THOMPSON GROUPS FOR SYSTEMS OF GROUPS, AND THEIR FINITENESS PROPERTIES

STEFAN WITZEL AND MATTHEW C. B. ZAREMSKY

Abstract. We describe a general procedure for constructing a Thompson-like group out of a system of groups that is equipped with what we call a cloning system. Existing examples include the trivial group, symmetric groups, braid groups and pure braid groups, which yield the Thompson groups F, V, V_{br} and F_{br}, respectively.

We give new examples of systems of groups that admit a cloning system and study how the finiteness properties of the resulting Thompson group depend on those of the original groups. The main new examples here include upper triangular matrix groups over rings of S-integers, and also mock reflection groups.

Introduction

In 1965 Richard Thompson introduced three groups that today are usually denoted F, T, and V. These have received a lot of recent attention for their interesting and often surprising properties. Most prominently, T and V are finitely presented, infinite, simple groups, and F is not elementary amenable and contains no non-abelian free subgroups. It is an ongoing problem to determine whether F is amenable or not. As far as finiteness properties are concerned, which is what we will be most interested in, F is an example of a torsion-free group that is of type F_∞ without being of type F.

For these and other reasons, numerous generalizations of Thompson's groups have been introduced in the literature; see for example [Hig74, Ste92, GS97, Röv99, Bri04, Hug09, MPN, BF]. Most of these constructions either generalize the way in which branching can occur, or mimic the self-similarity in some way. Here we introduce a more algebraic construction of Thompson-like groups that is based on the usual branching of the group F acting on tree diagrams but allows one to incorporate other groups besides trivial groups (as for F) or symmetric groups (as for V) to act on the strands. The construction is based on Brin's description on the braided Thompson group V_{br} [Bri07]. Another example is the braided Thompson group F_{br} introduced by Brady, Burillo, Cleary and Stein in [BBCS08]. The input is a directed system of groups $(G_n)_{n\in\mathbb{N}}$ together with a *cloning system*. A cloning system consists of morphisms $G_n \to S_n$, where S_n are the symmetric groups, and cloning maps $\kappa^n_k \colon G_n \to G_{n+1}$, $1 \le k \le n$ subject to certain conditions (see Definition 2.15). The output is a group $\mathscr{T}(G_*)$ which is Thompson-like in many ways. Perhaps this is best illustrated by the fact that there are morphisms $F \hookrightarrow \mathscr{T}(G_*) \to V$ whose composition is the inclusion $F \hookrightarrow V$.

The finiteness properties of being of type F_n generalize the properties of being finitely generated (type F_1) and of being finitely presented (type F_2). Recall that a group G is of *type F_n* if there is a $K(G,1)$ with finite n-skeleton. Most of the generalizations of Thompson's groups mentioned above are known to be of type F_∞, that is, of type F_n for all n. Even though there are many examples of groups with separating finiteness properties [AB87, BB97, Bux04, BKW13], finiteness properties are not generally well understood. It

Date: December 19, 2013.

2010 *Mathematics Subject Classification.* Primary 20F65; Secondary 57M07.

Key words and phrases. Thompson's group, upper triangular matrix, mock reflection group, finiteness properties.

is therefore interesting to investigate how the finiteness properties of a group change when it is subject to a certain "operation." One such operation is braiding; the question of which finiteness properties the braided versions of F and V have was answered in [BFS⁺]. In this article we reinterpret these examples as "Thompsonifications" of the pure braid groups and braid groups respectively. We investigate how the finiteness properties of the Thompson group of a cloning system depends on the finiteness properties of the input groups. For certain systems of groups G_* we show the limiting behavior

$$\phi(\mathscr{T}(G_*)) = \liminf_n \phi(G_n) \tag{1}$$

where $\phi(G)$ is the finiteness length of G, i.e., the supremum over all n such that G is of type F_n. In other words (1) states that $\mathscr{T}(G_*)$ is of type F_n if and only if all but finitely many G_n are of type F_n. In some sense $\mathscr{T}(G_*)$ may be thought of as a limit of the groups G_*, for example it contains all of them. It is rather remarkable though, that it should be so well behaved with respect to finiteness properties. Indeed, (1) fails for most other limiting processes, for example an infinite direct limit of finite groups will not even be finitely generated.

A relatively well-developed machinery is available to prove finiteness properties of existing Thompson-like groups (namely that, so far, they are all of type F_∞); see [Bro92, Ste92, Bro06, Far03, FMWZ13, BFS⁺]. As far as proving positive finiteness properties, this will also work nicely for our groups. The Thompson group acts on a contractible cube complex called the *Stein space* with stabilizers coming from the cloning system. The space has a natural cocompact filtration. To show that the Thompson group is of type F_n (assuming all the stabilizers are) amounts to showing that the descending links in this filtration are eventually $(n-1)$-connected. This has to be done for every cloning system individually, and depends on the nature of the examples being considered. It is less clear how to analyze the negative finiteness properties; the Stein space does not seem to be the right space to study in this regard, and we have not yet developed a general framework.

The first main examples we consider in the present work are groups of upper triangular matrices. These are particularly interesting because they include on the one hand the Abels groups $\mathrm{Ab}_n(\mathbb{Z}[\frac{1}{p}])$, whose finiteness length tends to infinity with n, and on the other hand groups $B_n(\mathcal{O}_S)$ of upper triangular matrices of S-integers in positive characteristic, whose finiteness length is constant as a function of n. We prove here that for any R, $\mathscr{T}(B_*(R))$ inherits any positive finiteness properties of $B_n(R)$, so it satisfies the inequality "$\geq$" of (1):

Theorem 7.4. $\phi(\mathscr{T}(B_*(R))) \geq \liminf_n(\phi(B_n(R)))$.

We are currently not able to show the converse inequality in general, but we have some evidence:

Proposition 7.14. *Let k be a field and $R = k[t]$ its polynomial ring. Then $\mathscr{T}(B_*(k[t]))$ is not finitely generated.*

The argument for upper triangular matrices also applies to the Abels groups. Here it shows the full equation (1) because the right hand side is infinite:

Theorem 7.13. *$\mathscr{T}(\mathrm{Ab}_*(\mathbb{Z}[\frac{1}{p}]))$ is of type F_∞.*

This last fact is especially interesting since individually none of the $\mathrm{Ab}_n(\mathbb{Z}[\frac{1}{p}])$ themselves are of type F_∞. The second main example we consider is a Thompson group V_{mock} for the family of mock symmetric groups S_n^{mock}, which was proposed to us by Januszkiewicz. Since mock symmetric groups are F_∞ it is expected that the same is true for the associated Thompson group.

Conjecture 8.4. *V_{mock} is of type F_∞.*

The paper is organized as follows. In Section 1 we recall some basics about monoids and the Zappa–Szép product. In Section 2 we introduce cloning systems (Definition 2.15) and explain how they give rise to Thompson groups. Section 3 collects some group theoretic consequences that follow directly from the construction. To study finiteness properties, the Stein space is introduced in Section 4. The filtration and its descending links are described in Section 5, and we discuss some background on Morse theory and other related techniques for proving high connectivity. Up to this point everything is mostly generic. The following sections discuss examples. Section 6 gives an elementary example where $G_n = H^n$ for some group H. Section 7 discusses cloning systems for groups of upper triangular matrices. We also prove that the corresponding Thompson group has finiteness length at least the limit infimum of those of the matrix groups. Section 8 introduces the group V_{mock} which is built out of mock symmetric groups.

Acknowledgments. We are grateful to Matt Brin and Kai-Uwe Bux for helpful discussions, and to Tadeusz Januszkiewicz for proposing to us the group V_{mock}. Both authors also gratefully acknowledge support of the SFB 878 in Münster, and in the case of the second author also the SFB 701 in Bielefeld.

1. Preliminaries

Much of the material in this section is taken from [Bri07].

1.1. **Monoids.** A *monoid* is an associative binary structure with a two-sided identity. A monoid M is called *left cancellative* if for all $x, y, z \in M$, we have that $xy = xz$ implies $y = z$. Elements $x, y \in M$ have a *common left multiple* m if there exist $z, w \in M$ such that $zx = wy = m$. This is the *least common left multiple* if for all $p, q \in M$ such that $px = qy$, we have that px is a left multiple of m. There are the obvious definitions of *right cancellative*, *common right multiples* and *least common right multiples.* We say that M has common right/left multiples if any two elements have a common right/left multiple. It is said to have least common right/left multiples if any two elements that have a common right/left multiple have a least common right/left multiple. When we write that M has (least) common right multiples with least in brackets, we mean that it has common right multiples as well as least common right multiples (and analogously for left multiples and factors). Finally, we say M is *cancellative* if it is both left and right cancellative. The importance of these notions lies in the following classical theorem (see [CP61, Theorems 1.23, 1.25]):

Theorem 1.1 (Ore)**.** *A cancellative monoid with common right multiples has a unique group of right fractions.*

Recall that for every monoid M there exists a group G_M and a monoid morphism $\omega\colon M \to G_M$ such that every monoid morphism from M to a group factors through ω (the group generated by all the elements of M subject to all the relations that hold in M). This is *the group of fractions of* M. The morphism ω will be injective if and only if M embeds into a group. A group G is called *a group of right fractions of* M if it contains M and every element of G can be written as $m \cdot n^{-1}$ with $m, n \in M$. A group of right fractions exists precisely in the situation of Ore's theorem and is unique up to isomorphism; see [CP61, Section 1.10] for details. We call a monoid satisfying the hypotheses of Theorem 1.1 an *Ore monoid.* The group of right fractions of an Ore monoid is its group of fractions (see for example [KS06, Theorem 7.1.16]):

Lemma 1.2. *Let M be an Ore monoid, let G be its group of right fractions and let H be any group. Let $\varphi\colon M \to H$ be a monoid morphism. Then the map $\tilde{\varphi}\colon G \to H$ defined by $\tilde{\varphi}(mn^{-1}) = \varphi(m) \cdot \varphi(n)^{-1}$ is a group homomorphism and $\varphi = \tilde{\varphi}|_M$.*

Proof. That inverses map to inverses is clear. Let $m_1, m_2, n_1, n_2 \in M$ and let $n_1 \cdot x = m_2 \cdot y$ be a common right multiple so that $m_1 n_1^{-1} m_2 n_2^{-1} = m_1 x y^{-1} n_2^{-1}$. We have to check that

$$\varphi(m_1)\varphi(n_1)^{-1}\varphi(m_2)\varphi(n_2)^{-1} = \varphi(m_1 x)\varphi(n_2 y)^{-1}. \tag{1.1}$$

The fact that φ is a monoid morphism means that $\varphi(n_1)\varphi(x) = \varphi(m_2)\varphi(y)$ which entails $\varphi(n_1)^{-1}\varphi(m_2) = \varphi(x)\varphi(y)^{-1}$. Extending by $\varphi(m_1)$ from the left and by $\varphi(n_2)^{-1}$ from the right gives (1.1). □

1.2. **Posets from monoids.** Throughout this section let M be an Ore monoid and let G be its group of right fractions. The notions of left/right multiple/factor are uninteresting for G as a monoid because it is a group. Instead we introduce these notions relative to the monoid M. Concretely, assume that elements $a, b, c \in G$ satisfy

$$ab = c.$$

If $a \in M$ then we call b a *right factor* of c and c a *left multiple* of b. If $b \in M$ then we call a a *left factor* of c and c a *right multiple* of a. If g is a left factor (respectively right multiple) of both h and h' then we say that it is a *common left factor* (respectively *common right multiple*). If g is a common left factor of h and h' and any other left factor of h and h' is also a left factor of g then g is called a *greatest common left factor*. If g is a common right multiple of h and h' and every other right multiple is also a right multiple of g then g is called a *least common right multiple* of h and h'. Thus we obtain notions of when G has (least) common right/left multiples and (greatest) common right/left factors. We say that two elements have *no common right factor* if they have least common right factor 1.
Under a moderate additional assumption, having least common right multiples is inherited by G from M:

Lemma 1.3. *Let M have least common right multiples. Let $n, n', m, m' \in M$ be such that n and m have no common right factor and neither do n' and m'. Let $nv = n'u$ be a least common right multiple of n and n'. Then $nv = n'u$ is a least common right multiple of nm^{-1} and $n'm'^{-1}$.*

We call a monoid homomorphism $\mathrm{len}\colon M \to \mathbb{N}_0$ a *length function* if every element of the kernel is a unit. It induces a length function $\mathrm{len}\colon G \to \mathbb{Z}$. Note that if M admits a length function then every element of G can be written as mn^{-1} where m and n are elements of M with no common right factor.
The following is an extension of [Bri07, Lemma 2.3] to G.

Lemma 1.4. *Assume that M admits a length function. Then G has least common right multiples if and only if it has greatest common left factors.*

One reason for our interest in least common right multiples and greatest common left factors is order theoretic. Define a relation on G by declaring $g \le h$ if g is a left factor of h. This relation is reflexive and transitive but fails to satisfy antisymmetry if M has non-trivial units. We denote the relation induced on $G/M^\times$ also by $\le$. It is an order relation so $G/M^\times$ becomes a partially ordered set (poset). Spelled out, the relation is given by $gM^\times \le hM^\times$ if $g^{-1}h \in M$.
The algebraic properties discussed before immediately translate into order theoretic properties:

Observation 1.5. *If M has (least) common right multiples and (greatest) common left factors then $M/M^\times$ is a lattice. Similarly, if G has (least) common right multiples and (greatest) common left factors then $G/M^\times$ is a lattice.*

Putting everything together, we find:

Corollary 1.6. *Let M be a cancellative monoid with (least) common right multiples and length function. Let G be its group of right fractions. Then $G/M^\times$ is a lattice.*

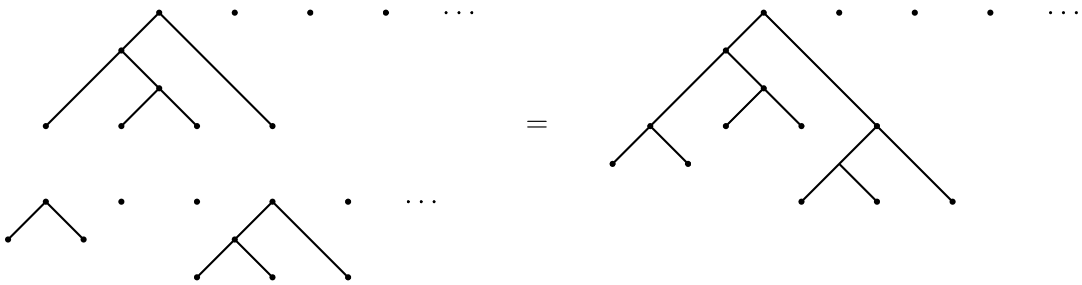

FIGURE 1. Multiplication of forests.

1.3. **The monoid of forests.** Since we are interested in Thompson's groups, an important monoid in all that follows will be the *monoid of forests*, which we define in this subsection. For us, a *tree* is always a finite rooted full binary tree. In other words, every vertex has either no outgoing edges or a left and right outgoing edge, and every vertex other than the root has an incoming edge. The vertices without outgoing edges are called *leaves*. The distinction between left and right induces a natural order on the leaves. If a tree has only one leaf, then the leaf is also its root and the tree is the *trivial tree*.

By a *forest* we mean a sequence of trees $E = (T_i)_{i\in\mathbb{N}}$ such that all but finitely many T_i are trivial. The roots are numbered in the obvious way, i.e., the i^{th} root of E is the root of T_i. If all the T_i are trivial we call E *trivial*. If the T_i are trivial for $i > 1$ then the forest is called *semisimple* (here we deviate from Brin's notation; what we call "semisimple" is called "simple" in [Bri07], and what we will later call "simple", Brin calls "simple and balanced"). The *rank* of E is the least index i such that T_j is trivial for $j > i$. So E is semisimple if it has rank at most 1. The leaves of all the T_i are called the *leaves* of E. The order on the leaves of the trees induces an order on the leaves of the forest by declaring that any leaf of T_i comes before any leaf of T_j, whenever $i < j$. We may equivalently think of the leaves as numbered by natural numbers. The number of *feet* of a semisimple forest $(T_i)_{i\in\mathbb{N}}$ is the number of leaves of T_1 .

Let $\mathscr{F}$ be the set of forests. Define a multiplication on $\mathscr{F}$ as follows. Let $E = (T_k)$ and $E' = (T'_k)$ be forests, and set EE' to be the forest obtained by identifying the i^{th} leaf of E with the i^{th} root of E', for each i. This product is associative, and the trivial forest is a left and right identity, so $\mathscr{F}$ is a monoid. Some more details on $\mathscr{F}$ can be found in Section 3 of [Bri07]. Figure 1 illustrates the multiplication of two elements.

There is an obvious set of generators of $\mathscr{F}$, namely the set of single-caret forests. Such a forest can be characterized by the property that there exists $k \in \mathbb{N}$ such that for $i < k$, the i^{th} root is also the i^{th} leaf, and for $i > k$, the i^{th} root is also the $(i + 1)$st leaf. Denote this forest by λ_k. Every tree in λ_k is trivial except for the k^{th} tree, which is a single caret.

Proposition 1.7 (Presentation of the forest monoid)**.** [Bri07, Proposition 3.3] *$\mathscr{F}$ is generated by the λ_k, and defining relations are given by*

$$\lambda_j\lambda_i = \lambda_i\lambda_{j+1} \quad \textit{for} \quad i < j. \tag{1.2}$$

Every element of $\mathscr{F}$ can be uniquely expressed as a word of the form $\lambda_{k_1}\lambda_{k_2}\cdots\lambda_{k_r}$ for some $k_1 \leq \cdots \leq k_r$.

A consequence is that the number of carets is an invariant of a forest, and is exactly the length of the word in the λ_k representing the forest. The following is part of [Bri07, Lemma 3.4].

Lemma 1.8. *The monoid $\mathscr{F}$ has the following properties.*

(1) It is cancellative.
(2) It has common right multiples.

(3) It has no non-trivial units.
(4) There is a monoid homomorphism $\mathrm{len}\colon \mathscr{F} \to \mathbb{N}_0$ *sending each generator to* 1
(5) It has greatest common right factors and least common left multiples
(6) It has greatest common left factors and least common right multiples.

In view of Theorem 1.1 properties (1) and (2) imply $\mathscr{F}$ has a unique group of right fractions which we denote $\widehat{F}$.

1.4. **Zappa–Szép products.** In this section we recall the background on Zappa–Szép products of monoids. Our main reference is [Bri07, Section 2.4], and also see [Bri05]. When the monoids are groups, Zappa–Szép products generalize semidirect products, with the extra flexibility that normality is no longer required.

