mathcal{P}}$ and $(u, v) \in \widetilde{\mathcal{L}}$. Then

$$\langle (wc, w\bar{s}), (u, v)\rangle = \langle wc, u\rangle + \langle w\bar{s}, v\rangle$$

$$
\begin{aligned}
&= \langle w, uc\rangle + \langle w, vs\rangle \\
&= \langle w, uc + vs\rangle \\
&\leqslant |w||uc + vs| = \frac{1}{\sqrt{2}},
\end{aligned}
$$

(the last equality holds because $|w| = 1$, and because $(us \mid vc) = 0$). By Cauchy–Schwarz, the inequality becomes an equality if and only if $w = \sqrt{2}(uc + vs)$. Translating this back into the quadrangle $H_n(\mathbb{F})$, we find that this is equivalent with

$$
\begin{aligned}
(c, s, \sqrt{2}(uc + vs))\mathbb{F} &= ((1, 0, \sqrt{2}u)c + (0, 1, \sqrt{2}v)s)\mathbb{F} \\
&\subseteq (1, 0, \sqrt{2}u)\mathbb{F} \oplus (0, 1, \sqrt{2}v)\mathbb{F}.
\end{aligned}
$$

This shows that the Euclidean inner product $\langle -, - \rangle$ in $\mathbb{R}^{2(n-1)d}$ contains all the information about the incidence. This is the *Veronese representation* of $H_n(\mathbb{F})$. The sets $\widetilde{\mathcal{P}}$, $\widetilde{\mathcal{L}}$ are the focal manifolds of an isoparametric hypersurface $\widetilde{\mathcal{F}}$ which can be identified with $\mathcal{F}$.

The main ingredients in this description are the Euclidean inner product $\langle -, - \rangle$ and the positive definite Hermitian form $(-|-)$. The idea for the Clifford quadrangles is to replace $(-|-)$ by a bilinear map $[\![-, -]\!]$ on V,

$$
[\![-, -]\!]: V \otimes V \to \mathrm{End}_{\mathbb{R}}(V)
$$

with special properties. *This is a purely algebraic process; thus, we assume for the next two sections only that* **R** *is a real closed field, i.e. an ordered field (necessarily of charcteristic* 0*), where every positive element is a square, and where every polynomial of odd degree has a zero.* The algebraic closure of such a real closed field **R** is the field $\mathbf{C} = \mathbf{R}[\sqrt{-1}]$. See Jacobson [9] Chapter 5 for properties of these fields. By a *Euclidean vector space* V over **R** we mean a vector space endowed with a positive definite symmetric bilinear form $\langle -, - \rangle$; put $|x| = \sqrt{\langle x, x\rangle}$. Note that one can perform Gram–Schmidt orthonormalization over real closed fields. We call the set of all unit vectors in $\mathbf{R}^{k+1}$ a k-sphere.

Let $V \cong \mathbf{R}^n$ be a Euclidean **R**-vector space, with the standard Euclidean inner product $\langle -, - \rangle$. Suppose that $E_1, \ldots, E_{m-1}$ are orthogonal matrices which satisfy the relations

$$
E_i E_j + E_j E_i = \begin{cases} 0 & \text{for } i \neq j, \\ -2\mathrm{id} & \text{for } i = j. \end{cases}
$$

In other words, the matrices anti-commute and are skew symmetric. Such a set of matrices is called a *Clifford system*. Put $E_0 = \mathrm{id}$ and let $\mathbb{A} \subseteq \mathrm{End}_{\mathbf{R}}(V)$ denote the **R**-vector space spanned by $E_0, E_1, \ldots, E_{m-1}$. We identify **R** with $\mathbf{R} \cdot E_0 \subseteq \mathbb{A}$, and we define the *Clifford bracket* of $x, y \in V$ by

$$
[\![x, y]\!] = \sum_{\nu=0}^{m-1} \langle x, E_\nu y\rangle E_\nu.
$$

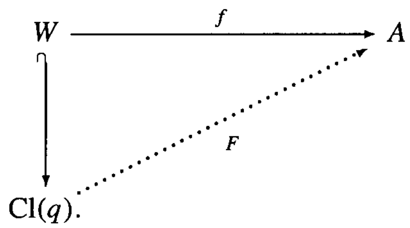
Thus we obtain a bilinear map $V \otimes V \to \mathbb{A}$, with the property $[\![x, x]\!] = |x|^2$. Two vectors x, y are *Clifford orthogonal* if $[\![x, y]\!] = 0$. The *Clifford Stiefel manifold* of 2-frames is defined as

$$V_2([\![-, -]\!]) = \{(x, y) \in \mathbf{R}^n \times \mathbf{R}^n \mid [\![x, y]\!] = 0, |x|^2 = |y|^2 = 1\},$$

cf. Pinkall–Thorbergsson [18]. Here is an example: let $\mathbf{H}$ denote the unique quaternion division algebra over $\mathbf{R}$, let $\mathbf{R}^{4k} = \mathbf{H}^k$, and let E_1, E_2, E_3 denote right multiplication by the imaginary units $\mathbf{i}, \mathbf{j}, \mathbf{k}$. Then $\mathbb{A} = \mathbf{H}$ and $V_2([\![-, -]\!]) = V_2(\mathbf{H}^k)$, as is easily checked, cf. Jacobson [9] p. 404, Example 3. The general construction of such sets of matrices is a matter of Clifford modules. We explain this briefly; cp. Jacobson [10] Chapter 4.8, Atiyah–Bott–Shapiro [2], Lawson–Michelsohn [16] Chapter 1 for more details.

9.2. CLIFFORD ALGEBRAS

Let $q: W \to \mathbf{R}$ be a quadratic form on an $\mathbf{R}$-vector space W. The *Clifford algebra* $\mathrm{Cl}(q)$ is the associative algebra (with unit 1) generated by W, subject to the relations $v^2 - q(v)1 = 0$, for all $v \in W$. It has the following universal property: if A is an associative $\mathbf{R}$-algebra with unit 1, and if $f: W \to A$ is a linear map such that $f(v)^2 = q(v)1$, then there exists a unique extension $F: \mathrm{Cl}(q) \to A$ which makes the following diagram commute.



There is an injection $W \hookrightarrow \mathrm{Cl}(q)$; if $U \subseteq W$ is a subspace, then the inclusion $U \hookrightarrow W$ extends to an inclusion $\mathrm{Cl}(q|_U) \hookrightarrow \mathrm{Cl}(q)$, cf. Jacobson [10] p. 235.

The map $-\mathrm{id}: W \to W$ extends uniquely to an involutive automorphism $x \mapsto x^*$ of $\mathrm{Cl}(q)$. Consider the anti-isomorphism $\mathrm{id}: \mathrm{Cl}(q) \to \mathrm{Cl}(q)^{\mathrm{opp}}$ onto the opposite algebra. By the universal property of Clifford algebras, its restriction to W extends to an isomorphism $\mathrm{Cl}(q) \cong \mathrm{Cl}(q)^{\mathrm{opp}}$. Therefore, $\mathrm{Cl}(q)$ has an anti-automorphism $x \mapsto x^\tau$ given by $v_1 \ldots v_r \mapsto v_r \ldots v_1$, for $v_1, \ldots, v_r \in W$. The composition $x \mapsto \bar{x} = (x^*)^\tau$ is the anti-automorphism which we are really interested in; it is given by $v_1 \ldots v_r \mapsto (-1)^r v_r \ldots v_1$, for $v_1, \ldots, v_r \in W$. The involution $x \mapsto x^*$ induces a $\mathbb{Z}/2$-grading of the Clifford algebra

$$\mathrm{Cl}(q) = \mathrm{Cl}(q)^\bullet = \mathrm{Cl}(q)^0 \oplus \mathrm{Cl}(q)^1,$$

where $\mathrm{Cl}(q)^0$ consists of the fixed elements of the involution and $\mathrm{Cl}(q)^1$ consists of the elements which are mapped to their negatives. The elements of $\mathrm{Cl}(q)^0$ are the elements which can be written as linear combinations of products of an even number of elements of W.

