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Projective Planes and Isoparametric Hypersurfaces

Dedicated to Professor Helmut Salzmann on his 65th birthday

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Abstract. The isoparametric hypersurfaces in spheres with three distinct principal curvatures were classified by Cartan in 1939. We give a new proof for this result by showing that every such hypersurface can be naturally identified with the flag space of a compact connected Moufang plane. Our approach also leads to a uniform and explicit description of these hypersurfaces.

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Cartan's classification of isoparametric hypersurfaces in the sphere with three distinct principal curvatures is one of the earliest results in the theory of isoparametric submanifolds. His proof is long and computational, and relies on the classification of the polynomials that define the hypersurface. *After* he obtained his classification result, he observed that the focal submanifolds are symmetric spaces of rank one, and in fact are the projective planes over the four real alternative division algebras $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, $\mathbb{O}$, cp. [2]. For another proof using normed division algebras see Karcher [10].

In this paper, we give first a uniform and explicit construction for these hypersurfaces and their focal sets. After this, we give two simple geometric proofs for Cartan's theorem. The first (and shorter) proof uses tools from topological geometry. The second proof is more along the lines of classical differential geometry and uses only techniques that were already known by the time Cartan proved his result, i.e. the classification of symmetric spaces of rank one.

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1. Isoparametric Hypersurfaces

A compact hypersurface $\mathcal{F}^n$ in the sphere $\mathbb{S}^{n+1}$ is called *isoparametric*, if it has constant principal curvatures, see Cartan [2], Münzner [14], Cecil–Ryan [3], Palais–Terng [15], or [11]. Let g be the number of distinct principal curvatures, and let $T_x\mathcal{F} = E_1(x) \oplus E_2(x) \oplus \cdots \oplus E_g(x)$ be the eigenspace decomposition of the tangent space with respect to the Weingarten map. The leaf S_i of the curvature distribution E_i is a totally geodesic sphere of dimension m_i, cp. [15, 6.2.9].

Let $\mathcal{P}, \mathcal{L} \subseteq \mathbb{S}^{n+1}$ be the focal submanifolds of $\mathcal{F}$ with the canonical focal maps $\mathcal{P} \xleftarrow{\mathrm{pr}_1} \mathcal{F} \xrightarrow{\mathrm{pr}_2} \mathcal{L}$, see [3, Ch. 2.4]. We may assume that $S_i(x)$ is the fiber of pr_i over $\mathrm{pr}_i(x)$ for $i = 1, 2$.

If the number g of distinct principal curvatures is odd, then $-\mathcal{P} = \{-p \mid p \in \mathcal{P}\} = \mathcal{L}$; furthermore, we may assume that $-\mathcal{F} = \mathcal{F}$ by passing to a suitable parallel hypersurface.

2. The Veronese Embeddings of the Classical Projective Planes

In this section we construct the isoparametric hypersurfaces associated with the classical projective planes. Let $\mathbb{D}$ denote one of the four alternative real division algebras $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, or $\mathbb{O}$ of real dimension $m = 1, 2, 4$, or 8, respectively. Basic information on these algebras can be found in [4]. We need to know the following facts:

$\mathbb{D}$ admits an involutorial antiautomorphism $^{-}\colon \mathbb{D} \to \mathbb{D}$ with fixed field $\mathbb{R}$. The usual inner product on the vector space $\mathbb{D} = \mathbb{R}^m$ is given by $\langle x, y\rangle = \frac{1}{2}(x\bar{y} + y\bar{x})$.

2.1. LEMMA. *For any three elements $x, y, z \in \mathbb{D}$ we have*

$$\langle x, yz\rangle = \langle \bar{y}x, z\rangle = \langle x\bar{z}, y\rangle.$$