The internal Zappa–Szép product is straightforward to define. Let M be a monoid with submonoids U and A such that every $m \in M$ can be written in a unique way as $m = u\alpha$ for $u \in U$ and $\alpha \in A$. In particular, for $\alpha \in A$ and $u \in U$ there exist $u' \in U$ and $\alpha' \in A$ such that $\alpha u = u'\alpha'$, and the u' and α' are uniquely determined by α and u, so we denote them $u' = \alpha \cdot u$ and $\alpha' = \alpha^u$, following [Bri07]. The maps $(\alpha, u) \mapsto \alpha \cdot u$ and $(\alpha, u) \mapsto \alpha^u$ should be thought of as mutual actions of U and A on each other. Then we can define a multiplication on $U \times A$ via

$$(u, \alpha)(v, \beta) := (u(\alpha \cdot v), \alpha^v \beta), \tag{1.3}$$

for $u, v \in U$ and $\alpha, \beta \in A$, and the map $(u, \alpha) \mapsto u\alpha$ is a monoid isomorphism from $U \times A$ (with this multiplication) to M; see [Bri07, Lemma 2.7]. We say that M is the (internal) Zappa–Szép product of U and A, and write $M = U \bowtie A$.

Example 1.9 (Semidirect product)**.** Suppose G is a group that is a semidirect product $G = U \ltimes A$ for $U, A \leq G$. Then for $u \in U$ and $\alpha \in A$ we have $\alpha u = u(u^{-1}\alpha u)$, and $u^{-1}\alpha u \in A$, so the actions defined above are just $\alpha \cdot u = u$ and $\alpha^u = u^{-1}\alpha u$.

We actually need to use the *external* Zappa–Szép product. This is discussed in detail in [Bri07, Section 2.4] (and in even more detail in [Bri05]).

Definition 1.10 (External Zappa–Szép product)**.** Let U and A be monoids with maps $(\alpha, u) \mapsto \alpha \cdot u$ and $(\alpha, u) \mapsto \alpha^u$ satisfying the following eight properties for all $u, v \in U$ and $\alpha, \beta \in A$:

1) $1_A \cdot u = u$ (Identity acting on U)
2) $(\alpha\beta) \cdot u = \alpha \cdot (\beta \cdot u)$ (Product acting on U)
3) $\alpha^{1_U} = \alpha$ (Identity acting on A)
4) $\alpha^{(uv)} = (\alpha^u)^v$ (Product acting on A)
5) $(1_A)^u = 1_A$ (U acting on identity)
6) $(\alpha\beta)^u = \alpha^{(\beta \cdot u)}\beta^u$ (U acting on product)
7) $\alpha \cdot 1_U = 1_U$ (A acting on identity)
8) $\alpha \cdot (uv) = (\alpha \cdot u)(\alpha^u \cdot v)$. ($A$ acting on product)

Then the maps are called a *Zappa–Szép action.* The set $U \times A$ together with the multiplication defined by (1.3) is called the *(external) Zappa–Szép product* of U and A, denoted $U \bowtie A$.

It is shown in Lemma 2.9 in [Bri07] that the external Zappa–Szép product turns $U \bowtie A$ into a monoid and coincides with the internal Zappa–Szép product of U and A with respect to the embeddings $u \mapsto (u, 1_A)$ and $\alpha \mapsto (1_U, \alpha)$.

Some pedantry about the use of the word "action" might now be advisable. The action of U on A is a right action described by a homomorphism of monoids $U \to \mathrm{Symm}(A)$, where $\mathrm{Symm}(A)$ is the symmetric group on A (and is *not* the group of monoid automorphisms). The action of A on U is a left action described by a homomorphism of monoids $A \to$

Symm(U), again *not* to Aut(U). In a phrase, both actions are actions *of* monoids as monoids, but not *on* monoids as monoids.
Brin [Bri07] regards the action $(\alpha, u) \mapsto \alpha^u$ of U on A as a family of maps from A to itself parametrized by U and defines properties of this family. For brevity we apply the same adjectives to the action itself but one should think of the family of maps. The action is called *injective* if $\alpha^u = \beta^u$ implies $\alpha = \beta$. It is *surjective* if for every $\alpha \in A$ and $u \in U$ there exists a $\beta \in A$ with $\beta^u = \alpha$. The action is *strongly confluent* if the following holds: if $u, v \in U$ have a least common left multiple $ru = sv$ and $\alpha = \beta^u = \gamma^v$ for some $\beta, \gamma \in A$ then there should be a $\theta \in A$ such that $\theta^r = \beta$ and $\theta^s = \gamma$. Note that if the action is injective then for this to happen it is sufficient that $\theta^{ru} = \alpha$. The notions for the action of A on U are defined by analogy.
The following lemmas can be found as Lemma 2.12 in [Bri07], or as Lemmas 3.15 in [Bri05].

Lemma 1.11. *Let U be a cancellative monoid with least common left multiples and let A be a group. Let U and A act on each other via Zappa-Szép actions. Assume that the action $(\alpha, u) \mapsto \alpha^u$ of U on A is strongly confluent. Then $M = U \bowtie A$ has least common left multiples.*
A least common left multiple $(r, \alpha)(u, \theta) = (s, \beta)(v, \phi)$ of (u, θ) and (v, ϕ) in M can be constructed so that $r(\alpha \cdot u) = s(\beta \cdot v)$ is the least common left multiple of $(\alpha \cdot u)$ and $(\beta \cdot v)$ in U. If M is cancellative, every least common left multiple will have that property.

Being actions of monoids, Zappa–Szép actions are already determined by the actions of generating sets. It is not obvious, but also true, that they are often also determined by the actions of generating sets *on* generating sets. This means that, in order to define the actions, we need only define $\alpha \cdot u$ and u^α where both α and u come from generating sets. Brin [Bri07, pp. 768–769] gives a sufficient condition for such partial actions to extend to well defined Zappa–Szép actions, which we restate here. Given sets X and Y, let X^* and Y^* denote the free monoids generated respectively by them. Suppose maps $Y \times X \to Y^*, (\alpha, u) \mapsto \alpha^u$ and $Y \times X \to X, (\alpha, u) \mapsto \alpha \cdot u$ are given. Let W be the set of relations $(\alpha u, (\alpha \cdot u)(\alpha^u))$ with $\alpha \in Y, u \in X$. Then

$$\langle X \cup Y \mid W \rangle$$

is a Zappa–Szép product of X^* and Y^*. In particular, the above maps extend to Zappa–Szép actions $Y^* \times X^* \to Y^*$ and $Y^* \times X^* \to X^*$.

Lemma 1.12 ([Bri07, Lemma 2.14])**.** *Let $U = \langle X \mid R \rangle$ and $A = \langle Y \mid T \rangle$ be presentations of monoids (with $X \cap Y = \emptyset$). Assume that functions $Y \times X \to Y^*, (\alpha, u) \mapsto \alpha^u$ and $Y \times X \to X, (\alpha, u) \mapsto \alpha \cdot u$ are given. Let $\sim_R$ and $\sim_T$ denote the equivalence relations on X^* and Y^* imposed by the relation sets R and T.*
Extend the above maps to $Y^* \times X^*$ as above. Assume that the following are satisfied. If $(u, v) \in R$ then for every $\alpha \in Y$ we have $(\alpha \cdot u, \alpha \cdot v) \in R$ or $(\alpha \cdot v, \alpha \cdot u) \in R$, and also $\alpha^u \sim_T \alpha^v$. If $(\alpha, \beta) \in T$ then for all $u \in X$ we have $\alpha \cdot u = \beta \cdot u$ and $\alpha^u \sim_T \beta^u$.
Then the lifted maps induce well-defined Zappa–Szép actions and the restriction of the map $A \times U \to U$ to $A \times X$ has its image in X. A presentation for $U \bowtie A$ is

$$\langle X \cup Y \mid R \cup T \cup W \rangle$$

where W consists of all pairs $(\alpha u, (\alpha \cdot u)(\alpha^u))$ for $(\alpha, u) \in Y \times X$.

2. Data defining generalized Thompson's groups

2.1. **Brin–Zappa–Szép products and cloning systems.** To construct Thompson-like groups we now consider Zappa–Szép products $\mathscr{F} \bowtie G$ of the forest monoid $\mathscr{F}$ with a group G.

Definition 2.1 (BZS products)**.** Suppose we have Zappa–Szép actions $(E, g) \mapsto g \cdot E$ and $(E, g) \mapsto g^E$ on $\mathscr{F} \times G$, for G a group. For each standard generator λ_k of $\mathscr{F}$ the map $\kappa_k = \kappa_{\lambda_k} \colon G \to G$ given by $g \mapsto g^{\lambda_k}$ is called the k^{th} *cloning map*. If every such cloning map is injective, we call the actions *Brin–Zappa–Szép actions (BZS actions)* and call the monoid $\mathscr{F} \bowtie G$ the *Brin–Zappa–Szép product (BZS product)*.

Since the action of $\mathscr{F}$ on G is a right action we will also write the cloning maps κ_k on the right.
The monoid $\mathscr{F}$ is cancellative and has common right multiples, and the same is true of G, being a group. Since G is a group these properties are inherited by $\mathscr{F} \bowtie G$:

Observation 2.2. *A BZS product $\mathscr{F} \bowtie G$ is cancellative and has (least) common right multiples. In particular it has a group of right fractions.*

Proof. This follows easily from the statements about $\mathscr{F}$ using the unique factorization in Zappa–Szép products and that E is a right multiple and left factor of (E, g). □

In Definition 2.1 we have already simplified the data needed to describe BZS products by using the fact that $\mathscr{F}$ is generated by the λ_k. In a similar fashion the following lemma reduces the data needed to describe the action of G on $\mathscr{F}$. We denote by S_ω the group $\operatorname{Symm}(\mathbb{N})$ of permutations of $\mathbb{N}$ and by $S_\infty \leq S_\omega$ the subgroup of permutations with finite support.

Lemma 2.3 (Carets to carets)**.** *Let $\mathscr{F} \bowtie G$ be a BZS product. The action of G on $\mathscr{F}$ preserves the set $\Lambda = \{\lambda_k\}_{k \in \mathbb{N}}$ and so induces a homomorphism $\rho \colon G \to S_\omega$. Conversely, the action of G on $\mathscr{F}$ is completely determined by ρ and $(\kappa_k)_{k \in \mathbb{N}}$.*

Proof. For $g \in G$ and $E, F \in \mathscr{F}$, we know that $g \cdot (EF) = (g \cdot E)(g^E \cdot F)$ by Definition 1.10. We show that the action of G preserves Λ. If $g \cdot \lambda_k = E_1 E_2$ then $g^{-1} \cdot (EF) = \lambda_k$, so one of $g^{-1} \cdot E$ or $(g^{-1})^E \cdot F$ equals $1_{\mathscr{F}}$. Again by Definition 1.10, we see that either $E = 1_{\mathscr{F}}$ or $F = 1_{\mathscr{F}}$. We conclude that $g \cdot \lambda_k$ equals λ_ℓ for some ℓ depending on k and g. The map ρ then is defined via $\rho(g)k = \ell$.
To see that the action of G on $\mathscr{F}$ is determined by ρ and (κ_k), we use repeated applications of the equation $g \cdot (\lambda_k E) = \lambda_{\rho(g)k}((g)\kappa_k \cdot E)$. □

As a consequence we see that the action of G on $\mathscr{F}$ preserves the length of an element:

Corollary 2.4. *There is a monoid homomorphism* $\operatorname{len} \colon \mathscr{F} \bowtie G \to \mathbb{N}_0$ *taking (E, g) to the length of E in the standard generators. The kernel of* len *is $G = (\mathscr{F} \bowtie G)^\times$.*

In particular, len is a length function in the sense of Section 1.2. The induced morphism from the group of right fractions to $\mathbb{Z}$ (Lemma 1.2) is also denoted len.
The next result is a technical lemma that tells us that ρ and the cloning maps always behave well together, in any BZS product.

Lemma 2.5 (Compatibility)**.** *Let $\mathscr{F} \bowtie G$ be a BZS product. The homomorphism $\rho \colon G \to S_\omega$ and the maps $(\kappa_k)_{k \in \mathbb{N}}$ satisfy the following compatibility condition for $k < \ell$:*
If $\rho(g)k < \rho(g)\ell$ then $\rho((g)\kappa_\ell)k = \rho(g)k$ and $\rho((g)\kappa_k)(\ell + 1) = \rho(g)\ell + 1$.
If $\rho(g)k > \rho(g)\ell$ then $\rho((g)\kappa_\ell)k = \rho(g)k + 1$ and $\rho((g)\kappa_k)(\ell + 1) = \rho(g)\ell$.

Proof. For $k < \ell$ we know that

$$g \cdot (\lambda_\ell \lambda_k) = g \cdot (\lambda_k \lambda_{\ell+1}).$$

Writing this out using the axioms for Zappa–Szép products we obtain that

$$(g \cdot \lambda_\ell)(g^{\lambda_\ell} \cdot \lambda_k) = (g \cdot \lambda_k)(g^{\lambda_k} \cdot \lambda_{\ell+1})$$

which can be rewritten using the action morphism ρ as

$$\lambda_{\rho(g)\ell} \lambda_{\rho(g^{\lambda_\ell})k} = \lambda_{\rho(g)k} \lambda_{\rho(g^{\lambda_k})(\ell+1)}.$$

Using the normal form for $\mathscr{F}$ (see Proposition 1.7) we can distinguish cases for how this could occur. The first case is that both pairs of indices

$$(\rho(g)\ell, \rho(g^{\lambda_\ell})k) \text{ and } (\rho(g)k, \rho(g^{\lambda_k})(\ell+1))$$

are ordered increasingly and coincide. But this is impossible because $\rho(g)\ell \neq \rho(g)k$. The second case is that both pairs are ordered strictly decreasingly and coincide, which is impossible for the same reason. The remaining two cases have that one pair is ordered increasingly and the other strictly decreasingly. In either case the monoid relation now yields a relationship among the indices, namely either

$$\rho(g^{\lambda_k})(\ell+1) - 1 = \rho(g)\ell > \rho(g^{\lambda_\ell})k = \rho(g)k$$

or

$$\rho(g)\ell = \rho(g^{\lambda_k})(\ell+1) < \rho(g)k = \rho(g^{\lambda_\ell})k - 1.$$

Finally, replacing the action of λ_k by the map κ_k yields the result. □

The compatibility condition can also be rewritten as

$$\rho((g)\kappa_\ell)(k) = \begin{cases} \rho(g)(k) & k < \ell, \rho(g)k < \rho(g)\ell, \\ \rho(g)(k) + 1 & k < \ell, \rho(g)k > \rho(g)\ell, \\ \rho(g)(k-1) & k > \ell, \rho(g)(k-1) < \rho(g)\ell, \\ \rho(g)(k-1) + 1 & k > \ell, \rho(g)(k-1) > \rho(g)\ell. \end{cases} \tag{2.1}$$

Lemma 2.3 said that the action of G on $\mathscr{F}$ is uniquely determined by ρ and the cloning maps. The action of $\mathscr{F}$ on G is also uniquely determined by the cloning maps, simply because $\mathscr{F}$ is generated by the λ_k. Our findings can be summarized as:

Proposition 2.6 (Uniqueness). *A BZS product $\mathscr{F} \bowtie G$ induces a homomorphism $\rho\colon G \to S_\omega$ and injective maps $\kappa_k\colon G \to G, k \in \mathbb{N}$ satisfying the following conditions for $k, \ell \in \mathbb{N}$ with $k < \ell$ and $g, h \in G$:*

(CS1) $(gh)\kappa_k = (g)\kappa_{\rho(h)k}(h)\kappa_k$. *(Cloning a product)*
(CS2) $\kappa_\ell \circ \kappa_k = \kappa_k \circ \kappa_{\ell+1}$. *(Product of clonings)*
(CS3) If $\rho(g)k < \rho(g)\ell$ *then* $\rho((g)\kappa_\ell)k = \rho(g)k$ *and* $\rho((g)\kappa_k)(\ell+1) = \rho(g)\ell + 1$.
If $\rho(g)k > \rho(g)\ell$ *then* $\rho((g)\kappa_\ell)k = \rho(g)k + 1$ *and* $\rho((g)\kappa_k)(\ell+1) = \rho(g)\ell$. *(Compatibility)*

The BZS product is uniquely determined by these data. □

The converse is also true:

Proposition 2.7 (Existence). *Let G be a group, $\rho\colon G \to S_\omega$ a homomorphism and $(\kappa_k)_{k\in\mathbb{N}}$ a family of injective maps from G to itself. Assume that for $k < \ell$ and $g, h \in G$ the conditions* (CS1), (CS2) *and* (CS3) *in Proposition 2.6 are satisfied. Then there is a well-defined BZS product $\mathscr{F} \bowtie G$ corresponding to these data.*

Proof. We will verify the assumptions of Lemma 1.12. This will produce a Zappa–Szép action, which will be a Brin–Zappa–Szép action by construction. We take U to be $\mathscr{F}$ with the presentation

$$\langle \lambda_k \text{ for } k \in \mathbb{N} \mid (\lambda_\ell\lambda_k, \lambda_k\lambda_{\ell+1}) \text{ for } k < l\rangle.$$

Let R denote the set of relations used here and let R^{sym} be the symmetrization. We take A to be G with the presentation

$$\langle g \text{ for } g \in G \mid (gh, g') \text{ for } gh = g'\rangle.$$

The maps on generators are defined as $g^{\lambda_k} := (g)\kappa_k$ and $g \cdot \lambda_k := \lambda_{\rho(g)k}$.
First, for $k < \ell$ and $g \in G$ we need to verify that

$$(g \cdot (\lambda_\ell\lambda_k), g \cdot (\lambda_k\lambda_{\ell+1})) \in R^{\text{sym}} \quad \text{and} \quad g^{\lambda_\ell\lambda_k} = g^{\lambda_k\lambda_{\ell+1}}.$$

The latter of these is just condition (CS2). The former condition means that

$$(\lambda_{\rho(g)\ell}\lambda_{\rho((g)\kappa_\ell)k}, \lambda_{\rho(g)k}\lambda_{\rho((g)\kappa_k)(\ell+1)})$$

should lie in R^{sym}. If $\rho(g)k > \rho(g)\ell$ we can use condition (CS3) to rewrite this as

$$(\lambda_{\rho(g)\ell}\lambda_{\rho(g)k+1}, \lambda_{\rho(g)k}\lambda_{\rho(g)\ell})$$

which is in R^{sym}. If $\rho(g)k < \rho(g)\ell$ then the tuple is

$$(\lambda_{\rho(g)\ell}\lambda_{\rho(g)k}, \lambda_{\rho(g)k}\lambda_{\rho(g)\ell+1})$$

which already lies in R.

