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The topology of smooth projective planes

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It is an open problem whether the point space of a compact connected projective plane is always homeomorphic to the point space of the real, the complex, the quaternion or the octonion projective plane. The following partial results are known:

(i) If the (covering) dimension n of the point space $\mathscr{P}$ is finite, then $\mathscr{P}$ is a generalized manifold, and $\mathscr{P}$ has the same cohomology as one of the four classical point spaces $\mathbf{P}_2\mathbb{R}$, $\mathbf{P}_2\mathbb{C}$, $\mathbf{P}_2\mathbb{H}$ or $\mathbf{P}_2\mathbb{O}$, and thus $n \in \{2, 4, 8, 16\}$, cp. Löwen [11]. This holds in particular, if the point rows and the pencils of lines are topological manifolds. In this case, the point rows and pencils of lines are $n/2$-spheres, cp. Salzmann [14, 7.12], Breitsprecher [2, 2.1]. It is not known whether the point rows have to be topological manifolds even if the point space is a topological manifold.

(ii) If the (covering) dimension of the point space $\mathscr{P}$ is 2 or 4, then the point rows and the pencils of lines are topological manifolds, cp. Salzmann [14, 2.0] [15, 1.1], and $\mathscr{P}$ is homeomorphic to $\mathbf{P}_2\mathbb{R}$ or $\mathbf{P}_2\mathbb{C}$, respectively, cp. Salzmann [14, 2.0], Breitsprecher [2, 2.5].

Much more is known for compact planes with large automorphism groups, cp. Salzmann et al. [16], or for planes with special ternary fields, cp. Buchanan [3], Otte [13].

On the other hand, there are manifolds that have similar properties as projective planes, but that are not homeomorphic or homotopy equivalent to any of the four classical projective planes, cp. Eells-Kuiper [4].

We prove the following result: the point space of a smooth projective plane is always homeomorphic to the point space of one of the four classical projective planes. By duality, the same holds of course for the line space. Notice that a smooth projective plane need not be isomorphic (in the geometrical sense) to one of the four classical projective planes, cp. Otte [13].

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Smooth projective planes. In this section, we collect some basic facts about smooth projective planes.

1. Definition. A projective plane is a triple $\mathfrak{P} = (\mathscr{P}, \mathscr{L}, \mathscr{F})$, consisting of a set of *points* $\mathscr{P}$, a set of *lines* $\mathscr{L}$ and a set of *flags* $\mathscr{F} \subseteq \mathscr{P} \times \mathscr{L}$ such that the axioms of a projective plane are satisfied, i.e. two distinct points p, q can be joined by a unique line $p \vee q$, and two distinct lines l, h intersect in a unique point $l \wedge h$, and there exists a non-degenerate quadrangle. A *point row* $L \subseteq \mathscr{P}$ is the set of all points that are incident with a certain line l. Similarly, a *pencil of lines* $\mathscr{L}_p \subseteq \mathscr{L}$ is the set of all lines that are incident with a certain point p.

The projective plane is called *smooth* if $\mathscr{P}$ and $\mathscr{L}$ are smooth manifolds, and if the maps $\vee : \mathscr{P} \times \mathscr{P} - \Delta(\mathscr{P}) \to \mathscr{L}$ and $\wedge : \mathscr{L} \times \mathscr{L} - \Delta(\mathscr{L}) \to \mathscr{P}$ are smooth.

We exclude the case that $\mathscr{P}$ or $\mathscr{L}$ are discrete spaces.