Proof. Freudenthal ([6, 1.3.1 and 1.3.2]). □

Consider now the following set

$$\begin{aligned}\mathbf{M} &= \{(x_1, x_2, x_3) \in \mathbb{D}^3 \mid |x_1|^2 + |x_2|^2 + |x_3|^2,\\ &\qquad x_i \in \mathbb{R} \text{ for at least one } i\}\\ &= 1,\end{aligned}$$

and define $\mathbf{S}_i = \{(x_1, x_2, x_3) \in \mathbf{M} \mid x_i \in \mathbb{R}\}$ for $i = 1, 2, 3$. Note that $\mathbf{S}_i$ is a $2m$-dimensional sphere. We define mappings $f_k\colon \mathbf{M} \to \mathbb{D}^3 \times \mathbb{R}^2, k = 1, 2$, as follows

$$\begin{aligned}&f_1(x, y, z)\\ &= \left(\sqrt{3}\,\bar{x}y, \sqrt{3}\,\bar{y}z, \sqrt{3}\,\bar{z}x, \frac{\sqrt{3}}{2}(|x|^2 - |y|^2), |z|^2 - \tfrac{1}{2}(|x|^2 + |y|^2)\right),\\ &f_2(a, b, c)\\ &= \left(-\sqrt{3}\,a\bar{b}, -\sqrt{3}\,b\bar{c}, -\sqrt{3}\,c\bar{a}, \frac{\sqrt{3}}{2}(|b|^2 - |a|^2), \tfrac{1}{2}(|a|^2 + |b|^2) - |c|^2\right).\end{aligned}$$

For $m \leq 4$, i.e. for $\mathbb{D} \neq \mathbb{O}$, these mappings were used by E. Cartan ([2]) to parametrize the focal submanifolds of isoparametric hypersurfaces in spheres with three distinct principal curvatures. He defined them on the whole unit sphere in $\mathbb{D}^3$ and showed that this does not work in the case $m = 8$ since the octonions are not associative ([2, p. 355]). It turns out that Cartan's approach *does* work in the octonion case if we restrict the domain of definition of f_k to $\mathbf{M}$.

2.2. LEMMA. *For all* $(x, y, z) \in \mathbf{M}$ *we have*

$$|f_1(x, y, z)| = |f_2(x, y, z)| = 1.$$

Proof. This is a straightforward calculation. □

The following result is fundamental for our investigations.

2.3. LEMMA. *For all* $(x, y, z) \in \mathbf{S}_i$ *and* $(a, b, c) \in \mathbf{S}_j, i \neq j$, *we have*

$$\langle f_1(x, y, z), f_2(a, b, c)\rangle = \tfrac{1}{2} - \tfrac{3}{2}|xa + yb + zc|^2$$

Proof. It is sufficient to treat the case $i = 1, j = 3$, i.e. that $x, c \in \mathbb{R}$.

$$\begin{aligned}
&\langle f_1(x, y, z), f_2(a, b, c)\rangle \\
&= -3\langle \bar{x}y, a\bar{b}\rangle - 3\langle \bar{y}z, b\bar{c}\rangle - 3\langle \bar{z}x, c\bar{b}\rangle + \tfrac{3}{4}(|x|^2 - |y|^2)(|b|^2 - |a|^2) \\
&\quad -|z|^2|c|^2 + \tfrac{1}{2}|z|^2(|a|^2 + |b|^2) + \tfrac{1}{2}(|x|^2 + |y|^2)|c|^2 \\
&\quad -\tfrac{1}{4}(|x|^2 + |y|^2)(|a|^2 + |b|^2) \\
&= -3\langle yb, xa\rangle - 3\langle zc, yb\rangle - 3\langle xb, zc\rangle + \tfrac{3}{4}(|xb|^2 - |xa|^2 - |yb|^2 + |ya|^2) \\
&\quad -|zc|^2 + \tfrac{1}{2}|z|^2(|a|^2 + |b|^2) + \tfrac{1}{2}(|x|^2 + |y|^2)|c|^2 \\
&\quad -\tfrac{1}{4}(|xa|^2 + |xb|^2 + |ya|^2 + |yb|^2)
\end{aligned}$$