Second, for every relation (gh, g') of G and every $k \in \mathbb{N}$ we have to verify that

$$(gh) \cdot \lambda_k = g' \cdot \lambda_k \quad \text{and} \quad (gh)^{\lambda_k} = (g')^{\lambda_k}$$

for $k \in \mathbb{N}$. The former is not really a condition because the partial action was already defined using G (rather than the free monoid spanned by G). The latter means that we need

$$(g')^{\lambda_k} = g^{\lambda_{\rho(h)k}} h^{\lambda_k}$$

which is just condition (CS1). □

Definition 2.8. Let G be a group, $\rho\colon G \to S_\omega$ a homomorphism and $(\kappa_k)_{k\in\mathbb{N}}\colon G \to G$ a family of maps, also denoted κ_* for brevity. The triple (G, ρ, κ_*) is called a *cloning system* if the data satisfy conditions (CS1), (CS2) and (CS3) above. We may also refer to ρ and κ_* as a forming a *cloning system on G*.

Example 2.9 (Symmetric groups)**.** Let $G = S_\infty$. Let $\rho\colon S_\infty \to S_\omega$ just be inclusion. The action of G on $\mathscr{F}$ is thus given by $g \cdot \lambda_k = \lambda_{\rho(g)k} = \lambda_{gk}$.

Since we will use the specific cloning maps in this example even in the future general setting, we will give them their own name, ς_ℓ. They are defined by the formula

$$((g)\varsigma_k)(m) = \begin{cases} gm & m \le k, gm \le gk, \\ gm+1 & m < k, gm > gk, \\ g(m-1) & m > k, g(m-1) < gk, \\ g(m-1)+1 & m > k, g(m-1) \ge gk. \end{cases} \tag{2.2}$$

It is immediate that the compatibility condition (CS3) in the formulation (2.1) is satisfied. To aid in checking condition (CS1), we define two families of maps, $\pi_k\colon \mathbb{N} \to \mathbb{N}$ and $\tau_k\colon \mathbb{N} \to \mathbb{N}$, for $k \in \mathbb{N}$:

$$\pi_k(m) = \begin{cases} m & m \le k, \\ m-1 & m > k \end{cases} \quad \text{and} \quad \tau_k(m) = \begin{cases} m & m \le k, \\ m+1 & m > k. \end{cases} \tag{2.3}$$

Note that $\pi_k \circ \tau_k = \text{id}$ and $\tau_k \circ \pi_k(m) = m$, unless $m = k+1$ in which case it equals $m-1$. In the $m = k+1$ case, we see that

$$(gh)\varsigma_k(k+1) = gh(k)+1 = (g)\varsigma_{hk}(hk+1) = (g)\varsigma_{hk}(h)\varsigma_k(k+1),$$

by repeated use of the last case in the definition. It remains to check condition (CS1) in the $m \ne k+1$ case. According to the definitions, we have

$$((g)\varsigma_k)(m) = \tau_{gk}(g\pi_k(m))$$

whenever $m \ne k+1$. Using this we see that

$$\begin{aligned} ((g)\varsigma_{hk}) \circ ((h)\varsigma_k)(m) &= \tau_{ghk} g\pi_{hk} \circ \tau_{hk} h\pi_k(m) \\ &= \tau_{ghk} gh\pi_k(m) \\ &= ((gh)\varsigma_k)(m) \end{aligned}$$

for $m \ne k+1$.

To check condition (CS2), we consider $k < \ell$. We first verify, from the definition, the special cases

$$((g)\varsigma_\ell \circ \varsigma_k)(k+1) = gk+1 = ((g)\varsigma_k \circ \varsigma_{\ell+1})(k+1) \quad \text{and}$$
$$((g)\varsigma_\ell \circ \varsigma_k)(\ell+2) = g\ell+2 = ((g)\varsigma_k \circ \varsigma_{\ell+1})(\ell+2).$$

For the remaining case, when $m \neq k+1, \ell+2$, we have

$$((g)\varsigma_\ell \circ \varsigma_k)(m) = \tau_k\tau_\ell g \pi_\ell \pi_k(m) \quad \text{and}$$
$$((g)\varsigma_k \circ \varsigma_{\ell+1})(m) = \tau_{\ell+1}\tau_k g \pi_k \pi_{\ell+1}(m)$$

and it is straightforward to check that

$$\pi_\ell\pi_k = \pi_k\pi_{\ell+1} \text{ and } \tau_k\tau_\ell = \tau_{\ell+1}\tau_k. \tag{2.4}$$

We conclude that $(S_\infty, \rho, (\varsigma_k)_k)$ is a cloning system.

Observation 2.10. *Condition* (CS3) *in Proposition 2.6 can equivalently be rewritten as*

$$\rho((g)\kappa_k) = (\rho(g))\varsigma_k.$$

We finish by discussing least common left multiples. Let κ_* be the cloning maps of a cloning system. For $E = \lambda_{k_1} \cdots \lambda_{k_r}$ define $\kappa_E := \kappa_{k_1} \circ \cdots \circ \kappa_{k_r}$. Note that this is well defined by condition (CS2) and is just the map $g \mapsto g^E$.

Observation 2.11. *Let G be a group and let (ρ, κ_*) be a cloning system on G. The action of $\mathscr{F}$ on G defines a strongly confluent family if and only if* $\operatorname{im}(\kappa_{E_1}) \cap \operatorname{im}(\kappa_{E_2}) = \operatorname{im}(\kappa_F)$ *whenever E_1 and E_2 have least common left multiple F.*
In particular the BZS product $\mathscr{F} \bowtie G$ has least common left multiples in that case.

Proof. This is just unraveling the definition and using the remark before Lemma 1.11. Assume that the above condition holds. Write $F = F_1E_1 = F_2E_2$. Assume that $g = g_1^{E_1} = g_2^{E_2}$, that is, $g \in \operatorname{im}(\kappa_{E_1}) \cap \operatorname{im}(\kappa_{E_2})$. By assumption there is an $h \in G$ such that $g = (h)\kappa_F$. That is $g = h^F = h^{F_1E_1} = g_1^{E_1}$. Injectivity of the action of $\mathscr{F}$ on G now implies $h^{F_1} = g_1$. A similar argument shows $h^{F_2} = g_2$.
Conversely assume that the action of $\mathscr{F}$ on G is strongly confluent and write F as before. Let $g \in \operatorname{im}(\kappa_{E_1}) \cap \operatorname{im}(\kappa_{E_2})$. Write $g = (g_1)\kappa_{E_1}$ and $g = (g_2)\kappa_{E_2}$, that is $g = g_1^{E_1}$ and $g = g_2^{E_2}$. By strong confluence there is an $h \in G$ such that $h^{F_1} = g_1$ and $h^{F_2} = g_2$. Then $g = h^F = (h)\kappa_F$ as desired. □

To check this global confluence condition one either needs a good understanding of the action of $\mathscr{F}$ on G (as was the case for braided V [Bri07, Section 5.3]) or one has to reduce it to local confluency statements.

2.2. **Interlude: hedges.** In the above example of the symmetric group, the action of $\mathscr{F}$ on S_ω factors through an action of a proper quotient. This amounts to a further relation being satisfied in addition to the product of clonings relation (CS2). The quotient turns out to be what Brin [Bri07] called the monoid of *hedges*. Without going into much detail we want to explain the action of the hedge monoid on S_ω.

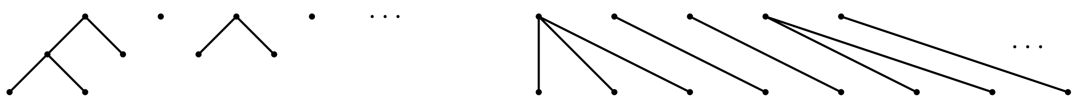

FIGURE 2. A forest and the corresponding hedge.

The *hedge monoid* $\mathscr{H}$ is the monoid of monotone surjective maps $\mathbb{N} \to \mathbb{N}$. Multiplication is given by composition: $f \cdot h = f \circ h$. There is an action of S_ω on $\mathscr{H}$ given by the property that, for $g \in S_\omega$ and $f \in \mathscr{H}$, the cardinality of $(g \cdot f)^{-1}(i)$ is that of $f^{-1}(g^{-1}i)$. There is an obvious equivariant morphism $c\colon \mathscr{F} \to \mathscr{H}$ given by $c(\lambda_k) = \eta_k$ where

$$\eta_k(m) = \begin{cases} m & m \leq k, \\ m-1 & m > k. \end{cases}$$

This morphism is surjective but not injective, in fact (see [Bri07, Proposition 4.4]):

Lemma 2.12. *The monoid $\mathscr{H}$ has the presentation*

$$\langle \eta_k, k \in \mathbb{N} \mid \eta_\ell \eta_k = \eta_k \eta_{\ell+1}, \ell \geq k \rangle.$$

Observe that the only difference between this and the presentation of $\mathscr{F}$ is that the relation also holds for $\ell = k$, rather than only for $\ell > k$. It turns out that the action of $\mathscr{F}$ on S_ω defined in Example 2.9 factors through c:

Observation 2.13. *The maps ς_k defined in* (2.2) *satisfy $\varsigma_k \varsigma_k = \varsigma_k \varsigma_{k+1}$. Thus they define an action of $\mathscr{H}$ on S_ω.*

Proof. The verification of (CS2) above extends to the case $k = \ell$. □

2.3. **Filtered cloning systems.** Typically one will want to think of Thompson's group V not as built from S_∞ but rather from the family $(S_n)_{n\in\mathbb{N}}$. We will now describe this approach. We regard S_∞ as the direct limit $\varinjlim S_n$ where the maps $\sigma_{m,n}\colon S_m \to S_n$ are induced by the inclusions $\{1,\dots,m\} \hookrightarrow \{1,\dots,n\}$.

Let $(G_n)_{n\in\mathbb{N}}$ be a family of groups with monomorphisms $\iota_{m,n}\colon G_m \to G_n$ for each $m \leq n$. For convenience will sometimes write G_* for $(G_n)_{n\in\mathbb{N}}$; note that in this case the index set is always $\mathbb{N}$. The maps $\iota_{m,n}$ will be written on the right, e.g., $(g)\iota_{m,n}$ for $g \in G_m$. Suppose that $\iota_{m,m} = \mathrm{id}$ and $\iota_{m,n} \circ \iota_{n,\ell} = \iota_{m,\ell}$ for all $m \leq n \leq \ell$. Then $((G_n)_{n\in\mathbb{N}}, (\iota_{m,n})_{m\leq n})$ is a directed system of groups with a direct limit $G := \varinjlim G_n$. Since all the $\iota_{m,n}$ are injective, we may equivalently think of a group G filtered by subgroups G_n.

Consider injective maps $\kappa_k^n\colon G_n \to G_{n+1}$ for $k, n \in \mathbb{N}, k \leq n$. We call such maps a *family of cloning maps* for the directed system $(G_n)_{n\in\mathbb{N}}$ if for $m, k \leq n$ they satisfy

$$\iota_{m,n} \circ \kappa_k^n = \begin{cases} \kappa_k^m \circ \iota_{m+1,n+1} & \text{if } k \leq m \\ \iota_{m,n+1} & \text{if } m < k. \end{cases} \tag{2.5}$$

This amounts to setting $\kappa_k^n = \iota_{n,n+1}$ for $k > n$ and requiring that

$$\iota_{m,n} \circ \kappa_k^n = \kappa_k^m \circ \iota_{m+1,n+1},$$

i.e., that the family $(\kappa_k^n)_{n\in\mathbb{N}}$ defines a morphism of directed systems of sets. From that it is clear that a family of cloning maps induces a family of injective maps $\kappa_k\colon G \to G$ by setting

$$(g)\iota_n \circ \kappa_k = (g)\kappa_k^n \circ \iota_{n+1}$$

for $g \in G_n$. Here $\iota_n\colon G_n \to G$ denotes the map given by the universal property of G.

We say that the cloning maps are *properly graded* if the following strong confluence condition holds: if $g \in G_{n+1}$ can be written as $(h)\kappa_k^n = g = (\bar{g})\iota_{n,n+1}$ then there is an $\bar{h} \in G_{n-1}$ with $(\bar{h})\kappa_k^{n-1} = h$ and $(\bar{h})\iota_{n-1,n} = h$. In view of the injectivity of all maps involved this is equivalent to saying that

$$\operatorname{im} \kappa_k^n \cap \operatorname{im} \iota_{n,n+1} \subseteq \operatorname{im}(\iota_{n-1,n} \circ \kappa_k^n) \tag{2.6}$$

(where the converse inclusion is automatic). Note that a filtered cloning system satisfying the confluence condition of Observation 2.11 is automatically properly graded. Note also that being properly graded is equivalent to the diagram

$$\begin{array}{ccc} G_{n-1} & \xrightarrow{\iota_{n-1,n}} & G_n \\ \Big\downarrow{\scriptstyle \kappa_k^{n-1}} & & \Big\downarrow{\scriptstyle \kappa_k^{n}} \\ G_n & \xrightarrow{\iota_{n,n+1}} & G_{n+1} \end{array}$$

being a pullback diagram of sets.
Suppose further that we have a family of homomorphisms $\rho_n \colon G_n \to S_n$ for each $n \in \mathbb{N}$ that are compatible with the directed systems, i.e., $\rho_n((g)\iota_{m,n}) = (\rho_m(g))\sigma_{m,n}$ for $m < n$ and $g \in G_m$. Let $\rho \colon G \to S_\infty$ be the induced homomorphism.

Example 2.14. Take $G_n = S_n$ and $\iota_{m,n} = \sigma_{m,n}$ as in Example 2.9. A family of cloning maps ς_k^n is obtained by restriction of the maps from Example 2.9:

$$\varsigma_k^n := \varsigma_k|_{S_n}^{S_{n+1}}. \tag{2.7}$$

This family of cloning maps is properly graded: if $g \in \mathrm{im}_{n,n+1}$ then g fixes $n+1$; if moreover $g = (h)\kappa_k$ then it follows from (2.2) that h fixes n so $h \in \mathrm{im}_{n-1,n}$.

We are of course interested in the case when ρ and the family $(\kappa_k)_{k\in\mathbb{N}}$ define a cloning system on G. The corresponding defining formulas are obtained by adding decorations to the formulas from Section 2.1.

Definition 2.15 (Cloning system)**.** Let $((G_n)_{n\in\mathbb{N}}, (\iota_{m,n})_{m\le n})$ be an injective directed system of groups. Let $(\rho_n)_{n\in\mathbb{N}} \colon G_n \to S_n$ be a homomorphism of directed systems of groups and let $(\kappa_k^n)_{k\le n} \colon G_n \to G_{n+1}$ be a family of cloning maps. The quadruple

$$((G_n)_{n\in\mathbb{N}}, (\iota_{m,n})_{m\le n}, (\rho_n)_{n\in\mathbb{N}}, (\kappa_k^n)_{k\le n})$$

is called a *cloning system* if the following hold for all $k \le n$ and all $g, h \in G_n$:
(FCS1) $(gh)\kappa_k^n = (g)\kappa_{\rho(h)k}^n (h)\kappa_k^n$. (Cloning a product)
(FCS2) $\kappa_\ell^n \circ \kappa_k^{n+1} = \kappa_k^n \circ \kappa_{\ell+1}^{n+1}$. (Product of clonings)
(FCS3) $\rho_{n+1}((g)\kappa_k^n) = (\rho_n(g))\varsigma_k^n$ (Compatibility)
We may also refer to ρ_* and $(\kappa_k^n)_{k\le n}$ as forming a *cloning system on* the directed system G_*. The cloning system is *properly graded* if the cloning maps are properly graded.

Note that condition (FCS3) is phrased more concisely than (CS3), but this is just in light of Observation 2.10.

Observation 2.16. *Let $(G_n)_{n\in\mathbb{N}}$ be an injective directed system of groups. A cloning system on $(G_n)_{n\in\mathbb{N}}$ gives rise to a cloning system on $G := \varinjlim G_n$. Conversely a cloning system on G gives rise to a cloning system on $(G_n)_{n\in\mathbb{N}}$ provided $(G_n)\kappa_k^n \subseteq G_{n+1}$ and $\rho_n(G_n) \subseteq S_n$.*

We will usually not distinguish explicitly between a cloning system on G_* and a cloning system on $\varinjlim G_*$ that preserves the filtration. In particular, given a cloning system on a directed system of groups we will implicitly define $\rho := \varinjlim \rho_n$ and $\kappa_k := \varinjlim \kappa_k^n$.

2.4. **Thompson groups from cloning systems.** Let $(G, \rho, (\kappa_k)_{k\in\mathbb{N}})$ be a cloning system and let $\mathscr{F} \bowtie G$ be the associated BZS product.

Definition 2.17 (Thompson group of a cloning system)**.** The group of right fractions of $\mathscr{F} \bowtie G$ is denoted by $\widehat{\mathscr{T}}(G)$ and is called the *Thompson group* of G. If more context is required we denote it $\widehat{\mathscr{T}}(G, \rho, (\kappa_k)_k)$ and call it the Thompson group of the cloning system $(G, \rho, (\kappa_k)_k)$.

By Observation 2.2 and Theorem 1.1 every element t of of $\widehat{\mathscr{T}}(G)$ can be written as $t = (E_-, g)(E_+, h)^{-1}$ for some $E_-, E_+ \in \mathscr{F}$ and $g, h \in G$. If it can also be written $t = (E_-, g')(E_+, h')^{-1}$ then $gh^{-1} = g'h'^{-1}$. It therefore makes sense to represent it by just the triple (E_-, gh^{-1}, E_+). Of course, this representation is still not unique, for example $(E, 1_G, E)$ represents the identity element for every $E \in \mathscr{F}$.

Now assume that $G = \varinjlim G_n$ is an injective direct limit of groups $(G_n)_{n\in\mathbb{N}}$ and that the cloning system is a cloning system on $(G_n)_{n\in\mathbb{N}}$. Recall from Subsection 1.3 that a forest E is called semisimple if all but its first tree are trivial and in that case its number of feet is the number of leaves of the first tree.

We collect some facts about semisimple elements of $\mathscr{F} \bowtie G$. We start with semisimple elements of $\mathscr{F}$.