9.3. CLIFFORD MODULES

A *Clifford module* is an $\mathbf{R}$-vector space V, endowed with a homomorphism $\mathrm{Cl}(q) \to \mathrm{End}_{\mathbf{R}}(V)$. We are particularly interested in the Clifford algebra Cl_k associated to the quadratic form $v \mapsto -|v|^2$ on $W = \mathbf{R}^k$ and its modules. If $e_1, \dots, e_k$ is an orthonormal basis of $\mathbf{R}^k$, then we have the relations

$$e_i e_j + e_j e_i = \begin{cases} 0 & \text{for } i \neq j, \\ -2 & \text{for } i = j \end{cases}$$

in Cl_k. The e_i generate a finite group F_k with 2^{k+1} elements, the *Clifford group*. If V is a Cl_k-module, then there exists a F_k-invariant positive definite inner product $\langle -, - \rangle$ on V. With respect to this inner product, all $a \in \mathbf{R}^k$, $w \in V$ satisfy $|aw|^2 = |a|^2|w|^2$; in particular, the elements a of unit length $|a|^2 = 1$ in $\mathbf{R}^k$ act as orthogonal maps, cp. Lawson–Michelsohn I.5.16. We call this an *orthogonal representation* of Cl_k. The images $E_1, \dots, E_k$ of the e_i in Cl_k form a Clifford system in $\mathrm{End}_{\mathbf{R}}(V)$. Therefore, the classification of all Clifford systems is equivalent to the classification of all modules of the algebras Cl_k.

We will also have to consider the Clifford algebras $\mathrm{Cl}_{k,0}$ associated to the quadratic form $v \mapsto |v|^2$. Here, similar remarks apply. There is an isomorphism $\mathrm{Cl}^0_{k+1,0} \cong \mathrm{Cl}_k$, which can be seen as follows. Let $b_0, \dots, b_k$ be an orthonormal basis of W. Then $e_1 = b_0 b_1, \dots, e_k = b_0 b_k$ generate a Clifford algebra Cl_k, cp. Jacobson [10] 4.14.

The first Clifford algebras are $\mathrm{Cl}_0 = \mathbf{R}$ and $\mathrm{Cl}_1 = \mathbf{C}$ (with $e_1 = \mathbf{i} = \sqrt{-1}$). The next Clifford algebra is the quaternion division algebra $\mathrm{Cl}_2 = \mathbf{H}$ over $\mathbf{R}$, generated by the imaginary elements $e_1 = \mathbf{i}$, $e_2 = \mathbf{j}$, $\mathbf{i}^2 = -1 = \mathbf{j}^2$, $\mathbf{ij} + \mathbf{ji} = 0$. The next Clifford algebra is $\mathrm{Cl}_3 = \mathbf{H} \oplus \mathbf{H}$ (with $e_1 = \mathbf{i} \oplus (-\mathbf{i})$, $e_2 = \mathbf{j} \oplus (-\mathbf{j})$, $e_3 = \mathbf{ij} \oplus (-\mathbf{ij})$). The irreducible modules over the first three Clifford algebras are $\mathbf{R}$, $\mathbf{C}$, and $\mathbf{H}$; the modules over these algebras are the finite-dimensional vector spaces over $\mathbf{R}$, $\mathbf{C}$ and $\mathbf{H}$. We denote these representations by μ_1, μ_2, μ_3 (note that there is a shift in the subscripts). The Clifford algebra Cl_3 is not simple; it has two nonequivalent irreducible four-dimensional representations μ_4^+ and μ_4^-. A Cl_3-module is therefore given by a sum $a\mu_4^+ + b\mu_4^-$. Given a skew field F, put $F(n) = \mathrm{End}_F(F^n)$. The next Clifford algebras are $\mathrm{Cl}_4 \cong \mathbf{H}(2)$, $\mathrm{Cl}_5 = \mathbf{C}(4)$, $\mathrm{Cl}_6 = \mathbf{R}(8)$ and $\mathrm{Cl}_7 = \mathbf{R}(8) \oplus \mathbf{R}(8)$. For $k \geqslant 8$, there is a general periodicity isomorphism $\mathrm{Cl}_{k+8} \cong \mathrm{Cl}_k \otimes \mathbf{R}(16)$. Using this, one obtains the following classification of the Clifford algebras Cl_k and their modules. Put $k = 8r + s$, with $0 \leqslant s \leqslant 7$.

9.4. The table gives the Clifford algebras, the real dimensions, and the centralizers of their irreducible representations.

s	0	1	2	4	5	6
Cl_k	$\mathbf{R}(16^r)$	$\mathbf{C}(16^r)$	$\mathbf{H}(16^r)$	$\mathbf{H}(2{\cdot}16^r)$	$\mathbf{C}(4{\cdot}16^r)$	$\mathbf{R}(8{\cdot}16^r)$
$\dim_{\mathbf{R}} \mu_{k+1}$	16^r	$2{\cdot}16^r$	$4{\cdot}16^r$	$8{\cdot}16^r$	$8{\cdot}16^r$	$8{\cdot}16^r$
	$\mathbf{R}$	$\mathbf{C}$	$\mathbf{H}$	$\mathbf{H}$	$\mathbf{C}$	$\mathbf{R}$

For these values of s, an arbitrary Cl_k module is given by a sum $a\mu_{k+1}$, for some $a \in \mathbb{N}$. If $s = 3, 7$, then Cl_k is not simple, and there are two inequivalent irreducible representations $\mu_{k+1}^{\pm}$.

s	3	7
Cl_k	$\mathbf{H}(16^r) \oplus \mathbf{H}(16^r)$	$\mathbf{R}(8{\cdot}16^r) \oplus \mathbf{R}(8{\cdot}16^r)$
$\dim_{\mathbf{R}} \mu_{k+1}^{\pm}$	$4{\cdot}16^r$	$8{\cdot}16^r$
	$\mathbf{H}$	$\mathbf{R}$

The Cl_k-modules are therefore of the form $a\mu_{k+1}^{+} + b\mu_{k+1}^{-}$, for $a, b \in \mathbb{N}$. The *index* of such a module is the number $\iota = a - b$. The automorphism $x \mapsto x^*$ maps $a\mu_k^+ + b\mu_k^-$ to $b\mu_k^+ + a\mu_k^-$; therefore, the resulting modules (and Clifford systems) are quasi-equivalent under an automorphism of the Clifford algebra, and the absolute value $|\iota|$ is a more important invariant. These results can be found e.g., in Lawson–Michelsohn [16], and in Atiyah–Bott–Shapiro [2].

10. The Clifford Quadrangles

First we derive some properties of the Clifford bracket.

