Geometriae Dedicata **55**: 95–114, 1995.

Flag-Homogeneous Compact Connected Polygons

T. GRUNDHÖFER[1], N. KNARR[2] and L. KRAMER[1]
[1]*Mathematisches Institut, Universität Würzburg, Am Hubland, 97074 Würzburg, Germany*
[2]*Institut für Analysis, TU Braunschweig, Pockelstraße 14, 38106 Braunschweig, Germany*

(Received: 15 September 1993; revised version: 2 March 1994)

Abstract. The flag-homogeneous compact connected polygons with equal topological parameters $p = q$ are classified explicitly. These polygons turn out to be Moufang polygons.

Mathematics Subject Classifications (1991): 51H10, 51E12, 51A10.

According to Löwen [34], every point-homogeneous compact connected projective plane is isomorphic to one of the four classical (Moufang) planes over $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ or $\mathbb{O}$. In contrast, there exist point-homogeneous compact connected generalized quadrangles which are not Moufang quadrangles, see 3.9. In this paper, we determine all compact connected polygons which are flag-homogeneous and have equal topological parameters $p = q$, as well as all flag-transitive groups; these polygons are Moufang polygons.

Compact connected polygons are closely related to isoparametric hypersurfaces in spheres. One conjectures that each hypersurface of this kind is the flag manifold of some compact connected polygon. This conjecture is true for all examples presently known, see Thorbergsson [45]. The homogeneous isoparametric hypersurfaces in spheres have been classified by Hsiang and Lawson [26]. These hypersurfaces turn out to be precisely the flag manifolds of the compact connected Moufang polygons. Note that the 'real' hexagon is erroneously omitted from Table II in Hsiang and Lawson [26].

The finite flag-homogeneous generalized n-gons have not yet been classified completely. For $n = 3$, i.e. for projective planes, it is known that such a plane is either pappian, or its full automorphism group is sharply flag-transitive (see Kantor [28] [29], Feit [13], Fink [15], Lunardon and Pasini [35]). For $n = 4$, one knows precisely two finite flag-homogeneous generalized quadrangles which are not Moufang quadrangles. These two quadrangles have parameters (3, 5) and (15, 17), respectively, cp. Lunardon and Pasini [35].

The paper is organized as follows: Sections 1 and 2 contain basic information on the topology of generalized polygons and on compact transformation groups. The classification results are proved in Section 3, and the Appendix describes the algebraic topology of compact connected polygons.

1. Topology of Generalized Polygons

In this section we obtain some basic topological properties of compact (connected) polygons. Our main tool are the 'addition' and 'multiplication' introduced in 1.4 and 1.1.

A generalized n-gon (or n-gon for short) $\mathfrak{P} = (\mathcal{P}, \mathcal{L}, \mathcal{F})$ is a building of rank 2, considered as an incidence structure (cp. Tits [47, 3.34], Ronan [39, 3.2]); the set $\mathcal{F}$ of flags is a subset of the product $\mathcal{P} \times \mathcal{L}$ of the point set $\mathcal{P}$ and the line set $\mathcal{L}$. We consider only thick generalized n-gons with $n \geq 3$.

If both $\mathcal{P}$ and $\mathcal{L}$ are compact (connected) Hausdorff spaces such that $\mathcal{F}$ is closed in the product space $\mathcal{P} \times \mathcal{L}$, then we say that $\mathfrak{P} = (\mathcal{P}, \mathcal{L}, \mathcal{F})$ is a *compact (connected) n-gon*. This definition implies that the usual geometric operations f_{n-1} of $\mathfrak{P}$ are continuous (cp. [22, 2.1]); in particular, projectivities are continuous, hence any two point rows and any two pencils of lines are homeomorphic; if n is odd, then point rows and pencils of lines are homeomorphic (cp. [46, 3.30]). Furthermore, point rows and pencils of lines are doubly-homogeneous closed subspaces in $\mathcal{P}$ and $\mathcal{L}$, respectively (cp. [30, 1.2], [22, 2.1b]). The compact 3-gons are precisely the compact projective planes in the sense of Salzmann [40, §1], (cp. [19, 2.1]).

1.1 PROPOSITION. *Let $\mathfrak{P} = (\mathcal{P}, \mathcal{L}, \mathcal{F})$ be a compact n-gon, and let $x_0, x_1, \ldots, x_{2n-1}, x_{2n} = x_0$ be the points and lines of an ordinary n-gon in $\mathfrak{P}$, with x_i and x_{i+1} incident for all i. Put $\infty_K = x_{2n-1}$, $0_K = x_1$ and $\infty_L = x_{n-2}$, $0_L = x_n$. Let $K \subseteq \mathcal{P} \cup \mathcal{L}$ be the set of all elements incident with x_0, excluding ∞_K, and let $L \subseteq \mathcal{P} \cup \mathcal{L}$ denote the set of elements incident with x_{n-1}, excluding ∞_L. Choose an element $1_L \in L - \{0_L\}$. Then there exists a continuous map*

$$\circ \colon K \times L \to K$$

with the following properties: $x \circ 1_L = x$ for all $x \in K$, the map $x \mapsto x \circ a$ is a homeomorphism of K onto K for every $a \in L - \{0_L\}$, but $x \circ 0_L = 0_K \circ y = 0_K$ for all $x \in K$, $y \in L$.

Proof. As in [31, 1.2], let f_{n-1} denote the continuous geometric operation which is characterized as follows: $f_{n-1}(x, y) = z$ precisely if $x, y \in \mathcal{P} \cup \mathcal{L}$ have distance $n - 1$ in the incidence graph of $\mathfrak{P}$, whereas z has distance 1 from y and distance $n - 2$ from x. Define $\circ$ by

$$x \circ y = f_{n-1}(f_{n-1}(f_{n-1}(f_{n-1}(x, y), x_{2n-2}), 1_L), x_0)$$

for $x \in K$, $y \in L$ (cp. the proof of [31, 2.1]). It is easy to check that $\circ$ is well-defined. Furthermore,

$$f_{n-1}(f_{n-1}(f_{n-1}(x, 1_L), x_{2n-2}), 1_L) = f_{n-1}(x, 1_L),$$

hence $x \circ 1_L = f_{n-1}(f_{n-1}(x, 1_L), x_0) = x$ for $x \in K$, and for $a \in L - \{\infty_L\}$ the map $x \mapsto x \circ a$ is the projectivity $[x_0, a, x_{2n-2}, 1_L, x_0]$ in the notation of [30], hence a homeomorphism. Finally,

$$f_{n-1}(x, 0_L) = x_{n+1} \quad \text{for} \quad x \in K,$$

hence $x \circ 0_L = 0_K$, and $f_{n-1}(0_K, y) = x_{n-1}$ for $y \in L$, hence $0_K \circ y = 0_K$. □

1.2 COROLLARY. *Let $\mathfrak{P}$ be a compact n-gon. If one point row of $\mathfrak{P}$ is non-discrete, then all point rows and all pencils of lines are non-discrete as well* (cp. Burns and Spatzier [8, 1.14]). *In particular, if point rows are finite, then pencils of lines are finite, hence $\mathfrak{P}$ is finite.*

If the point rows are zero-dimensional, then the pencils of lines are zero-dimensional as well.

Recall that a space is zero-dimensional if every point has arbitrarily small neighbourhoods with empty boundaries. A locally compact space is zero-dimensional if and only if it is totally disconnected (cp. [11, 6.2.10]).

Proof. We have to consider only the case that n is even, because otherwise point rows and pencils of lines are homeomorphic via projectivities (cp. [30]).

We assume that point rows are not discrete. Then punctured point rows are not discrete. Let L be a non-discrete punctured point row and let K be a punctured pencil of lines as in the proof of 1.1. Since $\{0_L\}$ is not open in L, there exists a net y_ν converging to 0_L such that $y_\nu \neq 0_L$ for all ν. Choose $z \in K - \{0_K\}$. Now $z \circ y_\nu$ converges to $z \circ 0_L = 0_K$, but $z \circ y_\nu \neq 0_K \circ y_\nu = 0_K$, because $z \mapsto z \circ y_\nu$ is a bijection for every ν. Thus $\{0_K\}$ is not open in K. Since the group of projectivities is transitive on K, both K and $K \cup \{\infty_K\}$ are not discrete. Finally, any two point rows and any two pencils of lines are homeomorphic via projectivities.

Now assume that the punctured point row L has positive dimension. Since L is not totally disconnected and homogeneous, we find a connected set $A \subseteq L$ that contains 0_L and 1_L. For $k \in K - \{0_K\}$, the set $k \circ A$ is connected and contains $0_K = k \circ 0_L$ and $k = k \circ 1_L$, hence K is not totally disconnected. □

Note that the existence of generalized polygons with finite point rows and infinite pencils of lines is an open problem (cp. Brouwer [7]).

1.3 LEMMA. *Let X be a topological Hausdorff space endowed with two continuous binary operations $\pm\colon X^2 \to X$ which satisfy $(x+y)-y = (x-y)+y = x$ for all $x,\ y \in X$. Assume that some element $0 \in X$ satisfies $0 + y = y$ for all $y \in X$. If X has countable neighbourhood bases, then every compact subset C of X has a countable basis.*

Proof. Let $U_i,\ i \in \mathbb{N}$, be a neighbourhood basis at 0 consisting of open sets. The maps $x \mapsto x \pm y$ are homeomorphisms for fixed y. For every $i \in \mathbb{N}$, the compact space $0 + C = C$ is covered by the collection $\{U_i + c | c \in C\}$ of open sets, hence $C \subseteq \bigcup_{c \in C_i} U_i + c$ for finite sets $C_i \subseteq C$. We show that the sets $C \cap (U_i + c)$ with $i \in \mathbb{N},\ c \in C_i$ form a (clearly countable) basis of C.