2. Theorem. *Let $\mathfrak{P} = (\mathscr{P}, \mathscr{L}, \mathscr{F})$ be a smooth projective plane. Then the point rows and the pencils of lines are smoothly embedded (possibly exotic) m-spheres for some $m \in \{1, 2, 4, 8\}$. The spaces $\mathscr{P}$, $\mathscr{L}$ and $\mathscr{F} \subseteq \mathscr{P} \times \mathscr{L}$ are compact connected smooth manifolds of dimension $2m$, $2m$ and $3m$, respectively, and $\mathrm{pr}_1 : \mathscr{F} \to \mathscr{P}$ is a locally trivial smooth m-sphere bundle.*

In order to prove this theorem, we need the following lemmata:

3. Lemma. *Let $o \in \mathscr{P}$ be a point, and consider the smooth map $f : \mathscr{P} - \{o\} \to \mathscr{L}$, $p \mapsto p \vee o$. There exists a nonempty open set $U \subseteq \mathscr{P} - \{o\}$, such that for every point row L containing p, the intersection $U \cap L$ is either empty or a smooth m-dimensional submanifold.*

Proof. Let U be the set of all points where the rank of the derivative f_* is maximal, say k. Now rank $(f_*) \geqq k$ means that there is a nonsingular $k \times k$ submatrix of the Jacobian of f. This is clearly an open condition, and thus U is open. The restriction $f|_U$ has constant rank k, hence the fibers of $f|_U$ are $(2m - k)$-dimensional smooth submanifolds of U. On the other hand, the fibers of f are exactly the (m-dimensional) point rows through o. $\square$

4. Lemma. *Let X, Z be two point rows, and let $o \in X$, $p \in Z$. Then there exist open neighborhoods U of o and V of p and a diffeomorphism $(U, o) \to (V, p)$ that maps $U \cap X$ homeomorphically onto $V \cap Z$.*

Proof. We may assume that the flags (o, X) and (p, Z) are in general position. Thus we can choose a triangle X, Y, L such that $X \cap Y = \{o\}$, $X \cap Z = X \cap L = \{u\}$ and $Y \cap L = \{v\}$, and such that o, u, v, p is a non-degenerate quadrangle. On the open set $\mathscr{P} - L$ we may define a smooth 'coordinate function' $\phi(q) = (x, y) = ((q \vee v) \wedge X, (q \vee u) \wedge Y))$ and on the open set $(\mathscr{P} - L) \times (\mathscr{P} - L)$ an 'inverse coordinate function' $\psi(x, y) = q = (x \vee v) \wedge (y \vee u)$. Clearly $\psi(\phi(q)) = q$. We put $(a, b) = \phi(p)$.

The composite

$$q \overset{\phi}{\mapsto} (x, y) \mapsto (x', y') = ((((x \vee b) \wedge L) \vee p) \wedge X, (((y \vee a) \wedge L) \vee p) \wedge Y) \overset{\psi}{\mapsto} q'$$

is the desired diffeomorphism, with inverse

$$q' \overset{\phi}{\mapsto} (x', y') \mapsto (x, y) = ((((x' \vee p) \wedge L) \vee b) \wedge X, (((y' \vee p) \wedge L) \vee a) \wedge Y) \overset{\psi}{\mapsto} q$$

(draw a picture). $\square$

P r o o f o f t h e T h e o r e m. By the two lemmata above, every point row is a smooth submanifold. The other claims follow now from [2] or [9]. $\square$

The cohomology of $\mathscr{F} \to \mathscr{P}$. We need the cohomology ring of the m-sphere bundle $\mathrm{pr}_1 \colon \mathscr{F} \to \mathscr{P}$. Consider the following commutative diagram of maps and spaces:

$$\begin{array}{ccc} \{p\} \times \mathscr{L}_p & \hookrightarrow & \mathscr{F} \\ \downarrow & & \downarrow \\ \{p\} & \hookrightarrow & \mathscr{P} \hookleftarrow L \end{array}$$

Passing to rational cohomology, we get the following diagram (for $m > 1$, cp. Breitsprecher [2, 2.3]. For $m = 1$, the same is true with $\mathbb{Z}_2$-coefficients):

$$\begin{array}{ccccc} \mathbb{Q}[y]/(y^2) & \leftarrow & \mathbb{Q}[x, y]/(x^3, y^3, x^2 + y^2 - xy) & & \\ \uparrow & & \uparrow & & \\ \mathbb{Q} & \leftarrow & \mathbb{Q}[x]/(x^3) & \rightarrow & \mathbb{Q}[x]/(x^2) \end{array}$$

For $m > 1$, none of these spaces has torsion.