PROPOSITION 10.1. *Let $E_1, \ldots, E_{m-1}$ be a Clifford system given by an orthogonal representation of* Cl_{m-1} *on the Euclidean vector space $V = \mathbf{R}^n$, and let*

$$[\![-,-]\!]\colon V \otimes V \to \mathbb{A} \subseteq \mathrm{End}_{\mathbf{R}}(V)$$

denote the Clifford bracket. The involution $x \mapsto \bar{x}$ of Cl_{m-1} *descends to an involution on $\mathbb{A}$ which is the same as matrix transposition. The Clifford bracket satisfies the following identities, for $a \in \mathbb{A}$ and $x, y \in V$.*

$$[\![x, y]\!] = \overline{[\![y, x]\!]}, \tag{1}$$

$$[\![ax, x]\!] = a|x|^2, \tag{2}$$

$$[\![x, y]\!] + \overline{[\![x, y]\!]} = 2\langle x, y\rangle, \tag{3}$$

$$\langle x, ay\rangle = \langle [\![x, y]\!], a\rangle, \tag{4}$$

$$\langle x, ay\rangle = \langle \overline{a}x, y\rangle, \tag{5}$$

$$[\![ax, \overline{a}y]\!] = a[\![x, y]\!]a. \tag{6}$$

Proof. Identity (1) is clear. Put $a = \sum_{\nu=0}^{m-1} a_\nu E_\nu$. Then

$$\langle x, ay\rangle = \sum_\nu a_\nu \langle x, E_\nu y\rangle = \langle a, [\![x, y]\!]\rangle,$$

and this shows (4). Similarly,

$$\begin{aligned} [\![ax, y]\!] &= \sum_\nu \langle ax, E_\nu y\rangle E_\nu \\ &= \sum_\nu \langle x, \overline{a}E_\nu y\rangle E_\nu \\ &= \sum_\nu \langle x, (2a_\nu - \overline{E}_\nu a)y\rangle E_\nu \\ &= \sum_\nu 2\langle x, y\rangle a_\nu E_\nu - \sum_\nu \langle E_\nu x, ay\rangle E_\nu \\ &= 2a\langle x, y\rangle - [\![ay, x]\!], \end{aligned}$$

so $[\![ax, y]\!] + [\![ay, x]\!] = 2a\langle x, y\rangle$, and (2), (3) follow. Equation (5) is easy. Finally, we prove the 'Moufang identity' (6). For $a, b \in \mathbb{A}$ we have $aba = 2a\langle a, \bar{b}\rangle - |a|^2\bar{b}$, as is easily checked, hence

$$\begin{aligned} [\![ax, \overline{a}y]\!] &= \sum_\nu \langle ax, E_\nu \bar{a}y\rangle E_\nu \\ &= \sum_\nu \langle x, \bar{a}E_\nu \bar{a}y\rangle E_\nu \\ &= \sum_\nu \langle x, (2\bar{a}a_\nu - |a|^2\bar{E}_\nu)y\rangle E_\nu \\ &= 2\sum_\nu \langle x, \bar{a}y\rangle a_\nu E_\nu - \sum_\nu \langle x, \bar{E}_\nu y\rangle |a|^2 E_\nu \\ &= 2a\langle x, \bar{a}y\rangle - |a|^2[\![y, x]\!]. \end{aligned}$$

On the other hand,

$$a[\![x, y]\!]a = 2a\langle a, \overline{[\![x, y]\!]}\rangle - |a|^2\overline{[\![x, y]\!]}$$

$$= 2a\langle a, [\![y,x]\!]\rangle - |a|^2[\![y,x]\!]$$

$$= 2a\langle x, \bar{a}y\rangle - |a|^2[\![y,x]\!]. \qquad \square$$

10.2. CAUCHY–SCHWARZ PRINCIPLE

Let $x, y \in V$ be nonzero vectors. Then $|[\![x,y]\!]| \leqslant |x|\cdot|y|$, and equality holds if and only if $y = ax$ for some $a \in \mathbb{A}$. In this case, $[\![x,y]\!] = \bar{a}|x|^2$.

Proof. Let $x, y \in V$, $x \neq 0$. Then there is a unique decomposition $y = ax + z$, where $a \in \mathbb{A}$ and $z \in (\mathbb{A}\cdot x)^{\perp}$. Hence $[\![x,y]\!] = [\![x,ax]\!] = |x|^2\bar{a}$. Therefore $|[\![x,y]\!]| \leqslant |x||y|$, and equality holds if and only if $y = ax$ for some $a \in \mathbb{A}$; in that case, $[\![x,y]\!] = [\![x,ax]\!] = \bar{a}|x|^2$. $\square$

Note that in general $[\![ax,y]\!] \neq a[\![x,y]\!]$ (consider for example $x = 1$, $y = \mathbf{ij}$ and $a = \mathbf{i}$ in Cl_2; then $[\![x,y]\!] = 0$ while $[\![ax,y]\!] = a$), hence the Clifford bracket behaves not always like a Hermitian form.

10.3. THE CLIFFORD QUADRANGLES

Let $E_0, \ldots, E_{m-1}$ and $\mathbb{A}$ be as above and put

$$\mathcal{P} = \{(x,y) \in \mathbf{R}^n \times \mathbf{R}^n \mid |x|^2 + |y|^2 = 1, |[\![x,y]\!]| = |x||y|\},$$

$$\mathcal{L} = \{(u,v) \in \mathbf{R}^n \times \mathbf{R}^n \mid |u| = |v| = 1/\sqrt{2}, [\![u,v]\!] = 0\}.$$

Put

$$\mathsf{S} = \mathsf{S}^m = \{(c,s) \mid c^2 + |s|^2 = 1\} \subseteq \mathbf{R} \oplus \mathbb{A}.$$

By Cauchy–Schwarz,

$$\mathcal{P} = \{(cw, sw) \in \mathbf{R}^n \times \mathbf{R}^n \mid w \in \mathbf{R}^n, |w| = 1, (c,s) \in \mathsf{S}\}$$

$$= \{(sw, cw) \in \mathbf{R}^n \times \mathbf{R}^n \mid w \in \mathbf{R}^n, |w| = 1, (c,s) \in \mathsf{S}\}.$$

Let $(cw, \bar{s}w) \in \mathcal{P}$ and $(u,v) \in \mathcal{L}$. Then

$$\begin{aligned}\langle (cw,\bar{s}w),(u,v)\rangle &= \langle cw,u\rangle + \langle \bar{s}w, v\rangle\\ &= \langle w, cu\rangle + \langle w, sv\rangle\\ &= \langle w, cu+sv\rangle\\ &\leqslant |w||cu+sv| = 1/\sqrt{2},\end{aligned}$$

because $[\![u,v]\!] = 0$ implies that $\langle cu, sv\rangle = 0$. We define the incidence by the Euclidean inner product as in 9.1 by requiring that $\langle (cx,sx),(u,v)\rangle = 1/\sqrt{2}$. Then $(cw,\bar{s}w)\,\mathtt{I}\,(u,v)$ if and only if $w = \sqrt{2}(cu+sv)$, i.e. if and only if

$$cw = 1/\sqrt{2}\left((1 + (c^2 - |s|^2))u + (2sc)v)\right)$$

and

$$\bar{s}w = 1/\sqrt{2}\left((2\bar{s}c)u + (1-(c^2-|s|^2))v\right).$$

Note that $(c^2-|s|^2, 2cs) \in \mathsf{S}$. The resulting incidence structure is denoted by $\mathrm{FKM}(E_1,\dots,E_{m-1}) = (\mathcal{P},\mathcal{L},\mathtt{I})$. If $m \not\equiv 0 \pmod 4$, then $E_1,\dots E_{m-1}$ is determined by the numbers m and n, and we put $\mathrm{FKM}(E_1,\dots,E_{m-1}) = \mathrm{FKM}(m,n)$; for $m \equiv 0 \pmod 4$ we put $\mathrm{FKM}(E_1,\dots,E_{m-1}) = \mathrm{FKM}(m,n,\iota)$, where ι is the index of the representation, cp. 9.4.