(here we use that $x, c \in \mathbb{R}$ and apply 2.1)

$$\begin{aligned}
&= -3\langle xa, yb\rangle - 3\langle yb, zc\rangle - 3\langle zc, xb\rangle - |xa|^2 - |yb|^2 - |zc|^2 \\
&\quad +\tfrac{1}{2}(|xb|^2 + |ya|^2 + |za|^2 + |zb|^2 + |xc|^2 + |yc|^2) \\
&= -3\langle xa, yb\rangle - 3\langle yb, zc\rangle - 3\langle zc, xa\rangle - \tfrac{3}{2}(|xa|^2 + |yb|^2 + |zc|^2) \\
&\quad +\tfrac{1}{2}(|xb|^2 + |ya|^2 + |za|^2 + |zb|^2 + |xc|^2 \\
&\quad +|yc|^2 + |xa|^2 + |yb|^2 + |zc|^2) \\
&= -\tfrac{3}{2}|xa + yb + zc|^2 + \tfrac{1}{2}
\end{aligned}$$

since (x, y, z) and (a, b, c) are unit vectors. □

Remark. In the smallest case $\mathbb{D} = \mathbb{R}$ we have $\mathbf{M} = \mathbb{S}^2$ and $f_1 = -f_2 \colon \mathbb{S}^2 \to \mathbb{S}^4$. So Lemma 2.3 says that

$$\langle f_1(x, y, z), f_1(a, b, c)\rangle = \tfrac{3}{2}\langle (x, y, z), (a, b, c)\rangle^2 - \tfrac{1}{2}.$$

Hence the spherical distance between $f_1(x, y, z)$ and $f_1(a, b, c)$ depends only on the spherical distance between (x, y, z) and (a, b, c), but it is not invariant.

Now we construct an incidence structure $(\mathcal{P}, \mathcal{L}, \mathcal{F})$ as follows

$$\mathcal{P} = f_1(\mathbf{M}), \quad \mathcal{L} = f_2(\mathbf{M}) \quad \text{and} \quad \mathcal{F} = \{(p,\ell) \in \mathcal{P} \times \mathcal{L} \mid \langle p, \ell \rangle = \tfrac{1}{2}\}.$$

Note that by Lemma 2.3 a point $p = f_1(x, y, z)$ is incident with a line $\ell = f_2(a, b, c)$ if and only if $xa + yb + zc = 0$, provided that $(x, y, z) \in \mathbf{S}_i$ and $(a, b, c) \in \mathbf{S}_j$ with $i \neq j$. Moreover, p and ℓ are incident if and only if $\mathrm{dist}(p, \ell) = \mathrm{dist}(\mathcal{P}, \mathcal{L})$, where dist denotes the spherical distance.

2.4. LEMMA. *We have* $\mathcal{P} = f_1(\mathbf{M}) = f_1(\mathbf{S}_i)$ *and* $\mathcal{L} = f_2(\mathbf{M}) = f_2(\mathbf{S}_i)$ *for* $i = 1, 2, 3$.

Proof. Let $i, j \in \{1, 2, 3\}$ and let $(x_1, x_2, x_3) \in \mathbf{S}_i$. By Artin's theorem, the subalgebra of $\mathbb{D}$ generated by the elements x_1, x_2 and x_3 is associative since at least one of them is real. If $x_j = 0$ then $(x_1, x_2, x_3) \in \mathbf{S}_j$. If $x_j \neq 0$ we define $c = \bar{x}_j/|x_j|$. Then we have $|c| = 1$ and $(cx_1, cx_2, cx_3), (x_1c, x_2c, x_3c) \in \mathbf{S}_j$. Since c is contained in the associative subalgebra generated by x_1, x_2 and x_3 it follows that $f_1(x_1, x_2, x_3) = f_1(cx_1, cx_2, cx_3)$ and $f_2(x_1, x_2, x_3) = f_2(x_1c, x_2c, x_3c)$ as is easily seen. This shows that $f_k(\mathbf{S}_i) = f_k(\mathbf{S}_j)$, and since $\mathbf{M} = \mathbf{S}_1 \cup \mathbf{S}_2 \cup \mathbf{S}_3$ the result follows. □

2.5. THEOREM. *The incidence structure* $(\mathcal{P}, \mathcal{L}, \mathcal{F})$ *is isomorphic to the projective plane over the division algebra* $\mathbb{D}$.