Recall that the signature σ of a $4k$-manifold M is defined as the index of the rational quadratic form $u \mapsto \langle u^2, [M] \rangle$, $\mathbf{H}^{2k}(M; \mathbb{Q}) \to \mathbb{Q}$, cp. Milnor-Stasheff [12, pg. 224]. Hence the signature $\sigma(\mathscr{P})$ of the point space is 1 for $m > 1$.

Characteristic classes of $\mathscr{P}$. In this section we always assume that the dimension m of the point rows of the smooth projective plane $(\mathscr{P}, \mathscr{L}, \mathscr{F})$ is at least 2. We calculate the Pontrjagin classes of the point space $\mathscr{P}$. The main tool is Knarr's embedding theorem [9], which implies that the Pontrjagin classes of the flag space $\mathscr{F}$ vanish. Using the topological invariance of the rational Pontrjagin classes, Hirzebruch's signature theorem, and the fact that $\mathscr{F}$ is a sphere bundle over $\mathscr{P}$, we obtain the total Pontrjagin class $\mathbf{p}(\mathscr{P})$ (up to a sign). Our reference for characteristic classes is the book by Milnor and Stasheff [12].

Recall that a *topological* $\mathbb{R}^m$*-bundle* is a locally trivial fiber bundle with fiber $\mathbb{R}^m$. The structure group of such a bundle is the group $\mathbf{TOP}(m)$ of all base-point preserving homeomorphisms of $\mathbb{R}^m$.

5. D e f i n i t i o n. Let θ be the m-plane subbundle of the tangent bundle $\tau(\mathscr{F})$ tangent to the fiber, i.e. $\theta = \ker \mathrm{pr}_{1,*}$. Choose a Riemannian metric on $\mathscr{F}$. Then there is a vector bundle isomorphism $\theta^{\perp} \cong \mathrm{pr}_1^*(\tau(\mathscr{P}))$, and hence $\tau(\mathscr{F}) \cong \theta \oplus \mathrm{pr}_1^*(\tau(\mathscr{P}))$.

6. Proposition. *The (rational) total Pontrjagin class* $\mathbf{p}(\mathscr{F}) = 1 + \sum_{j \geq 1} \mathbf{p}_j(\mathscr{F})$ *of* $\mathscr{F}$ *is trivial. Hence* $\mathbf{p}(\theta)\, \mathrm{pr}_1^*\, \mathbf{p}(\mathscr{P}) = \mathbf{p}(\theta)\, \mathbf{p}(\mathrm{pr}_1^*(\tau(\mathscr{P}))) = \mathbf{p}(\mathscr{F}) = 1$ *by* [12, 15.2, 15.3].

Proof. According to Knarr [9], $\mathscr{F}$ can be embedded as a topological compact hypersurface in $\mathbb{S}^{3m+1}$, with trivial normal bundle ε, and therefore the bundle $\tau(\mathscr{F}) \oplus \varepsilon$, considered as a topological $\mathbb{R}^{3m+1}$-bundle, is trivial, because the (topological) tangent bundle of $\mathbb{R}^{3m+1}$ is trivial.

Since the fiber **TOP/O** of the map between the classifying spaces $\mathbf{BO} \to \mathbf{BTOP}$ has finite homotopy groups in every dimension, cp. Kirby-Siebenmann [7, Essay V, § 5], this map yields an isomorphism $\mathbf{H}^\bullet(\mathbf{BO}; \mathbb{Q}) \cong \mathbf{H}^\bullet(\mathbf{BTOP}; \mathbb{Q})$, i.e. the rational Pontrjagin classes are topological invariants, cp. [12, Epilogue].

