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Results in Mathematics

SPLITTING OFF THE REAL LINE AND PLANE

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Abstract: We show that $S \times \mathbb{R}^m \cong \mathbb{R}^{m+k}$ implies $S \cong \mathbb{R}^k$ for $k \leq 2$.

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J. Rätz [4] and J. Tabor [5] prove that $S \times \mathbb{R}^m \cong \mathbb{R} \times \mathbb{R}^m$ implies $S \cong \mathbb{R}$, and mention that this was posed by M.C. Zdun as an open problem. However, a more general result follows easily from old theorems of A. Borel [2] and G. Young [6]. Since the result is of some interest in compact transformation groups and topological geometry, we give a direct proof of the more general statement.

Lemma *Let R be a principal ideal domain and let X be a connected, separable and metrizable n-cm_R (cohomology n-manifold over R, see Bredon [3] V.16.7). If X factors as $X \cong S \times T$, and if $\dim_R(S) = k \leq 2$ (equivalently, if $k = \dim_R(X) - \dim_R(T) \leq 2$), then S is a topological k-manifold.*

Proof. The factors S and T are k- and $(n-k)$-cm_Rs, respectively, see [3] V.16.11. A connected k-cm_R is a k-hm_R (homology k-manifold over R) [3] V.16.8, and a separable metrizable k-hm_R is a topological manifold [3] V.16.32, provided that $k \leq 2$. □

Corollary *Let $X \cong S \times T$ be as above. Suppose that X is 1-connected. If $k = 1$, then $S \cong \mathbb{R}$; if $k = 2$, then $S \cong \mathbb{R}^2$ or $S \cong \mathbb{S}^2$. In particular, if $S \times \mathbb{R}^m \cong \mathbb{R}^{m+k}$, for $k \leq 2$, then $S \cong \mathbb{R}^k$.*

Proof. By the Lemma, S is a 1-connected k-manifold. It is well-known that every 1-connected 1-manifold is homeomorphic to the real line. Similarly, it follows from the classification of surfaces that a 1-connected surface is either homeomorphic to $\mathbb{R}^2$ or to the sphere $\mathbb{S}^2$. □

The result does not carry over to higher dimensions: there is a 3-cm_R E such that $E \times \mathbb{R} \cong \mathbb{R}^4$, but $E \not\cong \mathbb{R}^3$ [1].

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