2 The Compact-Open Topology and Transformation Groups

A *topological transformation group* consists of a Hausdorff topological group G and a nonempty Hausdorff space X on which G acts (usually from the left) in such a way that the map

$$G \times X \longrightarrow X, \quad (g, x) \longmapsto gx$$

is continuous. The *orbit* of $x \in X$ is denoted by

$$G(x) = \{gx \mid g \in G\} \subseteq X$$

and the stabilizer of x is the subgroup

$$G_x = \{g \in G \mid gx = x\} \subseteq G.$$

The *orbit space* of the action is the set of orbits

$$G\backslash X = \{G(x) \mid x \in X\}.$$

The orbit space is endowed with the quotient topology with respect to the map

$$q : x \longmapsto G(x), \quad x \longmapsto G(x).$$

If $U \subseteq G$ is open, then $q^{-1}(q(U)) = \bigcup\{g(U) \mid g \in G\}$ is open, hence q is an open map.

In this chapter we study basic properties of topological transformation groups. We first consider the compact-open topology on sets of mappings. The universal properties of this topology are then applied to transformation groups acting on locally compact spaces. Then we study proper actions and in particular actions of compact groups and their orbit spaces.

The Compact-Open Topology

We recall that a *basis* for a topology $\mathcal{T}$ on a set X is a subset $\mathcal{B} \subseteq \mathcal{T}$, such that for every $U \in \mathcal{T}$ and every $x \in U$ there exists $V \in \mathcal{B}$ with $x \in V \subseteq U$. In other words, every open set is a union of elements of $\mathcal{B}$. For example, $\mathcal{T}$ itself is a basis for $\mathcal{T}$. In a metric space, the collection of all open balls is a basis. A subset $\mathcal{S} \subseteq \mathcal{T}$ is called a *subbasis* for the topology $\mathcal{T}$ if for every open set $U \subseteq X$ and every $x \in X$ there exist $V_1, \dots, V_m \in \mathcal{S}$, for $m \geq 1$, with $x \in V_1 \cap \cdots \cap V_m \subseteq U$. Then the collection of all finite intersections of members of $\mathcal{S}$ is a basis for $\mathcal{T}$. If $Y \subseteq X$, then $\mathcal{B}|Y$ is a basis for the subspace topology on Y, and $\mathcal{S}|Y$ is a subbasis for the subspace topology.

Suppose that $\mathcal{S}$ is any collection of subsets of a set X, with $\bigcup \mathcal{S} = X$. The latter condition can always be met by enlarging $\mathcal{S}$ to $\mathcal{S} \cup \{X\}$. If we let $\mathcal{B}$ denote the collection of all finite intersections of members of $\mathcal{S}$,

$$\mathcal{B} = \{\bigcap \mathcal{F} \mid \varnothing \neq \mathcal{F} \subseteq \mathcal{S} \text{ is finite }\},$$

then the set

$$\mathcal{T} = \{\bigcup \mathcal{C} \mid \mathcal{C} \text{ is any subset of } \mathcal{B}\}$$

is a topology on X with basis $\mathcal{B}$, and $\mathcal{S}$ is a subbasis for this topology. One calls $\mathcal{T}$ the *topology generated by the subbasis* $\mathcal{S}$. If $\mathcal{S}$ is a subbasis of a topology $\mathcal{T}$, then this process recovers the topology $\mathcal{T}$ from its subbasis $\mathcal{S}$. The following observation is elementary.

Lemma 2.1. *Let $\mathcal{T}$ be a topology on a set Y, with a subbasis $\mathcal{S} \subseteq \mathcal{T}$. Let X be a topological space and let $f : X \longrightarrow Y$ be a map. Then f is continuous at $x \in X$ if and only if for every $V \in \mathcal{S}$ with $f(x) \in V$ there exists a neighborhood U of x with $f(U) \subseteq V$.*

Proof. If f is continuous at x, then this condition is satisfied because $\mathcal{S} \subseteq \mathcal{T}$. Conversely, suppose that the condition is satisfied. If W is any neighborhood of $f(x)$, then there exist $V_1, \dots, V_m \in \mathcal{S}$ with $f(x) \in V_1 \cap \cdots \cap V_m \subseteq W$. For each $i = 1, \dots, m$ there exists a neighborhood U_i of x with $f(U_i) \subseteq V_i$. Then $U = U_1 \cap \cdots \cap U_m$ is a neighborhood of x with $f(U) \subseteq W$. □

Now we get to the central definition of this chapter.