Proof. Let the point and the line space of this projective plane be called $\mathcal{P}'$ and $\mathcal{L}'$, respectively. Then we have

$$\mathcal{P}' = \{(x, y) \mid x, y \in \mathbb{D}\} \cup \{(s) \mid s \in \mathbb{D}\} \cup \{(\infty)\}$$

and

$$\mathcal{L}' = \{[a, b] \mid a, b \in \mathbb{D}\} \cup \{[c] \mid c \in \mathbb{D}\} \cup \{[\infty]\}.$$

A point (x, y) is incident with a line $[a, b]$ if and only if $y = xa + b$ for $x, y, a, b \in \mathbb{D}$. Furthermore, the point (x, y) is incident with the line $[x]$ and the point (x) is incident with the lines $[x, b]$ and $[\infty]$ for $x, y, b \in \mathbb{D}$. Finally, (∞) is incident with $[c]$ for $c \in \mathbb{D} \cup \{\infty\}$ and there are no further incidences.

We define mappings $g_1 : \mathcal{P}' \to \mathbf{M}$ and $g_2 : \mathcal{L}' \to \mathbf{M}$ as follows

$$g_1((x,y)) = \frac{1}{\sqrt{1 + |x|^2 + |y|^2}}(x, y, 1) \qquad g_2([a,b]) = \frac{1}{\sqrt{1 + |a|^2 + |b|^2}}(a, -1, b)$$

$$g_1((s)) = \frac{1}{\sqrt{1 + |s|^2}}(1, s, 0) \qquad g_2([c]) = \frac{1}{\sqrt{1 + |c|^2}}(-1, 0, c)$$

$$g_1((\infty)) = (0, 1, 0) \qquad g_2([\infty]) = (0, 0, -1).$$

Using Lemma 2.4 we see that $f_1 \circ g_1 : \mathcal{P}' \to \mathcal{P}$ and $f_2 \circ g_2 : \mathcal{L}' \to \mathcal{L}$ are bijections and by Lemma 2.3 they are incidence preserving. □

Remark. The last result and its proof show that the elements of $\mathbf{M}$ can serve as a sort of restricted homogeneous coordinates for the projective plane over $\mathbb{D}$. The embedding $\mathcal{P} \hookrightarrow \mathbb{S}^{3m+1}$ is called the *Veronese embedding* of $\mathcal{P}$. See also [13].

2.6. PROPOSITION. *The sets $\mathcal{P}$ and $\mathcal{L}$ are $2m$-dimensional smooth submanifolds of the sphere $\mathbb{S}^{3m+1}$. The tangent space at a point $p \in \mathcal{P}$ is given by*

$$T_p\mathcal{P} = \{x \in \mathbb{D}^3 \times \mathbb{R}^2 \,|\, \langle x, \ell\rangle = 0 = \langle x, p\rangle \text{ for all } \ell \in \mathcal{L} \text{ with } \langle p, \ell\rangle = \tfrac{1}{2}\}.$$

Dually, the tangent space at $\ell \in \mathcal{L}$ is given by

$$T_\ell\mathcal{L} = \{y \in \mathbb{D}^3 \times \mathbb{R}^2 \,|\, \langle y, p\rangle = 0 = \langle y, \ell\rangle \text{ for all } p \in \mathcal{P} \text{ with } \langle p, \ell\rangle = \tfrac{1}{2}\}.$$

Proof. It is easy to check that the restriction of f_k to the set $\{(x_1, x_2, x_3) \in \mathbf{S}_j \mid x_j > 0\}$ is an embedding for $k = 1, 2$ and $j = 1, 2, 3$. Thus we get coverings of $\mathcal{P}$ and $\mathcal{L}$ by three subsets which are homeomorphic to $\mathbb{R}^{2m}$. The coordinate changes are given by multiplication with elements of $\mathbb{D}$ (cp. the proof of the Lemma 2.4), and hence they are smooth.