Let V be a neighbourhood of $a \in C$. For every $i \in \mathbb{N}$ we find $c_i \in C_i$ with $a \in U_i + c_i$. If suffices to show that $U_i + c_i \subseteq V$ for some $i \in \mathbb{N}$. Otherwise we find $u_i \in U_i$ with $u_i + c_i \notin V$ for every $i \in \mathbb{N}$. In view of $a - c_i,\ u_i \in U_i$, the sequences

$a - c_i$ and u_i converge to 0. By compactness of C, every subsequence of c_i has a convergent subsequence, and the latter has limit a because $a = (a - c_i) + c_i$. Thus c_i converges to a, and $u_i + c_i$ converges to $0 + a = a$, a contradiction to $u_i + c_i \notin V$. □

1.4 PROPOSITION. *Let $x_0, x_1, \ldots, x_{2n-1}, x_{2n} = x_0$ be an ordinary n-gon in $\mathfrak{P}$ as in* 1.1. *Since $\mathfrak{P}$ is thick, we find an element $a \in \mathcal{P} \cup \mathcal{L}$ which is incident with x_n and distinct from x_{n-1}, x_{n+1}. Put $0_K = x_0$, $\infty_K = x_2$ and let $K \subseteq \mathcal{P} \cup \mathcal{L}$ denote the set of all elements incident with x_1, excluding $\infty_K = x_2$. Then there exist continuous maps $\pm\colon K \times K \to K$ with $(x + y) - y = (x - y) + y = x$ and $0_K + y = y = y + 0_K$ for all $x, y \in K$.*

Proof. For $x, y \in K$ we define

$$x + y = x^{\pi} \quad \text{and} \quad x - y = x^{\pi^{-1}},$$

where π denotes the projectivity $\pi = [x_1, x_{n+1}, f_{n-1}(a, x_0), x_{n-1}, f_{n-1}(a, y), x_{n+1}, x_1]$ in the notation of [30]. Thus for every $y \in K$, the map $x \mapsto x + y$ is a homeomorphism of K, with inverse map $x \mapsto x - y$ (note that $\infty_K^{\pi} = \infty_K$). Furthermore $0_K + y = y = y + 0_K$ for every $y \in K$. □

1.5 THEOREM. *Let $(\mathcal{P}, \mathcal{L}, \mathcal{F})$ be a compact n-gon. Then the point space $\mathcal{P}$ has a countable basis; hence $\mathcal{P}$ is separable and metrizable. The same holds for $\mathcal{L}$, for every point row, and for every pencil of lines.*

Proof. (1) The point space $\mathcal{P}$ is locally homeomorphic to a product of (finitely many) point rows and pencils of lines.

This is a consequence of the topological labelling (coordinatization) of $\mathfrak{P}$ (see [31, 2.4, 2.5], [22, proof of 2.1]): the space $\mathcal{P}$ is covered by n open pieces, where each piece is geometrically and topologically a product of $n - 1$ factors, and each factor is a punctured point row or a punctured pencil. In the notation of [31, 2.4 (P3)], these open pieces are sets of shape $\Gamma_{n-1}(x, y)$, and the punctured point rows and pencils have form $\Gamma(x, y) = \Gamma_1(x, y)$. See also [21, 2.7] for the special case $n = 4$. The fact that the pieces $\Gamma_{n-1}(x, y)$ are open follows from [22, 2.1].

A compact space which is covered by open subspaces with countable bases has itself a countable basis and is therefore separable and metrizable (cp. Dugundji [10, VIII.7.3, IX.4.1], Engelking [11, 4.2.8]). Thus by (1) and by duality it suffices to prove the assertions on point rows (and pencils).

(2) If a point row is finite, then the polygon is finite and discrete by 1.2, hence we consider only the case that point rows and pencils of lines are not discrete.

Each point row L of a line l (and each pencil) is a doubly homogeneous compact space. Indeed, the point row L is the image of the compact set $(\mathcal{P} \times \{l\}) \cap \mathcal{F}$ under the projection $\mathcal{F} \to \mathcal{P}$ and therefore compact (cp. also [22, 2.1b]).

(3) Each point row (and each pencil) has countable neighbourhood bases.

This can be proved as follows, using the notation from 1.1. Let x_0 be a line. By (double) homogeneity, it suffices to prove that 0_K has a countable neighbourhood basis in the punctured point row K. Since L is homogeneous and infinite, we find elements $b_i \in L - \{0_L\}$, $i \in \mathbb{N}$, such that the sequence b_i accumulates at 0_L. Let U be a compact neighbourhood of 0_K in K. By 1.1, the sets $U \circ b_i$ with $i \in \mathbb{N}$ are neighbourhoods of $0_K = 0_K \circ b_i$. We claim that these sets form a neighbourhood basis at 0_K. Let V be an open subset of K containing 0_K. For every $a \in U$, we have $a \circ 0_L = 0_K$, hence there exist open neighbourhoods V_a and W_a of a in U and 0_L in L, respectively, such that $V_a \circ W_a \subseteq V$. The compact set U is covered by finitely many sets V_a. The intersection W of the corresponding sets W_a is a neighbourhood of 0_L in L with $U \circ W \subseteq V$. Thus $b_i \in W$ and $U \circ b_i \subseteq V$ for infinitely many $i \in \mathbb{N}$.

(4) Each point row L (and each pencil) has a countable basis.

By regularity, L contains two compact disjoint neighbourhoods U_1, U_2, hence $L = (L - U_1) \cup (L - U_2)$ is a union of two open subsets which are not dense in L. Therefore it suffices to show that compact subsets of punctured point rows have countable bases. This follows from 1.3 and 1.4. □

1.6 THEOREM. *Let $\mathfrak{P} = (\mathcal{P}, \mathcal{L}, \mathcal{F})$ be a compact connected n-gon. Then $\mathcal{P}$, $\mathcal{L}$ and $\mathcal{F}$, each point row and each pencil of lines is locally contractible. In fact, each punctured point row and each punctured pencil of lines is locally and globally contractible.*

Proof. (1) Each point row (and each pencil) is locally connected.

From step (1) of the proof above we infer that the punctured pencils of lines and the punctured point rows cannot both be zero-dimensional. Hence neither the point rows nor the pencils of lines are zero-dimensional by 1.2. By duality, it suffices to prove assertion (1) for point rows. Again we use the notation from 1.1. Let x_0 be a line. No neighbourhood in L is zero-dimensional, hence there exists a compact connected subset C of L containing 0_L and at least one more element. Let U be a neighbourhood of 0_K in K. Then

$$U \circ C = \bigcup_{u \in U} u \circ C$$

is connected, since each of the connected sets $u \circ C$ contains $u \circ 0_L = 0_K$. Furthermore,

$$U \circ C = \bigcup_{c \in C - \{0_L\}} U \circ c$$

is open in K, because $x \mapsto x \circ c \colon K \to K$ is a homeomorphism. Hence $U \circ C$ is a connected open neighbourhood of 0_K in K. If U is chosen sufficiently small, then $U \circ C$ will be an arbitrarily small neighbourhood, in view of $0_K \circ L = \{0_K\}$. Thus the punctured point row K is locally connected.

(2) Each punctured point row K (and each punctured pencil) is locally and globally contractible.

Every compact connected locally connected metric space is arcwise connected and locally arcwise connected (see Engelking and Sieklucki [12, 6.5.17, 6.5.19], Kuratowski [33, Chap. VI, §50, II, Th.1, p. 254], Ball [3], Schurle [42, 4.2.5]). Thus, using again the notation of 1.1, we find an arc $\alpha : [0, 1] \to L$ with $\alpha(0) = 0_L$, $\alpha(1) = 1_L$. Then $H: K \times [0, 1] \to K$ with $H(x, t) = x \circ \alpha(t)$ is a homotopy which contracts K to 0_K. A sufficiently small neighbourhood of 0_K can be contracted within a given neighbourhood (by a compactness argument as in step (3) of the proof of the previous Theorem), hence K is also locally contractible. □

The following key result was proved in [31].

1.7 THEOREM. *Let $\mathfrak{P} = (\mathcal{P}, \mathcal{L}, \mathcal{F})$ be a compact connected n-gon such that the lines, considered as point rows, and the pencils of lines are topological manifolds. Then $n \in \{3, 4, 6\}$, i.e. $\mathfrak{P}$ is a projective plane, a quadrangle or a hexagon.*

Furthermore, each point row is homeomorphic to a p-sphere $\mathbb{S}_p$, and each pencil of lines is homeomorphic to a q-sphere $\mathbb{S}_q$, with the following restrictions on p and q:

If $n = 3$, then $p = q \in \{1, 2, 4, 8\}$.
If $n = 4$ and $p, q > 1$, then $p + q$ is odd or $p = q \in \{2, 4\}$.
If $n = 6$, then $p = q \in \{1, 2, 4\}$.
Moreover, $\dim\mathcal{F} = n(p + q)/2 = q + \dim\mathcal{P} = p + \dim\mathcal{L}$. □

This theorem is proved by constructing a 'topological Veronese imbedding' of $\mathcal{P}$, $\mathcal{L}$, and $\mathcal{F}$ into the sphere $\mathbb{S}_{\dim\mathcal{F}+1}$ such that $\mathcal{F}$ is the sphere bundle of normal disk bundles over $\mathcal{P}$ and over $\mathcal{L}$. This leads to a topological problem studied by Münzner [38]. A careful examination of Münzner's proof yields the structure of the cohomology rings of $\mathcal{P}$, $\mathcal{L}$, and $\mathcal{F}$ (the rings as well as the fundamental groups are listed in the Appendix); this in turn leads to the restrictions on the parameters p and q. The restrictions for the case $p = q$ are not given in Münzner [38]; they can be obtained as follows: for $p = q$ even, there are elements $x \in \mathbf{H}^p(\mathcal{P}; \mathbb{Z}_2)$ with $x^2 \neq 0$, hence $p \in \{2, 4, 8\}$ by Adams and Atiyah [1, Th.A]. In the case of the quadrangles and hexagons, there are also elements $x \in \mathbf{H}^p(\mathcal{P}; \mathbb{Z}_3)$ with $x^3 \neq 0$, and thus $p \in \{2, 4\}$, see Adams and Atiyah [1, Th.B], and also Grove and Halperin [18, 6.4].

1.8 AUTOMORPHISM GROUPS. An automorphism of a compact connected n-gon $\mathfrak{P} = (\mathcal{P}, \mathcal{L}, \mathcal{F})$ is a continuous collineation of $\mathfrak{P}$. We endow the group Σ of all automorphisms of $\mathfrak{P}$ with the compact-open topology derived from the action on $\mathcal{P}$ (or on $\mathcal{L}$, which amounts to the same, see [20, Lemma 1]). Then Σ is a topological group with countable basis, acting on $\mathcal{P}$ and on $\mathcal{L}$ as a topological transformation group (cp. Arens [2]).

In fact, Σ is a locally compact group, as Burns and Spatzier [8, 2.1] have shown (note that $n > 2$, and that $\mathcal{P}$, $\mathcal{L}$ are metrizable by 1.5).

1.9 THE BUILDING OF A SIMPLE LIE GROUP. Let Γ be a simple non-compact Lie group with Lie algebra $\mathfrak{g}$, and let K be a maximal compact subgroup with Lie algebra $\mathfrak{k}$. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the Cartan decomposition of $\mathfrak{g}$ (cp. Helgason [25, III.7]), and let $\mathfrak{g} = \mathfrak{k} + \mathfrak{a} + \mathfrak{n}$ be the Iwasawa decomposition of $\mathfrak{g}$ with respect to some maximal abelian subalgebra $\mathfrak{a} < \mathfrak{p}$ and some ordering on the roots (cp. Helgason [25, VI.3]). Let $\Gamma = \mathsf{KAN}$ be the corresponding Iwasawa decomposition of Γ, and let $Z_{\mathsf{K}}(\mathfrak{a})$ and $N_{\mathsf{K}}(\mathfrak{a})$ be the centralizer and the normalizer of $\mathfrak{a}$ in K, respectively. Then $\mathsf{B} = Z_{\mathsf{K}}(\mathfrak{a})\mathsf{AN}$ is a minimal parabolic subgroup of Γ, and $(\Gamma, \mathsf{B}, N_{\mathsf{K}}(\mathfrak{a}))$ is a Tits-system corresponding to a building $\mathfrak{B}(\Gamma)$, cp. Warner [50, 1.2.3]. The flag space of the building is $\Gamma/\mathsf{B} \approx \mathsf{K}/Z_{\mathsf{K}}(\mathfrak{a})$, and the Weyl group of the building $\mathfrak{B}(\Gamma)$ is $N_{\mathsf{K}}(\mathfrak{a})/Z_{\mathsf{K}}(\mathfrak{a})$, hence the rank of the building is the real rank of Γ.

