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Compact Ovoids in Quadrangles II: the Classical Quadrangles

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Abstract. In this second part we consider ovoids in the classical compact connected quadrangles. We solve the problem whether closed ovoids or spreads exist in these quadrangles. In fact we prove a slightly more general result: we determine whether the normal sphere bundles of the point- or line space admit sections, or whether they are topologically trivial. We also give explicit geometric constructions for spreads and ovoids. Some of these spreads are apparently new.

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6. Ovoids in Compact Moufang Quadrangles

By Burns–Spatzier [3] a compact connected quadrangle with a BN-pair in its topological automorphism group arises from a noncompact simple real Lie group of real rank 2 or from a simple complex Lie group of rank 2, so a complete list of all compact connected Moufang quadrangles, their parameters (which can be read off from the dimensions of the root groups) and their Tits diagrams (called there Satake diagrams) can be extracted from Table VI, Ch. X in Helgason [7] or from Table 9, p. 312 in Onichshik–Vinberg [18]; cp. Grundhöfer–Knarr [4].

We briefly review the classical compact quadrangles. Let $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ denote the reals, the complex numbers, or the quaternions, endowed with the usual topology. An *involution* on $\mathbb{F}$ is a continuous anti-automorphism σ with $\sigma^2 = \mathrm{id}$. If f is a non-singular symplectic or σ-hermitian form of Witt index 2 on an $\mathbb{F}$-vector space V of sufficently large (finite) dimension $n+1$, then the collection of all totally isotropic subspaces of V forms a compact quadrangle. Up to conjugation, the only continuous involutions are id (for $\mathbb{F} = \mathbb{R}, \mathbb{C}$), the 'standard' involution $x \mapsto \bar{x}$ on $\mathbb{C}$, $\mathbb{H}$, and the involution $\alpha: x \mapsto -i\bar{x}i$ on $\mathbb{H}$. The form f is determined up to a scalar factor by the involution σ, the given Witt index (namely 2), and the vector space dimension $\dim_{\mathbb{F}} V = n+1$. An α-hermitian form on $\mathbb{H}^{n+1}$ has automatically the maximal possible Witt index, therefore $n = 3, 4$ in this case. Thus, we obtain the following classical compact quadrangles.

- The real and complex symplectic quadrangles $W(\mathbb{R})$ and $W(\mathbb{C})$ in $\mathbb{R}^4$ and $\mathbb{C}^4$, with parameters $(1, 1)$ and $(2, 2)$, respectively.
- The real orthogonal quadrangles $Q_n(\mathbb{R})$ in $\mathbb{R}^{n+1}$, $n \geqslant 4$, with parameters $(1, n-3)$.
- The complex orthogonal quadrangle $Q_4(\mathbb{R})$ in $\mathbb{C}^5$, with parameters $(2, 2)$.
- The complex hermitian quadrangles $H_n(\mathbb{C})$ in $\mathbb{C}^{n+1}$, $n \geqslant 3$, with parameters $(2, 2n-5)$.
- The quaternion (standard) hermitian quadrangles $H_n(\mathbb{H})$ in $\mathbb{H}^{n+1}$, $n \geqslant 3$, with parameters $(4, 4n-9)$.
- The quaternion α-hermitian quadrangles $H_n^\alpha(\mathbb{H})$ in $\mathbb{H}^{n+1}$, for $n = 3, 4$, with parameters $(4, 1)$ and $(4, 5)$, respectively.

It is convenient to put also $H_n(\mathbb{R}) = Q_n(\mathbb{R})$. By the result of Burns–Spatzier [3] and the classification of the simple real and complex Lie groups (cp. Helgason [7], Onishchik–Vinberg [18]), these quadrangles together with the exceptional Moufang quadrangle $Q(E_6, \mathbb{R})$ belonging to the simple real Lie group $\mathrm{PE}_{6(-14)}$ form (up to dualtiy) a complete list of all compact connected Moufang quadrangles. There are the following anti-isomorphisms (and no others)

$$H_3(\mathbb{C}) \cong Q_5(\mathbb{R})^{\mathrm{dual}},$$
$$H_4^\alpha(\mathbb{H}) \cong Q_7(\mathbb{R})^{\mathrm{dual}}$$
$$W(\mathbb{R}) \cong Q_4(\mathbb{R})^{\mathrm{dual}}$$
$$W(\mathbb{C}) \cong Q_4(\mathbb{C})^{\mathrm{dual}}.$$

Now we look more closely into the different types of classical quadrangles.