Assume now that $p \in \mathcal{P}$ and let $\gamma: (-\varepsilon, \varepsilon) \to \mathcal{P}$ be a smooth curve with $\gamma(0) = p$. Let $\ell \in \mathcal{L}$ with $\langle p, \ell\rangle = \frac{1}{2}$. Then Lemma 2.3 implies that $\langle \gamma(t), \ell\rangle \leq \frac{1}{2}$. Hence we get $\langle \gamma(t) - \gamma(0), \ell\rangle \leq 0$ for $-\varepsilon < t < \varepsilon$. Dividing by t gives us

$$\frac{1}{t}\langle \gamma(t) - \gamma(0), \ell\rangle \begin{cases} \leq 0 & \text{if } t > 0 \\ \geq 0 & \text{if } t < 0 \end{cases}$$

Letting $t \to 0$ we see that $\langle \dot\gamma(0), \ell\rangle = 0$. Let V be the vector space on the right-hand side of the stated equation. Then it follows that $T_p\mathcal{P} \subseteq V$ since $\mathcal{P}$ is also contained in $\mathbb{S}^{3m+1}$. The set $\{\ell \in \mathcal{L} \,|\, \langle p, \ell\rangle = \frac{1}{2}\}$ is the line pencil through p and hence is homeomorphic to an m-dimensional sphere. This implies that V is at most $2m$-dimensional, and the claim follows. The same argument applies to $\ell \in \mathcal{L}$. □

2.7. PROPOSITION. *Let $\mathfrak{n}$ be a unit normal at $p \in \mathcal{P} \subseteq \mathbb{S}^{3m+1}$. The Weingarten map $A_\mathfrak{n}$ has eigenvalues $-1/\sqrt{3}, 1/\sqrt{3}$, each with multiplicity m. Hence $\mathcal{P}$ is the focal manifold of an isoparametric hypersurface with $g = 3$ distinct principal curvatures*, see [3, Ch. 2.3] or [9, Th. A].

Proof. Put $\ell = \exp_p(\pi/3)\mathfrak{n} \in \mathcal{L}$. By 2.6, the point row $L \subseteq \mathcal{P}$ corresponding to the line ℓ is a round m-sphere of spherical radius $\pi/3$, hence the Weingarten map $A_\mathfrak{n}$ has an eigenspace of dimension $\geq m$ with eigenvalue $1/\sqrt{3}$. Now $A_{-\mathfrak{n}} = -A_\mathfrak{n}$. We may apply the same argument to $\ell' = \exp_p(-(\pi/3)\mathfrak{n})$, and the claim follows from the fact that $\dim T_p\mathcal{P} = 2m$. □

We may define a mapping $G: \mathcal{F} \times [-\pi/6, \pi/6] \to \mathbb{S}^{3m+1}$ as follows

$$G(p, \ell, t) = \left(\frac{1}{\sqrt{3}}\cos t - \sin t\right) p + \left(\frac{1}{\sqrt{3}}\cos t + \sin t\right) \ell.$$

Note that $G(p,\ell,-\pi/6)=p$ and $G(p,\ell,\pi/6)=\ell$. Since $\langle p,\ell\rangle=\frac{1}{2}$, i.e. since the spherical distance between p and ℓ is $\pi/3$, the mapping $G(p,\ell,\cdot)$ joins p and ℓ by a great circle segment.

Using the explicit description of the Weingarten map of a tube over a submanifold given in [3, Th. 3.2] we obtain the following corollary.

2.8. COROLLARY. *The mapping G describes an isoparametric foliation of the sphere $\mathbb{S}^{3m+1}$. For each $t\in(-\pi/6,\pi/6)$ the set $\mathcal{F}_t=G(\mathcal{F},t)$ is an isoparametric hypersurface with three distinct principal curvatures equal to* $\cot(\pi/6-t)$, $\cot(\pi/2-t)$ *and* $\cot(5\pi/6-t)$. *The focal submanifolds are $\mathcal{P}=G(\mathcal{F},-\pi/6)$ and $\mathcal{L}=G(\mathcal{F},\pi/6)$.* □

3. The Classification

In this section we prove Cartan's classification result. Let $\mathcal{F}^n\subseteq\mathbb{S}^{n+1}$ be a compact isoparametric hypersurface with focal manifolds $\mathcal{P},\mathcal{L}$. First, we associate an incidence structure with the triple $(\mathcal{P},\mathcal{L},\mathcal{F})$ as follows

3.1. DEFINITION. We may embed the hypersurface $\mathcal{F}$ into the product $\mathcal{P}\times\mathcal{L}$ by means of the map $x\mapsto(\mathrm{pr}_1(x),\mathrm{pr}_2(x))$, and hence we may consider the triple $(\mathcal{P},\mathcal{L},\mathcal{F})$ as an incidence structure: the elements of $\mathcal{P}$ are called *points*, the elements of $\mathcal{L}$ are called *lines*, the elements of $\mathcal{F}$ are called *flags*, and a point p and a line ℓ are incident if and only if $\mathrm{pr}_1^{-1}(p)\cap\mathrm{pr}_2^{-1}(\ell)\neq\emptyset$.