Observation 2.18. *Let $E, E_1, E_2, F \in \mathscr{F}$.*

1. *The number of feet of a non-trivial semisimple element of $\mathscr{F}$ is its length plus one.*
2. *Any two semisimple elements of $\mathscr{F}$ have a semisimple common right multiple. More generally, any two elements of rank at most m have a common right multiple of rank at most m.*
3. *If E is semisimple with n feet then EF is semisimple if and only if F has rank at most n. More generally, if E is non-trivial of rank m and length $n - m$ then EF has rank m if and only if F has rank at most n.*
4. *If E_1, E_2 are semisimple with n feet then E_1F is semisimple if and only if E_2F is.*

Now we upgrade these facts to $\mathscr{F} \bowtie G$. We say that an element $(E, g) \in \mathscr{F} \bowtie G$ has n *feet* if E is semisimple with n feet and $g \in G_n$.

Lemma 2.19. *Let $E, E_1, E_2, F \in \mathscr{F}$ and $g, h \in G$.*

1. *The number of feet of a semisimple element of $\mathscr{F} \bowtie G$ is its length plus one.*
2. *Any two semisimple elements of $\mathscr{F} \bowtie G$ have a semisimple common right multiple.*
3. *$(E, g)F = (E(g \cdot F), g^F)$ is semisimple if and only if $E(g \cdot F)$ is semisimple.*
4. *If (E, g) is semisimple with n feet then $(E, g)F$ is semisimple if and only if F has rank at most n.*
5. *If (E_1, g) and (E_2, h) are semisimple with same number of feet then $(E_1, g)E$ is semisimple if and only if $(E_2, g)E$ is semisimple.*

Proof. The first statement is clear by definition. The second statement can be reduced to the corresponding statement in $\mathscr{F}$ because E is a right multiple of (E, g).

In the third statement only the implication from right to left needs justification, namely that $g^F \in G_n$ where n is the number of feet of $E(g \cdot F)$. This is because if $g \in G_m$ and $\operatorname{len} E = k$ then $g^E \in G_{m+k}$ as can be seen by induction on $\operatorname{len} E$ using $\kappa_k(G_n) \subseteq G_{n+1}$.

For (4) note that $g \in G_n$. But $\rho(G_n) \subseteq S_n$ so having rank at most n is preserved under the action of G_n, i.e., $\operatorname{rk}(g \cdot F) \leq n \Leftrightarrow \operatorname{rk} F \leq n$. Thus the statement follows from the one for $\mathscr{F}$. The last statement is immediate from (4). □

Definition 2.20 (Simple)**.** A triple (E_-, g, E_+) (and the element represented by it) is said to be *simple* if E_- and E_+ are semisimple, both of them with n feet and $g \in G_n$. This is the case if it can be written as $(E_-, g)(E_+, h)^{-1}$ with both factors semisimple with same number of feet.

Proposition 2.21. *The set of simple elements in $\widehat{\mathscr{T}}(G)$ is a subgroup.*

Proof. We sketch the argument, which closely follows [Bri07, Section 7]. Let us write the element represented by (E_-, g, E_+) as $[E_-, g, E_+]$. Clearly $[E_-, g, E_+]^{-1} = [E_+, g^{-1}, E_-]$. Now consider two elements $s = [E_-, g, E_+], t = [F_-, h, F_+]$ represented by simple triples. Let

$$E_+E = F_-F \tag{2.8}$$

be a semisimple common right multiple of E_+ and F_- (Observation 2.18 (2)). Then

$$\begin{aligned} st &= E_- g E F^{-1} h F_+^{-1} \\ &= (E_-(g \cdot E), g^E)(F_+(h^{-1} \cdot F), (h^{-1})^F)^{-1} \\ &= [E_-(g \cdot E), g^E h^{h^{-1} \cdot F}, F_+(h^{-1} \cdot F)]. \end{aligned} \tag{2.9}$$

In the last line we used that $(h^F)^{-1} = (h^{-1})^{h \cdot F}$ so that $((h^{-1})^F)^{-1} = h^{h^{-1} \cdot F}$.
We claim that the last expression of (2.9) is simple. Indeed, (E_-, g) and E_+ are semisimple with same number of feet and E_+E is semisimple so $(E_-, g)E = (E_-(g \cdot E), g^E)$ is semisimple by Lemma 2.19 (5). Similar reasoning applies to $(F_+(h^{-1} \cdot F), h^{-1^F})$. Moreover, we can use Corollary 2.4 to compute

$$\operatorname{len}(E_-, g) + \operatorname{len} E \overset{s \text{ simple}}{=} \operatorname{len} E_+ + \operatorname{len} E \overset{(2.8)}{=} \operatorname{len} F_- + \operatorname{len} F \overset{t \text{ simple}}{=} \operatorname{len}(F_+, (h^{-1})^F) + \operatorname{len} F.$$

By Lemma 2.19 (1) this shows that the last expression of (2.9) is simple. □

Definition 2.22 (Thompson group of a filtered cloning system)**.** The group of simple elements in $\widehat{\mathscr{T}}(G)$ is denoted $\mathscr{T}(G_*)$ and called the *Thompson group* of G_*. If we need to be more precise, as with $\widehat{\mathscr{T}}(G)$ we can include other data from the cloning system in the notation as in $\mathscr{T}(G_*, \rho_*, (\kappa_k^*)_k)$.

Recall from the discussion after Corollary 2.4 that there is a length morphism $\operatorname{len}\colon \widehat{\mathscr{T}}(G) \to \mathbb{Z}$ which takes an element $[E, g, F]$ to $\operatorname{len}(E) - \operatorname{len}(F)$. The group $\mathscr{T}(G_*)$ lies in the kernel of that morphism, that is, simple elements have length 0.

Remark 2.23. Constructing $\mathscr{T}(G_*)$ as the subgroup of simple elements of $\widehat{\mathscr{T}}(G)$ is somewhat artificial as can be seen in some of the proofs above. The more natural approach would be to have each element of $\mathscr{F}$ "know" on which level it can be applied. This amounts to considering the category of forests $\mathcal{P}$ that has objects the natural numbers and morphisms $\lambda_k^n\colon n \to n+1, 1 \le k \le n$ subject to the forest relations (1.2), cf. [Bel04, Section 7]. Let $\mathcal{G}$ be another category which also has objects the natural numbers and such that the morphisms in n form a group G_n. So while $\mathcal{P}$ has only "vertical" arrows, $\mathcal{G}$ has only "horizontal" arrows. One would then want to form the Zappa–Szép product $\mathcal{P} \bowtie \mathcal{G}$ which would be specified by commutative squares of the form $\gamma\lambda_k^n = \lambda_{\rho(\gamma)k}^n \gamma^{\lambda_k}$ with $\gamma \in G_n$ and $\gamma^{\lambda_k} \in G_{n+1}$. Localizing everywhere one would obtain a groupoid of fractions $\mathcal{Q}$ and $\mathscr{T}(G_*)$ should be just $\operatorname{Hom}_{\mathcal{Q}}(1,1)$.
The reason that we have not chosen that description is simply that Zappa–Szép products for categories are not well-developed to our knowledge, while for monoids all the needed statements were already available thanks to Brin's work [Bri05, Bri07].
Artifacts of this approach, which should be overcome by the general approach above, include the maps $\iota_{n,n+1}$, the property of being properly graded, and the fact that our construction does not allow one to construct all the Thompson-like groups one might want to cover (for example T).

2.5. **Morphisms.** Let $(G, \rho^G, (\kappa_k^G)_{k\in\mathbb{N}})$ and $(H, \rho^H, (\kappa_k^H)_{k\in\mathbb{N}})$ be cloning systems. A homomorphism $\varphi\colon G \to H$ is a *morphism of cloning systems* if

(1) $(\varphi(g))\kappa_k^H = \varphi((g)\kappa_k^G)$ for all $k \in \mathbb{N}$ and $g \in G$, and
(2) $\rho^H \circ \varphi = \rho^G$.

Observation 2.24. *Let $\varphi\colon G \to H$ be a morphism of cloning systems. There is an induced homomorphism $\widehat{\mathscr{T}}(\varphi)\colon \widehat{\mathscr{T}}(G) \to \widehat{\mathscr{T}}(H)$. If φ is injective or surjective then so is $\widehat{\mathscr{T}}(\varphi)$. In particular, there is always a homomorphism $\widehat{\mathscr{T}}(G) \to \widehat{\mathscr{T}}(S_\omega)$.*

Proof. We show that a morphism of cloning systems induces a morphism $\mathscr{F} \bowtie G \to \mathscr{F} \bowtie H$. The statement then follows from Lemma 1.2. Naturally, $\widehat{\mathscr{T}}(\varphi)$ is defined by $\widehat{\mathscr{T}}(\varphi)(Eg) = E\varphi(g)$. Well definedness amounts to $\widehat{\mathscr{T}}(\varphi)((g \cdot E)g^E) = (\varphi(g) \cdot E)(\varphi(g)^E)$ which follows from (1) and (2) by writing E as a product of λ_ks and inducting on the length.
The injectivity and surjectivity statements are clear. □

Similarly let $(G_n)_{n\in\mathbb{N}}$ and $(H_n)_{n\in\mathbb{N}}$ be injective direct systems equipped with cloning systems. A morphism of directed systems of groups $\varphi_*\colon G_* \to H_*$ is a *morphism of cloning systems* if

(1) $(\varphi_n(g))\kappa_k^{H,n} = \varphi_{n+1}((g)\kappa_k^{G,n})$ for all $1 \le k \le n$ and $g \in G_n$, and
(2) $\rho_n^H \circ \varphi_n = \rho_n^G$ for all $n \in \mathbb{N}$.

Observation 2.25. *Let $\varphi_*\colon G_* \to H_*$ be a morphism of cloning systems. There is an induced homomorphism $\mathscr{T}(\varphi)\colon \mathscr{T}(G_*) \to \mathscr{T}(H_*)$. If φ is injective or surjective then so is $\mathscr{T}(\varphi)$. In particular, there is always a homomorphism $\mathscr{T}(G_*) \to \mathscr{T}(S_*)$, the latter being Thompson's group V.*

Proof. We have to show that if $Eg \in \mathscr{F} \bowtie G$ is semisimple with n feet then so is $\widehat{\mathscr{T}}(\varphi)(Eg) = E\varphi(g)$. But this follows since E is semisimple with n feet and $g \in G_n$, so $\varphi(g) \in H_n$. □

Functoriality is straightforward:

Observation 2.26. *If $\varphi\colon G \to H$ and $\psi\colon H \to K$ are morphisms of cloning systems then $\widehat{\mathscr{T}}(\psi\varphi) = \widehat{\mathscr{T}}(\psi)\widehat{\mathscr{T}}(\varphi)\colon \widehat{\mathscr{T}}(G) \to \widehat{\mathscr{T}}(K)$. If φ and ψ are morphisms of filtered cloning systems then $\mathscr{T}(\psi\varphi) = \mathscr{T}(\psi)\mathscr{T}(\varphi)\colon \mathscr{T}(G) \to \mathscr{T}(K)$.*

3. Basic properties

Throughout this section let $\mathscr{T}(G_*)$ be the Thompson group of a cloning system on an injective directed system of groups $(G_n)_{n\in\mathbb{N}}$ and let $G = \varinjlim G_n$. We collect some properties of $\mathscr{T}(G_*)$ that follow directly from the construction.

Observation 3.1. *Let $E \in \mathscr{F}$ be semisimple with n feet. The map $g \mapsto [E, g, E]$ is an injective homomorphism $G_n \to \mathscr{T}(G_*)$.*

Proof. The maps $G_n \to G \to \mathscr{F} \bowtie G \to \widehat{\mathscr{T}}(G)$ are all injective. The element $[E, g, E]$ is simple, so the image lies in $\mathscr{T}(G_*)$. The map is visibly a homomorphism. □

3.1. **A short exact sequence.** Let $\mathscr{T}(\rho_*)\colon \mathscr{T}(G_*) \to V$ denote the morphism from Observation 2.25. The kernel, which we denote $\mathscr{K}(G_*)$, consists of elements $[E, g, E]$ where E is a semisimple forest with n feet and and $g \in \ker(\rho_n)$. If $W \cong \mathscr{T}(G_*)/\mathscr{K}(G_*)$ is the image of $\mathscr{T}(\rho_*)$ we have the short exact sequence

$$1 \to \mathscr{K}(G_*) \to \mathscr{T}(G_*) \to W \to 1.$$

Note that W contains Thompson's group F.
In what follows we will concentrate on the case where $\rho = \mathrm{id}$.

Observation 3.2. *Suppose $\rho = \mathrm{id}$. Then $\mathscr{T}(G_*) = \mathscr{K}(G_*) \rtimes F$.*

Proof. Since each $\rho_n = \mathrm{id}$, we have $W = F$, which is $\mathscr{T}(\{1\})$. Then the splitting map $F \to \mathscr{T}(G_*)$ is $\mathscr{T}(\iota_*)$ where $\iota_*\colon \{1\} \to G_*$ is the trivial homomorphism. □

Continue to assume $\rho = \mathrm{id}$. For E a semisimple forest with n feet, let G_E denote the subgroup $\{[E, g, E] \mid g \in G_n\}$ of $\mathscr{K}(G_*)$. Note that $G_E \cong G_n$. If E is a right multiple of F then conjugation by FE^{-1} (in $\widehat{\mathscr{T}}(G)$) induces an injective homomorphism $G_E \to G_F$. This turns the family of G_E with semisimple E into a directed system.

Observation 3.3. *Suppose* $\rho = \mathrm{id}$*. Then* $\mathscr{K}(G_*) = \varinjlim G_E$*.*

The question of whether F is amenable or not is probably the most famous question about Thompson's groups. The following observation does not purport to be deep, but it seems worth recording nonetheless.

Observation 3.4 (Amenability)**.** *Suppose* $\rho = \mathrm{id}$*. Then* $\mathscr{T}(G_*)$ *is amenable if and only if* F *and every* G_n *is amenable.*

Proof. By the previous observation $\mathscr{K}(G_*)$ is a direct limit of copies of G_n. Since amenability is preserved under taking subgroups and direct limits, this tells us that $\mathscr{K}(G_*)$ is amenable if and only if every G_n is. Then since $\mathscr{T}(G_*) = \mathscr{K}(G_*) \rtimes F$, the conclusion follows since amenability is also closed under group extensions. □

Observation 3.5 (Free group-free)**.** *Suppose* $\rho = \mathrm{id}$*. If none of the* G_n *contains a non-abelian free group then neither does* $\mathscr{T}(G_*)$*.*

Proof. Suppose $H \leq \mathscr{T}(G_*)$ is free. If $H \cap \mathscr{K}(G_*) = \{1\}$ then H embeds into F, and so H must be abelian, since F does not contain a non-abelian free group. Now suppose there is some $1 \neq x \in H \cap \mathscr{K}(G_*)$. For any $y \in H$, the conjugate x^y is in $H \cap \mathscr{K}(G_*)$. Since $\mathscr{K}(G_*)$ is a direct limit of copies of the G_n, it does not contain a non-abelian free group by assumption, and so $\langle x, x^y \rangle$ is abelian. But $y \in H$ was arbitrary, so H must already be abelian. □

3.2. **Truncation.** For $g \in G_n$ and $k \leq n$ we have the equation $g\lambda_k = (g \cdot \lambda_k)g^{\lambda_k}$ in $\mathscr{F} \bowtie G$ where $g^{\lambda_k} \in G_{n+1}$. In $\widehat{\mathscr{T}}(G)$ this implies

$$g = (g \cdot \lambda_k)g^{\lambda_k}\lambda_k^{-1}. \tag{3.1}$$

This elementary observation has an interesting consequence. Let $N \in \mathbb{N}$ be arbitrary and define a directed system of groups $(G'_n)_{n\in\mathbb{N}}$ by $G'_n := \{1\}$ for $n \leq N$ and $G'_n := G_n$ for $n > N$. Define a cloning system on G'_* by letting $(\kappa')^n_k \colon G'_n \to G'_{n+1}$ be the trivial homomorphism when $n \leq N$, and $(\kappa')^n_k = \kappa^n_k$ and $\rho'_n = \rho_n$ when $n > N$. We call G'_* the *truncation* of G_* at N and $((\rho'_n)_n, ((\kappa')^n_k)_{k\leq n})$ the truncation of $((\rho_n)_n, (\kappa^n_k)_{k\leq n})$ at N.

Proposition 3.6 (Truncation isomorphism)**.** *Let* G'_* *be the truncation of* G_* *at* N*. The morphism* $\mathscr{T}(G'_*) \to \mathscr{T}(G_*)$ *induced by the obvious homomorphism* $G'_* \to G_*$ *is an isomorphism.*

Proof. The morphism $G'_* \to G_*$ is injective hence so is $\mathscr{T}(G'_*) \to \mathscr{T}(G_*)$. To show that it is surjective let $[E, g, F] \in \mathscr{T}(G_*)$ be semisimple with n feet. If $n > N$ there is nothing to show. Otherwise use (3.1) to write

$$[E, g, F] = [E(g \cdot \lambda_k), g^{\lambda_k}, F\lambda_k]$$

for some $k \leq n$. The right hand side expression is semisimple with $n + 1$ feet. Proceeding inductively, we obtain an element that is semisimple with $N + 1$ feet and therefore in $\mathscr{T}(G'_*)$. □

This Proposition justifies thinking of $\mathscr{T}(G_*)$ as a sort of limit of G_* since it does not depend on an initial segment of data.

4. Spaces for Thompson's groups

The goal of this section is to produce for each Thompson group $\mathscr{T}(G_*)$ a space on which it acts. The space will be contractible and have stabilizers isomorphic to the groups G_n. The ideas used in the construction were used before in [Ste92, Bro92, Far03, Bro06, FMWZ13, BFS+]. Throughout let G_* be an injective directed system of groups equipped with a properly graded cloning system and let $G = \varinjlim G_*$.

As a starting point we note that Corollary 1.6, Observation 2.2 and Corollary 2.4 imply that $\widehat{\mathscr{T}}(G)/G$ is a lattice under the relation $xG \le yG$ if $x^{-1}y \in \mathscr{F} \bowtie G$. Later on it will be convenient to have a symbol for the quotient relation so we let $x \sim_G y$ if $x^{-1}y \in G$.

4.1. **Semisimple group elements.** We generalize some of the notions that were introduced in Sections 1.3 and 2.4. We say that an arbitrary (not necessarily semisimple) element E of $\mathscr{F}$ has *n feet* if it has rank m and length $n-m$. Visually this means that the last leaf that is not a root is numbered n. An element (E,g) of $\mathscr{F} \bowtie G$ has *n feet* if E is of type at most n and $g \in G_n$. Finally, we call an element $[E,g,F]$ of $\widehat{\mathscr{T}}(G)$ *semisimple* if (E,g) is semisimple with n feet and F has at most n feet. This is actually not an abuse of terminology: If an element of the group $\widehat{\mathscr{T}}(G)$ is semisimple in this sense, and is an element of the monoid $\mathscr{F} \bowtie G$, then it must be semisimple in the monoid. We let $\widetilde{\mathcal{P}}_1$ denote the set of all semisimple elements of $\widehat{\mathscr{T}}(G)$.