7. Standard hermitian quadrangles

Let $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$, and let $\sigma\colon x \mapsto \bar{x}$ be the standard involution. On $\mathbb{F}^{n-1}$, consider the positive definite σ-hermitian form

$$(x|y) = \sum_{\nu=1}^{n-1} \bar{x}_\nu y_\nu.$$

Let f be a σ-hermitian form of Witt index 2 on the $(n+1)$-dimensional vector space V. Then we can decompose V into an f-orthogonal direct sum

$$V = \mathbb{F} \oplus \mathbb{F} \oplus V_+,$$
$$v = (v_0, v_1, v_+),$$

such that f (or a form proportional to f) is given by

$$\begin{aligned} f(u, v) &= f((u_0, u_1, u_+), (v_0, v_1, v_+)) \\ &= -\bar{u}_0 v_0 - \bar{u}_1 v_1 + (u_+|v_+). \end{aligned}$$

Note that the restriction of f to V_+ is given by the positive definite form $(-|-)$. Let $(\mathcal{P}, \mathcal{L}, \mathcal{F})$ denote the corresponding classical quadrangle.

The following construction of ovoids from hyperplanes is due to Thas. Since every standard hermitian quadrangle is contained as a full subquadrangle in a bigger one, it coincides also with the construction given in Part I [15] 3.1.

7.1. HYPERPLANES AND THAS-OVOIDS

Let $w = (w_0, w_1, 0) \in V$ be a nonzero vector and let H denote the hyperplane $H = w^\perp = \{v \in V \mid f(v, w) = 0\}$. The restriction $f|_{H\times H}$ has Witt index 1, so the set

$$\begin{aligned} \mathcal{O}_H &= \{p \in \mathcal{P} \mid p \subseteq H\} \\ &= \{p \in \mathcal{P} | f(p, w) = 0\} \end{aligned}$$

is an ovoid: every line of the quadrangle meets H in some point (since codim $H = 1$), and H contains no line of the quadrangle, because $f|_{H\times H}$ has Witt index 1. Moreover, $\mathcal{O}_H$ is the fixed point set of the involution

$$v \mapsto v - w\frac{2}{f(w, w)} f(w, v).$$

Clearly, the stabilizier of the hyperplane H in the automorphism group acts on $\mathcal{O}_H$. See Part I [15] 4.11 and 4.12. Note also that for two distinct hyperplanes $H \neq H'$ of this type the intersection $H \cap H' = V_+$ does not contain any nonzero totally isotropic subspaces, hence $\mathcal{O}_H \cap \mathcal{O}_{H'} = \emptyset$. Using the correspondence $H = w\mathbb{F}^\perp \mapsto H^\perp = w\mathbb{F}$, one sees that the set of hyperplanes of this type is parametrized by the projective line $\mathbb{F}\mathrm{P}^1$. □

7.2. THEOREM *The point set of the quadrangles* $Q_n(\mathbb{R})$, $H_n(\mathbb{C})$ *and* $H_n(\mathbb{H})$ *can be partitioned into a family of ovoids* $\mathcal{O}_H$*, indexed by the projective line* $\mathbb{F}\mathrm{P}^1$ *over* $\mathbb{F}$*. The map* $\mathcal{P} \to \mathbb{F}\mathrm{P}^1$ *that sends a point* $v\mathbb{F} \in \mathcal{P}$ *to the ovoid containing it is given by* $v\mathbb{F} \mapsto v^\perp \cap (\mathbb{F} \oplus \mathbb{F}) \subseteq \mathbb{F} \oplus \mathbb{F} \oplus V_+$*; the corresponding ovoid is* $\mathcal{O}_{v\mathbb{F}\oplus V_+}$*. This map defines a fibre bundle* $\mathcal{P} \to \mathbb{F}\mathrm{P}^1$*.*

Proof. The only thing that remains to show is the fact that this is a locally trivial bundle; this can easily be seen using homogeneous coordinates in V. □

Now we construct spreads which are, apparently, new. Let $\{b_2, \dots, b_n\}$ be an orthonormal basis of V_+ and put $b_0 = (1, 0)$, $b_1 = (0, 1) \in \mathbb{F} \oplus \mathbb{F} \subseteq V$.