We imbed $\mathbf{R}\oplus\mathbb{A}$ into $\mathrm{End}_{\mathbf{R}}(\mathbf{R}^{2n})$ by the map

$$B_{t,a}\colon (x,y) \mapsto (tx+ay, \bar{a}x - ty).$$

These are symmetric orthogonal operators, and $B_{t,a}^2 = (t^2+|a|^2)\mathrm{id}$; thus, we have a representation of the Clifford algebra $\mathrm{Cl}_{m+1,0}$ generated by the Euclidean vector space $\mathbf{R}\oplus\mathbb{A}$ on $\mathbf{R}^{2n}$. Let $K \subseteq \mathrm{O}(2n)$ denote the group generated by $B(\mathsf{S})$. To each point $p = (x,y) = (cw, sw) \in \mathcal{P}$ we assign the element

$$\phi(x,y) = (|x|^2-|y|^2, 2[\![y,x]\!]) = (c^2-|s|^2, 2c\bar{s}) \in \mathsf{S}.$$

For $(c,s) \in \mathsf{S}$ put $\mathcal{O}_{c,s} = \phi^{-1}(c,s)$. A short calculation shows that

$$\mathcal{O}_{c,s} = \{p \in \mathbf{R}^{2n} \mid |p|^2 = 1, B_{c,s}(p) = p\}$$

is the set of all unit vectors p fixed by $B_{c,s}$ (note that $B_{c,s}$ is symmetric with eigenvalues ± 1, both of multiplicity n).

LEMMA 10.4. *We have a partition of $\mathcal{P}$ into disjoint sets*

$$\begin{aligned}\mathcal{P} &= \bigsqcup\{\mathcal{O}_{c,s} \mid (c,s) \in \mathsf{S}\}\\ &= \{p \in \mathbb{S}^{2n-1} \subseteq \mathbf{R}^{2n} \mid Q(p) = p \text{ for some } Q \in B(\mathsf{S})\}.\end{aligned}$$ □

LEMMA 10.5. *The group $K = \langle Q \mid Q \in B(\mathsf{S})\rangle$ acts as a group of automorphisms on* $\mathrm{FKM}(E_1,\dots,E_{m-1})$.

Proof. Let $Q \in B(\mathsf{S})$. For $p \in \mathcal{P}$ and $P \in B(\mathsf{S})$ with $P(p) = p$ we have $QPQ \in B(\mathsf{S})$ and $Q(p) = QPQ(Q(p))$, so $Q(p) \in \mathcal{P}$ by 10.4. Let $(u,v) \in \mathcal{L}$ be a line and put $Q = B_{c,s}$. Then $Q(u,v) = (cu+sv, \bar{s}u - cv)$, and

$$[\![cu+sv, \bar{s}u-cv]\!] = cs|u|^2 - c[\![u,v]\!]c + s[\![u,v]\!]s - cs|v|^2 = 0.$$

Also, $|cu+sv|^2 = \frac{1}{2} = |\bar{s}u - cv|^2$, so $Q(\mathcal{L}) = \mathcal{L}$. Finally, $Q \in \mathrm{O}(2n)$, and the incidence is defined by the Euclidean inner product, so $Q \in \mathrm{Aut}(\mathrm{FKM}(E_1,\dots,E_{m-1}))$. □

10.6. AUTOMORPHISMS INDUCED BY THE CLIFFORD ALGEBRA

The group K acts also on $B(\mathbf{R} \oplus \mathbb{A}) \subseteq \mathrm{End}_{\mathbf{R}}(\mathbf{R}^{2n})$: let $P \in B(\mathbf{R} \oplus \mathbb{A})$ and $Q \in B(\mathsf{S})$, and put $Q\colon P \mapsto QPQ$. Up to the sign, this is a reflection, and thus the image of K contains $\mathrm{SO}(m+1)$; identifying $\mathbf{R} \oplus \mathbb{A}$ with $B(\mathbf{R} \oplus \mathbb{A})$, we obtain an orthogonal action of K on $\mathbf{R} \oplus \mathbb{A}$. Put $B_p = B_{\phi(p)}$. Then $B_p(p) = p$, and the map $\phi\colon \mathcal{P} \to \mathsf{S}$ is K-equivariant, $QB_pQ = B_{Q(p)}$. Thus, the group K acts transitively on the collection $\{\mathcal{O}_{c,s} \mid (c,s) \in \mathsf{S}\}$. The subgroup $K^0 = K \cap \mathrm{Cl}^0_{m+1,0}$ is isomorphic to $\mathrm{Spin}(m+1)$ and maps onto $\mathrm{SO}(m+1)$, with kernel ± 1; it is generated by the set $B_{1,0} \cdot \mathsf{S}$. Note that K^0 still acts transitively on S.

10.7. STRUCTURE OF POINT ROWS

Let $\ell = (u, v) \in \mathcal{L}$ and $P \in B(\mathsf{S})$. Then $p = (1/\sqrt{2})(1+P)\ell \in \mathcal{P}$ is incident with ℓ, and conversely every point incident with ℓ is obtained in this way. Therefore the point row corresponding to ℓ is the m-sphere

$$\left\{ \frac{1}{\sqrt{2}}(1+P)\ell \mid P \in B(\mathsf{S}) \right\} \cong \mathbb{S}^m.$$

Now let h, ℓ be distinct lines. Then h, ℓ are confluent if and only if there exists a $P \in B(\mathsf{S})$ such that $(1+P)\ell = (1+P)h$. This is equivalent with the condition $(-P)(\ell - h) = \ell - h$, cp. Thorbergsson [20], or with $\mathcal{P} \cap (\ell - h)\mathbf{R} \neq \emptyset$, i.e. with the condition $(\ell - h)/|\ell - h| \in \mathcal{P}$. (Note however that $(\ell - h)/|\ell - h|$ need not be a point incident with the lines h, ℓ).

10.8. STRUCTURE OF LINE PENCILS

Let $p \in \mathcal{P}$. Since K acts transitively on S, there is no loss of generality in assuming that $\phi(p) = (1, 0)$, i.e. that $p = (w, 0)$ for some unit vector $w \in \mathbf{R}^n$. Then every line through p is of the form (u, v), with $u = (1/\sqrt{2})w$. Therefore, the line pencil through p is the $(n-m-1)$-sphere

$$\{(u,v) \mid |v|^2 = \tfrac{1}{2},\ [\![u,v]\!] = 0\} \cong \mathbb{S}^{n-m-1}.$$

Thus, every line pencil is an m'-sphere, $m' = n - m - 1$.

Now we want a criterion for collinearity of points $p, q \in \mathcal{P}$, $p \neq q$. Put $P = B_p$, $Q = B_q$. If p, q are incident with ℓ, then $p = 1/\sqrt{2}(1+P)\ell$ and $q = 1/\sqrt{2}(1+Q)\ell$, whence $p - q = (1/\sqrt{2})(P-Q)\ell$. Note that this relation implies

$$|p - q| = \frac{1}{\sqrt{2}}|P - Q| \quad \text{and} \quad \frac{p-q}{|p-q|} \in \mathcal{L}.$$

The first equality follows by taking absolute values; the second one follows from $\ell = \sqrt{2}R(p-q)/|p-q|$, where $R = (P-Q)/|P-Q| \in B(\mathsf{S})$. Conversely, if

$p-q = 1/\sqrt{2}(P-Q)\ell$ for some $\ell \in \mathcal{L}$, then its easy to see that $p = 1/\sqrt{2}(1+P)\ell$ and $q = 1/\sqrt{2}(1+Q)\ell$. The next result was proved first by Thorbergsson [20] by completely different (topological) methods; cp. also Eschenburg–Schröder [4].

THEOREM 10.9. *If* $m, n-m-1 > 0$, *then the geometry* $\mathrm{FKM}(E_1, \ldots, E_{m-1})$ *is a generalized quadrangle.*

Proof. We have to show the following. If $p \in \mathcal{P}$ and $\ell \in \mathcal{L}$, $p \not I \ell$, then there exists a unique point q which is collinear with p and incident with ℓ.