Lemma 4.1. *If $[E_1, g_1, F_1]$ is simple and $[E_2, g_2, F_2]$ is semisimple then $[E_1, g, F_1][E_2, g, F_2]$ is semisimple. As a consequence, $\widehat{\mathscr{T}}(G_*)$ acts on $\widetilde{\mathcal{P}}_1$.*

Proof. This is shown analogously to Proposition 2.21. □

If $[E,g,F]$ is semisimple we say that it has $\operatorname{len}([E,g,F]) + 1 = \operatorname{len}(E) - \operatorname{len}(F) + 1$ *feet*, which is well defined by Corollary 2.4. This can be visualized as the number of roots of F that can be "reached" from the first root of E. We let $\widetilde{\mathcal{P}}_{1,n}$ denote the set of all semisimple elements with at most n feet. We define $\mathcal{P}_{1,n}$ to be the quotient $\widetilde{\mathcal{P}}_{1,n}/{\sim_G}$ and call the passage from $\widetilde{\mathcal{P}}_{1,n}$ to $\mathcal{P}_{1,n}$ *dangling*. Note that $\mathcal{P}_{1,n}$ is a subposet of $\widehat{\mathscr{T}}(G)/G$. We also denote $\widetilde{\mathcal{P}}_1/{\sim_G}$ by $\mathcal{P}_1$.

Lemma 4.2. *If $x, y \in \widetilde{\mathcal{P}}_{1,n}$ are semisimple then $x \sim_G y$ if and only if $x^{-1}y \in G_n$.*

Proof. What needs to be shown is that if $x^{-1}y \in G$ then $x^{-1}y \in G_n$. Write $x = [E_1, g^{-1}, F_1]$ and $y = [E_2, h^{-1}, F_2]$. Let $E = E_1E_1' = E_2E_2'$ be a common right multiple so that $x^{-1}y = [F_1(g \cdot E_1'), g^{E_1'}(h^{E_2'})^{-1}, F_2(h \cdot E_2')] =: [A, b, C]$. For this to equal some $d \in G$ it is necessary that $Ab = dC$ in $\mathscr{F} \bowtie G$, that is, $A = d \cdot C$ and $b = d^C$.
Say that E has length m. Then we compute that A and C have length $m - n + 1$, and that b lies in G_{m+1}. Since the cloning system is properly graded, the fact that $b = d^C$ implies that d has to lie in G_n. □

For context, the term "dangling" comes from the case when G_* is the system of braid groups B_*, and the elements of $\mathcal{P}_{1,n}$ can be pictured as "dangling braided strand diagrams" [BFS+].

4.2. **Poset structure.** Consider the geometric realization $|\mathcal{P}_1|$. This is the simplicial complex with a k-simplex for each chain $x_0 \le \cdots \le x_k$ of elements of $\mathcal{P}_1$, and face relation given by subchains.

Lemma 4.3. *The poset $\mathcal{P}_1$ is a lattice, in particular $|\mathcal{P}_1|$ is contractible.*

Proof. We already know that $\mathscr{T}(G_*)/G$ is a lattice so it remains to show that $\mathcal{P}_1$ is closed under taking suprema and infima. In other words, it suffices to show that least common right multiples of semisimple elements are semisimple and that left factors of semisimple elements are semisimple. The first is similar to the proof of Proposition 2.21 and the second is easy. □

The space $|\mathcal{P}_1|$ is not ideal; for one thing, every vertex is contained in a simplex of arbitrarily large dimension. It has therefore proven historically helpful to consider a subspace called the *Stein space*, which we introduce next.

4.3. **The Stein space.** The preorder on $\widetilde{\mathcal{P}}_1$ was defined by declaring that $x \leq y$ if $y = x(E, g)$ for some $(E, g) \in \mathscr{F} \bowtie G$. The basic idea in constructing the Stein space is to regard this relation as a transitive hull of a finer relation $\preceq$ and to use this finer relation in constructing the space. It is defined by declaring $x \preceq y$ if $y = x(E, g)$ for some $(E, g) \in \mathscr{F} \bowtie G$ with the additional assumption that E is elementary. An *elementary* forest is one in which every tree has at most two leaves. That is, a forest is elementary if it can be written as $\lambda_{k_1} \cdots \lambda_{k_r}$ with $k_{i+1} > k_i + 1$ for $i < r$.
Again it is clear that $\preceq$ is invariant under dangling and we also write $\preceq$ for the relation induced on $\mathcal{P}_1$. Note that $\preceq$ is not transitive, but it is true that if $x \preceq z$ and $x \leq y \leq z$ then $x \preceq y \preceq z$. Given a simplex $x_0 \leq \cdots \leq x_k$ in $|\mathcal{P}_1|$, call the simplex *elementary* if $x_0 \preceq x_k$. The property of being elementary is preserved under passing to subchains, so the elementary simplices form a subcomplex.

Definition 4.4 (Stein space)**.** The subcomplex of elementary simplices of $|\mathcal{P}_1|$ is denoted by $\mathscr{X}(G_*)$ and called the *Stein space* of $\mathscr{T}(G_*)$.

The Stein space has the structure of a cubical complex, which we now describe. The key point is:

Observation 4.5. *If E is elementary then the set of right factors of E forms a boolean lattice under $\preceq$.*

For $x \preceq y$ in $\mathcal{P}_1$ we consider the closed interval $[x, y] := \{z \in \mathcal{P}_1 \mid x \leq z \leq y\}$ as well as the open and half open intervals (x, y), $[x, y)$ and $(x, y]$ that are defined analogously. As a consequence of Observation 4.5 we obtain that the interval $[x, y] := \{z \in \mathcal{P}_1 \mid x \leq z \leq y\}$ is a boolean lattice and so $|[x, y]|$ has the structure of a cube. The intersection of two such cubes $|[x, y]|$ and $|[z, w]|$ is $|[\sup(x, z), \inf(y, w)]|$ (which may be empty if the supremum is larger than the infimum). In particular the intersection of cubes is either empty or is again a cube. Hence $\mathscr{X}(G_*)$ is a cubical complex in the sense of Definition 7.32 of [BH99]. Since there are only finitely many elementary forests of a given rank, we get:

Observation 4.6. *For any vertex x in $\mathscr{X}(G_*)$, there are only finitely many vertices y in $\mathscr{X}(G_*)$ with $x \preceq y$.*

The next step is to show that $\mathscr{X}(G_*)$ is itself contractible. The argument is similar to that given in Section 4 of [Bro92]. We follow the exposition in [BFS+].

Lemma 4.7. *For $x < y$ with $x \not\prec y$, $|(x, y)|$ is contractible.*

Proof. For any $z \in (x, y]$ let z_0 be the unique largest element of $[x, z]$ such that $x \preceq z_0$. By hypothesis $z_0 \in [x, y)$, and by the definition of $\preceq$ it is clear that $z_0 \in (x, y]$, so in fact $z_0 \in (x, y)$. Also, $z_0 \leq y_0$ for any $z \in (x, y)$. The inequalities $z \geq z_0 \leq y_0$ then imply that $|(x, y)|$ is contractible, by Section 1.5 of [Qui78]. ☐

Proposition 4.8. *$\mathscr{X}(G_*)$ is contractible.*

Proof. We know that $|\mathcal{P}_1|$ is contractible by Lemma 4.3. We can build up from $\mathscr{X}(G_*)$ to $|\mathcal{P}_1|$ by attaching new subcomplexes, and we claim that this never changes the homotopy type, so $\mathscr{X}(G_*)$ is contractible. Given a closed interval $[x, y]$, define $r([x, y]) := \text{len}(y) - \text{len}(x)$. As a remark, if $x \preceq y$ then $r([x, y])$ is the dimension of the cube given by $[x, y]$. We attach the contractible subcomplexes $|[x, y]|$ for $x \not\preceq y$ to $\mathscr{X}(G_*)$ in increasing order of r-value. When we attach $|[x, y]|$ then, we attach it along $|[x, y)| \cup |(x, y]|$. But this is the suspension of $|(x, y)|$, and so is contractible by the previous lemma. We conclude that attaching $|[x, y]|$ does not change the homotopy type, and since $|\mathcal{P}_1|$ is contractible, so is $\mathscr{X}(G_*)$. ☐

Finally we show that cell stabilizers are essentially copies of the G_n.

Lemma 4.9 (Stabilizers)**.** *The stabilizer in $\mathscr{T}(G_*)$ of a vertex in $\mathscr{X}(G_*)$ with n feet is isomorphic to G_n. The stabilizer in $\mathscr{T}(G_*)$ of an arbitrary cell is isomorphic to a finite index subgroup of some G_n.*

Proof. First consider the stabilizer of a vertex x with n feet. We claim that $\mathrm{Stab}_{\mathscr{T}(G_*)}(x) \cong G_n$. Choose $\tilde{x} \in \widetilde{\mathcal{P}}_1$ representing x and let $g \in \mathrm{Stab}_{\mathscr{T}(G_*)}(x)$. By definition of dangling, and by Lemma 4.2, there is a (unique) $h \in G_n$ such that $g\tilde{x} = \tilde{x}h$. Then the map $g \mapsto h = \tilde{x}^{-1}g\tilde{x}$ is a group isomorphism.
Now let $\sigma = |[x,y]|$, $x \preceq y$ be a an arbitrary cube. Since the action of $\mathscr{T}(G_*)$ preserves the number of feet, the stabilizer of σ fixes x and y. Hence G_σ is contained in G_x and contains the kernel of the map $G_x \to \mathrm{Symm}(\{w \mid x \preceq w \preceq y\})$, the image of which is finite by Observation 4.6. □

5. Finiteness properties

One of our main motivations for defining the functor $\mathscr{T}(-)$ is to study how it behaves with respect to finiteness properties. Recall that a group G if said to be *of type F_n* if there is a $K(G,1)$ whose n-skeleton is compact. Most of the known Thompson's groups are of type F_∞, that is, of type F_n for all n. To efficiently speak about groups that are not of type F_∞ recall that the *finiteness length* of G, denoted $\phi(G)$, is the supremum over all $n \in \mathbb{N}$ such that G is of type F_n.
We will see below that proofs of the finiteness properties of $\mathscr{T}(G_*)$ depend on the finiteness properties of the individual groups G_k as well as on the asymptotic connectivity of certain descending links, which is infinite in many cases. Since finite initial intervals of G_* can always be ignored by Proposition 3.6 we ask:

Question 5.1. *For which directed systems of groups G_* equipped with cloning systems do we have*
$$\phi(\mathscr{T}(G_*)) = \liminf \phi(G_*)\,?$$

Note that for any directed system of groups G_* one can take all ρ_k to trivial and all κ_k^n to be ι_n. In this case $\mathscr{T}(G_*) = \lim_n G_n$ so the answer to Question 5.1 will not encompass all cloning systems. Our hope is that all sufficiently natural cloning systems qualify.

5.1. **Morse theory.**

One of the main tools to study connectivity properties, and thus to study finiteness properties, is combinatorial Morse theory. We collect here the main ingredients that will be needed later on.
Let X be a Euclidean cell complex. A map $h\colon X^{(0)} \to \mathbb{N}_0$ is called a *Morse function* if the maximum of h over the vertices of a cell of X is attained in a unique vertex. We typically think of h as assigning a *height* to each vertex. If h is a Morse function and $r \in \mathbb{R}$, the sublevel set $X_r = X^{\le r}$ consists of all cells of X whose vertices have height at most n. For a vertex $x \in X^{(0)}$ of height n, the *descending link* $\mathrm{lk}{\downarrow}(x)$ of x is the subcomplex of $\mathrm{lk}(x)$ spanned by all vertices of strictly lower height. The main observation that makes Morse theory work is that keeping track of the connectivity of descending links allows one to deduce global (relative) connectivity statements:

Lemma 5.2 (Morse Lemma)**.** *Let X be a Euclidean cell complex and let $h\colon X^{(0)} \to \mathbb{N}_0$ be a Morse function on X. Let $s,t \in \mathbb{R} \cup \{\infty\}$ with $s < t$. If $\mathrm{lk}{\downarrow}(x)$ is $(k-1)$-connected for every vertex in $X_t \setminus X_s$ then the pair (X_t, X_s) is k-connected.*

The connection between connectivity of spaces and finiteness properties of groups is most directly made using Brown's criterion. A Morse function on X gives rise to a filtration $(X_n)_{n\in\mathbb{N}_0}$ by subcomplexes. We say that the filtration is *essentially k-connected* if for every $i \in \mathbb{N}_0$ there exists a $j \ge i$ such that $\pi_\ell(X_i \to X_j)$ is trivial for all $\ell \le k$.

Now assume that a group G acts on X. If h is G-invariant then so is the filtration $(X_n)_n$. We say that the filtration is *cocompact* if the quotient $G\backslash X_n$ is compact for all n. This is the setup for Brown's criterion, see [Bro87, Theorems 2.2, 3.2].

Theorem 5.3 (Brown's criterion)**.** *Let $n \in \mathbb{N}$ and assume a group G acts on an $(n-1)$-connected CW-complex X. Assume that the stabilizer of every p-cell of X is of type F_{n-p}. Let $(X_n)_{n\in\mathbb{N}_0}$ be a G-cocompact filtration of X. Then G is of type F_n if and only if $(X_n)_n$ is essentially $(n-1)$-connected.*

Putting both statements together we obtain the version that we will mostly use.

Corollary 5.4. *Let G act on a contractible Euclidean cell complex X and let $h\colon X^{(0)} \to \mathbb{N}_0$ be a G-invariant Morse function with cocompact sublevel sets. Assume that the stabilizer of every p-cell of X is of type F_{n-p} and that the sublevel sets X_n are cocompact. If there is an $s \in \mathbb{R}$ such that $\mathrm{lk}{\downarrow}(x)$ is $(n-1)$-connected for all vertices $x \in X^{(0)} \setminus X_s$ then G is of type F_n.*

If G_* is a system of groups equipped with a cloning system then $\mathscr{T}(G_*)$ acts on the Stein space $\mathscr{X}(G_*)$, which is contractible (Proposition 4.8) with stabilizers from G_* (Lemma 4.9). Our next goal is to define an invariant, cocompact Morse function and to describe the descending links.

5.2. **The Morse function.** Recall that the vertices of $\mathscr{X}(G_*)$ are classes $[E, g, F]$ of semisimple elements modulo dangling. The height function we will be using assigns to such a vertex its number of feet (see Section 4.1). That is, $\mathscr{X}(G_*)_n = |\mathcal{P}_{1,n}| \cap \mathscr{X}(G_*)$. This height function is G-invariant because it is induced by the morphism $\mathrm{len}\colon \widehat{\mathscr{T}}(G) \to \mathbb{Z}$ and every element of $\mathscr{T}(G_*)$ has length 0.

Lemma 5.5 (Cocompactness)**.** *The action of $\mathscr{T}(G_*)$ is transitive on vertices of $\mathscr{X}(G_*)$ with a fixed number of feet. Consequently the action of $\mathscr{T}(G_*)$ on $\mathscr{X}(G_*)_n$ is cocompact for every n.*

Proof. Let $\tilde{x} = [E_-, g, E_+]$ and $\tilde{y} = [F_-, h, F_+]$ be semisimple with n feet. We know $\tilde{x}\tilde{y}^{-1}$ takes $\tilde{y}$ to $\tilde{x}$, so it suffices to show that $\tilde{x}\tilde{y}^{-1}$ is simple. Note that E_+ and F_+ have rank at most n. By Observation 2.18 (2) they admit a common right multiple $E_+E = F_+F$ of rank at most n. Let the length of this multiple be m, so it has at most $m+n$ feet. Then

$$\tilde{x}\tilde{y}^{-1} = [E_-(g \cdot E), g^E(h^F)^{-1}, F_-(h \cdot F)]$$

and both $E_-(g \cdot E)$ and $F_-(h \cdot F)$ are semisimple by Observation 2.18 (3). They have $m+n$ feet and both g^E and h^F lie in G_{n+m}. Thus $\tilde{x}\tilde{y}^{-1}$ is simple.
The second statement now follows from Observation 4.6. □

5.3. **Descending links.** Let x be a vertex in $\mathscr{X}(G_*)$, with n feet. We want to describe the descending link of x. A vertex y is in the link of x if either $x \preceq y$ or $y \preceq x$. But in the first case y is ascending so the descending link is spanned by vertices y with $y \preceq x$. These are by definition of the form $x(E, g)^{-1}$ for E an elementary forest and $g \in G_n$. In particular, for a fixed n, the descending links of any vertices of height n look the same, and are all isomorphic to the simplicial complex of products gE^{-1} where $g \in G_n$ and E is an elementary forest with n feet, modulo the relation $\sim_G$.

It is helpful to describe this complex somewhat more explicitly. In doing so we slightly shift notation by making use of the fact that elementary forests can be parametrized by subgraphs of linear graphs.

Let L_n be the graph with n vertices, labeled 1 through n, and a single edge connecting i to $i+1$, for each $1 \le i \le n-1$. This is the *linear graph* with n vertices. Denote the edge from i to $i+1$ by e_i. We will exclusively consider *spanning* subgraphs of L_n, that is, subgraphs whose vertex set is $\{1, \dots, n\}$. We call the spanning subgraph without edges *trivial.* A *matching* on a graph is a spanning subgraph in which no two edges share a vertex. For an elementary forest E with n feet, define $\Gamma(E)$ to be the spanning subgraph of L_n that has an edge from i to $i+1$ if and only if the i^{th} and $(i+1)$st leaves of E are leaves of a common caret. Note that this is a matching. Conversely, given a matching Γ of L_n, there is an elementary forest $E(\Gamma) = \lambda_{i_k} \cdots \lambda_{i_1}$ where Γ has edges $e_{i_1}, \dots, e_{i_k}$. Both operations are inverse to each other so we conclude:

Observation 5.6. *There is a one-to-one correspondence between matchings of L_n and elementary forests with at most n feet.*

In particular, if Γ is a matching with m edges and n vertices we obtain a cloning map $\kappa_\Gamma \colon G_m \to G_n$ which is just the cloning map of $E(\Gamma)$ as defined before Observation 2.11. We also get an action of G_m on Γ which is given by the action of $\rho(G_m)$ on connected components. For future reference we also note:

Observation 5.7. *There is a one-to-one correspondence between spanning subgraphs of L_n and hedges with at most n feet.*

Now define a simplicial complex $\mathscr{L}(G_n)$ as follows. A simplex in $\mathscr{L}(G_n)$ is represented by a pair (g, Γ), where $g \in G_n$ and Γ is a non-trivial matching of L_n. Two such pairs (g_1, Γ_1), (g_2, Γ_2) are *equivalent (under dangling)* if the following conditions hold:

(1) Γ_1 and Γ_2 both have m edges for some $1 \le m \le n/2$,
(2) $g_2^{-1} g_1$ lies in the image of κ_{Γ_1}, and
(3) $\Gamma_2 = (g_2^{-1} g_1)\kappa_{\Gamma_1}^{-1} \cdot \Gamma_1$.