Let $\phi: \mathscr{F} \to \mathbf{BO}(3m+1)$ be the classifying map of $\tau(\mathscr{F}) \oplus \varepsilon$, and consider the commutative diagram

$$\begin{array}{ccccc} \mathbf{H}^\bullet(\mathscr{F};\mathbb{Q}) & \overset{\phi^*}{\leftarrow} & \mathbf{H}^\bullet(\mathbf{BO}(3m+1);\mathbb{Q}) & \leftarrow & \mathbf{H}^\bullet(\mathbf{BO};\mathbb{Q}) \\ & & \uparrow \iota^* & & \uparrow \cong \\ & & \mathbf{H}^\bullet(\mathbf{BTOP}(3m+1);\mathbb{Q}) & \leftarrow & \mathbf{H}^\bullet(\mathbf{BTOP};\mathbb{Q}). \end{array}$$

The composite $\iota \circ \phi$ is homotopic to a constant map, because the topological $\mathbb{R}^{3m+1}$-bundle $\tau(\mathscr{F}) \oplus \varepsilon$ is trivial. Hence the induced map $\mathbf{H}^\bullet(\mathbf{BO}; \mathbb{Q}) \to \mathbf{H}^\bullet(\mathscr{F}; \mathbb{Q})$ is trivial, and since $\mathbf{H}^\bullet(\mathbf{BO}; \mathbb{Q})$ is generated by the Pontrjagin classes, $\mathbf{p}(\mathscr{F}) = \mathbf{p}(\tau(\mathscr{F}) \oplus \varepsilon) = 1$. □

We put $\mathbf{p}(\mathscr{P}) = 1 + bx + cx^2$, with $b, c \in \mathbb{Q}$. Then $\mathbf{p}(\theta) = 1 - bx + (b^2 - c)x^2$.

7. Lemma. *The Euler class* $\mathbf{e}(\theta)$ *is of the form* $2y + ax$ *for some* $a \in \mathbb{Q}$.

Proof. The restriction of θ to a fiber is just the tangent bundle of the even-dimensional sphere $\mathbb{S}^m$, thus the restriction of $\mathbf{e}(\theta)$ to a fiber is $2y$, cp. Milnor-Stasheff [12, 11.12]. □

8. Lemma. *The Euler class of* θ *is* $2y - x$, *and the* $m/2$-th *Pontrjagin class of* θ *is* $-3x^2$. *Thus* $\mathbf{p}(\mathscr{P}) = 1 + bx + (b^2 + 3)x^2$.

Proof. Since m is even, we have the relation $\mathbf{e}(\theta)^2 = \mathbf{p}_{m/2}(\theta)$ by [12, 15.8]. This yields $(a^2 + 4a)x^2 + 4(1 + a)y^2 = (b^2 - c)x^2$, and hence $a = -1$ (cp. the proof of 6.4 in Grove-Halperin [5]). □

9. Theorem. *The total Pontrjagin class of* $\mathscr{P}$ *is* $1 + 3x^2$ *for* $m = 2$, $1 \pm 2x + 7x^2$ *for* $m = 4$, *and* $1 \pm 6x + 39x^2$ *for* $m = 8$.

Proof. The signature $\sigma(\mathscr{P})$ is 1. From the Hirzebruch signature theorem, cp. [12, 19.4], we get the equations

$$\begin{array}{rl} b^2 + 3 = 3 & (m = 2) \\ 7(b^2 + 3) - b^2 = 45 & (m = 4) \\ 381(b^2 + 3) - 19b^2 = 14175 & (m = 8) \end{array}$$

and the claim follows. □

The classification. Let $\mathscr{P}$ be a smooth $2m$-dimensional projective plane. Pick a point $o \in \mathscr{P}$ and a point row L that does not meet o. The map $h\colon \mathscr{P} - \{o\} \to L$, $p \mapsto (o \vee p) \wedge L$ yields a smooth, locally trivial bundle $\eta = (\mathscr{P} - \{o\} \to L)$ over the m-sphere L, and the fibers of this bundle are the point rows through o, with o removed, and thus homeomorphic to $\mathbb{R}^m$. Note that in the case of the four classical Moufang planes, η is just the Hopf bundle of the underlying division algebra.