7.3. THE J-SPREADS

We assume $\mathbb{F} = \mathbb{R}, \mathbb{C}$ and that $\dim_{\mathbb{F}} V = n + 1$ is even. Let J denote the σ-linear map

$$J\colon \sum_{\nu=0}^{n} b_\nu \beta_\nu \mapsto \sum_{\mu=0}^{(n-1)/2} (b_{2\mu+1}\bar{\beta}_{2\mu} - b_{2\mu}\bar{\beta}_{2\mu+1}).$$

Note that $J^2 = -\mathrm{id}$, and that J induces an involutive automorphism of the quadrangle. To each point $p = w\mathbb{F}$ of the quadrangle we assign the line

$$r(p) = w\mathbb{F} \oplus J(w)\mathbb{F}.$$

Given a line $r(p) = \ell = w\mathbb{F} \oplus J(w)\mathbb{F}$, a point $q \in D_1(\ell)$ is of the form $q = (wc + J(w)s)\mathbb{F}$, for $c, s \in \mathbb{F}$ with $|c|^2 + |s|^2 = 1$. Then

$$\begin{aligned} r(q) &= (wc + J(w\bar{s}))\mathbb{F} \oplus J(wc + J(w\bar{s}))\mathbb{F} \\ &= (wc + J(w)s)\mathbb{F} \oplus (J(w)c - ws)\mathbb{F} \\ &= w\mathbb{F} \oplus J(w)\mathbb{F} \\ &= r(p). \end{aligned}$$

Hence $r(\mathcal{P}) = \mathcal{S}_J$ is indeed a spread, consisting of the fixed lines of the automorphism induced by J. □

This construction depends on the commutativity of $\mathbb{F}$, and the even dimension of V. Note that for $\mathbb{F} = \mathbb{R}$, the map J makes $V = \mathbb{R}^{n+1}$ into a $\mathbb{C}$-module $\mathbb{C}^{(n+1)/2}$, and for $\mathbb{F} = \mathbb{C}$ into an $\mathbb{H}$-module $\mathbb{H}^{(n+1)/2}$, i.e., we introduced a complex or quaternionic structure on V. It is clear that the centralizer of J in the automorphism group acts on the spread $\mathcal{S}_J$. For $\mathbb{F} = \mathbb{R}$ we obtain a 2-transitive action of $\mathrm{PSU}_{(n-1)/2,1}\mathbb{C}$, and for $\mathbb{F} = \mathbb{C}$ a 2-transitive action of $\mathrm{PU}_{(n-1)/2,1}\mathbb{H}$.

We want to show that in the remaining cases ($\mathbb{F} = \mathbb{H}$ or n even) no spreads exist. In case of the real orthogonal quadrangles $Q_n(\mathbb{R})$, n even, this follows from Part I, Theorem 4.5. For the other classical quadrangles we have to use different methods.

7.4. DEFINITION The *Stiefel manifold* of two-frames is by definition the set

$$V_2(\mathbb{F}^k) = \{(x, y) \in \mathbb{F}^k \times \mathbb{F}^k \mid |x|^2 = |y|^2 = 1, (x \mid y) = 0\}$$

of all pairs of orthogonal unit vectors in $\mathbb{F}^k$. Let $G_2(V)$ denote the Grassmann manifold of projective lines in V, and consider the map

$$\begin{aligned} &V_2(V_+) \to G_2(V) \\ &(x, y) \mapsto (1, 0, x)\mathbb{F} \oplus (0, 1, y)\mathbb{F}. \end{aligned}$$

It's easy to check that this map is a bijection onto the line set $\mathcal{L}$ of the quadrangle; being continuous, it is a even a homeomorphism. Put $d = \dim_{\mathbb{R}} \mathbb{F}$. The map $(x, y) \mapsto x$ defines a fibre bundle

$$\begin{array}{ccc} \mathbb{S}^{d(k-1)-1} & \longrightarrow & V_2(\mathbb{F}^k) \\ & & \big\downarrow \\ & & \mathbb{S}^{dk-1}, \end{array}$$

which has been thoroughly studied by topologists; in particular, the problem whether this bundle admits a section is completely solved.