The principal orbits of the action of K on $\mathfrak{p}$ are isoparametric submanifolds diffeomorphic to Γ/B, and the induced map is an isomorphism of buildings [32, Th. 5.28]. In particular, $\mathfrak{B}(\Gamma)$ is a compact connected building, and Γ is the identity component of the automorphism group of $\mathfrak{B}(\Gamma)$.

In the special case of groups Γ of real rank 2, the building is a compact connected polygon. By Cartan's classification, we get the following polygons and groups:

n	Γ^1	(p, q)	n	Γ^1	(p, q)
3	$\mathrm{PGL}_3\mathbb{R}$	$(1, 1)$	4	$\mathrm{PSp}_4\mathbb{C}$	$(2, 2)$
	$\mathrm{PGL}_3\mathbb{C}$	$(2, 2)$		$\mathrm{PU}_{\alpha,5}\mathbb{H}$	$(4, 5)$
	$\mathrm{PGL}_3\mathbb{H}$	$(4, 4)$		$\mathrm{E}_{6(-14)}/Z$	$(9, 6)$
	$\mathrm{E}_{6(-26)}$	$(8, 8)$		$\mathrm{PSO}_{m+2}\mathbb{R}(2)^1$	$(1, m-2)$
6	$\mathrm{G}_{2(2)}/Z$	$(1, 1)$		$\mathrm{PSU}_{m+2}\mathbb{C}(2)$	$(2, 2m-3)$
	$\mathrm{G}_2^{\mathbb{C}}$	$(2, 2)$		$\mathrm{PU}_{m+2}\mathbb{H}(2)$	$(4, 4m-5)$

The parameters (p, q) are given up to duality, i.e. up to exchanging $\mathcal{P}$ and $\mathcal{L}$. Their values can be found in Takagi and Takahashi [44, Table I, II]. The isomorphisms $\mathrm{PSO}_5\mathbb{R}(2)^1 \cong \mathrm{PSp}_4\mathbb{R}$, $\mathrm{PSO}_5\mathbb{C} \cong \mathrm{PSp}_4\mathbb{C}$, $\mathrm{PSO}_6\mathbb{R}(2)^1 \cong \mathrm{PSU}_4\mathbb{C}(2)$, and $\mathrm{PU}_{\alpha,4}\mathbb{H} \cong \mathrm{PSO}_8\mathbb{R}(2)^1$ induce isomorphisms of the corresponding polygons (see 2.1 for the terminology for the groups). For more details see [32, Kap. 5].

2. Compact Transformation Groups

In this section we collect some facts on topological transformation groups for later use.

2.1 DEFINITION. Let Γ be a topological group. The identity component of Γ is denoted by Γ^1.

The groups $\mathrm{GL}_n\mathbb{K}$, $\mathrm{SL}_n\mathbb{K}$, with $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ and $\mathrm{U}_n\mathbb{C}$, $\mathrm{U}_n\mathbb{H}$, $\mathrm{SU}_n\mathbb{C}$, $\mathrm{SO}_n\mathbb{R}$, $\mathrm{SO}_n\mathbb{C}$ are defined as usual. A number in parentheses indicates the index of the hermitian form, eg. $\mathrm{U}_n\mathbb{C}(2)$ is the group that preserves the form $\Sigma_{i\leq n-2}||z_i||^2 - ||z_{n-1}||^2 - ||z_n||^2$. The symplectic groups $\mathrm{Sp}_{2n}\mathbb{K}$, $\mathbb{K} = \mathbb{R}, \mathbb{C}$ are the groups that leave the symplectic form $\Sigma_{i\leq n} x^i \wedge x^{n+i}$ invariant. We put $\mathrm{USp}_{2n}\mathbb{C} = \mathrm{U}_{2n}\mathbb{C} \cap \mathrm{Sp}_{2n}\mathbb{C}$; there is an isomorphism $\mathrm{USp}_{2n}\mathbb{C} \cong \mathrm{U}_n\mathbb{H}$ (Helgason [25] denotes this group by Sp_n).

For the exceptional groups we adopt the following convention: $\mathrm{G}_2, \mathrm{F}_4$ etc. are the compact exceptional groups, $\mathrm{G}_2^{\mathbb{C}}$ etc. are the complex exceptional groups, and the noncompact real forms are denoted by $\mathrm{G}_{2(2)}$ etc. as in Helgason [25]. A prefix P denotes the group induced on the corresponding projective space.

2.2 THEOREM (Szenthe). *Let Γ be a locally compact group with a countable basis, acting effectively and transitively on a connected, locally contractible space X. Then Γ is a Lie group, and X is a (real-analytic) manifold.*

Proof. The approximation theorem implies that Γ has an open subgroup Δ such that Δ/Δ^1 is compact (see Montgomery and Zippin [37, IV], Gluškov [17, Th. 9]). Evaluation of Γ at a point of X is open (see Freudenthal [16], Hohti [27]), hence also Δ is transitive on the connected space X. Furthermore, Γ and Δ are σ-compact, (cp. Dugundji [10, XI.6.3]). We infer from Szenthe [43, Th. 4] that Δ, and hence Γ as well, are Lie groups, and that X is a homogeneous space of Γ, hence a real-analytic manifold. Actually, it is not clear whether Szenthe's hypothesis of σ-compactness suffices for his proof, because he uses approximation by a well-ordered chain of Lie groups. In our situation, Δ has a countable basis, hence we can approximate even by *sequences* of Lie groups, and this resolves the problems in Szenthe's proof (cp. also Bickel [4, Kap. 6]). □

2.3 THEOREM. *Let Γ be a transitive Lie transformation group on the connected space $X \approx \Gamma/\Gamma_x$. Then the identity component Γ^1 of Γ acts also transitively, because the evaluation map is open. If X is compact and has a finite fundamental group, then any maximal compact subgroup of Γ acts transitively on X* (cp. Montgomery [36, Cor. 3]). □

Let Γ be a compact, connected Lie group. The rank m of Γ is the dimension of a maximal torus $\mathbb{T}^m$ in Γ. Let $N(\mathbb{T}^m)$ be the normalizer of this torus; the Weyl group of Γ is $W(\Gamma) = N(\mathbb{T}^m)/\mathbb{T}^m$.

There exists a compact covering group $\tilde{\Gamma}$ or Γ such that $\tilde{\Gamma} = \mathbb{T}^n \times \Phi_1 \times \cdots \times \Phi_k$ is a direct product of an n-torus $\mathbb{T}^n$ and of almost simple, simply connected Lie groups Φ_j.

2.4 THEOREM. *Let Γ be a compact, connected, effective and transitive Lie transformation group on $X \approx \Gamma/\Gamma_x$. Assume moreover that the Euler number of X is positive, and that X is simply connected, i.e. that the isotropy group Γ_x is connected. Then we have the following facts:*

(i) $\Gamma = \Gamma_1 \times \cdots \times \Gamma_k$ *is a direct product of simple Lie groups* Γ_i, *and* $\Gamma_x = \Gamma_{1,x} \times \cdots \times \Gamma_{k,x}$. *Moreover, rank* $\Gamma_i = \operatorname{rank} \Gamma_{i,x}$.

(ii) *For the Euler number we have the formula*

$$\chi(X) = W(\Gamma) \colon W(\Gamma_x) = \textstyle\prod_i (W(\Gamma_i) \colon W(\Gamma_{i,x})).$$

(iii) *The number k cannot exceed the number of the prime factors of $\chi(X)$; in particular, if $\chi(X)$ is a prime, then Γ is simple.*

Proof. (i) follows from the proof of Theorem 8.10.16 in Wolf [51]. The formula for the Euler number can be found in [48]. For (iii), it suffices to check the inequality $W(\Gamma): W(\Gamma_x) > 1$ for simple groups Γ and for maximal connected subgroups Γ_x of maximal rank. This can be seen from Table I below. □

By (ii), the investigation of this kind of homogeneous spaces is reduced to the case where Γ is a simple Lie group and Γ_x is a connected subgroup of the same rank as Γ. The maximal connected subgroups of maximal rank have been classified by Borel and De Siebenthal (cp. [6], [51, 8.10.9]); they are given by Table I. Each pair of Lie algebras corresponds with exactly *one* pair consisting of the simple Lie group and the conjugacy class of the isotropy group, i.e. up to conjugation, the imbedding of the subgroup is unique (Wolf [51, 8.10.8]).

3. Flag-Transitive Groups

In this section we classify all compact connected polygons which admit a flag-transitive group of automorphisms and have equal parameters $p = q$. Note that for projective planes and for generalized hexagons, the condition $p = q$ is automatically satisfied (see 1.7). As a byproduct, we obtain also a list of all (closed) flag-transitive groups on these polygons by using the fact that every maximal compact subgroup of an almost simple Lie group is a maximal subgroup (see Helgason [25, Chap. VI, Ex. A3(iv), pp. 276, 567]).

In this section we use the following notation: $\mathfrak{P} = (\mathcal{P}, \mathcal{L}, \mathcal{F})$ is a compact connected n-gon, Σ is the automorphism group of $\mathfrak{P}$, and Γ is a closed subgroup of Σ acting transitively on the flag space $\mathcal{F}$. Notice that Σ and every closed subgroup of Σ is a Lie group by 1.6, 2.2. Given a flag $(x, l) \in \mathcal{F}$, we have $\Gamma/\Gamma_x \approx \mathcal{P}$, $\Gamma/\Gamma_l \approx \mathcal{L}$, and $\Gamma/\Gamma_{x,l} \approx \mathcal{F}$. Let L be the point row corresponding to l, and $\mathcal{L}_x$ be the pencil of lines through x. Then $\Gamma_l/\Gamma_{x,l} \approx L$ and $\Gamma_x/\Gamma_{x,l} \approx \mathcal{L}_x$ are topological (in fact real-analytic) submanifolds of $\mathcal{P}$ and $\mathcal{L}$, respectively. Therefore we can apply 1.7.

Our classification method is roughly as follows: Given an incidence structure $(\mathcal{P}, \mathcal{L}, \mathcal{F})$ with a flag-transitive (not necessarily effective) group G that acts incidence-preserving, we may pick any flag (x, l) and replace the incidence structure $\mathfrak{P} = (\mathcal{P}, \mathcal{L}, \mathcal{F})$ by the isomorphic coset geometry $(G/G_x, G/G_l, G/G_{x,l})$ (Recall that in a coset geometry, two cosets are called incident, if they have non-empty intersection).