We want to make $\mathscr{L}(G_n)$ a simplicial complex by saying the face relation is given by passing to subgraphs of the second term in the pair. Denote the equivalence class of (g, Γ) under dangling by $[g, \Gamma]$. In summary,

$$\mathscr{L}(G_n) \text{ has simplex set } \{[g, \Gamma] \mid \Gamma \text{ is a matching of } L_n \text{ and } g \in G_n\}.$$

Observation 5.8. *If x has n feet, the correspondence $(g, \Gamma) \mapsto xgE_\Gamma^{-1}$ induces an isomorphism $\mathscr{L}(G_n) \to \operatorname{lk}{\downarrow}(x)$.*

In particular, $\mathscr{L}(G_*)$ is indeed a simplicial complex, since $\mathscr{X}(G_*)$ is a cubical complex. We now have all the pieces together to apply Brown's criterion to our setting.

Proposition 5.9. *Let G_* be equipped with a cloning system. If G_k is eventually of type F_n and $\mathscr{L}(G_k)$ is eventually $(n-1)$-connected then $\mathscr{T}(G_*)$ is of type F_n.*

Proof. Suppose first that all G_k are of type F_n. Let $X = \mathscr{X}(G_*)$ which is contractible by Proposition 4.8. Our Morse function "number of feet" has cocompact sublevel sets by Lemma 5.5. The stabilizer of any cell is a finite-index subgroup of some G_k by Lemma 4.9. Since finiteness properties are inherited by finite-index subgroups, our assumption implies that all stabilizers are of type F_n. By the second assumption there is an s such that $\mathscr{L}(G_k)$ is $(n-1)$-connected for $k > s$ which by Observation 5.8 means that descending links are $(n-1)$-connected from s on. Applying Corollary 5.4 we conclude that $\mathscr{T}(G_*)$ is of type F_n.
If the G_k are of type F_n only from t on, we use Proposition 3.6 to replace $\mathscr{T}(G_*)$ by the isomorphic group $\mathscr{T}(G'_*)$ where $G'_k = G_k$ for $k \ge t$ and $G_k = \{1\}$ for $k < t$. In particular, all of the G_k are of type F_n.
Of course $\mathscr{X}(G'_*)$ is not isomorphic to $\mathscr{X}(G_*)$ and neither are the $\mathscr{L}(G'_*)$ isomorphic to $\mathscr{L}(G_*)$. However, the k-skeleton of $\mathscr{L}(G'_m)$ is isomorphic to the k-skeleton of $\mathscr{L}(G_m)$ once $m > k + t$. Since $(n-1)$-connectivity only depends on the n-skeleton, if the $\mathscr{L}(G_*)$ are eventually $(n-1)$-connected then so are the $\mathscr{L}(G'_*)$. $\square$

A negative counterpart to this statement, to show that $\mathscr{T}(G_*)$ is not of type F_n, would need stabilizers with good finiteness properties and a filtration that is not essentially $(n-1)$-connected – at least as long as it is based on Brown's criterion. However, Question 5.1 suggests that the failure to be of type F_n rather corresponds to the failure of the stabilizers to be of type F_n.

Inspecting the homotopy type of $\mathscr{L}(G_n)$ does not seem possible uniformly. Instead, most of the remainder of the article will be concerned with proving instances of $\mathscr{L}(G_n)$ to be highly connected. In the case where G_n are braid groups, these complexes were modeled by arc complexes in [BFS+]. In Section 7 below we will directly work with the combinatorial description. General tools that have turned out to be helpful will be collected in Section 5.4.

We can make one positive statement without knowing much at all about G_*. Before stating this as a lemma, we need to define the *matching complex* of L_n. This is a simplicial complex, denoted $\mathcal{M}(L_n)$, whose simplices are matchings on L_n and with face relation given by passing to subgraphs. It is well-known that $\mathcal{M}(L_n)$ is $(\lfloor \frac{n-2}{3} \rfloor - 1)$-connected, see for example [BLVŽ94] (a more precise description of the homotopy type is given in [Koz08, Proposition 11.16] where $\mathcal{M}(L_n)$ arises as the independence complex $\mathrm{Ind}(L_{n_1})$).

Lemma 5.10 (Finite generation)**.** *Let G_* be a family of groups equipped with a cloning system, with cloning maps κ_k^n. Suppose that for n sufficiently large, all G_n are finitely generated and also are generated by the images of the cloning maps with codomain G_n. Then $\mathscr{T}(G_*)$ is finitely generated.*

Proof. By the above discussion, we need only show that the $\mathscr{L}(G_n)$ are connected, for large enough n. Suppose n is large enough that: (a) G_n is generated by images of cloning maps, and (b) $n \geq 5$ so $\mathcal{M}(L_n)$ is connected. Given a vertex $[g, E]$ in $\mathscr{L}(G_n)$, write $g = s_1 \cdots s_r$, where the s_i are generators coming from images of cloning maps $s_i \in \mathrm{im}(\kappa_{k_i})$ for some k_i. Since $\mathcal{M}(L_n)$ is connected, there is a path in $\mathscr{L}(G_n)$ from $[s_1 \cdots s_r, E]$ to $[s_1 \cdots s_r, E_{k_r}] = [s_1 \cdots s_{r-1}, E_{k_r}]$. Repeating this r times, we connect to $[1, E_k]$ for some k, and then to $[1, E_1]$. □

5.4. **Proving high connectivity.** As we have seen, Morse theory is a tool that allows one to show that a pair (X, X_0) is highly connected. We will want to inductively apply this to the situation where $X = \mathscr{L}(G_n)$ and $X_0 = \mathscr{L}(G_{n-k})$ for some $k \in \mathbb{N}$. This is insufficient to conclude that the connectivity tends to infinity though, because we would be trying to get X to be more highly connected than X_0. The following lemma expresses by how much it is insufficient. The lemma is straightforward to prove but can be seen as a roadmap for the argument that follows.

Lemma 5.11. *Let (X, X_0) be a k-connected CW-pair. Assume that X_0 is $(k-1)$-connected. Then X is k-connected if and only if $\pi_k(X_0 \to X)$ is trivial.*

Proof. Consider the part of the homotopy long exact sequence associated to (X, X_0):

$$\pi_{j+1}(X, X_0) \to \pi_j(X_0) \xrightarrow{\iota_j} \pi_j(X) \to \pi_j(X, X_0)$$

for $j < k$ the map ι_j is an isomorphism and $\pi_j(X_0)$ trivial. For $j = k$ it is an epimorphism and $\pi_k(X)$ is trivial if and only if ι_k is. □

In our applications we will know X_0 to be $(k-1)$-connected by induction and (X, X_0) will be seen to be k-connected using Morse theory. To show that $\pi_k(X_0 \to X)$ is trivial we will use a relative variant of the Hatcher flow that was shown to us by Andrew Putman (Proposition 5.13 below). Before we can prove it we need some technical preliminaries.

A *combinatorial k-sphere (respectively k-disk)* is a simplicial complex that can be subdivided to be isomorphic to a subdivision of the boundary of a $(k+1)$-simplex (respectively to a subdivision of a k-simplex). An m-dimensional *combinatorial manifold* is an m-dimensional simplicial complex in which the link of every simplex σ of dimension k is a combinatorial

$(m-k-1)$-sphere. In an m-dimensional *combinatorial manifold with boundary* the link of a k-simplex σ is allowed to be homeomorphic to a combinatorial $(m-k-1)$-disk; its *boundary* consists of all the simplices whose link is indeed a disk.
A simplicial map is called *simplexwise injective* if its restriction to any simplex is injective. The following is Lemma 3.8 of [BFS$^+$] cf. also the proof of Proposition 5.2 in [Put].

Lemma 5.12. *Let Y be an k-dimensional combinatorial manifold. Let X be a simplicial complex and assume that the link of every d-simplex in X is $(k-2d-2)$-connected for $d \geq 0$. Let $\psi\colon Y \to X$ be a simplicial map whose restriction to ∂Y is simplexwise injective. Upon changing the simplicial structure of Y, ψ is homotopic relative ∂Y to a simplexwise injective map.*

In practice Y will be a sphere, so the lemma allows us to restrict attention to simplexwise injective combinatorial maps when collapsing spheres.

Proposition 5.13. *Let $X_0 \subseteq X_1 \subseteq X$ be simplicial complexes. Assume that (X, X_0) is k-connected, that X_0 is $(k-1)$-connected and that the link of every d-simplex is $(k-2d-2)$-connected for $d \geq 0$. Further assume the following "exchange condition":*

(EXC1) There is a vertex $w \in X$ such that for every vertex $v \in X_0$ that is not in $\operatorname{st} w$ there is a vertex $v' \in \operatorname{st}_{X_1} w$ such that $\operatorname{lk}_{X_1} v \subseteq \operatorname{lk}_{X_1} v'$ and $\operatorname{lk}_{X_1} v$ is $(k-1)$-connected.

Then X is k-connected.

By taking $X_1 = X_0$ or $X_1 = X$ respectively we obtain the following special cases of (EXC1):

(EXC0) There is a vertex $w \in X$ such that for every vertex $v \in X_0$ that is not in $\operatorname{st} w$ there is a vertex $v' \in \operatorname{st}_{X_0} w$ such that $\operatorname{lk}_{X_0} v \subseteq \operatorname{lk}_{X_0} v'$ and $\operatorname{lk}_{X_0} v$ is $(k-1)$-connected.

(EXC) There is a vertex $w \in X$ such that for every vertex $v \in X_0$ that is not in $\operatorname{st} w$ there is a vertex $v' \in \operatorname{st} w$ such that $\operatorname{lk}_X v \subseteq \operatorname{lk}_X v'$ and $\operatorname{lk}_X v$ is $(k-1)$-connected.

Proof. Let $\iota\colon X_0 \to X$ denote the inclusion. In view of Lemma 5.11 what remains to be shown is that if $\varphi\colon S^k \to X_0$ is a map from a k-sphere then $\bar{\varphi} := \iota \circ \varphi$ is homotopically trivial.
By simplicial approximation [Spa66, Theorem 3.4.8] we may assume φ (and thus $\bar{\varphi}$) to be a simplicial map $Y \to X_0$ and by our assumptions and Lemma 5.12 we may assume it to be simplexwise injective. Our goal is to homotope $\bar{\varphi}$ to a map to $\operatorname{st} w$. Once we have achieved that, we are done since $\operatorname{st} w$ is contractible.
The simplicial sphere Y contains finitely many vertices x whose image $v = \bar{\varphi}(x)$ does not lie in $\operatorname{st} w$. We define $\bar{\varphi}'\colon Y \to X$ to be the map that coincides with $\bar{\varphi}$ outside the open star of x and takes x to the vertex v' from the statement. We claim that $\bar{\varphi}$ is homotopic to $\bar{\varphi}'$. Inductively replacing vertices then finishes the proof.
Finally, to see that $\bar{\varphi}|_{\operatorname{st} x}$ and $\bar{\varphi}'|_{\operatorname{st} x}$ are homotopic relative to $\operatorname{lk} x$ note that $\bar{\varphi}(\operatorname{lk} x) \subseteq \operatorname{lk} v$ by simplexwise injectivity. Further the complex spanned by v, v' and $\operatorname{lk} v$ is the suspension $\Sigma(\operatorname{lk} v)$ of $\operatorname{lk} v$ (unless v and v' are adjacent in which case there is nothing to show). So both $\bar{\varphi}|_{\operatorname{st} x}$ and $\bar{\varphi}'|_{\operatorname{st} x}$ are maps $(D^k, S^{k-1}) \cong (\operatorname{st} x, \operatorname{lk} x) \to (\Sigma(\operatorname{lk} v), \operatorname{lk} v)$. But $\operatorname{lk} v$ is $(k-1)$-connected by assumption so $(\Sigma(\operatorname{lk} v), \operatorname{lk} v)$ is k-connected and both maps are homotopic. ☐

6. A Thompson group for direct products of a group

The examples in this section have been constructed independently by S. Tanusevski using entirely different techniques, and in discussions with him we have determined that his groups are identical to those discussed here.
Fix a group G. Let G_n be the direct product G^n. We declare that ρ_n is trivial for all n, and define cloning maps via $(g_1, \ldots, g_k, \ldots, g_n)\kappa_k^n := (g_1, \ldots, g_k, g_k, \ldots, g_n)$. This makes rather literal the word "cloning." To verify that this defines a cloning system, observe that since

the ρ_n are trivial, we need only check that the cloning maps are homomorphisms (which they are) and that $\kappa_\ell^n \circ \kappa_k^{n+1} = \kappa_k^n \circ \kappa_{\ell+1}^{n+1}$ for $1 \le k < \ell \le n$ (which is visibly true). These respectively handle conditions FCS1 and FCS2 of Definition 2.15, and condition FCS3 is trivial.

If G is finite, $\mathscr{T}(G^*)$ is a group with finite similarity structure in the sense of [FH] and hence of type F_∞. More generally, it turns out that this cloning system is an example answering Question 5.1, that is, the finiteness length of $\mathscr{T}(G^*)$ is exactly that of G. The proof is due to Tanusevski, and we sketch a version of it here, using our setup and language. For the positive finiteness properties, we just need that the complexes $\mathscr{L}(G^n)$ becomes increasingly highly connected. This follows by noting that every simplex fiber of the projection $\mathscr{L}(G^n) \to \mathcal{M}(L_n)$ is the join of its vertex fibers, and applying [Qui78, Theorem 9.1]. For the negative finiteness properties, we claim that there is a sequence of homomorphisms $G \to \mathscr{T}(G^*) \to G$ that composes to the identity. This is sufficient by the Bieri–Eckmann criterion [BE74, Proposition 1.2]; see [Bux04, Proposition 4.1]. The first map in the claim is $g \mapsto (1, g, 1)$, and the second map sends $(T_-, (g_1, \dots, g_n), T_+)$ to g_1. One must check that this second map is well defined on equivalence classes under reduction and expansion, and is a homomorphism, but this is not hard to see.

7. Thompson groups for matrix groups

Let R be a unital ring and consider the algebra of n-by-n matrices $M_n(R)$. We will define a family of injective functions $M_n(R) \to M_{n+1}(R)$, which will become cloning maps after we restrict to the subgroups of upper triangular matrices $B_n(R)$. Consider the map κ_k defined by

$$\left(\begin{pmatrix} A_{<,<} & A_{<,k} & A_{<,>} \\ A_{k,<} & A_{k,k} & A_{k,>} \\ A_{>,<} & A_{>,k} & A_{>,>} \end{pmatrix}\right)\kappa_k = \begin{pmatrix} A_{<,<} & A_{<,k} & A_{<,k} & A_{<,>} \\ A_{k,<} & A_{k,k} & 0 & 0 \\ 0 & 0 & A_{k,k} & A_{k,>} \\ A_{>,<} & A_{>,k} & A_{>,k} & A_{>,>} \end{pmatrix}$$

where the matrix has a block structure that makes the middle column and row be the kth column and row of the full matrix respectively. Given the block structure it is not hard to see that κ_k is a morphism of monoids. But it generally fails to map invertible elements to invertible elements. We therefore restrict to the groups $B_n(R)$ of invertible upper triangular matrices. Let $B_\infty(R) = \varinjlim B_n(R)$.

Lemma 7.1. *The trivial morphism ρ_n and the maps κ_k^n defined above describe a properly graded cloning system on $B_*(R)$.*

It may be noted that the action of $\mathscr{F}$ on $B_\infty(R)$ factors through $\mathscr{H}$, that is $\kappa_\ell \kappa_k = \kappa_k \kappa_{\ell+1}$ even for $\ell = k$.

Proof. Since ρ_* is trivial, condition FCS1 asks that the cloning maps be group homomorphisms. That κ_k is multiplicative and takes 1 to 1 is straightforward to check. Also, A is invertible only if all the $A_{i,i}$ are units, in which case $(A)\kappa_k$ is also invertible.

To check condition (FCS2) it is helpful to note that $((A)\kappa_k)_{i,j} = A_{\pi_k(i),\pi_k(j)}$ unless $i = k$ or $i > j$. One can now distinguish cases similar to Example 2.9. The compatibility condition (FCS3) is vacuous for trivial ρ_*.

To see that the cloning system is properly graded note that $g \in \operatorname{im} \iota_{n,n+1}$ if and only if the last column of g is the vector e_{n+1}. If at the same time $g = (h)\kappa_k$ then by the definition of κ_k the last column of h has to be e_n. Hence $h \in \operatorname{im} \iota_{n-1,n}$. □

Having equipped $B_*(R)$ with a cloning system, we get a Thompson group $\mathscr{T}(B_*(R))$. Elements are represented by triples (T_-, A, T_+) for trees $T_\pm$ with n leaves and matrices $A \in B_n(R)$, up to reduction and expansion. Figure 3 gives an example of an element of $\mathscr{T}(B_*(R))$, represented as a triple and an expansion of that triple.

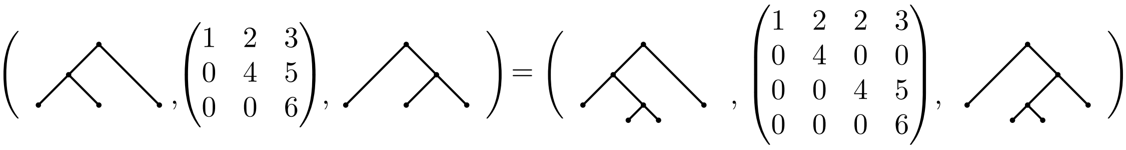

FIGURE 3. An example of expansion in $\mathscr{T}(B_*(\mathbb{Q}))$.