Since $K \subseteq \mathrm{Aut}(\mathrm{FKM}(E_1, \ldots, E_{m-1}))$ acts transitively on the sphere S, there is no loss of generality in assuming that $p \in \mathcal{O}_{1,0}$, i.e. that $p = (w, 0)$ for some unit vector $w \in \mathbf{R}^n$. Let $\ell = (u, v)$. The non-incidence of p, ℓ implies that $w \neq \sqrt{2}u$. A typical point q incident with ℓ is of the form

$$q = 1/\sqrt{2}((1+c)u + sv, \bar{s}u + (1-c)v).$$

If p, q are collinear, then $(q-p)\mathbf{R} \cap \mathcal{L} \neq \varnothing$, thus

$$\begin{aligned} 0 &= [\![(1+c)u + sv - \sqrt{2}w, \bar{s}u + (1-c)v]\!] \\ &= (1+c)s/2 + s(1-c)/2 - \sqrt{2}[\![w, \bar{s}u + (1-c)v]\!] \\ &= s - \sqrt{2}[\![w, \bar{s}u + (1-c)v]\!]. \end{aligned}$$

One solution of this equation is $(c, s) = (1, 0)$, but this is not allowed, since $\sqrt{2}u \neq w$. Consider the $\mathbf{R}$-linear map $\mathbf{R} \oplus \mathbb{A} \to \mathbb{A}$, $(c, s) \mapsto s - \sqrt{2}[\![w, \bar{s}u - cv]\!]$. The kernel L of this map has at least dimension 1. Restrict the map to the hyperplane $0 \oplus \mathbb{A}$. By Cauchy–Schwarz, $|\sqrt{2}[\![w, \bar{s}u]\!]| \leqslant |s|$, and $s = \sqrt{2}[\![w, \bar{s}u]\!]$ implies that $w = \sqrt{2}u$, for some $s \in \mathbb{A}$, $|s| = 1$. But this possibility was excluded. Thus, $L \cap (0 \oplus \mathbb{A}) = 0$. This implies that $\dim L = 1$, and that the inhomogeneous equation has precisely one other solution (c, s) on S besides $(1, 0)$. So we have established the uniqueness of q, and it remains to show that q is collinear with p.

There exists an element $g \in K$ such that $g(1, 0) = (1, 0)$ and $g(c, s) \in \mathbf{R} \oplus \mathbf{R} \subseteq \mathbf{R} \oplus \mathbb{A}$. Therefore we may assume that c and s are real numbers. Then the solution of the equation above takes the simpler form $s/\sqrt{2} = s[\![w, u]\!] + (1-c)[\![w, v]\!]$; since the left-hand side is a real number, this yields $s/\sqrt{2} = s\langle w, u\rangle + (1-c)\langle w, v\rangle$; since $s^2 = (1+c)(1-c)$, multiplication with $s/(1-c)$ yields

$$(1+c)/\sqrt{2} = (1+c)\langle w, u\rangle + s\langle w, v\rangle.$$

To show that $p - q$ is a scalar multiple of a vector in $\mathcal{L}$, it remains to check that

$$|1/\sqrt{2}((1+c)u + sv) - w|^2 = |1/\sqrt{2}(su + (1-c)v)|^2.$$

The right-hand side is $(2-2c)/4 = (1-c)/2$, and the left-hand side is

$$(2+2c)/4 + 1 - \sqrt{2}\langle(1+c)u + sv, w\rangle$$

$$= (3+c)/2 - \sqrt{2}((1+c)\langle w, u\rangle + s\langle w, v\rangle)$$
$$= (3+c)/2 - (1+c)$$
$$= (1-c)/2,$$

so $p-q$ is indeed a scalar multiple of an element of $\mathcal{L}$.

The last equation which we have to check is $\sqrt{2}|p-q| = |P-Q|$, or, equivalently, $2\langle p, q\rangle = 1 + \langle P, Q\rangle$. Now

$$\langle p, q\rangle = 1/\sqrt{2}\langle w, (1+c)u + sv\rangle = (1+c)/2,$$

and $\langle P, Q\rangle = c$. This shows that the geometry is indeed a quadrangle. □

COROLLARY 10.10. *Two distinct points $p, q \in \mathcal{P}$ are collinear if and only if*

$$\frac{p-q}{|p-q|} \in \mathcal{L}.$$

Proof. Let ℓ be a line incident with q, and not incident with p. In the first part of the proof of the theorem, we have seen that there is a unique point r incident with ℓ, such that $(p-r)/|p-r| \in \mathcal{L}$. Thus, $q = r$. In the second part of the proof we showed that this point is collinear with p. □

There is an isomorphism $\mathrm{FKM}(m, n, \iota) \cong \mathrm{FKM}(m, n, -\iota)$. Note also that

$$Q_n(\mathbf{R}) \cong \mathrm{FKM}(1, n-1),$$
$$H_n(\mathbf{C}) \cong \mathrm{FKM}(2, 2(n-1)),$$
$$H_n(\mathbf{H}) \cong \mathrm{FKM}(4, 4(n-1), n-1).$$

We derive some more properties of these quadrangles.

PROPOSITION 10.11. *The sets $\mathcal{O}_{c,s}$, $(c, s) \in \mathsf{S}$, are ovoids. Therefore, the point space of each Clifford quadrangle can be partitioned into ovoids.*

Proof. We showed already that every line ℓ meets $\mathcal{O}_{c,s}$ in the unique point $1/\sqrt{2}(1 + B_{c,s})\ell$. □

There is also a generalizations of the J-spreads, which was pointed out to me by Stephan Stolz. Suppose that J is an orthogonal skew symmetric map which anti-commutes with $E_1, \ldots, E_{m-1}$ (i.e. $\{E_1, \ldots, E_{m-1}, J\}$ is a Clifford system). Then $\langle E_i Jx, Jy\rangle = -\langle E_i x, y\rangle$ for $i = 1, \ldots, m-1$, while $\langle E_0 Jx, Jy\rangle = \langle x, y\rangle$. Therefore $[\![Jx, Jy]\!] = \overline{[\![x, y]\!]} = [\![y, x]\!]$. Similarly, one shows that $[\![Jx, x]\!] = [\![x, Jx]\!] = 0$ and that $aJ = J\bar{a}$ for $a \in \mathbb{A}$. Consider the orthogonal involution $(x, y) \mapsto (Jy, -Jx)$ on $\mathbf{R}^{2n}$. By the relations above, it preserves $[\![$-,-$]\!]$-orthogonality and permutes the line space $\mathcal{L}$ and the point space $\mathcal{P}$. If it fixes the vector (x, y), then $(x, y) = (x, -Jx)$. Note that $(x, -Jx) \in \mathcal{L}$ for $|x|^2 = \frac{1}{2}$. Let

$\mathcal{S} \subseteq \mathcal{L}$ denote the set of all fixed lines. We claim that this is a spread. Consider the map

$$(x, y) \mapsto 1/\sqrt{2}(x + Jy, -Jx + y)$$

and let σ denote its restriction to $\mathcal{P}$. A typical point incident with the line $(u, -Ju)$ is of the form

$$\begin{aligned}
&1/\sqrt{2}\left(1 + B_{c,s}\right)(u, -Ju)\\
&\quad = 1/\sqrt{2}\,((1 + c)u - sJu, \bar{s}u - (1 - c)Ju)\\
&\quad = 1/\sqrt{2}\,((1 + c - J\bar{s})u, (\bar{s} - J(1 - c))u)\,;
\end{aligned}$$

under σ, it is mapped to

$$\begin{aligned}
&\sigma(1/\sqrt{2}((1 + c - J\bar{s})u, (\bar{s} - J(1 - c))u))\\
&\quad = \tfrac{1}{2}((1 + c - J\bar{s} + J(\bar{s} - J(1 - c)))u,\\
&(-J(1 + c - J\bar{s}) + \bar{s} - J(1 - c))u)\\
&\quad = \tfrac{1}{2}((1 + c - J\bar{s} + J\bar{s} + 1 - c)u,\\
&(-J - Jc - \bar{s} + \bar{s} - J + Jc)u)\\
&\quad = (u, -Ju).
\end{aligned}$$

This shows that $\mathcal{S}$ is indeed a spread.