In order to prove the classification result, we have to show in each case that the triple (G, G_x, G_l) is uniquely determined (up to automorphisms of G) by the topological and geometrical properties of the polygon $\mathfrak{P}$.

TABLE I. We list the Lie algebras of the simple groups Γ as in 2.4, the Lie algebras of the isotropy groups Γ_x, the Euler numbers of the corresponding homogeneous spaces $X = \Gamma/\Gamma_x$, and their dimensions.

Simple algebra	Maximal subalgebra	$\chi(X)$	$\dim X = \dim \Gamma - \dim \Gamma_x$	
$\mathfrak{a}_n$	$\mathfrak{a}_i + \mathfrak{a}_{n-i-1} + \mathbb{R}$	$\binom{n+1}{i+1}$	$2(n-i)(i+1)$	$X = \mathcal{P}_n\mathbb{C}$ for $i = 0$
$\mathfrak{b}_n$	$\mathfrak{d}_n$	2	$2n$	$X = \mathbb{S}_{2n}$
	$\mathfrak{b}_i + \mathfrak{d}_{n-i}$ [a]	$2\binom{n}{i}$	$2(n-i)(2i+1)$	
	$\mathfrak{b}_{n-1} + \mathbb{R}$	$2n$	$4n-2$	
$\mathfrak{c}_n$	$\mathfrak{c}_i + \mathfrak{c}_{n-i}$ [b]	$\binom{n}{i}$	$4(n-i)i$	$X = \mathcal{P}_{n-1}\mathbb{H}$ for $i = 1$
	$\mathfrak{a}_{n-1} + \mathbb{R}$	2^n	$n(n+1)$	
$\mathfrak{d}_n$	$\mathfrak{d}_i + \mathfrak{d}_{n-i}$ [c]	$2\binom{n}{i}$	$4(n-i)i$	
	$\mathfrak{a}_{n-1} + \mathbb{R}$	2^{n-1}	$n(n-1)$	
	$\mathfrak{d}_{n-1} + \mathbb{R}$	$2n$	$4(n-1)$	
$\mathfrak{e}_6$	$\mathfrak{a}_1 + \mathfrak{a}_5$	36	40	
	$\mathfrak{a}_2 + \mathfrak{a}_2 + \mathfrak{a}_2$	240	54	
	$\mathfrak{d}_5 + \mathbb{R}$	27	32	
$\mathfrak{e}_7$	$\mathfrak{a}_1 + \mathfrak{d}_6$	63	64	
	$\mathfrak{a}_2 + \mathfrak{a}_5$	672	90	
	$\mathfrak{a}_7$	72	70	
	$\mathfrak{e}_6 + \mathbb{R}$	56	54	
$\mathfrak{e}_8$	$\mathfrak{d}_8$	135	128	
	$\mathfrak{a}_8$	1920	168	
	$\mathfrak{a}_4 + \mathfrak{a}_4$	48384	200	
	$\mathfrak{e}_6 + \mathfrak{a}_2$	2240	162	
	$\mathfrak{e}_7 + \mathfrak{a}_1$	120	112	
$\mathfrak{f}_4$	$\mathfrak{b}_4$	3	16	$X = \mathcal{P}_2\mathbb{O}$
	$\mathfrak{a}_2 + \mathfrak{a}_2$	32	36	
	$\mathfrak{a}_1 + \mathfrak{c}_3$	12	28	
$\mathfrak{g}_2$	$\mathfrak{a}_2$	2	6	$X = \mathbb{S}_6$
	$\mathfrak{a}_1 + \mathfrak{a}_1$	3	8	$X =$ 'fake $\mathcal{P}_2\mathbb{H}$'

[a] $0 < i < n-1$; [b] $0 < i < n$; [c] $1 < i < n-1$

The fact that a generalized n-gon is connected in the graph-theoretic sense (because any two elements are contained in some ordinary n-gon), translates into the condition that the stabilizers G_x and G_l generate G.

Below we require the following lemma.

3.1 LEMMA. *Let G be a group with subgroups $A,\ B \leq G$, and let $k \in \mathbb{N}$. Then the incidence structure $(G/A,\ G/B,\ G/A \cap B)$ contains an ordinary k-gon, if there exist elements $a_1,\ a_2, \ldots,\ a_k \in A - B,\ b_1,\ b_2, \ldots,\ b_k \in B - A$ such that the product $a_1 b_1 a_2 b_2 \cdots a_k b_k$ is contained in $A \cap B$.*

Proof. The cosets $B,\ A,\ a_1 B,\ a_1 b_1 A,\ a_1 b_1 a_2 B,\ a_1 b_1 a_2 b_2 A, \ldots,\ a_1 b_1 a_2 \cdots a_{k-1} b_{k-1} A$ are the points and lines of an ordinary k-gon, because $a_1 b_1 a_2 b_2 \cdots a_k B = B$. □

3.2 LEMMA. *Let $p = q = 1$ and let Φ be a closed subgroup of Σ fixing a flag. If Φ is compact, then Φ is finite. In particular, a compact subgroup of Σ is at most n-dimensional, and if it is n-dimensional, then it is transitive on $\mathcal{F}$.*

Proof. Let Φ^1 be the identity component of Φ, and let $(x,\ \ell)$ be a flag fixed by Φ^1. Since Φ^1 has a fixed point on the point row $L \approx \mathbb{S}_1$ corresponding to ℓ and on the pencil of lines $\mathcal{L}_x \approx \mathbb{S}_1$, the action both on L and $\mathcal{L}_x$ is trivial (the orbits are connected proper submanifolds of $\mathbb{S}_1$, and hence points). Hence Φ^1 fixes every flag of the form $(x,\ h)$ and $(y,\ \ell)$, that is, it fixes the panels through $(x,\ \ell)$ elementwise. Since a generalized polygon is a connected incidence structure (in the graph-theoretic sense), Φ^1 fixes every flag in $\mathcal{F}$, and thus $\Phi^1 = 1$. Therefore the orbits of Φ in $\mathcal{F}$ are submanifolds of the same dimension as Φ. □

3.3 THEOREM. *If $n = 3$ and $p = q = 1$, then $\mathfrak{P}$ is the real projective plane, and $\Gamma = \mathrm{SO}_3\mathbb{R}$ or $\Gamma = \mathrm{PGL}_3\mathbb{R} = \mathrm{SL}_3\mathbb{R} = \Sigma$. (This is a weak version of the main result of Salzmann* [41].)

Proof. Let Δ be a maximal compact connected subgroup of Γ. By 2.3, Δ is transitive on $\mathcal{F}$, and by Lemma 3.2, $\dim\ \Delta = \dim\ \mathcal{F} = 3$. Since the fundamental group of $\mathcal{F}$ is finite (cf. Appendix 3_1), the group Δ is of type $\mathfrak{a}_1$, hence the universal covering group $\tilde{\Delta} = \mathrm{U}_1\mathbb{H}$ operates almost effectively on $\mathfrak{P}$. Now $\mathcal{F} = \mathrm{U}_1\mathbb{H}/\langle i,\ j\rangle$, because $\pi_1(\mathcal{F}) = \langle i,\ j\rangle$ is the quaternion group of order 8. The stabilizers of points and lines are one-dimensional and have two components, hence they are conjugates of the group $\mathrm{U}_1\mathbb{C} \cup j \cdot \mathrm{U}_1\mathbb{C} \subseteq \mathrm{U}_1\mathbb{H}$.

For a suitable flag $(x,\ \ell)$ we have $\tilde{\Delta}_{x,\ell} = \langle i,\ j\rangle$, and this determines $\tilde{\Delta}_x$ and $\tilde{\Delta}_\ell$ up to automorphisms of $\mathbb{H}$ permuting $\langle i,\ j\rangle$. Hence $\mathfrak{P} = (\mathcal{P},\ \mathcal{L},\ \mathcal{F})$ is isomorphic to the real projective plane. Finally the center of $\tilde{\Delta}$ is contained in $\tilde{\Delta}_{x,\ell}$, thus $\Delta = \mathrm{SO}_3\mathbb{R}$. □

3.4 THEOREM. *If $n = 4$ and $p = q = 1$, then $\mathfrak{P}$ is the real symplectic quadrangle or its dual, and Γ is one of the groups $\mathrm{SO}_3\mathbb{R} \times \mathrm{SO}_2\mathbb{R}$, $\mathrm{S}(\mathrm{O}_3\mathbb{R} \times \mathrm{O}_2\mathbb{R})$, $\mathrm{SO}_5\mathbb{R}(2)^1$, $\Sigma = \mathrm{SO}_5\mathbb{R}(2)$.*

Proof. By duality we may assume that $\pi_1(\mathcal{P}) = \mathbb{Z}_2$ (cf. Appendix 4_1). Let $\tilde{\mathcal{P}}$ be the universal covering space of $\mathcal{P}$, and let Δ be a maximal compact connected subgroup of Γ. Then a suitable compact covering group $\tilde{\Delta}$ acts transitively on $\tilde{\mathcal{P}}$. Now $\dim\ \Delta \leq 4$ by 3.2, hence $\tilde{\Delta} = \mathrm{SU}_2\mathbb{C} \times \mathrm{T}$, and T is a 1-torus or trivial.

Hence $\mathcal{P}$ is homeomorphic to real projective 3-space $\mathcal{P}_3\mathbb{R}$. The image of $\mathrm{SU}_2\mathbb{C}$ in Δ is isomorphic to $\mathrm{SO}_3\mathbb{R}$, because Δ operates effectively. Therefore $\mathrm{SO}_3\mathbb{R} \leq \Delta$ is sharply transitive on $\mathcal{P} \approx \mathcal{P}_3\mathbb{R}$.