7.5. THEOREM *Let $k \geqslant 2$. The bundle $V_2(\mathbb{F}^k) \to \mathbb{S}^{dk-1}$ does not admit a section, except for the following cases: $\mathbb{F} = \mathbb{R}, \mathbb{C}$ and k even, or $\mathbb{F} = \mathbb{H}$ and $k \equiv 0$* (mod 24).

Proof. The real case is fairly easy, see James [13] p. 2. For the complex and quaternionic case cp. James [11], Adams–Walker [1], Sigrist-Suter [19] and in particular James [13] p. 76. □

7.6. THEOREM *Let $k \geqslant 2$. The real Stiefel manifold $V_2(\mathbb{R}^k)$ is homotopy equivalent to $\mathbb{S}^{k-1} \times \mathbb{S}^{k-2}$ if and only if if $k = 2, 4, 8$. The complex Stiefel manifold $V_2(\mathbb{C}^k)$ is homotopy equivalent to $\mathbb{S}^{2k-1} \times \mathbb{S}^{2k-3}$ if and only if if $k = 2, 4$. The quaternion Stiefel manifold $V_2(\mathbb{H}^k)$ is not homotopy equivalent to $\mathbb{S}^{4k-1} \times \mathbb{S}^{4k-5}$ for any $k \geqslant 2$. Hence, except for the given values of k, the bundle $V_2(\mathbb{F}^k) \to \mathbb{S}^{dk-1}$ is never topologically trivial.*

Proof. From the division algebras $\mathbb{C}, \mathbb{H}, \mathbb{O}$ it is easy to see that $V_2(\mathbb{R}^k) \cong \mathbb{S}^{k-1} \times \mathbb{S}^{k-2}$ for $k = 2, 4, 8$. Using the fact that $\mathbb{H}$ and $\mathbb{O}$ are $\mathbb{C}$-modules, one sees also that $V_2(\mathbb{C}^k) \cong \mathbb{S}^{2k-1} \times \mathbb{S}^{2k-3}$ for $k = 2, 4$. Put $d = \dim_{\mathbb{R}}(\mathbb{F})$. The following is proved in James–Whitehead [10] 1.12–1.22. If $V_2(\mathbb{F}^k) \simeq \mathbb{S}^{dk-1} \times \mathbb{S}^{d(k-1)-1}$, then $\pi_{2dk-1}(\mathbb{S}^{dk})$ contains an element of Hopf invariant 1; also one has $k \geqslant 3$ in the quaternionic case. On the other hand, it is known by Adams' result that $\pi_{2n-1}(\mathbb{S}^n)$ contains an element of Hopf invariant 1 if and only if $n = 2, 4, 8$, see eg. Husemoller [9] Ch. 15. The result follows. □

The complex case of the result above is incorrectly stated in James [12] and in [13] p. 154. I am indepted to Stephan Stolz for pointing out the correct proof.

We can use these facts to prove a first nonexistence result. Consider the hyperplane $H = b_1^\perp = b_0\mathbb{F} \oplus V_+$, and the corresponding ovoid $\mathcal{O} = \mathcal{O}_H$. A typical point $p \in \mathcal{O}$ is of the from $p = (1, 0, w)\mathbb{F}$, where $w \in V_+$ is a uniquely determined unit vector. Suppose that a map $s: \mathcal{P} \to \mathcal{L}$ defines a closed spread $\mathcal{S} \subseteq \mathcal{L}$. Then $s(1, 0, w) = (1, 0, u)\mathbb{F} \oplus (0, 1, v)\mathbb{F}$ is a line containing p. Thus $w = u$, and the composite $r\colon \mathcal{O} \xrightarrow{s} \mathcal{L} \xrightarrow{\cong} V_2(V_+)$, $w \mapsto v$ is a section into the Stiefel manifold. By 7.5 we have the following result.