Definition 2.2. Suppose that X and Y are Hausdorff spaces. The compact-open topology on the set $C(X, Y)$ of continuous maps from X to Y is defined as follows. For $A \subseteq X$ and $B \subseteq Y$ we put

$$\langle A; B \rangle = \{f \in C(X, Y) \mid f(A) \subseteq B\}.$$

The *compact-open topology* is the topology generated by the subbasis

$$\mathcal{S} = \{\langle K; V \rangle \mid K \subseteq X \text{ is compact and } V \subseteq Y \text{ is open}\}.$$

Example 2.3. If X is a discrete space, then the compact subsets of X are the finite subsets. Therefore the compact-open topology on $C(X,Y) = Y^X$ is the product topology. This topology is also called the *topology of pointwise convergence*. If X consists of a single point, $X = \{x\}$, then in particular the map $C(\{x\}, Y) = Y^{\{x\}} \longrightarrow Y$ that maps f to $f(x)$ is a homeomorphism.

Suppose that $h : Z \longrightarrow X$ is a continuous map between Hausdorff spaces Z, X. Then h induces a map $h^* : C(X,Y) \longrightarrow C(Z,Y)$ via $h^*(g) = g \circ h$.

Lemma 2.4. *Let X, Y, Z be Hausdorff spaces, and let $h : Z \longrightarrow X$ be continuous. Then the map*

$$h^* : C(X,Y) \longrightarrow C(Z,Y)$$

is continuous with respect to the compact-open topologies.

Proof. Suppose that $K \subseteq Z$ is compact, that $W \subseteq Y$ is open and that $g \in C(X,Y)$ with $h^*(g) = g \circ h \in \langle K; W\rangle$. Then $h(K)$ is compact and $h^*(\langle h(K); W\rangle) \subseteq \langle K; W\rangle$. Hence h^* is continuous at g by Lemma 2.1. □

If we put $Z = X$, endowed with the discrete topology, and $h = \mathrm{id}_X$, we obtain in particular that the injection

$$C(X;Y) \longrightarrow Y^X$$

is continuous, where the right-hand side carries the topology of pointwise convergence. Therefore $C(X,Y)$ is a Hausdorff space, and even regular if Y is regular. On the other hand, if we put $Z = \{x\}$ for some point $x \in X$ and $h(x) = x$, we obtain from Example 2.3 and Lemma 2.4 the continuity of the evaluation map

$$\mathrm{ev}_x : C(X,Y) \longrightarrow Y, \quad f \longmapsto f(x).$$

The continuous maps from a space to itself make up a monoid, i.e. a semigroup with a two-sided identity.

Definition 2.5. A *topological monoid* $(M, \cdot, \mathcal{T})$ is a monoid $(M, \cdot)$ with a topology $\mathcal{T}$ on M such that the monoid multiplication $M \times M \longrightarrow M$, $(x,y) \longmapsto x \cdot y$ is continuous.

Proposition 2.6. *Let X and Z be Hausdorff spaces and let Y be a locally compact space. Then the composition map*

$$c : C(X,Y) \times C(Y,Z) \longmapsto C(X,Z), \qquad (f,h) \longmapsto h \circ f$$

is continuous with respect to the compact-open topologies on $C(X,Y)$, $C(Y,Z)$ and $C(X,Z)$. In particular, the monoid $C(Y,Y)$ is a Hausdorff topological monoid with respect to the compact-open topology.

Proof. Suppose that $f \in C(X,Y)$ and $h \in C(Y,Z)$, and that $\langle K;W\rangle \subseteq C(X,Z)$ contains $h \circ f$. We put $U = h^{-1}(W)$. Since U is locally compact and contains $f(K)$, there exists by Corollary 1.23 an open neighborhood V of $f(K)$ in U with compact closure $\overline{V} \subseteq U$. Thus $f \in \langle K;V\rangle$ and $h \in \langle \overline{V};W\rangle$, and $c(\langle K;V\rangle \times \langle \overline{V};W\rangle) \subseteq \langle K;W\rangle$. By Lemma 2.1, the map c is continuous at (f,h). ☐

Corollary 2.7. *Suppose that Y is locally compact and that Z is a Hausdorff space. Then the joint evaluation map*

$$\mathrm{ev} : Y \times C(Y,Z) \longrightarrow Z, \qquad (y,h) \longmapsto \mathrm{ev}_y(h) = h(y)$$

is continuous with respect to the compact-open topology.

Proof. Let $X = \{0\}$ denote the one-point topological space. We noted in Example 2.3 that there is a homeomorphism $Y \longrightarrow C(X,Y)$ that maps $y \in Y$ to the map $y' = [0 \longmapsto y]$. Hence the map $(y,f) \longmapsto \mathrm{ev}_0(y' \circ f) = f(y)$ is continuous by the remark following Lemma 2.4 and by Proposition 2.6. ☐

Before we prove the main theorem about the compact-open topology, we need two results about subbases.