Now $\mathrm{SO}_3\mathbb{R}$ cannot act transitively on the three-dimensional line space $\mathcal{L}$, because $\pi_1(\mathcal{L}) = \mathbb{Z}$ is infinite. Thus $\mathrm{SO}_3\mathbb{R}$ cannot have a three-dimensional orbit on $\mathcal{L}$. Hence each stabilizer in $\mathrm{SO}_3\mathbb{R}$ of a line ℓ contains a one-dimensional subgroup $\Phi(\ell) \cong \mathrm{SO}_2\mathbb{R}$. Since $\mathrm{SO}_3\mathbb{R}$ is sharply transitive on the point space $\mathcal{P}$, the orbits of $\Phi(\ell)$ on $\mathcal{P}$ are all one-dimensional. In fact, they are projective lines in the real projective three-dimensional space $\mathrm{P}_3\mathbb{R} \approx \mathcal{P}$, because the action of $\mathrm{SO}_3\mathbb{R}$ on $\mathcal{P}$ is induced by the regular action of $\mathrm{U}_1\mathbb{H}$ on $\mathbb{H} = \mathbb{R}^4$. On the other hand, the point row corresponding to ℓ has to be among the orbits of $\Phi(\ell)$. Hence every point row of the quadrangle $\mathfrak{P}$ is a point row in $\mathrm{P}_3\mathbb{R} \approx \mathcal{P}$. By the result of Dienst [9] on quadrangles embedded into projective spaces, $\mathfrak{P}$ is the real symplectic quadrangle. □

3.5 THEOREM. *If $n = 6$ and $p = q = 1$, then $\mathfrak{P}$ is one of the two mutually dual 'real' hexagons associated to the group $\mathrm{G}_{2(2)}/Z$, and $\Gamma = \mathrm{SO}_4\mathbb{R}$ or $\Gamma = \mathrm{G}_{2(2)}/Z = \Sigma$.*

Proof. Let Δ be a maximal compact connected subgroup of Γ. By Theorem 2.3(i), Δ is transitive on $\mathcal{F}$, and by Lemma 3.2, $\dim \Delta = \dim \mathcal{F} = 6$. Since $\pi_1(\mathcal{F})$ is finite, (cf. Appendix 6_1), Δ is of type $\mathfrak{a}_1 + \mathfrak{a}_1$, hence $\tilde{\Delta} = \mathrm{U}_1\mathbb{H} \times \mathrm{U}_1\mathbb{H}$ operates almost effectively on $\mathfrak{P}$.

Let (x, ℓ) be a flag. The isotropy groups $\mathsf{A} = \tilde{\Delta}_x$ and $\mathsf{B} = \tilde{\Delta}_\ell$ are one-dimensional subgroups which are not connected, because their intersection $\mathsf{A} \cap \mathsf{B} = \tilde{\Delta}_{x,\ell}$ is isomorphic to the quaternion group Q_8, and hence not commutative.

Up to conjugation we may assume that $\mathsf{A}^1 = \{(e^{iat}, e^{ibt}) | t \in \mathbb{R}\}$ with integers $a, b \in \mathbb{Z}$. As $\mathfrak{P}$ is a connected incidence structure in the graph-theoretic sense, we have $\tilde{\Delta} = \langle \mathsf{A}, \mathsf{B} \rangle$, and this implies $a \neq 0$ and $b \neq 0$. Thus the normalizer of A^1 in $\tilde{\Delta}$ is the group $(\mathrm{U}_1\mathbb{C} \times \mathrm{U}_1\mathbb{C})\langle (j, j) \rangle$. By conjugation with elements from $\mathrm{U}_1\mathbb{C} \times \mathrm{U}_1\mathbb{C}$ we can achieve that $\mathsf{A} = \mathsf{A}^1\langle (j, j) \rangle$. One of the two elements of order 4 in A^1 has the form $(i, \varepsilon i)$, with $\varepsilon = \pm 1$. Up to conjugation by $(1, j)$ (which replaces b by $-b$), we may assume that $\varepsilon = +1$. Since all copies of the quaternion group Q_8 in A are conjugate in A, we may further assume that $\mathsf{A} \cap \mathsf{B} = \langle (i, i), (j, j) \rangle$.

Next we wish to determine the isotropy group B. The connected component B^1 is a conjugate of a group $\{(e^{jct}, e^{jdt}) | t \in \mathbb{R}\}$, with non-zero integers c, d. We have $(j, j) \in \mathsf{B}^1$ or $(ij, ij) \in \mathsf{B}^1$, because otherwise $(i, i) \in \mathsf{B}^1$, a contradiction to $\langle \mathsf{A}, \mathsf{B} \rangle = \tilde{\Delta}$. By conjugation with $(1/\sqrt{2})(1 + i, 1 + i)$ we can achieve that $\mathsf{B}^1 = \{(e^{jct}, e^{jdt}) | t \in \mathbb{R}\}$ and $\mathsf{B} = \mathsf{B}^1\langle (i, i) \rangle$.

Now we aim at showing that there is (up to duality) essentially just one possibility for the numbers a, b, c, d. The comparison with the two mutually dual 'real' hexagons associated to $\mathrm{G}_{2(2)}$ then proves the assertion. Clearly we may assume that a, b are coprime, as well as c, d. Since $(i, i) \in \mathsf{A}^1$, the integers a, b are odd and congruent mod 4. Replacing (a, b) by $(-a, -b)$ does not change the group

A^1, hence we may assume that $a,\ b \in 1 + 4\mathbb{Z}$. By the same reasoning we may assume $c,\ d \in 1 + 4\mathbb{Z}$.

Now we claim that $a \in \{1,\ -3\}$. For the proof, we consider the product

$$(z,\ w) = (\mathrm{e}^{iat},\ \mathrm{e}^{ibt})(\mathrm{e}^{jcs},\ \mathrm{e}^{jds})(\mathrm{e}^{iat'},\ \mathrm{e}^{ibt'})(\mathrm{e}^{jcs'},\ \mathrm{e}^{jds'}) \in \mathsf{ABAB}$$

with $s,\ t,\ s',\ t' \in \mathbb{R}$. The real parts of the quaternions $z,\ w$ are given by

$$\begin{aligned} \operatorname{Re} z &= \cos\, cs\, \cos\, cs'\, \cos\, a(t+t') - \sin\, cs\, \sin\, cs'\, \cos\, a(t-t') \quad \text{and} \\ \operatorname{Re} w &= \cos\, ds\, \cos\, ds'\, \cos\, b(t+t') - \sin\, ds\, \sin\, ds'\, \cos\, b(t-t'). \end{aligned}$$

Specializing to $t = \pi/a,\ t' = -\pi/2a$, we obtain $t + t' = \pi/2a,\ t - t' = 3\pi/2a$, hence $\operatorname{Re} z = 0$, and

$$\operatorname{Re} w = \cos\, ds\, \cos\, ds'\, \cos\left(\frac{b\pi}{2a}\right) - \sin\, ds\, \sin\, ds'\, \cos\left(\frac{3b\pi}{2a}\right).$$

Assume that $a \notin \{1,\ -3\}$. Then $a \notin \{\pm 1,\ \pm 3\}$ in view of $a \in 1 + 4\mathbb{Z}$, hence neither b/a nor $3b/a$ is an integer. In particular, $\cos(b\pi/2a) \neq 0 \neq \cos(3b\pi/2a)$, and therefore we find solutions $s,\ s'$ of the equation

$$\tan\, ds\, \tan\, ds'\, \cos\left(\frac{3b\pi}{2a}\right) = \cos\left(\frac{b\pi}{2a}\right) \neq 0.$$

For such solutions $s,\ s'$, we have $\operatorname{Re} w = 0$. Note that $ds,\ ds' \notin \frac{\pi}{2}\mathbb{Z}$. Because $z,\ w$ are quaternions of norm 1, we infer that $(z, w)^2 = (z^2,\ w^2) = (-1,\ -1) \in \mathsf{A} \cap \mathsf{B}$. Since $(\mathrm{e}^{iat},\ \mathrm{e}^{ibt}),\ (\mathrm{e}^{iat'},\ \mathrm{e}^{ibt'}) \in \mathsf{A}$ and $(\mathrm{e}^{jcs},\ \mathrm{e}^{jds}),\ (\mathrm{e}^{jcs'},\ \mathrm{e}^{jds'}) \in \mathsf{B}$, and because the hexagon $\mathfrak{P}$ contains no ordinary quadrangles, we deduce from Lemma 3.1 that at least one of the four elements $(\mathrm{e}^{iat},\ \mathrm{e}^{ibt}),\ (\mathrm{e}^{jcs},\ \mathrm{e}^{jds}),\ (\mathrm{e}^{iat'},\ \mathrm{e}^{ibt'}),\ (\mathrm{e}^{jcs'},\ \mathrm{e}^{jds'})$ belongs to $\mathsf{A} \cap \mathsf{B} = \langle (i,\ i),\ (j,\ j) \rangle$. This is a contradiction to our choices for $t,\ t',\ s,\ s'$, and this contradiction proves that $a \in \{1,\ -3\}$.

Replacing the hexagon $\mathfrak{P}$ by its dual amounts to exchanging $(a,\ b)$ with $(c,\ d)$. Furthermore, permuting the two factors of $\tilde{\Delta}$ is a group automorphism of $\tilde{\Delta}$ which switches $(a,\ b)$ as well as $(c,\ d)$. Thus we infer from the preceding paragraph that $a,\ b,\ c,\ d \in \{1,\ -3\}$, and it remains to consider the four possibilities where $(a,\ b,\ c,\ d)$ is $(1,\ 1,\ 1,\ 1)$, $(1,\ -3,\ 1,\ -3)$, $(1,\ -3,\ -3,\ 1)$ or $(1,\ 1,\ 1,\ -3)$ (remember that $a,\ b$ and $c,\ d$ are coprime). In order to exclude the first three cases, we consider the product

$$(z,\ w) = (\mathrm{e}^{ia\pi/4},\ \mathrm{e}^{ib\pi/4})(\mathrm{e}^{jc\pi/4},\ \mathrm{e}^{jd\pi/4}) \in \mathsf{AB}.$$

We compute that in the first three cases $\operatorname{Re} z = \operatorname{Re} w \in \{\pm\frac{1}{2}\}$, hence the quaternions $z,\ w$ of norm 1 satisfy $z^3 = w^3 = \pm 1$. Thus $(z,\ w)^3 = \pm(1,\ 1) \in \mathsf{A} \cap \mathsf{B}$. Now Lemma 3.1 implies that the hexagon $\mathfrak{P}$ contains an ordinary triangle, a contradiction.

This contradiction shows $(a, b, c, d) = (1, 1, 1, -3)$, hence the pair (A, B) is determined uniquely (up to duality, and up to automorphisms of $\tilde{\Delta}$). □

3.6 PROPOSITION. *If $p, q > 1$, or if $p = q = 1$, then every maximal compact subgroup of a flag-transitive group is still flag transitive.*

Proof. If $p, q > 1$, then $\mathcal{F}$ is simply connected (see Appendix 3_2, 4_4 and 6_2), and the result follows from 2.3. The case $p = q = 1$ is covered by Theorems 3.3, 3.5 and 3.4. □

3.7 THEOREM. *If $p = q > 1$, then the identity component of Γ is simple, and one of the following cases occurs:*

(i) *$n = 3$ and $p = q = 2$. Then $\mathfrak{P}$ is the complex projective plane and Γ is one of the groups* $\mathrm{PSU}_3\mathbb{C}$, $\langle \mathrm{PSU}_3\mathbb{C}, x \mapsto \bar{x}\rangle$, $\mathrm{PGL}_3\mathbb{C}$, $\Sigma = \langle \mathrm{PGL}_3\, \mathbb{C}, x \mapsto \bar{x}\rangle$.
(ii) *$n = 3$ and $p = q = 4$. Then $\mathfrak{P}$ is the projective plane over the quaternion skew field $\mathbb{H}$, and Γ is one of the groups* $\mathrm{PU}_3\mathbb{H}$, $\Sigma = \mathrm{PGL}_3\mathbb{H}$.
(iii) *$n = 3$ and $p = q = 8$. Then $\mathfrak{P}$ is the projective plane over the Cayley numbers $\mathbb{O}$, and Γ is one of the groups* F_4, $\Sigma = \mathrm{E}_{6(-26)}$.
(iv) *$n = 4$ and $p = q = 2$. Then $\mathfrak{P}$ is the complex symplectic quadrangle or its dual, and Γ is one of the groups* $\mathrm{PUSp}_4\mathbb{C}$, $\langle \mathrm{PUSp}_4\mathbb{C}, x \mapsto \bar{x}\rangle$, $\mathrm{PSp}_4\mathbb{C}$, $\Sigma = \langle \mathrm{PSp}_4\, \mathbb{C}, x \mapsto \bar{x}\rangle$.
(v) *$n = 6$ and $p = q = 2$. Then $\mathfrak{P}$ is one of the two mutually dual 'complex' hexagons associated to the group $\mathrm{G}_2^{\mathbb{C}}$, and Γ is one of the groups* G_2, $\langle \mathrm{G}_2, x \mapsto \bar{x}\rangle$, $\mathrm{G}_2^{\mathbb{C}}$, $\Sigma = \langle \mathrm{G}_2^{\mathbb{C}}, x \mapsto \bar{x}\rangle$.