7.7. THEOREM *The real orthogonal quadrangles $Q_n(\mathbb{R})$ and the complex hermitian quadrangles $H_n(\mathbb{C})$ have closed spreads if and only if n is odd. The quaternion hermitian quadrangles $H_n(\mathbb{H})$ have no closed spreads, except possibly if* $n \equiv 1 \pmod{24}$. □

In Part III we show by different methods that in the quaternionic case no spreads exist. Now suppose that $\mathcal{L}$ can be partitioned into closed spreads. Then $\mathrm{pr}_{\mathcal{P}}\colon \mathcal{F} \to \mathcal{P}$ is a trivial bundle by Part I [15] 5.8; in particular, the restriction $\mathcal{L} \cong \mathcal{F}_{\mathcal{O}} \to \mathcal{O}$ is a product bundle.

7.8. PROPOSITION *The real orthogonal quadrangles $Q_n(\mathbb{R})$ and the complex hermitian quadrangles $H_n(\mathbb{C})$ can be partitioned into spreads if and only if $m + m' = 3, 7$. The quaternion hermitian quadrangles $H_n(\mathbb{H})$ cannot be partitioned into spreads. Thus, the only standard hermitian quadrangles which can be partitioned into closed spreads are $Q_5(\mathbb{R})$, $Q_9(\mathbb{R})$, $H_3(\mathbb{C})$, and $H_5(\mathbb{C})$.*

Proof. If $\mathcal{L}$ can be partitioned into closed spreads, then the bundle $\mathcal{F} \to \mathcal{P}$ is topologically trivial; in particular, the subbundle $\mathcal{F}_{\mathcal{O}} \to \mathcal{O}$ is topologically trivial. We may identify $\mathcal{F}_{\mathcal{O}}$ with $\mathcal{L}$ by the map $\ell \mapsto (o(\ell), \ell)$; it follows that $\mathcal{L}$ is homeomorphic to $\mathcal{O} \times D_1(p) = \mathbb{S}^{m+m'} \times \mathbb{S}^{m'}$. Therefore, the 'only if' part follows from 7.6.

The fact that the $Q_5(\mathbb{R})$, $H_3(\mathbb{C})$, $Q_9(\mathbb{R})$, and $H_5(\mathbb{C})$ can indeed be partitioned into closed spreads follows from the isomorphisms $Q_5(\mathbb{R})^{\mathrm{dual}} \cong H_3(\mathbb{C})$, $Q_9(\mathbb{R})^{\mathrm{dual}} \cong \mathrm{FKM}(6, 8)$, $H_5(\mathbb{C})^{\mathrm{dual}} \cong \mathrm{FKM}(5, 8)$ (see Part III for the definition of the Clifford quadrangles FKM) and the fact that the quadrangles on the righthand side can be partitioned into ovoids. □

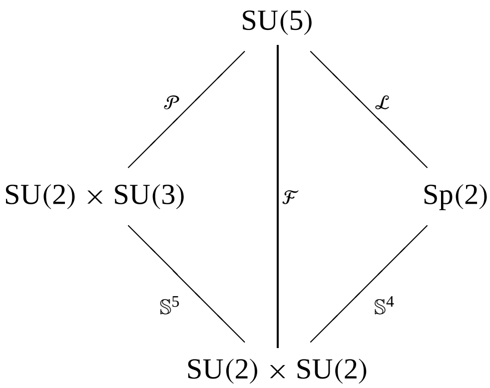
8. Summary: Ovoids and Spreads in Compact Moufang Quadrangles

We consider briefly the remaining compact connected Moufang quadrangles. The complex symplectic quadrangle $W(\mathbb{C})$ has neither closed ovoids nor closed spreads by Part I [12] 4.1; the real symplectic quadrangle $W(\mathbb{R})$ is the dual of $Q_4(\mathbb{R})$ and has therefore no closed ovoids by Part I [15] 5.1. The α-hermitian quadrangle $H_3^\alpha(\mathbb{H})$ is dual to $Q_7(\mathbb{R})$; therefore, it can be partitioned into closed spreads, and it has closed ovoids (but cannot be partitioned into closed ovoids), cp. 7.7.