10. Proposition. *Let ν denote the normal bundle of the smooth submanifold $L \subseteqq \mathscr{P}$. The bundles ν and η, considered as topological $\mathbb{R}^m$-bundles over L, are isomorphic. Thus the point space $\mathscr{P}$ is homeomorphic to the Thom space of the normal bundle ν.*

If $m \neq 4$, then the structure group of the smooth bundle η can be reduced to $\mathbf{GL}(m)$, *i.e. η is a vector bundle. In this case, the vector bundles ν and η are also isomorphic.*

Proof. The bundle η is a smooth microbundle. Hence there exists an open neighborhood U of L in $\mathscr{P} - \{o\}$ such that the restriction $h|U$ yields a smooth vector bundle $\eta' = (h|U\colon U \to L)$, cp. [7, Essay IV, § 1,1.1]. Since U is a tubular neighborhood of L, the vector bundle η' is isomorphic to the normal bundle ν of L. The topological $\mathbb{R}^m$-bundles η and η' are isomorphic because they represent the same microbundle, see Kister [8, Thm. 2].

Now suppose $m \neq 4$. The fibers of η are diffeomorphic to $\mathbb{R}^m$ by the result of Stallings [17]. The structure group of η is the group $\mathbf{Diff}(m)$ of all base-point preserving diffeomorphisms of $\mathbb{R}^m$. Since the inclusion $\mathbf{GL}(m) \to \mathbf{Diff}(m)$ is a homotopy equivalence, cp. Stewart [18], the structure group of this bundle can be reduced to $\mathbf{GL}(m)$. Clearly, the total space $\mathscr{P} - \{o\}$ of η is a tubular neighborhood of L, and thus the vector bundles ν and η are isomorphic. □

It is an open problem whether the point rows and the point space of a smooth projective plane always carry the standard differentiable structure. However, the differentiable structure is uniquely determined by the underlying topological plane, see [1].

By the Proposition above, it suffices to determine the m-plane bundle ν in order to classify the point space $\mathscr{P}$ up to homeomorphism.

11. Proposition. *The 2-plane bundles over $\mathbb{S}^2$ are classified by their Euler class. The 4-plane bundles over $\mathbb{S}^4$ are classified by their Euler class and by their first rational Pontrjagin class. The 8-plane bundles over $\mathbb{S}^8$ are classified by their Euler class and by their second rational Pontrjagin class.*

Proof. Let $x \in \mathbf{H}^m(\mathbf{BSO}(m);\mathbb{Q})$ and let $f \in [\mathbb{S}^m, \mathbf{BSO}(m)] = \pi_m \mathbf{BSO}(m)$. Let C denote the function that sends f to $f^\bullet(x) \in \mathbf{H}^m(\mathbb{S}^m;\mathbb{Q})$. We claim that $C\colon \pi_m \mathbf{BSO}(m) \to \mathbf{H}^m(\mathbb{S}^m;\mathbb{Q}) \cong \mathbb{Q}$ is a group homomorphism from the m-th homotopy group of $\mathbf{BSO}(m)$ into the m-th rational cohomology group of the m-sphere: consider the Kronecker index $\langle f^\bullet(x), [\mathbb{S}^m]\rangle = \langle x, f_\bullet[\mathbb{S}^m]\rangle \in \mathbb{Q}$. The term on the right-hand side is additive, since the Hurewicz homomorphism $f \mapsto f_\bullet[\mathbb{S}^m]$ is additive. Hence the map that assigns to an m-plane bundle over $\mathbb{S}^m$ the Euler class or some Pontrjagin class is additive.