For $n = 3$ see also Löwen [34] for a stronger result.

Proof. The spaces $\mathcal{P}$, $\mathcal{L}$, and $\mathcal{F}$ are simply connected, cp. Appendix 3_2, 4_4 and 6_2. Let Δ be a compact connected flag-transitive subgroup of Γ. Now $\chi(\mathcal{P}) = n \in \{3, 4, 6\}$, hence Δ is simple for $n = 3$, and Δ is a product at most two simple groups for $n = 4, 6$, see 2.4. By Table I, the simply connected homogeneous spaces with Euler number 2 are the even-dimensional spheres, and the simply connected homogeneous spaces with Euler number 3 are the classical projective planes of dimension 4, 8, 16, and in addition $\mathrm{G}_2/\mathrm{SO}_4\mathbb{R}$. The latter space has $1 + t^2 + t^3 + t^4 + t^5 + t^6 + t^8$ as $\mathbb{Z}_2$-Poincaré polynomial (Borel [5, 13.1]) which excludes this space for $n = 3$. Now it is clear from the structure of $\mathbf{H}^*(\mathcal{P}; R)$ that for $n = 4, 6$, the space $\mathcal{P}$ is not a product of a sphere with one of the spaces mentioned above. Hence Δ is simple also in these cases. Let Δ_x denote the stabilizer of a point $x \in \mathcal{P}$.

$n = 3$: We have shown that $(\Delta, \Delta_x) \in \{(\mathrm{PSU}_3\mathbb{C}, \mathrm{U}_2\mathbb{C}), (\mathrm{PU}_3\mathbb{H}), \mathrm{P}(\mathrm{U}_1\mathbb{H} \times \mathrm{U}_2\mathbb{H}), (\mathrm{F}_4, \mathrm{Spin}_9)\}$. Thus $\mathcal{P}$ and $\mathcal{L}$ are homeomorphic to the point space of one of the classical projective planes, and the stabilizers of arbitrary lines and points are conjugate. It remains to recover the point rows.

The stabilizer Δ_x of x has exactly one p-dimensional orbit $K \approx \mathbb{S}_p$ on $\mathcal{P}$, the other orbits are $\{x\}$ and a family of $(2p - 1)$-dimensional spheres. (This can be seen by using geodesic polar coordinates. The cut locus K is the polar line of x in the classical plane.) On the other hand, we know that $\Delta_x = \Delta_\ell$ for some line ℓ not incident with x. Hence K is contained in the point row L corresponding to ℓ, and for dimensional reasons $K = L$. This completes the proof for $n = 3$.

$n = 4$: There are two possibilities for Δ_x:

(1) Δ_x is a maximal connected subgroup. With the restriction $\dim \Delta - \dim \Delta_x \in \{6, 12\}$, we get $(\Delta, \Delta_x) \in \{(\mathrm{PU}_2\mathbb{H}, \mathrm{U}_2\mathbb{C}/\pm 1), (\mathrm{PSU}_4\mathbb{C}, \mathrm{U}_3\mathbb{C}), (\mathrm{PU}_4\mathbb{H}, \mathrm{P}(\mathrm{U}_1\mathbb{H} \times \mathrm{U}_3\mathbb{H}))\}$, up to conjugation.

(2) If Δ_x is not maximal, then there is some maximal connected subgroup Φ with $\Delta > \Phi > \Delta_x$, and Δ/Φ, Φ/Δ_x are even-dimensional spheres. The only possibility (up to conjugation) is $(\Delta, \Phi, \Delta_x) = (\mathrm{PU}_2\mathbb{H}, (\mathrm{U}_1\mathbb{H} \times \mathrm{U}_1\mathbb{H})/\pm 1, (\mathrm{U}_1\mathbb{H} \times \mathrm{U}_1\mathbb{C})/\pm 1)$.

The groups $\mathrm{PSU}_4\mathbb{C}$ and $\mathrm{PU}_4\mathbb{H}$ are excluded by the fact that they contain only one conjugacy class of possible stabilizers of points and lines, since $\mathcal{P}$ and $\mathcal{L}$ are not homeomorphic. Thus $\Delta = \mathrm{PU}_2\mathbb{H}$, and we may assume that Δ_x is a maximal subgroup.

The stabilizer of a flag (x, ℓ) is a 2-torus, hence conjugate to $(\mathrm{U}_1\mathbb{C} \times \mathrm{U}_1\mathbb{C})/\pm 1$. Thus we may assume $\Delta_\ell = (\mathrm{U}_1\mathbb{H} \times \mathrm{U}_1\mathbb{C})/\pm 1$, and $\Delta_{x,\ell} = \Delta_x \cap \Delta_\ell = (\mathrm{U}_1\mathbb{C} \times \mathrm{U}_1\mathbb{C})/\pm 1$. Examining the normalizer of the torus $\Delta_{x,\ell}$ we find two conjugates of $\mathrm{U}_2\mathbb{C}/\pm 1$ containing $\Delta_{x,\ell}$, namely $\mathrm{U}_2\mathbb{C}/\pm 1$ and $(\mathrm{U}_2\mathbb{C}/\pm 1)^{\left[\begin{smallmatrix} j & 0 \\ 0 & 1 \end{smallmatrix}\right]}$. Since $\left[\begin{smallmatrix} j & 0 \\ 0 & 1 \end{smallmatrix}\right] \in \Delta_x$, this does not affect the geometry of $\mathfrak{P}$.

$n = 6$: If Δ_x is a maximal connected subgroup, then $(\Delta, \Delta_x) \in \{(\mathrm{PSU}_6\mathbb{C}, \mathrm{U}_5\mathbb{C}\}), (\mathrm{PU}_6\mathbb{H}, \mathrm{P}(\mathrm{U}_1\mathbb{H} \times \mathrm{U}_5\mathbb{H})), (\mathrm{SO}_7\mathbb{R}, \mathrm{SO}_5\mathbb{R} \times \mathrm{SO}_2\mathbb{R})\}$. Otherwise, there is a maximal connected subgroup Φ with $\Delta > \Phi > \Delta_x$, and Δ/Φ, Φ/Δ_x have Euler number 2 or 3. There are only the following possibilities: $(\Delta, \Phi, \Delta_x) \in \{(\mathrm{G}_2, \mathrm{SU}_3\mathbb{C}, \mathrm{U}_2\mathbb{C}), (\mathrm{G}_2, \mathrm{SO}_4\mathbb{R}, \mathrm{U}_2\mathbb{C}), (\mathrm{PU}_3\mathbb{H}, \mathrm{P}(\mathrm{U}_1\mathbb{H} \times \mathrm{U}_2\mathbb{H}), (\mathrm{U}_1\mathbb{C} \times \mathrm{U}_2\mathbb{H})/\pm 1)\}$. The groups $\mathrm{PSU}_6\mathbb{C}$, $\mathrm{PU}_6\mathbb{H}$, $\mathrm{SO}_7\mathbb{R}$ and $\mathrm{PU}_3\mathbb{H}$ are again excluded by the fact that $\mathcal{P}$ and $\mathcal{L}$ are not homeomorphic. Thus $\Delta = \mathrm{G}_2$.

Let (x, ℓ) be a flag. We are given the following facts: Δ_x and Δ_ℓ are of type $\mathfrak{a}_1 + \mathbb{R}$, and their intersection $\Delta_{x,\ell}$ is of type $\mathbb{R} + \mathbb{R}$. Since $\mathcal{P}$ and $\mathcal{L}$ are not homeomorphic, the stabilizers of a point and a line cannot be conjugate in G_2. In order to understand this situation, we consider the group $\Delta = \mathrm{G}_2$ as the automorphism group of the Cayley numbers $\mathbb{O}$ (cp. Hähl [23, §3]).

(1) The stabilizer Δ_i of $i \in \mathbb{O}$ is $\mathrm{SU}_3\mathbb{C}$, and $\mathbb{C}^\perp \subseteq \mathbb{O}$ is a $\mathbb{C}$-module with $\mathbb{C}$-basis j, l, jl. The action of Δ_i on $\mathbb{C}^\perp = \mathbb{C}^3$ is the usual one. Thus the subgroups of type $\mathrm{U}_2\mathbb{C}$ in $\mathrm{SU}_3\mathbb{C}$ are the stabilizers of the one-dimensional complex subspaces, and every 2-torus in $\mathrm{SU}_3\mathbb{C}$ is contained in exactly three copies of $\mathrm{U}_2\mathbb{C} \subseteq \mathrm{SU}_3\mathbb{C}$, because it fixes three orthogonal complex one-dimensional subspaces.

(2) The stabilizer of $\mathbb{H} \subseteq \mathbb{O}$ is $\mathrm{SO}_4\mathbb{R}$. Put $\mathbb{O} = \mathbb{H} + l\mathbb{H}$, and for $a, b \in \mathbb{S}_3 \subseteq \mathbb{H}$ let $[a, b] = (u + lv \mapsto aua^{-1} + l(avb^{-1}))$. Thus $\mathrm{SO}_4\mathbb{R} = [\mathbb{S}_3, \mathbb{S}_3]$. Now there are obviously two kinds of groups of type $\mathrm{U}_2\mathbb{C}$ in $\mathrm{SO}_4\mathbb{R}$, namely $[\mathbb{S}_1, \mathbb{S}_3]$ and $[\mathbb{S}_3, \mathbb{S}_1]$, where $\mathbb{S}_1 = \mathbb{S}_3 \cap \mathbb{C}$. The group $[\mathbb{S}_1, \mathbb{S}_3]$ fixes $\mathbb{C}$ pointwise, hence it is contained in

$SU_3\mathbb{C}$; in fact, it is the stabilizer of $\mathbb{C}j$ in $SU_3\mathbb{C}$. The fixed field of $[\mathbb{S}_3, \mathbb{S}_1]$ is $\mathbb{R}$, hence this group is not conjugate to the ones considered before.