8.1. THE $(4, 5)$-MOUFANG QUADRANGLE $H_4^\alpha(\mathbb{H})$

It remains to inspect the $(4, 5)$-quadrangle $H_4^\alpha(\mathbb{H})$. The group $\mathrm{SU}(5)$ acts flag-transitively on the quadrangle $H_4^\alpha(\mathbb{H})$. In fact, there is an $\mathrm{SU}(5)$-equivariant imbedding of the homogeneous spaces $\mathcal{F}$, $\mathcal{P}$ and $\mathcal{L}$ into the sphere $\mathbb{S}^{19}$ such that

$\mathcal{F} \subseteq \mathbb{S}^{19}$ is an isoparametric hypersurface, with focal manifols $\mathcal{P}$ and $\mathcal{L}$ (the multiplicities are (4, 5)). The following lattice of subgroups gives the stabilizers of a point, a line and a flag.



The $\mathbb{F}_2$-cohomology of $\mathcal{P} \leftarrow \mathcal{F} \rightarrow \mathcal{L}$ is given by the inclusions of $\mathbb{F}_2$-algebras

$$\mathbb{F}_2[x_4]/(x_4^2) \otimes \bigwedge(x_9) \rightarrow \mathbb{F}_2[x_4]/(x_4^2) \otimes \bigwedge(x_9, y_5, y_9) \leftarrow \bigwedge(y_5, y_9),$$

where the lower subscripts indicate the degrees. cp. Hebda [6], Münzner [17], Grundhöfer–Knarr–Kramer [5], [14]. We need to compute the Steenrod squares of these algebras. Let y_3, y_5, y_7, y_9 denote the primitive generators of the $\mathbb{F}_2$-cohomology of SU(5). Then

$$\mathrm{Sq}^2 y_3 = y_5, \qquad \mathrm{Sq}^4 y_5 = y_9,$$

and all other Steenrod squares vanish, cp. Borel [2] 11.4, Hsiang-Su [8] 3.2. The $\mathbb{F}_2$-Serre spectral sequence of the fibration $\mathrm{Sp}(2) \rightarrow \mathrm{SU}(5) \rightarrow \mathcal{L}$ collapses, since the dimension of the E_2-term is already $\dim_{\mathbb{F}_2} E_2 = 2^4 = \dim \mathrm{H}^\bullet(\mathrm{SU}(5); \mathbb{F}_2)$. Therefore the map $\mathrm{SU}(5) \rightarrow \mathcal{L}$ induces in $\mathbb{F}_2$-cohomology an injection

$$\bigwedge(y_3, y_5, y_7, y_0) \longleftarrow \bigwedge(y_5, y_9)$$

of graded $\mathbb{F}_2$-algebras. This gives us the Steenrod algebra of $\mathcal{L}$. From the relation $x_9 + y_9 = x_4 y_5$ one sees that $\mathrm{Sq}^4 x_9 = x_4 x_9$, and all other Steenrod squares vanish on $\mathrm{H}^\bullet(\mathcal{P}; \mathbb{F}_2)$.

8.2. PROPOSITION *The map $\mathcal{F} \rightarrow \mathcal{P}$ in $H_4^\alpha(\mathbb{H})$ does not admit a section; in particular, this quadrangle has no closed spreads.*

Proof. Suppose that $r\colon \mathcal{P} \rightarrow \mathcal{F}$ is a section of $\mathrm{pr}_{\mathcal{P}}$. Then $r^\bullet(y_5) = 0$, and thus $r^\bullet(y_9) = r^\bullet(\mathrm{Sq}^4 y_5) = \mathrm{Sq}^4 r^\bullet(y_5) = 0$. We also have $r^\bullet(x_4 y_5) = 0$; thus $0 = r^\bullet(x_4 y_5 + y_9) = r^\bullet(x_9) = x_9$, a contradiction. □

8.3. PROPOSITION *The bundle $\mathcal{F} \to \mathcal{L}$ in $H_4^\alpha(\mathbb{H})$ is not topologically trivial; in particular, $H_4^\alpha(\mathbb{H})$ cannot be partitioned into closed ovoids.*