This definition agrees with Thorbergsson's construction of the building associated to an isoparametric submanifold of rank ≥ 3 (cp. [18],[11]).

Note that a point p and a line ℓ are incident if and only if $\mathrm{dist}(p,\ell)=\mathrm{dist}(\mathcal{P},\mathcal{L})$. Moreover, the following is true

3.2. PROPOSITION. *We have the identities*

$$\mathcal{F}=\left\{x\in\mathbb{S}^{n+1}\,|\,\mathrm{dist}(x,\mathcal{P})=\mathrm{dist}(\mathcal{F},\mathcal{P})=\frac{\pi}{2g}\right\}$$

and

$$\mathcal{L}=\left\{x\in\mathbb{S}^{n+1}\,|\,\mathrm{dist}(x,\mathcal{P})=\mathrm{dist}(\mathcal{L},\mathcal{P})=\frac{\pi}{g}\right\}.$$

Hence the geometry of $(\mathcal{P},\mathcal{L},\mathcal{F})$ is completely determined by one focal submanifold and by the spherical distance function. □

3.3. LEMMA. *If the number of principal curvatures is at least three, then the incidence structure $(\mathcal{P},\mathcal{L},\mathcal{F})$ contains no digons, i.e. two lines intersect in at most one point, and two points are joined by at most one line.*

Proof. (due to Thorbergsson, unpublished). Suppose there is a digon consisting of two distinct lines h,ℓ and two distinct points p,q. Let $x_1,\ldots,x_4\in\mathcal{F}$ be the elements corresponding to $(p,\ell),(q,\ell),(q,h)$ and (p,h), respectively. Consider the quadrangle obtained by joining x_1,x_2,x_3,x_4,x_1 by straight line segments. On one corner, say x_1, the angle must be $\leq \pi/2$. But this is impossible, because the tangent spaces E_k of the curvature spheres S_k are orthogonal, and because the radius vectors of the curvature spheres $S_1(x_1)$ and $S_2(x_1)$ meet at x_1 at an angle of $\pi[(g-1)/g] > \pi/2$. □

3.4. PROPOSITION. *If the number of principal curvatures is $g=3$, then $(\mathcal{P},\mathcal{L},\mathcal{F})$ is a compact connected projective plane in the sense of Salzmann* ([16]).

Proof. We show first that any two lines intersect, and that any two points are joined by a line. For $x\in\mathcal{F}$, let

$$S_{12}(x)=\{(y,z)\in\mathcal{F}\times\mathcal{F}\mid y\in S_1(x), z\in S_2(y)\}$$

and consider the map $\rho(y,z)=\mathrm{pr}_1(z)$. Note that $\rho(S_{12}(x))$ is precisely the set of all points that can be joined with the point $\mathrm{pr}_1(x)$ by a line. Thus, if we can show that $\rho(S_{12}(x))=\mathcal{P}$, then we have proved that any point in $\mathcal{P}$ is incident to some line through $\mathrm{pr}_1(x)$.