Lemma 2.8. *Let X and Y be Hausdorff spaces and let $\mathcal{S}$ be a subbasis for the topology on Y. Then the sets $\langle K;V\rangle$, for $K \subseteq X$ compact and $V \in \mathcal{S}$, form a subbasis for the compact-open topology on $C(X,Y)$.*

Proof. Suppose that $f \in \langle K;W\rangle$, for $K \subseteq X$ compact and $W \subseteq Y$ open. For each $x \in K$ there are sets $V_{x,1},\ldots,V_{x,n_x} \in \mathcal{S}$ with $f(x) \subseteq V_{x,1} \cap \cdots \cap V_{x,n_x} \subseteq W$. Since K is normal, there exists a closed neighborhood K_x of x in K such that $f(K_x) \subseteq V_{x,1} \cap \cdots \cap V_{x,n_x}$. Since K is compact, there is a finite set $F \subseteq K$ with $K \subseteq \bigcup\{K_x \mid x \in F\}$. Thus

$$f \in \bigcap_{x\in F}\bigcap_{j=1}^{n_x}\langle K_x;V_{x,j}\rangle \subseteq \langle K;W\rangle.$$ ☐

Lemma 2.9. *Let X, Y and Z be Hausdorff spaces. The the sets $\langle K\times L;W\rangle$, with $K \subseteq X$ and $L \subseteq Y$ compact and $W \subseteq Z$ open, form a subbasis for the compact-open topology on $C(X \times Y, Z)$.*

Proof. Let $M \subseteq X \times Y$ be compact and let $W \subseteq Z$ be open. Suppose that $f \in \langle M;W\rangle$. We put $K = \mathrm{pr}_1(M) \subseteq X$ and $L = \mathrm{pr}_2(M) \subseteq Y$ and we consider the restriction $f|_{K\times L} : K \times L \longrightarrow Z$. Since $K \times L$ is normal, every $p \in M$ has a neighborhood of the form $K_p \times L_p$ in $K \times L$, with K_p and L_p compact, with $f(K_p \times L_p) \subseteq W$. Since M is compact, there is a finite set $E \subseteq M$ with $M \subseteq \bigcup_{p\in E} K_p \times L_p$. Thus $f \in \bigcap_{p\in E}\langle K_p \times L_p;W\rangle \subseteq \langle M;W\rangle$. ☐

The following theorem captures the universal property of the compact-open topology.

Theorem 2.10. *Let X, Y and Z be Hausdorff spaces. The map*

$$C(X\times Y,Z)\longrightarrow C(X,C(Y,Z)),\quad f\longmapsto \hat f=[x\longmapsto f(x,-)]$$

is a topological embedding. If Y is locally compact, then this map is bijective.

Proof. We divide the proof into several steps.

Step 1. If $f:X\times Y\longrightarrow Z$ is continuous, then $\hat f:X\longrightarrow C(Y,Z)$ is continuous.

If $L\subseteq Y$ is compact and $W\subseteq Z$ is open, with $\hat f(x)\in\langle L;W\rangle$, then $f(\{x\}\times L)\subseteq W$. There exists by Wallace's Lemma 1.21 an open neighborhood of U of x with $f(U\times L)\subseteq W$, whence $\hat f(U)\in\langle L;W\rangle$. Thus $\hat f$ is continuous at x by Lemma 2.1.

Step 2. If Y is locally compact and if $h:X\longrightarrow C(Y,Z)$ is continuous, then the map $f:(x,y)\longmapsto h(x)(y)$ is continuous.

This map is the composite $X\times Y\longrightarrow C(Y,Z)\times Y\longrightarrow Z$ of the continuous map $(x,y)\longmapsto (h(x),y)$ and the joint evaluation map, which is continuous by Corollary 2.7.

Step 3. The sets $\langle K\times L;W\rangle$, with $K\subseteq X$ and $L\subseteq Y$ compact and $W\subseteq Z$ open, form a subbasis for the compact-open topology on $C(X\times Y,Z)$.

This is the content of Lemma 2.9.

Step 4. The sets $\langle K;\langle L;W\rangle\rangle$, with $K\subseteq X$ and $L\subseteq Y$ compact and $W\subseteq Z$ open, form a subbasis for the compact-open topology on $C(X,C(Y,Z))$.

This follows at once from Lemma 2.8.