Proof. Consider the principal bundle

$$\begin{array}{ccc} \mathrm{Sp}(2) & \longrightarrow & \mathrm{SU}(5) \\ & & \downarrow \\ & & \mathcal{L}. \end{array}$$

The bundle $\mathcal{F} \to \mathcal{L}$ can be viewed as the sphere bundle of the 5-plane normal bundle $\perp \mathcal{L}$ of $\mathcal{L} \subseteq \mathbb{S}^{19}$; therefore, there exists a real five-dimensional Sp(2)-module X such that $\perp \mathcal{L}$ is the associated vector bundle,

$$\perp \mathcal{L} = \mathrm{SU}(5) \times_{\mathrm{Sp}(2)} X.$$

The elements of this vector bundle are equivalence classes $\{[g, v] |\ g \in \mathrm{SU}(5), v \in X\}$, where $[g, v] = [gh^{-1}, hv]$ for $h \in \mathrm{Sp}(2)$. The unit sphere in the fibre over $\ell \in \mathcal{L}$ corresponds to the point row $D_1(\ell)$, and the action of Sp(2) on this four-sphere is transitive. Thus X is given by the representation of Sp(2) as SO(5) on $\mathbb{R}^5$. The question is whether this bundle is topologically trivial. Restricted to the line pencil

$$D_1(p) \cong \mathrm{SU}(3) \times \mathrm{SU}(2)/\mathrm{SU}(2) \times \mathrm{SU}(2) \subseteq \mathcal{L},$$

the bundle splits off the trivial line bundle given by the section $\ell \mapsto (p, \ell)$, and we obtain the four-plane bundle

$$(\mathrm{SU}(3) \times \mathrm{SU}(2)) \times_{\mathrm{SU}(2)\times\mathrm{SU}(2)} \mathbb{R}^4,$$

where $\mathrm{SU}(2) \times \mathrm{SU}(2)$ acts as $\mathrm{SO}(4)$ on $\mathbb{R}^4$. The associated unit sphere bundle is homogeneous, with stabilizer $\{(a, a^{-1}) \mid a \in \mathrm{SU}(2)\} \subseteq \mathrm{SU}(2) \times \mathrm{SU}(2)$. Thus, it is the Stiefel manifold $V_2(\mathbb{C}^3)$ (with its canonical bundle structure over $\mathbb{S}^5 \subseteq \mathbb{C}^3$). Adding the trivial bundle that splits of, we obtain the tangent bundle of the five-sphere. This bundle, which corresponds to $V_2(\mathbb{R}^6)$, is not topologically trivial, cp. 7.6. □

8.4. PROPOSITION *The classical quadrangle $H_4^\alpha(\mathbb{H})$ has no closed ovoids.*

Proof. Suppose that $\mathcal{O} \subseteq \mathcal{P}$ is a closed ovoid. Let $p \in \mathcal{P} \setminus \mathcal{O}$. Then the fibre bundle $D_2(p) \to D_1(p)$ has a section $\sigma_1 \colon \ell \mapsto o(\ell)$. Fix another point $q \in D_4(p) \cap \mathcal{O}$. Then there is another section $\sigma_2 \colon \ell \mapsto \mathrm{proj}_\ell q$. Note that $\sigma_1(\ell) \neq \sigma_2(\ell)$: the line $q\sigma_2(\ell)$ meets $\mathcal{O}$ in q, so $\sigma_2(\ell) \notin \mathcal{O}$. In the proof of 8.3 we identified the bundle $D_2(p) \to D_1(\ell)$ as the vector bundle

$$(\mathrm{SU}(3) \times \mathrm{SU}(2)) \times_{\mathrm{SU}(2)\times\mathrm{SU}(2)} \mathbb{R}^4 \longrightarrow \mathrm{SU}(3)/\mathrm{SU}(2) = \mathbb{S}^5.$$

Table I.