Now $S_{12}(x)$ is a compact m_1+m_2-manifold. The open subset $U=\{(y,z)\in S_{12}(x)\mid y\neq z\}$ is mapped homeomorphically onto its image $\rho(U)$, because two lines intersect in at most one point. Let $(y,z)\in U$ and consider the commutative diagram

$$\begin{array}{ccc}
\mathbb{Z}_2=\mathbf{H}_{m_1+m_2}(S_{12}(x);\mathbb{Z}_2) & \xrightarrow{\cong} & \mathbf{H}_{m_1+m_2}(S_{12}(x),S_{12}(x)-\{(y,z)\};\mathbb{Z}_2)\\
\downarrow{\scriptstyle\rho_\bullet} & & \downarrow{\scriptstyle\cong}\\
\mathbb{Z}_2=\mathbf{H}_{m_1+m_2}(\mathcal{P};\mathbb{Z}_2) & \xrightarrow{\cong} & \mathbf{H}_{m_1+m_2}(\mathcal{P},\mathcal{P}-\{\rho(y,z)\};\mathbb{Z}_2)
\end{array}$$

The vertical arrow at the right is an isomorphism, because the restriction of ρ to U is a homeomorphism, and because we can excise. If there were a point $q\in\mathcal{P}-\rho(S_{12}(x))$, then ρ would factor as $S_{12}(x)\to\mathcal{P}-\{q\}\to\mathcal{P}$, and hence $\rho_\bullet\colon\mathbf{H}_{m_1+m_2}(S_{12}(x);\mathbb{Z}_2)\to\mathbf{H}_{m_1+m_2}(\mathcal{P};\mathbb{Z}_2)$ could not be an isomorphism. By the same reasoning, any two lines have some point in common. Finally, $(\mathcal{P},\mathcal{L},\mathcal{F})$ is a compact connected projective plane, because $\mathcal{F}$ is compact and connected (see Grundhöfer ([7, 2.1])). □

3.5. LEMMA. *Let $\phi\in\mathrm{O}_{n+2}\mathbb{R}$ be an orthogonal map with $\phi(\mathcal{P})=\mathcal{P}$. Then ϕ is an isometry of the isoparametric hypersurface $\mathcal{F}$ and of the focal submanifold $\mathcal{L}$. Moreover, ϕ preserves the incidence structure, i.e. $\phi\in\mathrm{Aut}(\mathcal{P},\mathcal{L},\mathcal{F})$.*

Proof. This is clear from 3.2. □

From now on, we assume that the number g of distinct principal curvatures is three.

3.6. THEOREM. *Let $\mathcal{F}$ be an isoparametric hypersurface with three distinct principal curvatures. For every point p in the focal submanifold $\mathcal{P}$, there exists an* *involution* $\phi_p \in \mathrm{O}_{n+2}\mathbb{R}$ *that maps* $\mathcal{P}, \mathcal{L}, \mathcal{F}$ *onto themselves, and that has* p *as an isolated fixed point. In particular,* $\mathcal{P}$ *is an extrinsically symmetric space* (see, e.g., Ferus ([5])).

Moreover, the map ϕ_p *has the following properties*:

(i) *Every point row* L *through* p *is mapped isometrically onto itself.*

(ii) *The restriction* $\phi_p \mid L$ *is a reflection of the round sphere* $L \approx \mathbb{S}^{m_1}$ *that fixes* p.

Hence every point row L *is an extrinsically symmetric round sphere under the group generated by the reflections* $\{\phi_p \mid p \in L\}$.

Proof. Put

$$\phi_p = (\mathbf{1}_{N_p\mathcal{P}}) \oplus (-\mathbf{1}_{T_p\mathcal{P}}).$$

Then ϕ_p fixes p and every point in $p + N_p\mathcal{P}$. Since $\mathcal{P}$ is a projective plane, it suffices to show that ϕ_p maps every point row L through p isometrically onto itself. But this follows from the fact that the center of the round sphere L is contained in $p + N_p\mathcal{P}$, and from $\phi_p \mid T_pL = -\mathbf{1}$. □

3.7. DEFINITION. Let $\mathbf{\Phi} = \langle \phi_p \mid p \in \mathcal{P} \rangle$ be the group generated by the reflections ϕ_p.

Since $\mathcal{P}$ is a symmetric space, $\mathbf{\Phi}$ is a normal subgroup of the isometry group of $\mathcal{P}$, and $\mathbf{\Phi}$ is transitive on $\mathcal{P}$.

Now we can give our first proof for Cartan's classification result:

3.8. THEOREM. *Let* $\mathcal{F}^{3m}$ *be a compact isoparametric hypersurface in the sphere* $\mathbb{S}^{3m+1}$ *with three distinct principal curvatures. Then* $\mathcal{F}$ *is a tube around the Veronese embedding of one of the four classical projective planes, as given in Section* 2.