PROPOSITION 10.12. *The set $\mathcal{S}$ of fixed lines of the involution $(x, y) \mapsto (Jy, -Jx)$ is a spread.* □

Let $\{E_1, \ldots, E_{m-1}, E_m = J\}$ be a Clifford system on V. Restricting to the Clifford system $E_1, \ldots, E_{m-1}$, we can view V as a Cl_{m-1}-module. From the classification of Clifford modules 9.4 one sees the following. If $m \equiv 1, 2 \pmod 8$, then V is given by $a\mu_m$, where a is even, and conversely, all such Clifford systems are obtained as restrictions of larger Clifford systems. If $m \equiv 3, 5, 6, 7 \pmod 8$, then every Cl_{m-1}-module is obtained by restriction. If $m \equiv 0 \pmod 4$, then the module structure on V is obtained by restriction if and only if the index of V is $\iota = 0$.

COROLLARY 10.13. *The Clifford quadrangles do have J-spreads in the following cases*:

$$\begin{aligned}
&m \equiv 1 \pmod 8 \ \text{ and } \ n = (even) \cdot 2^{(m-1)/2},\\
&m \equiv 2 \pmod 8 \ \text{ and } \ n = (even) \cdot 2^{m/2},\\
&m \equiv 3, 5, 6, 7 \pmod 8,\\
&m \equiv 0 \pmod 4 \ \text{ and } \ \iota = 0.
\end{aligned}$$

□

11. Topological Properties of the Clifford Quadrangles

From now on we assume again that $\mathbf{R} = \mathbb{R}$ it the field of real numbers. It is clear from the definition that the Clifford quadrangles are compact connected $(m, n - m - 1)$-quadrangles, since the flag space is closed in $\mathcal{P} \times \mathcal{L}$. The ovoids $\mathcal{O}_{c,s}$ and the J-spreads constructed in the last section are obviously closed. The flag space can be identified with an isoparametric hypersurface in the sphere as follows.

11.1. THE ISOPARAMETRIC FUNCTION

Put $H(x, y) = (|x|^2 - |y|^2)^2 + 4|[\![x, y]\!]|^2$. This is a homogeneous polynomial of degree 4 on $\mathbb{R}^{2n}$, and

$$H^{-1}(0) \cap \mathbb{S}^{2n-1} = \mathcal{L},$$

$$H^{-1}(1) \cap \mathbb{S}^{2n-1} = \mathcal{P}.$$

The homogeneous polynomial $F(x, y) = (|x|^2 + |y|^2)^2 - 2H(x, y)$ is *isoparametric*, cf. Ferus–Karcher–Münzner [5] 4.1; the isoparametric hypersurface $F^{-1}(0) \cap \mathbb{S}^{2n-1}$ can be canonically identified with the flag space $\mathcal{F}$ by mapping the flag $((cx, sx), (u, v))$ to $1/\sqrt{2 + \sqrt{2}}(cx + u, sx + v) \in \mathbb{S}^{2n-1}$. This is essentially the description due to Ferus–Karcher–Münzner [5] Abschnitt 4. In their notation,

$$M_- = \mathcal{P}, \quad M_+ = \mathcal{L}, \quad \text{and} \quad M = \mathcal{F}.$$

The following (anti) isomorphisms were also proved by Ferus–Karcher–Münzner.

$$\mathrm{FKM}(1, 4) \cong Q_5(\mathbb{R}) \cong \mathrm{FKM}(2, 4)^{\mathrm{dual}} \cong H_3(\mathbb{C}),$$

$$\mathrm{FKM}(1, 8) \cong Q(\mathbb{R}, 9) \cong \mathrm{FKM}(6, 8)^{\mathrm{dual}},$$

$$\mathrm{FKM}(2, 8) \cong H_5(\mathbb{C}) \cong \mathrm{FKM}(5, 8)^{\mathrm{dual}},$$

$$\mathrm{FKM}(3, 8) \cong \mathrm{FKM}(4, 8, 0)^{\mathrm{dual}},$$

$$\mathrm{FKM}(9, 16) \cong Q(E_6, \mathbb{R}).$$

The isomorphisms with the Moufang quadrangles can be seen as follows. Ferus–Karcher–Münzner show that the corresponding isoparametric hypersurfaces are homogeneous; by Hsiang–Lawson [7] (or by representation theory), this implies that the geometry arises from the isotropy representation of a noncompact Riemannian symmetric space of rank 2. By general theory, the isoparametric foliation can be identified with the building of the corresponding noncompact simple Lie group, cp. Thorbergsson [19], Kramer [12]. In addition, we saw that there are isomorphisms

$$Q_n(\mathbb{R}) \cong \mathrm{FKM}(1, n - 1),$$

$$H_n(\mathbb{C}) \cong \mathrm{FKM}(2, 2(n - 1)),$$

$$H_n(\mathbb{H}) \cong \mathrm{FKM}(4, 4(n - 1), n - 1).$$

11.2. The map $\phi\colon \mathcal{P} \to \mathsf{S}$ is closely related to to the real K-theory of spheres. First, we claim that this is a locally trivial sphere bundle. For $-1 < c \leqslant 1$, consider the trivialization

$$(c, s, w) \mapsto 1/\sqrt{2(1+c)}((1+c)w, \bar{s}w),$$

and for $-1 \leqslant c < 1$ the trivialization

$$(c, s, w) \mapsto 1/\sqrt{2(1-c)}(sw, (1-c)w),$$

cp. Ferus–Karcher–Münzner [5] 4.2.i. The clutching map along the equator $c = 0$ is given by by $w \mapsto sw$; the structure group is therefore $\mathrm{Spin}_m\mathbb{R}$, cf. Husemoller [8] Chapter 10. The associated vector bundle ξ can also be described as a difference bundle. Let $X = \{(c, s) \in \mathsf{S} \mid 0 \leqslant c \leqslant 1\}$ and $A = \{(c, s) \in \mathsf{S} \mid c = 0\}$ denote the northern hemisphere and the equator of S, respectively. The sequence of (trivial) vector bundles over X

$$0 \to \underline{\mathbb{R}^n} \xrightarrow{s} \underline{\mathbb{R}^n} \to 0,$$

becomes exact when restricted to A and represents a vector bundle over $X/A \simeq \mathsf{S}$ isomorphic to ξ, cf. Section 8,9 in Atiyah–Bott–Shapiro [2]; cp. also Wang [21] Proposition 1 for a different proof. The upshot is that we obtain a map $\mu \mapsto \xi(\mu)$ from the collection of all finite-dimensional representations of the Clifford algebra Cl_{m-1} to vector bundles over S^m which is additive in the sense that $\xi(\mu + \mu') \cong \xi(\mu) \oplus \xi(\mu')$. One main result of [2] is that these vector bundles represent the real K-theory of S^m; see *loc. cit.* for a more precise statement.