For appropriate R, the finiteness properties of $\mathscr{T}(B_*(R))$ become very intriguing. For instance, when $R = \mathbb{F}_p[t, t^{-1}]$, every $B_n(\mathbb{F}_p[t, t^{-1}])$ is finitely generated but not finitely presented, for $n \geq 2$. More generally the following holds, see [Bux04, Theorem A, Remarks 3.6, 3.7]:

Theorem 7.2. *Let k be a global function field, let S be a finite nonempty set of places and $\mathcal{O}_S$ the ring of S-integers. Then $B_n(\mathcal{O}_S)$ is of type $\mathrm{F}_{|S|-1}$ but not of type $\mathrm{F}_{|S|}$ for any $n \geq 2$.*

Since the finiteness properties of these groups do not depend on the size of the matrices, Question 5.1 suggests that the group $\mathscr{T}(B_*(\mathcal{O}_S))$ has the same finiteness properties as all the $B_n(\mathcal{O}_S)$.
A different class of examples occurs as subgroups of groups of the form $B_n(R)$. Let $\mathrm{Ab}_n \leq B_{n+1}$ be the group of invertible upper triangular $n+1$-by-$n+1$ matrices whose upper left and lower right entry are 1. The groups $\mathrm{Ab}_n(\mathbb{Z}[\frac{1}{p}])$ were studied by Abels and others and we call them *Abels groups*. Their finiteness length tends to infinity with n [AB87, Bro87]:

Theorem 7.3. *For any prime p the group $\mathrm{Ab}_n(\mathbb{Z}[\frac{1}{p}])$ is of type F_{n-1} but not of type F_n for $n \geq 1$.*

For any ring R, the cloning system described above for $B_n(R)$ preserves the groups $\mathrm{Ab}_{n-1}(R)$. By restriction we obtain a Thompson group $\mathscr{T}(\mathrm{Ab}_{*-1}(R))$ which we will just denote by $\mathscr{T}(\mathrm{Ab}_*(R))$.

7.1. **Finiteness properties.** The first main result in this subsection is that the groups $B_*(R)$ satisfy half of what is needed to qualify for the answer of Question 5.1 .

Theorem 7.4. $\phi(\mathscr{T}(B_*(R))) \geq \liminf_n(\phi(B_n(R)))$.

In particular, together with Theorem 7.2 this implies:

Corollary 7.5. *$\mathscr{T}(B_*(\mathcal{O}_S))$ is of type $\mathrm{F}_{|S|-1}$.*

In view of Proposition 5.9 it suffices to show that the connectivity of $\mathscr{L}(B_n(R))$ goes to infinity with n. Define $\eta(m) := \lfloor \frac{m-1}{4} \rfloor$. Theorem 7.4 follows from:

Proposition 7.6. *$\mathscr{L}(B_n(R))$ is $(\eta(n-1)-1)$-connected.*

In order to prove the proposition we will induct. To do so we need to enlarge the class of complexes under consideration. For a spanning subgraph Δ of the linear graph L_n, define $\mathscr{L}(B_n(R);\Delta)$ to be the subcomplex of $\mathscr{L}(B_n(R))$ whose elements only use graphs that are subgraphs of Δ. Define $e(\Delta)$ to be the number of edges of Δ; we will induct on $e(\Delta)$ and prove the following strengthening of Proposition 7.6:

Proposition 7.7. *$\mathscr{L}(B_n(R);\Delta)$ is $(\eta(e(\Delta))-1)$-connected.*

The base case is that $\mathscr{L}(B_n(R);\Delta)$ is non-empty provided $e(\Delta) \geq 1$, which is clearly true. To work with simplices of $\mathscr{L}(B_n(R))$ it will be helpful to have simple representatives for dangling classes. To define them we have to recall some of the origins of $\mathscr{L}(B_n(R))$:

by Observation 5.6 matchings Γ of L_n correspond to elementary forests. Using this correspondence, it makes sense to denote the corresponding cloning map by κ_Γ. In fact, since our cloning maps factor through the hedge monoid, we even get a cloning map κ_Γ for any spanning subgraph Γ of L_n using Observation 5.7. For the sake of readability, we describe this map explicitly. Let $D_k(\lambda)$ be the k-by-k matrix with all diagonal entries λ and all other entries 0. Let $F_{k,\ell}(\lambda)$ be the k-by-ℓ matrix whose bottom row has all entries λ and all other entries are 0 and let $C_{k,\ell}(\lambda)$ be defined analogously for the top row. Assume that Γ has m connected components which we think of as numbered from left to right. Then

$$\kappa_\Gamma \colon B_m(R) \to B_n(R)$$

can be described as follows. The image of $\kappa(A)$ has a block structure where columns and rows are grouped together if their indices lie in a common component of Γ. So the (i,j)-block has k rows and ℓ columns if the ith (respectively jth) component of Γ has k (respectively ℓ) vertices. The block is $D_k(A_{i,i})$, $F_{k,\ell}(A_{i,j})$ or $C_{k,\ell}(A_{i,j})$ depending on whether $i = j$, $i < j$, or $i > j$ (see Figure 4).

$$\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{pmatrix} \overset{\kappa_\Gamma}{\mapsto} \begin{pmatrix} D_2(a_{1,1}) & F_{2,4}(a_{1,2}) & F_{2,3}(a_{1,3}) \\ C_{4,2}(a_{2,1}) & D_4(a_{2,2}) & F_{4,3}(a_{2,3}) \\ C_{3,2}(a_{3,1}) & C_{3,4}(a_{3,2}) & D_3(a_{3,3}) \end{pmatrix}$$

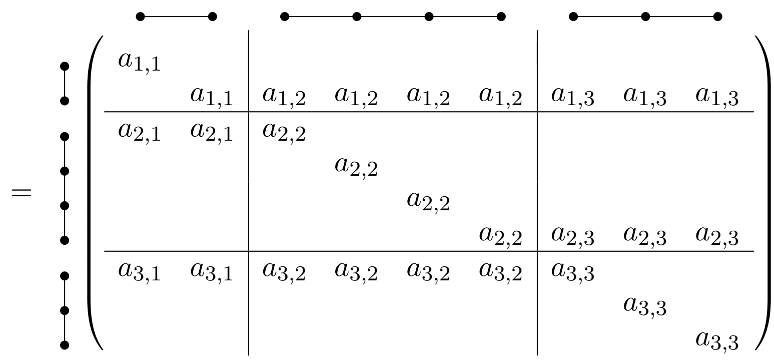

FIGURE 4. Visualization of the cloning map of a graph. The graph Γ is drawn on top and to the left of the last matrix.

Recall that we denote by e_k the kth edge of L_n. We denote by E_k the spanning subgraph of Γ whose only edge is e_k. For a subgraph Γ of L_n we say that an index i is *fragile* if $e_i \in \Gamma$ and we say that i is *stable* otherwise. In other words, i is stable if it is the rightmost vertex of its component in Γ. A matrix $A \in M_n(R)$ is said to be *modeled on* Γ if $A_{i,j} = 0$ whenever both i and j are stable in Γ (see Figure 5).

Lemma 7.8. *Let Γ be a spanning subgraph of L_n with m components and let $A \in B_n(R)$. There is a representative $B \in A(B_m(R))\kappa_\Gamma$ such that $B - I_n$ is modeled on Γ. Moreover, rows of zeroes in A (off the diagonal) can be preserved in B.*

Proof. We inductively multiply A on the right by matrices in $(B_m(R))\kappa_\Gamma$ to eventually obtain B. Let $E_{i,j}(\lambda)$ denote the matrix that coincides with the identity matrix in all entries but (i,j) and is λ there.
We begin by clearing the diagonal. Let i be the (stable) rightmost vertex of the kth component of Γ and let $\lambda = A_{i,i}^{-1}$. Then $A(E_{k,k}(\lambda))\kappa_\Gamma$ has (i,i)-entry one and no other entry with stable indices was affected.
Now we clear the region above the diagonal. We proceed inductively by rows and columns. Let (i,j) be the (lexicographically) minimal pair of stable indices of Γ such

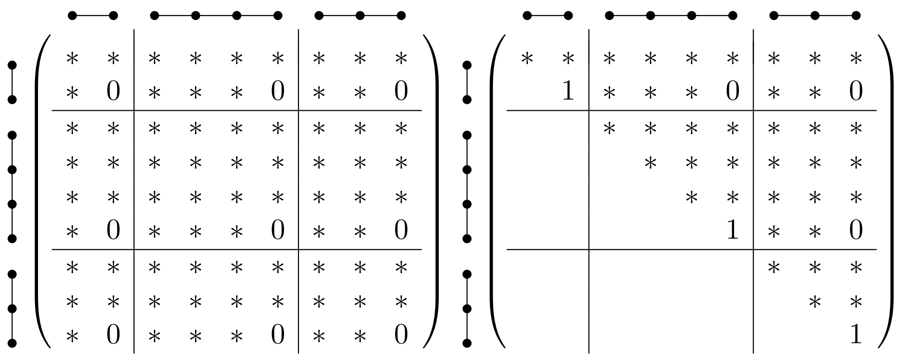

FIGURE 5. A matrix that is modeled on a graph (left) and an upper triangular matrix that is reduced relative to a graph (right).

that $0 \neq A_{i,j} =: -\lambda$. Let i and j lie in the kth respectively ℓth component of Γ. Then $A(E_{k,\ell}(\lambda))\kappa_m$ has (i,j)-entry zero and no other entry with stable indices was affected. For the last statement assume that the ith row of A was zero off the diagonal. Then none of the matrices by which we multiplied had a nonzero off-diagonal entry in the ith row. If i is fragile no such matrix even lies in $(B_k(R))\kappa_\Gamma$. If i is stable then the only matrices we might have used of this form were meant to clear the ith row, but since the entries there were zero, nothing happened in these steps. □

Corollary 7.9 (Reduced form)**.** *Every simplex in* $\mathscr{L}(B_n(R))$ *has a representative* (A,Γ) *such that the matrix* $A - I_n$ *is modeled on* Γ*.*

We will refer to a matrix $A \in B_n(R)$ as being *reduced relative* Γ if it satisfies the conclusion of Corollary 7.9.

Let $\Delta_0 := \Delta \setminus \{e_1 \cup e_2\}$, and consider $\mathscr{L}(B_n(R);\Delta_0)$ as a subcomplex of $\mathscr{L}(B_n(R);\Delta)$. For a vertex $[A,E_k] \in \mathscr{L}(B_n(R);\Delta_0)$ we write $\mathrm{lk}_0([A,E_k])$ for the link in $\mathscr{L}(B_n(R);\Delta_0)$ to differentiate from the link in $\mathscr{L}(B_n(R);\Delta)$ which is just denoted $\mathrm{lk}([A,E_k])$. To prove Proposition 7.7 we follow the strategy outlined by Proposition 5.13: we want to show that $\mathscr{L}(B_n(R);\Delta_0)$ is $(\eta(e(\Delta))-2)$-connected, that $(\mathscr{L}(B_n(R);\Delta),\mathscr{L}(B_n(R);\Delta_0))$ is $(\eta(e(\Delta))-1)$-connected and that there is a vertex satisfying condition (EXC1). That vertex is $w := [I_n, E_1]$ in our case. The following statements (up to the proof of Proposition 7.7) are part of an induction, so we assume that Proposition 7.7 has been proven for graphs Δ' with $e(\Delta') < e(\Delta)$ and intend to prove it for Δ.

Lemma 7.10 (Links are lower rank complexes)**.** *Let* σ *be a simplex of dimension* $d \geq 0$ *in* $\mathscr{L}(B_n(R);\Delta)$*. Then* $\mathrm{lk}(\sigma)$ *is isomorphic to a complex of the form* $\mathscr{L}(B_{n-(d+1)}(R);\Delta')$ *where* Δ' *is a spanning subgraph of* $L_{n-(d+1)}$ *with at least* $e(\Delta)-3d-3$ *edges. In particular, it is* $(\eta(e(\Delta)-3d-3)-1)$*-connected by induction.*

Proof. The simplex σ is of the form $[g,\Gamma]$ with $g \in B_n(R)$ and $\Gamma \subseteq \Delta$. If it has dimension d then Γ has $d+1$ edges, say $e_{i_1},\ldots,e_{i_{d+1}}$. Using the left action of $B_n(R)$ (which comes from the left action of $\mathscr{T}(B_*(R))$) we may assume that $g = 1$. Then $\mathrm{lk}(\sigma)$ is $\mathscr{L}((B_{n-(d+1)})\kappa_\Gamma;\Delta^\sharp)$, where $\Delta^\sharp$ is Δ with the edges $e_{i_j-1}, e_{i_j}, e_{i_j+1}$ removed for each $1 \leq j \leq d+1$. In particular $\Delta^\sharp$ has at least $e(\Delta)-3d-3$ edges. Now consider the map $b_\Gamma\colon L_n \to L_{n-(d+1)}$ given by blowing down the edges of Γ. The image of $\Delta^\sharp$ under b_Γ is what we will call Δ'. Note that Δ' still has at least $e(\Delta)-3d-3$ edges. Since κ_Γ is injective, we may now apply κ_Γ^{-1} paired with b_Γ to $\mathscr{L}((B_{n-(d+1)})\kappa_\Gamma;\Delta^\sharp)$ and get an isomorphism to $\mathscr{L}(B_{n-(d+1)}(R);\Delta')$. □

Lemma 7.11. *The pair* $(\mathscr{L}(B_n(R);\Delta),\mathscr{L}(B_n(R);\Delta_0))$ *is* $(\eta(e(\Delta))-1)$*-connected.*

Proof. Note that for any vertex of $\mathscr{L}(B_n(R);\Delta) \setminus \mathscr{L}(B_n(R);\Delta_0)$, the entire link of the vertex lies in $\mathscr{L}(B_n(R);\Delta_0)$. Hence the function sending vertices of the former to 1 and vertices of the latter to 0 yields a Morse function in the sense of Section 5, and to prove the statement we need only show that links of vertices in $\mathscr{L}(B_n(R);\Delta) \setminus \mathscr{L}(B_n(R);\Delta_0)$ are $(\eta(e(\Delta)) - 2)$-connected. By Lemma 7.10, each descending link is isomorphic to a complex of the form $\mathscr{L}(B_{n-1}(R);\Delta')$ for Δ' a graph with at least $e(\Delta) - 3$ edges. By induction, these are $(\eta(e(\Delta)) - 2)$-connected as desired. □

In addition to the subcomplex $\mathscr{L}(B_n(R);\Delta_0)$ we will soon need to consider $\mathscr{L}(B_n(R);\Delta_1)$ where $\Delta_1 := \Delta \setminus \{e_1\}$. We will write links in this complex using the symbol lk_1.

Lemma 7.12 (Shared links)**.** *Let $k > 2$ and let A be reduced relative E_k. Let A' be obtained from A by setting the $(1,k)$-entry to 0. Then $\mathrm{lk}_1([A, E_k]) \subseteq \mathrm{lk}_1([A', E_k])$ and $[A', E_k] \in \mathrm{lk}\, w$.*

Proof. As a first observation, note that since A is reduced relative E_k and $k > 2$, the $(1,1)$-entry and $(2,2)$-entry of A are both 1, and the entries of the top row of A past the first entry is all 0's except possibly in the kth column. Let $-\lambda$ be the $(1,k)$-entry of A, and note that $A' = AE_{1k}(\lambda)$. The first row of A' is now $(1,0,\dots,0)$ and the $(2,2)$-entry is 1, which tells us that $A' \in (B_{n-1})\kappa_1$. Hence $[A', E_k] \in \mathrm{lk}_0\, w$.
To see that $\mathrm{lk}_1([A, E_k]) \subseteq \mathrm{lk}_1([A', E_k])$ we first multiply by A^{-1} from the left and are reduced to showing that $\mathrm{lk}_1([I_n, E_k]) \subseteq \mathrm{lk}_1([E_{1k}(\lambda), E_k])$. An arbitrary simplex of $\mathrm{lk}_1([I_n, E_k])$ is of the form $[B, \Gamma]$, with $B \in \mathrm{im}(\kappa_k)$ and Γ not containing any of e_1, e_{k-1}, e_k, or e_{k+1}. Note that the kth row of B is zero off the diagonal. By Lemma 7.8 there is a $B' \in B\, \mathrm{im}(\kappa_\Gamma)$ that is reduced relative Γ and has kth row zero off the diagonal. We have $[B', \Gamma] = [B, \Gamma]$. Since $e_1 \notin \Gamma$ and B' is reduced relative Γ, the first column of B' is e_1.
We now claim that B' commutes with $E_{1k}(\lambda)$. Indeed, left multiplication by $E_{1k}(\lambda)$ is the row operation $r_1 \mapsto r_1 + \lambda r_k$, and right multiplication by $E_{1k}(\lambda)$ is the column operation $c_k \mapsto c_k + \lambda c_1$. For our B', both of these operations change the $(1,k)$-entry by adding λ to it, and change no other entries. This proves the claim.
Now we have

$$[B, \Gamma] = [B', \Gamma] = [E_{1k}(\lambda) B' E_{1k}(-\lambda), \Gamma] = [E_{1k}(\lambda) B', \Gamma] = [E_{1k}(\lambda) B, \Gamma].$$

The second to last step works since $E_{1k}(-\lambda) \in \mathrm{im}(\kappa_\Gamma)$ by virtue of $e_k \notin \Gamma$. This shows that our arbitrary simplex of $\mathrm{lk}_1([I_n, E_k])$ is also in $\mathrm{lk}_1([E_{1k}(\lambda), E_k])$. □

Proof of Proposition 7.7. We want to apply Proposition 5.13. The complexes are $X = \mathscr{L}(B_n(R);\Delta)$, $X_1 = \mathscr{L}(B_n(R);\Delta_1)$ and $X_0 = \mathscr{L}(B_n(R);\Delta_0)$ and $k = \eta(e(\Delta)) - 1$. We check the assumptions. The pair $(\mathscr{L}(B_n(R);\Delta), \mathscr{L}(B_n(R);\Delta_0))$ is k-connected by Lemma 7.11.
The complex $\mathscr{L}(B_n(R);\Delta_0)$ is $(\eta(e(\Delta_0)) - 1)$-connected by induction. This is sufficient because $\eta(e(\Delta_0)) - 1 \geq \eta(e(\Delta) - 2) - 1 \geq \eta(e(\Delta)) - 2 = k - 1$.
The link of a d-simplex is $(\eta(e(\Delta) - 3d - 3) - 1)$-connected by Lemma 7.10. This is sufficient because $\eta(e(\Delta) - 3d - 3) - 1 \geq \eta(e(\Delta)) - d - 2 = k - d - 1$.
Finally condition (EXC1) is satisfied by Lemma 7.12 where $\mathrm{lk}_1([A, E_k])$ is at least $(\eta(e(\Delta) - 4) - 1)$-connected and $\eta(e(\Delta) - 4) - 1 = \eta(e(\Delta)) - 2 = k - 1$ as desired. □

Shifting focus to the Abels groups, the above arguments also show high connectivity of $\mathscr{L}(\mathrm{Ab}_*(\mathbb{Z}[\frac{1}{p}]))$, and using Proposition 5.9 and Theorem 7.3 we conclude:

Theorem 7.13. *$\mathscr{T}(\mathrm{Ab}_*(\mathbb{Z}[\frac{1}{p}]))$ is of type F_∞.*

This, despite none of the $\mathrm{Ab}_n(\mathbb{Z}[\frac{1}{p}])$ individually being F_∞. The remaining question is whether $\phi(\mathscr{T}(B_*(R))) = \liminf_n(\phi(B_n(R)))$, that is whether negative finiteness properties of the $B_n(R)$ can impose negative finiteness properties on $\mathscr{T}(B_*(R))$. A clue in this direction is the following proposition, which we prove by simple algebraic means.