Proof. The group $\mathbf{\Phi}$ is a compact point-transitive collineation group on the compact connected projective plane $(\mathcal{P}, \mathcal{L}, \mathcal{F})$, hence by the results of Salzmann ([17]) and Löwen ([12]), $(\mathcal{P}, \mathcal{L}, \mathcal{F})$ is one of the four classical Moufang planes over the real or complex numbers, over the quaternions or over the octonions, and the identity component of $\mathbf{\Phi}$ is one of the groups $\mathrm{SO}_3\mathbb{R}$, $\mathrm{PSU}_3\mathbb{C}$, $\mathrm{PU}_3\mathbb{H}$ or F_4, respectively.

It remains to show that the embedding $\mathcal{P} \hookrightarrow \mathbb{S}^{3m+1}$ is uniquely determined by the group $\mathbf{\Phi}$. Now $\mathbf{\Phi}$ acts irreducibly on $\mathbb{R}^{3m+2}$, because every orbit is full, and the groups $\mathrm{SO}_3\mathbb{R}, \mathrm{PSU}_3\mathbb{C}$, $\mathrm{PU}_3\mathbb{H}$ and F_4 have only one irreducible representation in dimension 5, 8, 14 and 26, respectively. □

Note that $\mathbf{\Phi}$ is the elliptic motion group of $(\mathcal{P}, \mathcal{L}, \mathcal{F})$; the elliptic polarity is given by the antipodal map $-\mathbf{1}$.

This completes our first proof of Cartan's result. For the second proof, we need the following lemmata:

3.9. LEMMA. *The $\mathbb{Z}_2$-Poincaré polynomial of the focal manifold $\mathcal{P}$ is* $1 + t^{m_1} + t^{2m_1}$.

Proof. This was proved for compact projective planes whose lines are manifolds by Breitsprecher ([1, 2.3]). If we use the fact that $\mathcal{P}$ is the focal submanifold of an isoparametric hypersurface with three distinct principal curvatures we can also refer to Münzner ([14, Satz 5]) and Hebda ([8]). □

3.10. LEMMA. *The group Φ is transitive on $\mathcal{L}$ and hence on the point rows in $\mathcal{P}$.*

Proof. Clearly, Φ commutes with the map -1 that interchanges $\mathcal{P}$ and $\mathcal{L}$. □

3.11. THEOREM. *Let L be a point row. Then L is two-point homogeneous under the stabilizer Φ_L, and $\mathcal{P}$ is two-point homogeneous under the group Φ.*

Proof. The point rows are extrinsically symmetric round spheres under the group $\langle \phi_p \mid p \in L \rangle \leq \Phi_L$ by 3.6, and hence two-point homogeneous.

Since any two points are contained in a point row, and since Φ is transitive on the point rows, $\mathcal{P}$ is two-point homogeneous as well. □

Second proof of Cartan's Theorem. The compact two-point homogeneous symmetric spaces are precisely the compact symmetric spaces of rank 1, see Wolf ([19, 8.12.2]). Using 3.10, we find that $\mathcal{P}$ is isometric to one of the four classical projective planes $\mathcal{P}_2\mathbb{R} = \mathrm{SO}_3\mathbb{R}/\mathrm{O}_2\mathbb{R}$, $\mathcal{P}_2\mathbb{C} = \mathrm{PSU}_3\mathbb{C}/\mathrm{U}_2\mathbb{C}$, $\mathcal{P}_2\mathbb{H} = \mathrm{PU}_3\mathbb{H}/\mathrm{P}(\mathrm{U}_1\mathbb{H} \times \mathrm{U}_2\mathbb{H})$ or $\mathcal{P}_2\mathbb{O} = \mathrm{F}_4/\mathrm{Spin}_9$, and the identity component of Φ is one of the groups $\mathrm{SO}_3\mathbb{R}$, $\mathrm{PSU}_3\mathbb{C}$, $\mathrm{PU}_3\mathbb{H}$ or F_4. Thus the claim follows as in 3.8. □

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