Returning to the hexagon, we now may assume that $\Delta_x = [\mathbb{S}_3, \mathbb{S}_1]$, $\Delta_{x,\ell} = [\mathbb{S}_1, \mathbb{S}_1]$, and that Δ_ℓ is contained in a conjugate of $SU_3\mathbb{C}$. The fixed field of $\Delta_{x,\ell}$ is $\mathbb{C}$, hence $\Delta_\ell \subseteq SU_3\mathbb{C}$. So far, this leaves three possibilities for Δ_l, namely $(SU_3\mathbb{C})_l$, $(SU_3\mathbb{C})_{jl}$ and $(SU_3\mathbb{C})_j$, and the last group fixes $\mathbb{H}$ elementwise. Since $\mathfrak{P}$ is a connected incidence structure in the graph-theoretic sense, $\langle\Delta_x, \Delta_\ell\rangle = \Delta$, and we infer that $\Delta_\ell = (SU_3\mathbb{C})_l$ or $\Delta_\ell = (SU_3\mathbb{C})_{jl}$. But these two groups are conjugate under $[j, 1] \in \Delta_x$. This determines the geometry of $\mathfrak{P}$. □

In view of 1.7, the preceding results imply:

3.8 COROLLARY. *Let $\mathfrak{P}$ be a compact connected n-gon. If $\mathfrak{P}$ admits a flag-transitive automorphism group Γ, and if $n \neq 4$, or if the point rows and the pencils of lines have the same dimension $p = q$, then $\mathfrak{P}$ is one of the Moufang polygons listed in* 3.3, 3.5, 3.4 *and* 3.7, *and Γ contains a maximal compact connected subgroup Δ of the full automorphism group of $\mathfrak{P}$. The compact group Δ is also flag-transitive.*

3.9 REMARK. For projective planes, the last result holds even for point-transitive automorphism groups (see Löwen [34]). For generalized quadrangles, this is not true, i.e. there are non-Moufang quadrangles with point-transitive automorphism groups. (Cp. Ferus *et al.* [14, 6.4] in connection with Thorbergsson [45]; here, the incidence relation is given by the Euclidean distance, hence every (exterior) isometry of an isoparametric hypersurface is an automorphism of the corresponding incidence structure. In fact, isoparametric hypersurfaces are rigid, hence every interior isometry extends to an isometry of the ambient space [14, 2.7]).

4. Appendix

Let $(\mathcal{P}, \mathcal{L}, \mathcal{F})$ be as in 1.7. The maps $\mathcal{F} \xrightarrow{p_1} \mathcal{P}$ and $\mathcal{F} \xrightarrow{p_2} \mathcal{L}$ are locally trivial q- and p-sphere bundles, respectively. The induced maps of the cohomology rings are monomorphisms for any coefficient ring R (Münzner [38, 7]). In the following we list the fundamental groups and cohomology rings of $\mathcal{P}$, $\mathcal{L}$ and $\mathcal{F}$. The natural inclusions between the listed rings correspond to the monomorphisms

$$\mathbf{H}^\bullet(\mathcal{P}; R) \xrightarrow{p_1^*} \mathbf{H}^\bullet(\mathcal{F}; R) \xleftarrow{p_2^*} \mathbf{H}^\bullet(\mathcal{L}; R).$$

Compare also Hebda [24] for the $\mathbb{Z}_2$-cohomology. The subscripts indicate the degrees of the cohomology classes.

For quadrangles and hexagons, the assertions about the topological structure of $\mathcal{P}$ and $\mathcal{L}$ hold up to duality, i.e. up to exchanging $\mathcal{P}$ and $\mathcal{L}$.

$\mathbf{3_1}$: Projective planes with $p = q = 1$. In this case, $\pi_1(\mathcal{P}) = \mathbb{Z}_2 = \pi_1(\mathcal{L})$ and $\pi_1(\mathcal{F}) = Q_8$ (the quaternion group of order 8). The cohomology rings are

$$\begin{aligned}
\mathbf{H}^\bullet(\mathcal{P};\, \mathbb{Z}_2) &= \mathbb{Z}_2[x_1]/(x_1^3) \\
\mathbf{H}^\bullet(\mathcal{L};\, \mathbb{Z}_2) &= \mathbb{Z}_2[y_1]/(y_1^3) \\
\mathbf{H}^\bullet(\mathcal{F};\, \mathbb{Z}_2) &= \mathbb{Z}_2[x_1,\, y_1]/(x_1^3,\, y_1^3,\, x_1^2 + x_1 y_1 + y_1^2).
\end{aligned}$$

The $\mathbb{Z}_2$-bases of the cohomology modules are $\{1,\, x_1,\, x_1^2\}$, $\{1,\, y_1,\, y_1^2\}$, and $\{1,\, x_1,\, y_1,\, x_1^2,\, y_1^2,\, x_1 y_1^2\}$ respectively. The flag space $\mathcal{F}$ is orientable, but $\mathcal{P}$ and $\mathcal{L}$ as well as p_1, p_2 are not.

$\mathbf{3_2}$: Projective planes with $p = q \in \{2,\, 4,\, 8\}$. In this case, all three spaces are simply connected, and therefore $\mathcal{P}$, $\mathcal{L}$, $\mathcal{F}$ and p_1, p_2 are orientable. The cohomology rings are

$$\begin{aligned}
\mathbf{H}^\bullet(\mathcal{P};\, R) &= R[x_p]/(x_p^3) \\
\mathbf{H}^\bullet(\mathcal{L};\, R) &= R[y_p]/(y_p^3) \\
\mathbf{H}^\bullet(\mathcal{F};\, R) &= R[x_p,\, y_p]/(x_p^3,\, y_p^3,\, x_p^2 - x_p y_p + y_p^2).
\end{aligned}$$

The cohomology modules are free with R-bases $\{1,\, x_p,\, x_p^2\}$, $\{1,\, y_p,\, y_p^2\}$, and $\{1,\, x_p,\, y_p,\, x_p^2,\, y_p^2,\, x_p y_p^2\}$ respectively.

$\mathbf{4_1}$: Quadrangles with $p = q = 1$. In this case, $\pi_1(\mathcal{P}) = \mathbb{Z}_2$, $\pi_1(\mathcal{L}) = \mathbb{Z}$, and $\pi_1(\mathcal{F}) = \mathbb{Z} \oplus \mathbb{Z}_2$. The cohomology rings are

$$\begin{aligned}
\mathbf{H}^\bullet(\mathcal{P};\, \mathbb{Z}_2) &= \mathbb{Z}_2[x_1]/(x_1^4) \\
\mathbf{H}^\bullet(\mathcal{L};\, \mathbb{Z}_2) &= \mathbb{Z}_2[y_1,\, y_2]/(y_1^2,\, y_2^2) \\
\mathbf{H}^\bullet(\mathcal{F};\, \mathbb{Z}_2) &= \mathbb{Z}_2[x_1,\, y_1,\, y_2]/(x_1^4,\, y_1^2,\, y_2^2,\, y_2 + x_1^2 + x_1 y_1)
\end{aligned}$$

The $\mathbb{Z}_2$-bases of the cohomology modules are $\{1,\, x_1,\, x_1^2,\, x_1^3\}$, $\{1,\, y_1,\, y_2,\, y_1 y_2\}$, and $\{1,\, x_1,\, y_1,\, x_1^2,\, y_2, x_1^3,\, y_1 y_2,\, x_1^3 y_1\}$ respectively. $\mathcal{P}$, $\mathcal{F}$ and p_1 are orientable, but $\mathcal{L}$ and p_2 are not.

$\mathbf{4_2}$: Quadrangles with $p = 1$ and $q > 1$. In this case, $\pi_1(\mathcal{P}) = \mathbb{Z}$, $\pi_1(\mathcal{L}) = 1$, and $\pi_1(\mathcal{F}) = \mathbb{Z}$. The $\mathbb{Z}_2$-cohomology rings are

$$\begin{aligned}
\mathbf{H}^\bullet(\mathcal{P};\, \mathbb{Z}_2) &= \mathbb{Z}_2[x_1,\, x_{q+1}]/(x_1^2,\, x_{q+1}^2) \\
\mathbf{H}^\bullet(\mathcal{L};\, \mathbb{Z}_2) &= \mathbb{Z}_2[y_q,\, y_{q+1}]/(y_q^2,\, y_{q+1}^2) \\
\mathbf{H}^\bullet(\mathcal{F};\, \mathbb{Z}_2) &= \mathbb{Z}_2[x_1,\, x_{q+1},\, y_q,\, y_{q+1}]/(x_1^2,\, x_{q+1}^2,\, y_q^2, y_{q+1}^2,\, y_{q+1} \\
&\qquad + x_{q+1} + x_1 y_q)
\end{aligned}$$

The $\mathbb{Z}_2$-bases of the cohomology modules are $\{1,\, x_1,\, x_{q+1},\, x_1 x_{q+1}\}$, $\{1,\, y_q,\, y_{q+1},\, y_q y_{q+1}\}$, and $\{1,\, x_1,\, y_q,\, x_{q+1},\, y_{q+1},\, x_1 x_{q+1},\, y_q y_{q+1},\, x_1 x_{q+1} y_q\}$ respectively.

$\mathbf{4_3}$: Quadrangles with $p,\, q > 1$ and $p + q$ odd. The integral cohomology rings of $\mathcal{P}$, $\mathcal{L}$, and $\mathcal{F}$ are anticommutative graded $\mathbb{Z}$-algebras with generators

(x_p, x_{p+q}), (y_q, y_{p+q}) and $(x_p, x_{p+q}, y_q, y_{p+q})$ respectively, subject to the relations $x_p^2 = x_{p+q}^2 = 0$, $y_q^2 = y_{p+q}^2 = 0$, and $x_p^2 = x_{p+q}^2 = y_q^2 = y_{p+q}^2 = x_{p+q} + y_{p+q} - x_p y_q = 0$ (the subscripts indicate the degrees). All spaces are simply connected and hence orientable.

4_4: Quadrangles with $p = q \in \{2, 4\}$. All three spaces are simply connected, and therefore $\mathcal{P}$, $\mathcal{L}$, $\mathcal{F}$ and p_1, p_2 are orientable. The cohomology rings are

$$\begin{aligned} \mathbf{H}^\bullet(\mathcal{P};\, R) &= R[x_p]/(x_p^4) \\ \mathbf{H}^\bullet(\mathcal{L};\, R) &= R[y_p, y_{2p}]/(y_{2p}^2, y_p^2 - 2y_{2p}) \\ \mathbf{H}^\bullet(\mathcal{F};\, R) &= R[x_p, y_p, y_{2p}]/(x_p^4, y_{2p}^2, y_p^2 - 2y_{2p}, y_{2p} + x_p^2 - y_p x_p) \end{aligned}$$

The cohomology modules are free with R-bases $\{1, x_p, x_p^2, x_p^3\}$, $\{1, y_p, y_{2p}, y_p y_{2p}\}$, and $\{1, x_p, y_p, x_p^2, y_{2p}, x_p^3, y_p y_{2p}, x_p^3 y_p\}$ respectively.