Since $\pi_2 \mathbf{BSO}(2) = \pi_1 \mathbf{SO}(2) = \mathbb{Z}$, $\pi_4 \mathbf{BSO}(4) = \pi_3 \mathbf{SO}(4) = \mathbb{Z} \oplus \mathbb{Z}$, and $\pi_8 \mathbf{BSO}(8) = \pi_7 \mathbf{SO}(8) = \mathbb{Z} \oplus \mathbb{Z}$, it suffices to find enough vector bundles with linearly independent characteristic classes, see also [12, 20.10].

The Euler class of the tangent bundle of $\mathbb{S}^m$ is $2x$ by [12, 11.12], whereas the Pontrjagin classes vanish by [12, 15.2], because the tangent bundle of the sphere is stably trivial. The Hopf bundles over $\mathbb{S}^4$, $\mathbb{S}^8$ have nonvanishing Pontrjagin classes (see below), hence the Proposition follows. $\square$

Now we can prove our main result:

12. Theorem. *Let $\mathscr{P}$ be the point space of a smooth projective plane. Then $\mathscr{P}$ is homeomorphic to the point space $\mathbf{P}_2\mathbb{R}$, $\mathbf{P}_2\mathbb{C}$, $\mathbf{P}_2\mathbb{H}$ or $\mathbf{P}_2\mathbb{O}$ of one of the four classical Moufang planes.*

Proof. Let ν denote the normal bundle of L as in 10. If $m = 1$, then ν is either trivial or the Möbius bundle, because $\pi_1\,\mathbf{BO}(1) = \pi_0\,\mathbf{O}(1) = \mathbb{Z}_2$. Since $\mathscr{P}$ is a manifold, ν is nontrivial, and thus $\mathscr{P} \cong \mathbf{P}_2\mathbb{R}$.

Now let $m \in \{2, 4, 8\}$. Since the tangent bundle of the (possibly exotic) m-sphere L is stably trivial (at least considered as a topological bundle, but see also Kervaire-Milnor [6, 3.1]), the total Pontrjagin class of ν is just the restriction of the total Pontrjagin class of $\mathscr{P}$ to L by [12, 15.3]. Hence $\mathbf{p}(\nu) = 1,\ 1 \pm 2x,\ 1 \pm 6x$ for $m = 2, 4, 8$.

Now we want to calculate the Euler class of ν. From the Gysin exact sequence

$$\leftarrow \mathbf{H}^m(\mathscr{P} - \{o\} - L) \leftarrow \mathbf{H}^m(L) \xleftarrow{\cup\,\mathbf{e}(\nu)} \mathbf{H}^0(L) \leftarrow \mathbf{H}^{m-1}(\mathscr{P} - \{o\} - L) \leftarrow$$

and the fact that $\mathscr{P} - \{o\} - L = \mathbb{R}^{2m} - \{0\}$ we find $\mathbf{e}(\nu) = x$. Thus we have determined the Euler class and (up to a sign) the total Pontrjagin class of ν. The uncertainty about the sign in the Pontrjagin class does not affect the classification: ν is either the classical Hopf bundle or the bundle obtained from the opposite division algebra. The total spaces of these bundles are homeomorphic by the map that is induced on the base space by conjugation, cp. Tamura [19]. This finishes the proof. $\square$

Observe that for $m = 1, 2$, this proof works also for compact connected projective planes. The inclusion $\mathbf{BO}(m) \to \mathbf{BTOP}(m)$ is a homotopy equivalence for $m = 1, 2$, cp. Kneser [10], Kirby-Siebenmann [7, Essay V, § 5], hence in these dimensions the bundle η is a vector bundle. By the same argument as above, η has to be the Möbius bundle for $m = 1$. In the case $m = 2$, we may use the Gysin sequence of η to obtain the Euler class $\mathbf{e}(\eta)$. This yields a slightly different proof for Salzmann's and Breitsprecher's classification [14, 2.0], [2, 2.5] of the point space of compact connected 2- or 4-dimensional planes.

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