Proposition 7.14. *Let k be a field and $R = k[t]$ its polynomial ring. Then $\mathscr{T}(B_*(k[t]))$ is not finitely generated.*

Proof. Define a function $d\colon B_*(R) \to \mathbb{N}_0 \cup \{-\infty\}$ by sending A to

$$d(A) := \max\{\text{degree}(a_{i,i+1}) \mid i \geq 1\},$$

where degree$(a_{i,i+1})$ means the degree of the polynomial $a_{i,i+1} \in k[t]$ appearing as the $(i, i+1)$-entry of A. (Note that degree$(0) = -\infty$.) An element g of $\mathscr{T}(B_*(k[t]))$ is represented by a triple (T_-, A, T_+), and every triple representing g can be obtained from this triple via a finite sequence of reductions and expansions. The function d is invariant under any cloning map, so setting $d(T_-, A, T_+) := d(A)$ gives a well defined function $\mathscr{T}(B_*(k[t])) \to \mathbb{N}_0 \cup \{-\infty\}$.
Now for $N \in \mathbb{N}_0 \cup \{-\infty\}$, define

$$\mathscr{T}(B_*(k[t]))^{d \leq N} := \{g \in \mathscr{T}(B_*(k[t])) \mid d(g) \leq N\}.$$

It is an exercise to check that $d(gh) \leq \max(d(g), d(h))$ and $d(g^{-1}) = d(g)$, which tells us that $\mathscr{T}(B_*(k[t]))^{d \leq N}$ is in fact a (proper) subgroup. But any finite set of elements of $\mathscr{T}(B_*(k[t]))$ is contained in some $\mathscr{T}(B_*(k[t]))^{d \leq N}$, so $\mathscr{T}(B_*(k[t]))$ cannot be generated by any finite set. □

8. Thompson groups for mock-symmetric groups

The groups discussed in this section are instances of what Davis, Januszkiewicz and Scott call "mock reflection groups" [DJS03]. These are groups generated by involutions, and act on associated cell complexes very much like Coxeter groups, with the only difference being that some of the generators may be "mock reflections" that do not fix their reflection mirror pointwise. Here we will only be concerned with one family of groups consisting of the minimal blow up of Coxeter groups of type A_n. These Coxeter groups are symmetric groups and so we call their blow ups *mock symmetric groups*. For $n \in \mathbb{N}$ the mock symmetric group S_n^{mock} is given by the presentation

$$\begin{aligned} S_n^{\text{mock}} = \langle s_{i,j}, 1 \leq i < j \leq n \mid s_{i,j}^2 &= 1 \text{ for all } i,j \\ s_{i,j}s_{k,\ell} &= s_{k,\ell}s_{i,j} \text{ for } i < j < k < \ell \\ s_{k,\ell}s_{i,j} &= s_{k+\ell-j,k+\ell-i}s_{k,\ell} \text{ for } k \leq i < j \leq \ell\rangle. \end{aligned} \tag{8.1}$$

We also set $S_\infty^{\text{mock}} = \varinjlim S_n^{\text{mock}}$.

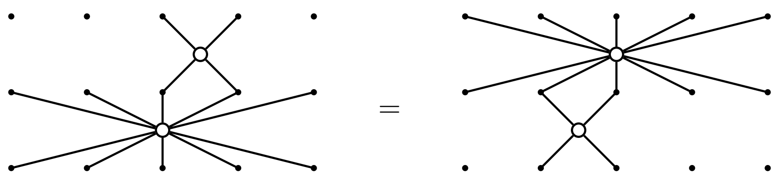

Figure 6. The relation $s_{i,j}s_{k,\ell} = s_{k,\ell}s_{k+\ell-j,k+\ell-i}$ of S_n^{mock} in the case $i = 3$, $j = 4$, $k = 1$, $\ell = 5$, $n = 5$.

Let $\bar{s}_{i,j} \in S_n$ be the involution $(i\ j)((i+1)\ (j-1)) \cdots (\lfloor \frac{i+j}{2} \rfloor\ \lceil \frac{i+j}{2} \rceil)$ (this is the longest element in the Coxeter group generated by $(i\ i+1), \ldots, (j-1\ j)$). Taking $s_{i,j}$ to

$\bar{s}_{i,j}$ defines a surjective homomorphism from $\rho\colon S_n^{\mathrm{mock}} \to S_n$. We define cloning maps $\kappa_k^n\colon S_n^{\mathrm{mock}} \to S_{n+1}^{\mathrm{mock}}$ by first defining them on the generators:

$$(s_{i,j})\kappa_k^n = \begin{cases} s_{i,j} & \text{for } j < k \\ s_{i,j+1}s_{k,k+1} & \text{for } i \le k \le j \\ s_{i+1,j+1} & \text{for } k < i. \end{cases} \tag{8.2}$$

Now we extend κ_k^n to a map $S_n^{\mathrm{mock}} \to S_{n+1}^{\mathrm{mock}}$ as in the paragraph leading up to Lemma 1.12.

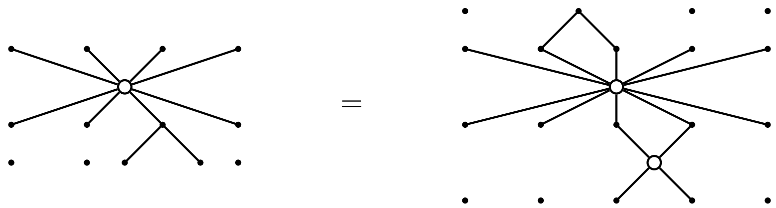

FIGURE 7. The relation $s_{1,4}\lambda_3 = \lambda_2 s_{1,5}s_{3,4}$ of $\mathscr{F} \bowtie S_\infty^{\mathrm{mock}}$.

Proposition 8.1. *The above data define a properly graded cloning system on S_*^{mock}.*

Proof. Note first that (8.1) is a presentation for S_n^{mock} as a monoid because all the generators are involutions by the first relation. Since we have such a nice presentation, we will apply Lemma 1.12 with this presentation rather than the trivial presentation used in Proposition 2.7.

We have to verify conditions coming from relations of $\mathscr{F}$ and conditions coming from relations of S_n^{mock}, after which the proof proceeds as that of Proposition 2.7. For the relations of $\mathscr{F}$ we must verify the conditions "product of clonings" and "compatibility."

$$(s_{i,j})\kappa_\ell\kappa_k = (s_{i,j})\kappa_k\kappa_{\ell+1} \qquad \text{for } k < \ell \text{ and } i < j \tag{8.3}$$

$$\rho((s_{i,j})\kappa_k) = (\rho(s_{i,j}))\varsigma_k \qquad \text{for } i < j. \tag{8.4}$$

For the relations of S_n^{mock} we have to check that ρ is a well defined homomorphism, and check that the following equations (standing in for "cloning of products") are satisfied:

$$(s_{i,j})\kappa_{\rho(s_{k,\ell})p}(s_{k,\ell})\kappa_p = (s_{k,\ell})\kappa_{\rho(s_{i,j})p}(s_{i,j})\kappa_p \qquad \text{for } i < j < k < \ell \tag{8.5}$$

$$(s_{k+\ell-j,k+\ell-i})\kappa_{\rho(s_{k,\ell})p}(s_{k,\ell})\kappa_p = (s_{k,\ell})\kappa_{\rho(s_{i,j})p}(s_{i,j})\kappa_p \qquad \text{for } k \le i < j \le \ell. \tag{8.6}$$

Note that the conditions coming from the relations $s_{i,j}^2 = 1$ are vacuous.

Condition (8.3) is easy to check if $k < i$ or $\ell > j$ so we consider the situation where $i \le k < \ell \le j$. In this case we have

$$\begin{aligned}(s_{i,j})\kappa_\ell\kappa_k &= (s_{i,j+1}s_{\ell,\ell+1})\kappa_k = (s_{i,j+1})\kappa_k(s_{\ell,\ell+1})\kappa_k = s_{i,j+2}s_{k,k+1}s_{\ell+1,\ell+2} \\ &= s_{i,j+2}s_{\ell+1,\ell+2}s_{k,k+1} = (s_{i,j+1})\kappa_{\ell+1}(s_{k,k+1})\kappa_{\ell+1} = (s_{i,j+1}s_{k,k+1})\kappa_{\ell+1} = (s_{i,j})\kappa_k\kappa_{\ell+1}\end{aligned}$$

since $\rho(s_{k,k+1})(\ell+1) = (\ell+1)$, $\rho(s_{\ell,\ell+1})k = k$ and $s_{k,k+1}$ and $s_{\ell+1,\ell+2}$ commute.

Condition (8.4) amounts to showing that

$$(\bar{s}_{i,j})\varsigma_k = \begin{cases} \bar{s}_{i+1,j+1} & k < i \\ \bar{s}_{i,j+1}\bar{s}_{k,k+1} & i \le k \le j \\ \bar{s}_{i,j} & k > j. \end{cases}$$

the cases $k < i$ and $k > j$ are clear. For the remaining case we first note that

$$\bar{s}_{i,j+1}\bar{s}_{k,k+1}(m) = \tau_{i+j-k}\bar{s}_{i,j}\pi_k(m) = ((\bar{s}_{i,j})\varsigma_k)(m)$$

for $m \ne k, k+1$ (which is also the same as $\bar{s}_{i,j+1}(m)$). Finally one checks that

$$\bar{s}_{i,j+1}\bar{s}_{k,k+1}(k) = i + j - k = (\bar{s}_{i,j})\varsigma_k(k)$$

and that

$$\bar{s}_{i,j+1}\bar{s}_{k,k+1}(k+1) = i+j-k+1 = (\bar{s}_{i,j})\varsigma_k(k+1).$$

That ρ is a well defined homomorphism amounts to saying that the defining relations of S_n^{mock} hold in S_n with $s_{i,j}$ replaced by $\bar{s}_{i,j}$, which they do.
Condition (8.5) is also easy to check unless $i \le p \le j$ or $k \le p \le \ell$. We treat the case $i \le p \le j$, the other remaining case being similar. We have

$$(s_{i,j})\kappa_{\rho(s_{k,\ell})p}(s_{k,\ell})\kappa_p = s_{i,j+1}s_{p,p+1}s_{k+1,\ell+1} = s_{k+1,\ell+1}s_{i,j+1}s_{p,p+1} = (s_{k,\ell})\kappa_{\rho(s_{i,j})p}(s_{i,j})\kappa_p.$$

Finally, the interesting case of condition (8.6) is the case where $i \le p \le j$. We have

$$\begin{aligned}(s_{k,\ell})\kappa_{\rho(s_{i,j})p}(s_{i,j})\kappa_p &= s_{k,\ell+1}s_{i+j-p,i+j-p+1}s_{i,j+1}s_{p,p+1} = s_{k,\ell+1}s_{i,j+1}\\ &= s_{k+\ell-j,k+\ell-i+1}s_{k+\ell-p,k+\ell-p+1}s_{k,\ell+1}s_{p,p+1} = (s_{k+\ell-j,k+\ell-i})\kappa_{\rho(s_{k,\ell})p}(s_{k,\ell})\kappa_p\end{aligned}$$

using the defining relations of S_n^{mock} several times.
To see that the cloning system is properly graded note that it suffices (by induction) to check condition (2.6) on generators in the following sense: if $s_{i,j} \in S_n^{\text{mock}}$ satisfies $(s_{i,j})\kappa_k \in \operatorname{im}\iota_{n,n+1}$ then $s_{i,j} \in \operatorname{im}\iota_{n-1,n}$. Looking at (8.2) we see that $(s_{i,j})\kappa_k \in \operatorname{im}\iota_{n,n+1}$ only if $j < n$ if and only if $s_{i,j} \in \operatorname{im}\iota_{n-1,n}$. □

Definition 8.2. We denote the group $\mathscr{T}(S_*^{\text{mock}})$ by V_{mock}.

Observation 8.3. *The natural morphism $V_{\text{mock}} \to V$ is surjective.*

Conjecture 8.4. *V_{mock} is of type F_∞.*

Since each S_n^{mock} is of type F_∞ [DJS03, Section 4.7, Corollary 3.5.4], to prove the conjecture it would suffice to show that the connectivity of the complexes $\mathscr{L}(S_n^{\text{mock}})$ goes to infinity as n goes to infinity. As a remark, one can calculate by hand that the hypotheses of Lemma 5.10 hold, and so V_{mock} is finitely generated.

References

[AB87] H. Abels and K. S. Brown. Finiteness properties of solvable S-arithmetic groups: an example. *J. Pure Appl. Algebra*, 44(1-3):77–83, 1987.

[BB97] M. Bestvina and N. Brady. Morse theory and finiteness properties of groups. *Invent. Math.*, 129(3):445–470, 1997.

[BBCS08] T. Brady, J. Burillo, S. Cleary, and M. Stein. Pure braid subgroups of braided Thompson's groups. *Publ. Mat.*, 52(1):57–89, 2008.

[BE74] R. Bieri and B. Eckmann. Finiteness properties of duality groups. *Comment. Math. Helv.*, 49:74–83, 1974.

[Bel04] J. Belk. *Thompson's Group F*. PhD thesis, Cornell University, 2004.

[BF] J. M. Belk and B. Forrest. A Thompson Group for the Basilica. arXiv:1201.4225.

[BFS+] K.-U. Bux, M. Fluch, M. Schwandt, S. Witzel, and M. C. B. Zaremsky. The braided Thompson's groups are of type F_∞. arXiv:1210.2931.

[BH99] M. R. Bridson and A. Haefliger. *Metric spaces of non-positive curvature*, volume 319 of *Grundlehren der Mathematischen Wissenschaften*. Springer-Verlag, Berlin, 1999.

[BKW13] K.-U. Bux, R. Köhl, and S. Witzel. Higher finiteness properties of reductive arithmetic groups in positive characteristic: the rank theorem. *Ann. of Math. (2)*, 177(1):311–366, 2013.

[BLVŽ94] A. Björner, L. Lovász, S. T. Vrećica, and R. T. Živaljević. Chessboard complexes and matching complexes. *J. London Math. Soc. (2)*, 49(1):25–39, 1994.

[Bri04] M. G. Brin. Higher dimensional Thompson groups. *Geom. Dedicata*, 108:163–192, 2004.

[Bri05] M. G. Brin. On the Zappa-Szép product. *Comm. Algebra*, 33(2):393–424, 2005.

[Bri07] M. G. Brin. The algebra of strand splitting. I. A braided version of Thompson's group V. *J. Group Theory*, 10(6):757–788, 2007.

[Bro87] K. S. Brown. Finiteness properties of groups. *J. Pure Appl. Algebra*, 44(1-3):45–75, 1987.

[Bro92] K. S. Brown. The geometry of finitely presented infinite simple groups. In *Algorithms and classification in combinatorial group theory (Berkeley, CA, 1989)*, volume 23 of *Math. Sci. Res. Inst. Publ.*, pages 121–136. Springer, New York, 1992.

[Bro06] K. S. Brown. The homology of Richard Thompson's group F. In *Topological and asymptotic aspects of group theory*, volume 394 of *Contemp. Math.*, pages 47–59. Amer. Math. Soc., Providence, RI, 2006.

[Bux04] K.-U. Bux. Finiteness properties of soluble arithmetic groups over global function fields. *Geom. Topol.*, 8:611–644, 2004.

[CP61] A. H. Clifford and G. B. Preston. *The algebraic theory of semigroups. Vol. I.* Mathematical Surveys, No. 7. American Mathematical Society, Providence, R.I., 1961.

[DJS03] M. Davis, T. Januszkiewicz, and R. Scott. Fundamental groups of blow-ups. *Adv. Math.*, 177(1):115–179, 2003.

[Far03] D. S. Farley. Finiteness and CAT(0) properties of diagram groups. *Topology*, 42(5):1065–1082, 2003.

[FH] D. S. Farley and Bruce Hughes. Finiteness properties of some groups of local similarities. arXiv:1206.2692.

[FMWZ13] M. G. Fluch, M. Marschler, S. Witzel, and M. C. B. Zaremsky. The Brin–Thompson groups sV are of type F_∞. *Pacific J. Math.*, 266(2):283–295, 2013.

[GS97] V. Guba and M. Sapir. Diagram groups. *Mem. Amer. Math. Soc.*, 130(620), 1997.

[Hig74] G. Higman. *Finitely presented infinite simple groups.* Department of Pure Mathematics, Department of Mathematics, I.A.S. Australian National University, Canberra, 1974. Notes on Pure Mathematics, No. 8 (1974).

[Hug09] B. Hughes. Local similarities and the Haagerup property. *Groups Geom. Dyn.*, 3(2):299–315, 2009. With an appendix by D. S. Farley.

[Koz08] D. Kozlov. *Combinatorial Algebraic Topology.* Springer, 2008.

[KS06] M. Kashiwara and P. Schapira. *Categories and sheaves*, volume 332 of *Grundlehren der Mathematischen Wissenschaften.* Springer, 2006.

[MPN] C. Martínez-Pérez and B. E. A. Nucinkis. Bredon cohomological finiteness conditions for generalisations of Thompson's groups. arXiv:1105.0189.

[Put] A. Putman. Representation stability, congruence subgroups, and mapping class groups. arXiv:1201.4876v2.

[Qui78] D. Quillen. Homotopy properties of the poset of nontrivial p-subgroups of a group. *Adv. in Math.*, 28(2):101–128, 1978.

[Röv99] C. E. Röver. Constructing finitely presented simple groups that contain Grigorchuk groups. *J. Algebra*, 220(1):284–313, 1999.

[Spa66] E. H. Spanier. *Algebraic Topology.* Springer, 1966.

[Ste92] M. Stein. Groups of piecewise linear homeomorphisms. *Trans. Amer. Math. Soc.*, 332(2):477–514, 1992.

Mathematical Institute, University of Münster, Einsteinstrasse 62, 48149 Münster, Germany
E-mail address: s.witzel@uni-muenster.de

Department of Mathematical Sciences, Binghamton University, Binghamton, NY 13902
E-mail address: zaremsky@math.binghamton.edu