6_1: Hexagons with $p = q = 1$. In this case, $\pi_1(\mathcal{P}) = \mathbb{Z}_2 = \pi_1(\mathcal{L})$ and $\pi_1(\mathcal{F}) = Q_8$ (the quaternion group of order 8). The cohomology rings are

$$\begin{aligned} \mathbf{H}^\bullet(\mathcal{P};\, \mathbb{Z}_2) &= \mathbb{Z}_2[x_1, x_3]/(x_1^3, x_3^2) \\ \mathbf{H}^\bullet(\mathcal{L};\, \mathbb{Z}_2) &= \mathbb{Z}_2[y_1, y_3]/(y_1^3, y_3^2) \\ \mathbf{H}^\bullet(\mathcal{F};\, \mathbb{Z}_2) &= \mathbb{Z}_2[x_1, x_3, y_1, y_3]/ \\ &\qquad (x_1^3, y_1^3, x_3^2, y_3^2, x_1^2 + y_1^2 + x_1 y_1, x_3 + y_3 + x_1^2 y_1) \end{aligned}$$

The $\mathbb{Z}_2$-bases of the cohomology modules are $\{1, x_1, x_1^2, x_3, x_1x_3, x_1^2x_3\}$, $\{1, y_1, y_1^2, y_3, y_1y_3, y_1^2y_3\}$, and $\{1, x_1, y_1, x_1^2, y_1^2, x_3, y_3, x_1x_3, y_1y_3, x_1^2x_3, y_1^2y_3, x_3y_3\}$ respectively. The flag space $\mathcal{F}$ is orientable, but $\mathcal{P}$ and $\mathcal{L}$ as well as p_1, p_2 are not.

6_2: Hexagons with $p = q \in \{2, 4\}$. In this case, all three spaces are simply connected, therefore $\mathcal{P}$, $\mathcal{L}$, $\mathcal{F}$ and p_1, p_2 are orientable. The cohomology rings are as follows:

$$\mathbf{H}^\bullet(\mathcal{P};\, R) = R[x_p, x_{3p}]/(x_{3p}^2, x_p^3 - 2x_{3p})$$

with R-basis $\{1, x_p, x_p^2, x_{3p}, x_{3p}x_p, x_{3p}x_p^2\}$. The rings $\mathbf{H}^\bullet(\mathcal{L};\, R)$ and $\mathbf{H}^\bullet(\mathcal{F};\, R)$ have bases $\{y_p, y_{2p}, y_{3p}, y_{4p}, y_p y_{4p}\}$ and $\{x_p, y_p, x_p^2, y_{2p}, x_{3p}, y_{3p}, x_p x_{3p}, y_{4p}, x_p^2 x_{3p}, y_p y_{4p}, x_{3p} y_{3p}\}$ respectively. The missing products can easily be calculated from the equations $y_p^2 = 3y_{2p}$, $y_p y_{2p} = 2y_{4p}$, $y_p y_{3p} = 3y_{4p}$, $y_p^2 y_{4p} = 0$, $x_p y_p = x_p^2 + y_{2p}$, and the fact that the integral cohomology modules are torsion-free. For example $3y_{2p}y_{3p} = y_p^2 y_{3p} = 3y_p y_{4p}$, hence $y_{2p}y_{3p} = y_p y_{4p}$.

Acknowledgements

The authors would like to thank Richard Bödi and Hermann Hähl for various helpful discussions, and Markus Stroppel for important remarks concerning the proof of 3.7.

References

1. Adams, J. F. and Atiyah, M. F.: K-theory and the Hopf invariant, *Quart. J. Math. Oxford (2)* **17** (1966), 31–38.
2. Arens, R.: Topologies for homeomorphism groups, *Amer. J. Math.* **68** (1946), 593–610.
3. Ball, B. J.: Arcwise connectedness and the persistence of errors, *Amer. Math. Monthly* **91** (1984), 431–433.
4. Bickel, H.: Lie-projektive Gruppen: Vergleich von Approximationsbegriffen, Diplomarbeit Braunschweig, 1992.
5. Borel, A.: La cohomologie mod 2 de certains espaces homogénes, *Comment. Math. Helv.* **27** (1953), 165–197.
6. Borel, A. and De Siebenthal, J.: Les sous-groupes fermés de rang maximum des groupes de Lie clos, *Comment. Math. Helv.* **23** (1949), 200–221.
7. Brouwer, A. E.: A non-degenerate generalized quadrangle with lines of size four is finite, in J. W. P. Hirschfeld *et al.* (eds) *Proc. Brighton 1990*, Oxford University Press 1991, pp. 47–49.
8. Burns, K. and Spatzier, R.: On topological Tits buildings and their classification, *Publ. Math. I.H.E.S.* **65** (1987), 5–34.
9. Dienst, K. J.: Verallgemeinerte Vierecke in projektiven Räumen, *Arch. Math.* **35** (1980), 177–186.
10. Dugundji, J.: *Topology*, Allyn and Bacon, Boston, 1966.
11. Engelking, R.: *General Topology*, Heldermann Verlag, Berlin, 1989.
12. Engelking, R. and Sieklucki, K.: *Topology. A Geometric Approach*, Heldermann Verlag, Berlin, 1992.
13. Feit, W.: Finite projective planes and a question about primes, *Proc. Amer. Math. Soc.* **108** (1990), 561–564.
14. Ferus, D., Karcher, H. and Münzner, H.-F.: Cliffordalgebren und neue isoparametrische Hyperflächen, *Math. Z.* **177** (1981), 479–502.
15. Fink, J. B.: Flag-transitive projective planes, *Geom. Dedicata* **17** (1985), 219–226.
16. Freudenthal, H.: Einige Sätze über topologische Gruppen, *Ann. Math.* **37** (1936), 44–56.
17. Gluškov, V. M.: The structure of locally compact groups and Hilbert's fifth problem, *Amer. Math. Soc. Transl. (2)*, **15** (1960), 55–93.
18. Grove, K. and Halperin, S.: Dupin hypersurfaces, group actions, and the double mapping cylinder, *J. Differential Geom.* **26** (1987), 429–459.
19. Grundhöfer, T.: Ternary fields of compact projective planes, *Abh. Math. Sem. Univ. Hamburg* **57** (1986), 87–101.
20. Grundhöfer, T.: Automorphism groups of compact projective planes, *Geom. Dedicata* **21** (1986), 291–298.
21. Grundhöfer, T. and Knarr, N.: Topology in generalized quadrangles, *Topology Appl.* **34** (1990), 139–152.
22. Grundhöfer, T. and Van Maldeghem, H.: Topological polygons and affine buildings of rank three, *Atti Sem. Mat. Fis. Univ. Modena* **38** (1990), 459–479.
23. Hähl, H.: Automorphismengruppen achtdimensionaler lokalkompakter Quasikörper, *Math. Z.* **149** (1976), 203–225.
24. Hebda, J. J.: The possible cohomology rings of certain types of taut submanifolds, *Nagoya Math. J.* **111** (1988), 85–97.
25. Helgason, S.: *Differential Geometry, Lie Groups, and Symmetric Spaces*, Academic Press, San Diego, 1978.
26. Hsiang, W.-Y. and Lawson, H. B.: Minimal submanifolds of low cohomogeneity, *J. Differential Geom.* **5** (1971), 1–38.
27. Hothi, A.: Another alternative proof of Effros' Theorem, *Topology Proc.* **12** (1987), 256–298.
28. Kantor, W. M.: Flag-transitive planes, in C. A. Baker and L. M. Batten (eds.), *Finite Geometries, Proc. Winnipeg 1984*, Dekker, New York, 1985, pp. 179–181.
29. Kantor, W. M.: Primitive permutation groups of odd degree, and an application to finite projective planes, *J. Algebra* **106** (1987), 15–45.
30. Knarr, N.: Projectivities of generalized polygons, *Ars Combin.* **25B** (1988), 265–275.
31. Knarr, N.: The nonexistence of certain topological polygons, *Forum Math.* **2** (1990), 603–612.

32. Kramer, L.: Gebäude auf isoparametrischen Untermannigfaltigkeiten, Diplomarbeit Tübingen, 1991.
33. Kuratowski, K.: *Topology*, Vol. II, Academic Press, New York, 1968.
34. Löwen, R.: Homogeneous compact projective planes, *J. reine angew. Math.* **321** (1981), 217–220.
35. Lunardon, G. and Pasini, A.: Finite C_n geometries: a survey, *Note di Matematica* **10** (1990), 1–35.
36. Montgomery, D.: Simply connected homogeneous spaces, *Proc. Amer. Math. Soc.* **1** (1950), 467–469.
37. Montgomery, D. and Zippin, L.: *Topological Transformation Groups*, Interscience Publishers, New York, 1955.
38. Münzner, H.-F.: Isoparametrische Hyperflächen in Sphären II, *Math. Ann.* **256** (1981), 215–232.
39. Ronan, M.: *Lectures on Buildings*, Academic Press, San Diego, 1989.
40. Salzmann, H.: Topologische projektive Ebenen, *Math. Z.* **67** (1957), 436–466.
41. Salzmann, H.: Kompakte zweidimensionale projektive Ebenen, *Math. Ann.* **145** (1962), 401–428.
42. Schurle, A. W.: *Topics in Topology*, Elsevier North Holland, New York, 1979.
43. Szenthe, J.: On the topological characterization of transitive Lie group actions, *Acta Sci. Math. (Szeged)* **36** (1974), 323–344.
44. Takagi, R. and Takahashi, T.: On the principal curvatures of homogeneous hypersurfaces in a sphere, in S. Kobayashi, M. Obata and T. Takahashi (eds), *Differential Geometry, in honor of Kentaro Yano*, Kinokuniya, Tokyo, 1972, pp. 469–481.
45. Thorbergsson, G.: Clifford algebras and polar planes, *Duke Math. J.* **67** (1992), 627–632.
46. Tits, J.: Sur la trialité et certaines groupes qui s'en déduisent, *Publ. Math. I.H.E.S.* **2** (1959), 14–60.
47. Tits, J.: *Buildings of Spherical Type and Finite BN-Pairs (2nd edn)*, Lecture Notes in Maths 386, Springer, Berlin, 1986.
48. Wang, H.-C.: Homogeneous spaces with non-vanishing Euler characteristic, *Ann. Math. (2)* **50** (1949), 469–481.
49. Wang, Q.-M.: On the topology of Clifford isoparametric hypersurfaces, *J. Diff. Geom.* **27** (1988), 55–66.
50. Warner, G.: *Harmonic Analysis on Semi-simple Lie Groups*, Springer, 1972.
51. Wolf, J. A.: *Spaces of Constant Curvature*, Publish or Perish, Wilmington, 1984.