Now we finish the proof. By Step 1, the map $\varphi:f\longmapsto\hat f$ takes its values in $C(X,C(Y,Z))$. Since $\hat f(x)(y)=f(x,y)$, the map φ is injective. By Step 2 the map φ is surjective if Y is locally compact. In any case, φ maps $C(X\times Y,Z)$ bijectively onto the set

$$Q=\{\hat f\mid f\in C(X\times Y,Z)\}\subseteq C(X,C(Y,Z)).$$

Let $\mathcal{S}_1$ denote the subbasis on $C(X\times Y,Z)$ provided by Step 3, and let $\mathcal{S}_2$ denote the subbasis on $C(X,C(Y,Z))$ provided by Step 4. We note that φ maps $\langle K\times L;W\rangle\in\mathcal{S}_1$ bijectively onto $\langle K;\langle L;W\rangle\rangle\cap Q\in\mathcal{S}_2|Q$. Therefore the corestriction $C(X\times Y,Z)\longrightarrow Q$ of φ is a homeomorphism, that is, φ is an embedding. ☐

Topological Transformation Groups

Now we turn to topological transformation groups. We first have to consider actions of monoids.

Proposition 2.11. *Let M be a Hausdorff topological monoid (e.g. a Hausdorff topological group) and let X be a locally compact space. Let $h:M\longrightarrow C(X,X)$ be an abstract homomorphism of monoids. Then the following are equivalent.*

(i) *The monoid homomorphism h is continuous with respect to the compact-open topology.*
(ii) *The action $M \times X \longrightarrow X$ that maps (m,x) to $h(m)(x)$ is continuous.*
In particular, the compact-open topology is the coarsest topology that turns $C(X,X)$ into a topological monoid in such a way that the action $C(M,M) \times M \longrightarrow M$ is continuous.

Proof. This is a direct consequence of Theorem 2.10. □

We are interested in actions of groups, rather than in actions of monoids. Given a topological space X, we let

$$\mathrm{Homeo}(X) \subseteq C(X,X)$$

denote the group of all homeomorphisms of X. If X is a locally compact space, then the previous corollary tells us that the multiplication in $\mathrm{Homeo}(X)$ is continuous with respect to the compact-open topology. It remains to inspect the inversion map $g \longmapsto g^{-1}$ more closely. The following is a general construction that deals with this issue.

Lemma 2.12. *Let G be a group and let $\mathcal{T}$ be a topology on G such that the multiplication*

$$m : G \times G \longrightarrow G, \quad (x,y) \longmapsto xy$$

is continuous with respect to $\mathcal{T}$, i.e. G is a topological monoid with respect to $\mathcal{T}$. If $\mathcal{S}$ is a subbasis for $\mathcal{T}$, then $\mathcal{S}^i = \mathcal{S} \cup \{V^{-1} \mid V \in \mathcal{S}\}$ is a subbasis for a group topology $\mathcal{T}^i$ on G that refines the topology $\mathcal{T}$. This topology $\mathcal{T}^i$ is the unique coarsest group topology on G that contains $\mathcal{T}$.

Proof. From the definition of $\mathcal{S}^i$, the inversion map $g \longmapsto g^{-1}$ is continuous with respect to $\mathcal{T}^i$ by Lemma 2.1. We have to check that the group multiplication $m : G \times G \longrightarrow G$ is continuous with respect to $\mathcal{T}^i$. Let $W \in \mathcal{S}^i$ and let $a, b \in G$ with $ab \in W$. If $W \in \mathcal{S}$, we choose $U_1, \ldots, U_k, V_1, \ldots, V_\ell \in \mathcal{S}$ with $a \in U_1 \cap \cdots \cap U_k$ and $b \in V_1 \cap \cdots \cap V_\ell$, such that

$$m((U_1 \cap \cdots \cap U_k) \times (V_1 \cap \cdots \cap V_\ell)) \subseteq W.$$

If $W^{-1} \in \mathcal{S}$, we choose $U_1, \ldots, U_k, V_1, \ldots, V_\ell \in \mathcal{S}$ with $a^{-1} \in U_1 \cap \cdots \cap U_k$ and $b^{-1} \in V_1 \cap \cdots \cap V_\ell$, such that

$$m((V_1 \cap \cdots \cap V_\ell) \times (U_1 \cap \cdots \cap U_k)) \subseteq W^{-1}.$$

Then

$$m((U_1^{-1} \cap \cdots \cap U_k^{-1}) \times (V_1^{-1} \cap \cdots \cap U_\ell^{-1})) \subseteq W$$

This shows by Lemma 2.1 that the multiplication is continuous with respect to $\mathcal{T}^i$ at (a,b).