We use the bundle $\phi\colon \mathcal{P} \to \mathsf{S}$ to obtain nonexistence result for spreads. This settles also the case of the Hermitian quadrangles $H_n(\mathbb{H})$, where $n \equiv 23 \pmod{24}$. Consider the $(n-1)$-sphere bundle

$$\begin{array}{ccc} \mathcal{O}_{c,s} & \longrightarrow & \mathcal{P} \\ & & \big\downarrow \phi \\ & & \mathsf{S}. \end{array}$$

Suppose that $\hat{\sigma}\colon \mathcal{P} \to \mathcal{F}$ is a section of $\mathrm{pr}_{\mathcal{P}}$. Let $\sigma = \mathrm{pr}_{\mathcal{L}}\hat{\sigma}\colon \mathcal{P} \to \mathcal{L}$ be the corresponding map, and put $f = o\sigma$, where $o\colon \mathcal{L} \to \mathcal{P}$ is the map corresponding to the ovoid $\mathcal{O}_{1,0} = \mathcal{P} \cap \mathbb{R}^n \oplus 0$. Then $\sigma|_{\mathcal{O}_{(c,s)}}$ is an injection, and so is the composite $o\sigma|_{\mathcal{O}_{(c,s)}}\colon \mathcal{O}_{(c,s)} \to \mathcal{O}_{1,0}$. Therefore, the map $\mathcal{P} \to \mathsf{S} \times \mathcal{O}_{1,0}$, $p \mapsto (\phi(p), f(p))$ is a homeomorphism and a bundle isomorphism,

$$\begin{array}{ccc} \mathcal{P} & \xrightarrow{(\phi, f)} & \mathsf{S} \times \mathcal{O} \\ & {\scriptstyle\phi}\searrow \quad \swarrow{\scriptstyle \mathrm{pr}_1} & \\ & \mathsf{S}. & \end{array}$$

We have proved the following.

LEMMA 11.3. *If* $\mathrm{pr}_{\mathcal{P}}: \mathcal{F} \to \mathcal{P}$ *has a section* (*in particular, if the quadrangle has a spread*), *then* $\phi: \mathcal{P} \to \mathsf{S}$ *is topologically trivial*. □

If σ is a sufficiently 'nice' map, e.g., arises from a J-spread as above, then the map (ϕ, f) above induces in fact a vector bundle isomorphism. A special case of this lemma is proved in Ferus–Karcher–Münzner [5] 4.2.i.

Let X be a compact connected Hausdorff space. Two vector bundles ξ_1, ξ_2 over a space X are called *stably equivalent* if they become isomorphic after adding a suitable number of trivial line bundles, i.e. if $\xi_1 \oplus r_1\underline{\mathbb{R}} \cong \xi \oplus r_2\underline{\mathbb{R}}$, for certain numbers r_1, r_2. A bundle which is stably isomorphic to the trivial bundle is called *stably trivial*. The reduced K-theory $\widetilde{\mathrm{KO}}(X)$ can be identified with the set of vector bundles over X modulo stable equivalence, by the map $\xi \mapsto [\xi - \mathrm{rk}(\xi)\underline{\mathbb{R}}]$.

A *fibre homotopy* between two bundles $E, E' \to X$ is a homotopy $F_t: E \to E'$ which commutes with the projection maps. A bundle map $f: E \to E'$ is a *fibre homotopy equivalence* if there exists a bundle map $g: E \to E'$ such that fg and gf are fibre homotopic to the identity map on E, E'. Clearly, two (topologically) isomorphic bundles are fibre homotopy equivalent; the converse need not be true. Consider the subgroup of $\mathrm{KO}(X)$ consisting of all differences $\xi - \eta$, where $S(\xi)$ and $S(\eta)$ fibre homotopy equivalent. The resulting quotient of the group $\mathrm{KO}(X)$ is the Abelian group $\mathrm{J}(X)$. This subgroup is in fact contained in the reduced K-theory $\widetilde{\mathrm{KO}}(X)$; the corresponding quotient is the group $\widetilde{\mathrm{J}}(X)$ (this is the notation of Adams [1]). Its elements may be viewed as vector bundles over X, modulo *stable fibre homotopy equivalence*. Thus, two vector bundles ξ, η are equivalent in $\widetilde{\mathrm{J}}(X)$ if and only if the sphere bundles $S(\xi_1 \oplus r_1\underline{\mathbb{R}})$ and $S(\xi_2 \oplus r_2\underline{\mathbb{R}})$ are fibre homotopy equivalent, for some numbers $r_1, r_2 \in \mathbb{N}$. Note that a vector bundle whose sphere bundle is topologically trivial is certainly trivial in $\widetilde{\mathrm{J}}(X)$.

LEMMA 11.4. *Let* $m \equiv 1, 2 \pmod 8$. *The vector bundle* $\xi(a\mu_m)$ *is trivial in* $\widetilde{\mathrm{J}}(\mathsf{S})$ *if and only if* a *is even*.

Proof. By Bott periodicity, $\widetilde{\mathrm{KO}}(\mathsf{S}^m) \cong \mathbb{Z}/2$, and $\xi = \xi(\mu_m)$ is a generator for this group. Therefore, $a\xi$ is stably trivial if and only if a is even (and in that case ξ is trivial by the construction of the spread above). For $m \equiv 1, 2 \pmod 8$, the map $\widetilde{\mathrm{KO}}(\mathsf{S}) \to \widetilde{\mathrm{J}}(\mathsf{S})$ is a group isomorphism by Adams [1], Part II, Ex. 6.4. Thus, $a\xi$ is stably fibre homotopically trivial if and only if a is even; in particular, $S(a\xi)$ is not topologically trivial if a is odd. But the sphere bundle of $a\xi$ is precisely $\mathcal{P} \to \mathsf{S}$. □

For $m \equiv 0 \pmod 4$ we cannot use this method, since there the map $\widetilde{\mathrm{KO}}(\mathsf{S}) \to \widetilde{\mathrm{J}}(\mathsf{S})$ is not an isomorphism; eg. $\widetilde{\mathrm{J}}(\mathbb{S}^4) \cong \mathbb{Z}/24 \neq \widetilde{\mathrm{KO}}(\mathbb{S}^4) \cong \mathbb{Z}$. Instead, we prove that the underlying disk bundle of ξ_m^+ is not stably trivial in the group STOP of orientation preserving based homeomorphisms by showing that the total rational Pontryagin class of ξ is not trivial. This is due to Wang [21] Corollary 1,2.

LEMMA 11.5. *Suppose that* $m \equiv 0 \pmod 4$. *Then the total rational Pontrjagin class of* $\xi(\mu_m^+)$ *is not trivial. Thus, the bundle* $\phi\colon \mathcal{P} \to \mathsf{S}$ *is topologically trivial if and only if* $\iota = 0$.

Proof. The bundle $\xi^\pm = \xi(\mu_m^\pm)$ is (together with the trivial bundle) a generator of $\mathrm{KO}(\mathsf{S})$. Consider the ring homomorphism

$$\mathrm{KO}(\mathsf{S}) \xrightarrow{cplx} \mathrm{K}(\mathsf{S}) \xrightarrow{\mathrm{ch}} \mathrm{H}^\bullet(\mathsf{S}) \hookrightarrow \mathrm{H}^\bullet(\mathsf{S};\mathbb{Q}),$$

where $cplx(\beta) = \beta \otimes_{\mathbb{R}} \mathbb{C}$ and where ch is the Chern character. The complexification $cplx$ is a monomorphism, and the Chern character is an isomorphism, given in this special case by

$$\mathrm{ch}(\gamma) = \mathrm{rk}\,\gamma + (-1)^{(m/2)}(m/2-1)!\mathrm{c}_{m/2}(\gamma),$$

where c_k is the kth Chern class. In fact,

$$\mathrm{ch} \circ cplx(\mathrm{KO}(\mathsf{S})) = \mathrm{H}^0(\mathsf{S}) \oplus d\mathrm{H}^m(\mathsf{S}),$$

where $d = 1$ for $m \equiv 0 \pmod 8$ and $d = 2$ for $m \equiv 4 \pmod 8$, cp. Hirzebruch [6] 1.4–1.6. Now there is the relation