Quadrangle	Parameters	Ovoids	$\mathcal{P}$ partitioned into ovoids	Spreads	$\mathcal{L}$ partitioned into spreads
$W(\mathbb{R})$	(1, 1)	no	no	yes	yes
$W(\mathbb{C})$	(2, 2)	no	no	no	no
$Q_5(\mathbb{R})$	(1, 2)	yes	yes	yes	yes
$Q_7(\mathbb{R})$	(1, 4)	yes	yes	yes	no
$Q_9(\mathbb{R})$	(1, 6)	yes	yes	yes	yes
$H_3(\mathbb{C})$	(2, 1)	yes	yes	yes	yes
$H_5(\mathbb{C})$	(2, 5)	yes	yes	yes	yes
$H_3^\alpha(\mathbb{H})$	(4, 1)	yes	no	yes	yes
$H_4^\alpha(\mathbb{H})$	(4, 5)	no	no	no	no
$Q(E_6, \mathbb{R})$	(9, 6)	yes	yes	no	no
$Q_{2k}(\mathbb{R}), k \geqslant 2$	$(1, 2k-3)$	yes	yes	no	no
$Q_{2k+1}(\mathbb{R}), k \geqslant 5$	$(1, 2k-2)$	yes	yes	yes	no
$H_{2k}(\mathbb{C}), k \geqslant 2$	$(2, 4k-7)$	yes	yes	no	no
$H_{2k+1}(\mathbb{C}), k \geqslant 3$	$(2, 4k-3)$	yes	yes	yes	no
$H_k(\mathbb{H}), k \geqslant 3$	$(4, 4k-9)$	yes	yes	no	no

Our ovoid yields via σ_1 a section of this bundle which is nowhere zero. If we add a trivial line bundle, then we end up with the tangent bundle of $\mathbb{S}^5$. So we obtain two linearly independent vector fields on $\mathbb{S}^5$, or, in other words, a section $\mathbb{S}^5 \to V_3(\mathbb{R}^6)$. By Adams' result [9] Chapter 16, 13.10, this is impossible. □

Maybe one can prove in a similar way that that the bundle $\mathrm{SU}(5)/\mathrm{SU}(2)\times\mathrm{SU}(2) \to \mathrm{SU}(5)/\mathrm{Sp}(2)$ does not admit a section.

The real E_6-quadrangle $Q(E_6, \mathbb{R})$ arising from the exceptional real Lie group $\mathrm{PE}_{6(-14)}$ belongs to the Clifford series and will be considered in Part III [16]. Recall the anti-isomorphisms

$$H_3(\mathbb{C}) \cong Q_5(\mathbb{R})^{\mathrm{dual}},$$
$$H_3^\alpha(\mathbb{H}, 3) \cong Q_7(\mathbb{R})^{\mathrm{dual}},$$
$$W(\mathbb{R}) \cong Q_4(\mathbb{R})^{\mathrm{dual}},$$
$$W(\mathbb{C}) \cong Q_4(\mathbb{C})^{\mathrm{dual}}.$$

Taking these anti-isomorphisms as well as the (anti) isomorphisms

$$\mathrm{FKM}(6, 8) \cong Q_9(\mathbb{R})^{\mathrm{dual}},$$
$$\mathrm{FKM}(5, 8) \cong H_5(\mathbb{C})^{\mathrm{dual}},$$
$$\mathrm{FKM}(1, n-1) \cong Q_n(\mathbb{R}),$$

$$\mathrm{FKM}(2, 2(n-1)) \cong H_n(\mathbb{C}),$$
$$\mathrm{FKM}(4, 4(n-1), n-1) \cong H_n(\mathbb{H}),$$
$$\mathrm{FKM}(9, 16) \cong Q(E_6, \mathbb{R}),$$

between Moufang quadrangles and Clifford quadrangles (see Part III [16]) into account we get Table 1 about closed ovoids and spreads in compact connected Moufang quadrangles.

Note that the nonexistence results that we actually proved are slightly stronger (except for the bundle $\mathcal{F} \to \mathcal{P}$ in $H_4^\alpha(\mathbb{H})$): instead of the nonexistence of spreads or ovoids, we proved that the maps $\mathrm{pr}_{\mathcal{P}}: \mathcal{F} \to \mathcal{P}$ or $\mathrm{pr}_{\mathcal{L}}: \mathcal{F} \to \mathcal{L}$ admit no sections. See also the remarks at the end of Part III [16].

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