$$\mathrm{p}_k(\beta) = (-1)^k \mathrm{c}_{2k}(cplx(\beta))$$

between the Chern classes and the Pontryagin classes, cf. Milnor and Stasheff [17] p. 174. This shows that the rational Pontryagin class $\mathrm{p}_{m/4}(\xi^\pm)$ is not trivial. Moreover, $\xi^+ \oplus \xi^-$ is trivial, whence $\mathrm{p}_{m/4}(a\xi^+ + b\xi^-) = (a-b)\mathrm{p}_{m/4}(\xi^+) \neq 0$ for $\iota = a - b \neq 0$. Consider the classifying map

$$\mathsf{S}^m \to \mathrm{BSO} \to \mathrm{BSTOP}.$$

The homotopy fibre STOP/SO has finite homotopy groups in every dimension, cp. Milnor–Stasheff [17] p. 250–251, Kirby–Siebenmann [11] p. 246, hence the map $\mathrm{BSO} \to \mathrm{BSTOP}$ induces an isomorphism in rational cohomology, i.e. the rational Pontryagin classes are topologically invariant. It follows that $\xi(a\mu_m^+ + b\mu_m^-)$ is not stably trivial in STOP for $\iota = a - b \neq 0$. On the other hand, if the bundle $\phi\colon\mathcal{P} \to \mathsf{S}$ is topologically trivial, then its underlying disk bundle also is trivial, hence the claim follows. □

THEOREM 11.6. *The map* $\mathrm{pr}_{\mathcal{P}}\colon \mathcal{F} \to \mathcal{P}$ *admits a section only for the following values of* (m, m') *and* ι.

(i) $m \equiv 1 \pmod 8$, $n = (even) \cdot 2^{(m-1)/2}$.
(ii) $m \equiv 2 \pmod 8$, $n = (even) \cdot 2^{m/2}$.
(iii) $m \equiv 3, 5, 6, 7 \pmod 8$.

(iv) $m \equiv 0 \pmod 4$ and $\iota = 0$.

By 10.13*, theses are exactly the parameters for which the Clifford quadrangles have closed spreads; for other values of* m, m', ι*, no closed spreads exist.* □

COROLLARY 11.7. *The Moufang quadrangles* $H_n(\mathbb{H})$*,* $n \geqslant 3$*, and* $Q(E_6, \mathbb{R})$ *do not have closed spreads.* □

LEMMA 11.8. *If the* $\mathbb{S}^{n-m-1}$*-bundle* $V_2([\![-,-]\!]) \to \mathbb{S}^{n-1}$ *is topologically trivial, then* $n = 2, 4, 8$.

Proof. If this bundle is trivial, then the tangent sphere bundle $V_2(\mathbb{R}^n)$ of $\mathbb{S}^{n-1}$ is trivial. This holds only for $n = 2, 4, 8$, cp. Part II [15] 6.6. □

PROPOSITION 11.9. *If* $n - 1 \neq 3, 7$*, then the bundle* $\mathcal{F} \to \mathcal{P}$ *is not trivial. In particular, no Clifford quadrangle is continuously dual to another one, except possibly if* $n - 1 = 3, 7$.

Proof. This follows from the previous lemma. Consider the subbundle

$$\begin{array}{ccc} \mathcal{F}_{\mathcal{O}_{1,0}} & \hookrightarrow & \mathcal{F} \\ \downarrow & & \downarrow \\ \mathcal{O}_{1,0} & \hookrightarrow & \mathcal{P}. \end{array}$$

If $\mathcal{F} \to \mathcal{P}$ is topologically trivial, then the same is true for the restriction $\mathcal{F}_{\mathcal{O}_{1,0}} \to \mathcal{O}_{1,0}$. This bundle can be identified with the bundle projection of the Clifford–Stiefel manifold as in the proof of Part II [15] 6.7. □

If $n - 1 = 3, 7$, then $m = 1, 2, 3, 4, 5, 6$. The quadrangle FKM(4, 8, 0) has J-spreads, whereas FKM(4, 8, 2) has no closed spreads; therefore, $\mathrm{FKM}(3, 8)^{\mathrm{dual}} \not\cong$ FKM(4, 8, 2). Thus, there are no (continuous) coincidences between Clifford quadrangles besides the ones mentioned in 11.1.

Remarks 11.10. We would like to point out some consequences for isoparametric hypersurfaces. Let $\mathcal{F}_1, \mathcal{F}_2$ be isoparametric hypersurfaces, and let $\mathcal{P}_i \leftarrow \mathcal{F}_i \to \mathcal{L}_i$ be the canonical maps onto the focal manifolds $\mathcal{P}_i, \mathcal{L}_i$, for $i = 1, 2$. Call $(\mathcal{P}_1, \mathcal{L}_1, \mathcal{F}_1)$ and $(\mathcal{P}_2, \mathcal{L}_2, \mathcal{F}_2)$ *topologically equivalent* if there exist homeomorphisms $f_\mathcal{P}, f_\mathcal{L}, f_\mathcal{F}$ which make the following diagram commute

$$\begin{array}{ccccc} \mathcal{P}_1 & \longleftarrow & \mathcal{F}_1 & \longrightarrow & \mathcal{L}_1 \\ \downarrow{\scriptstyle f_\mathcal{P}} & & \downarrow{\scriptstyle f_\mathcal{F}} & & \downarrow{\scriptstyle f_\mathcal{L}} \\ \mathcal{P}_2 & \longleftarrow & \mathcal{F}_2 & \longrightarrow & \mathcal{L}_2. \end{array}$$

Thus, if two isoparametric foliations are congruent by a rigid motion, then they are topologically equivalent in our sense. By Bödi–Kramer [3] 4.7, the vertical arrows in the diagram above are in fact automatically smooth.

In the terminology of Ferus–Karcher–Münzner, $(\mathcal{P}, \mathcal{L}, \mathcal{F}) = (M_-, M_+, M)$. Theorem 11.6 says under which conditions the map $M \to M_-$ (or, equivalently, the normal sphere bundle of M_-) admits a section. Theorem 11.9 says that, except for some low dimensional cases, $M \to M_-$ is never (topologically) a product bundle. In particular, no two distinct Clifford hypersurfaces are topologically equivalent, except for the low dimensional cases indicated in 11.9. This generalizes differential geometric results of Ferus–Karcher–Münzner [5]. Corresponding results for homogeneous isoparametric hypersurfaces with 4 distinct principal curvatures are contained in Part II [15] Table 1. If the number of principal curvatures is 3 or 6, then the normal sphere bundles of the focal manifolds admit no sections; this is proved in Kramer [13].

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Corrections to: Compact Ovoids in Quadrangles I–III

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The following mistakes in [1] and [2] were pointed out by M. Wolfrom and N. Rosehr.

In our Definition 1.1 [1] of a generalized quadrangle, we should, in addition to the axioms (GQ_1) and (GQ_2), require that no digons exist, i.e. two lines which have two points in common coincide. The same condition should be added to our definition of a weak generalized quadrangle.

In Theorem 4.5, the inequalities are stated the wrong way. The correct statement for Theorem 4.5 is thus as follows.

THEOREM 4.5. *Let $\mathfrak{Q}$ be a compact connected finite-dimensional quadrangle with parameters (m, m'). If $m \geqslant m'$, then $\mathfrak{Q}$ is line-minimal, and if $m \leqslant m'$ then $\mathfrak{Q}$ is point-minimal. If $m = m'$, then $\mathfrak{Q}$ has no full or ideal closed subquadrangles.*

This is exactly what we prove.

In [2] p. 336, the correct formula for the Chern character is

$$ch(\gamma) = \mathrm{rk}\gamma + \frac{(-1)^{(m/2)}}{(m/2-1)!}\mathrm{c}_{m/2}(\gamma),$$

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