If $\mathcal{T}'$ is any group topology on G containing $\mathcal{T}$, then $\mathcal{T}'$ also contains $\mathcal{S}^i$ and hence $\mathcal{T}^i \subseteq \mathcal{T}'$. □

Suppose that X is a topological space, that $K, W \subseteq X$ and that $g \in \mathrm{Homeo}(X)$. Then

$$\begin{aligned} g \in \langle K; W\rangle &\iff g(K) \subseteq W \\ &\iff K \subseteq g^{-1}(W) \\ &\iff X - K \supseteq g^{-1}(X - W) \\ &\iff g^{-1} \in \langle X - W; X - K\rangle. \end{aligned}$$

The following is an immediate consequence of this observation and Lemma 2.12.

Proposition 2.13. *Let X be a locally compact space. Then the topology on* $\mathrm{Homeo}(X)$ *generated by the subbasis*

$$\mathcal{S} = \{\mathrm{Homeo}(X) \cap \langle A; W\rangle \mid A \subseteq X \text{ is closed and } W \subseteq X \text{ is open and } A \text{ is compact or } X - W \text{ is compact}\}$$

is the unique coarsest group topology on $\mathrm{Homeo}(X)$ *that contains the compact-open topology.*

We call this group topology on $\mathrm{Homeo}(X)$ the *Arens topology*. It has the following universal property.

Proposition 2.14. *Let G be a Hausdorff topological group, let X be a locally compact space and let $h : G \longrightarrow$* $\mathrm{Homeo}(X)$ *be an abstract homeomorphism. Then the following are equivalent.*

(i) *The homomorphism h is continuous with respect to the Arens topology.*
(ii) *The action $G \times X \longrightarrow X$ that maps (g, x) to $h(g)(x)$ is continuous.*

Proof. If h is continuous, then the composite $G \xrightarrow{h} \mathrm{Homeo}(X) \longrightarrow C(X, X)$ is also continuous, for the compact-open topology on $C(X, X)$. Hence the action is continuous by Proposition 2.11. Conversely, suppose that the action is continuous. Then the homomorphism h is, by Proposition 2.11, continuous with respect to the compact-open topology on $\mathrm{Homeo}(X)$. If $K \subseteq X$ is compact and $W \subseteq X$ is open and if $V = \mathrm{Homeo}(X) \cap \langle K; W\rangle$, then $h^{-1}(V^{-1}) = (h^{-1}(V))^{-1}$ is open in G and thus h is, by Lemma 2.1, continuous with respect to the Arens topology. ☐

The following is immediate from the definition of the Arens topology.

Proposition 2.15. *If X is a compact space, then the Arens topology and the compact-open topology on* $\mathrm{Homeo}(X)$ *coincide and hence* $\mathrm{Homeo}(X)$ *is a Hausdorff topological group with respect to the compact-open topology.*

The following is another useful criterion for this.

Theorem 2.16 (Arens–Dijkstra). *Let X be a locally compact space. If every point $x \in X$ has a compact connected neighborhood, then the Arens topology and the compact-open topology on* $\mathrm{Homeo}(X)$ *coincide. In particular,* $\mathrm{Homeo}(X)$ *is a Hausdorff topological group with respect to the compact-open topology.*

Proof. In view of Proposition 2.13 it suffices to show the following.
Claim. If $A \subseteq X$ is closed and if $W \subseteq X$ has a compact complement K, then the set $Q = \mathrm{Homeo}(X) \cap \langle A; W\rangle$ is open with respect to the compact-open topology on $\mathrm{Homeo}(X)$.

Let $g \in Q$. For each $x \in X$ we choose a compact connected neighborhood C_x, with interior U_x. There is a finite subset $F \subseteq g^{-1}(K)$ such that $g^{-1}(K) \subseteq \bigcup_{x \in F} U_x$. We put

$$L = \bigcup_{x \in F} C_x.$$

Thus $L \supseteq g^{-1}(K)$ is a finite union of compact connected sets. Now we choose a compact set $M \subseteq X$ that contains L in its interior, using either a similar construction as for L, or Corollary 1.23. The topological boundary of M is the compact set $\partial M = M \cap \overline{X - M}$. We note that $\partial M \cap L = \varnothing$. The set

$$V = \mathrm{Homeo}(X) \cap \langle A \cap M; W\rangle \cap \langle \partial M; g(X - L)\rangle \cap \bigcap_{x \in F} \langle \{x\}; g(U_x)\rangle$$

is open in $\mathrm{Homeo}(X)$ and contains g. We claim that V is contained in Q. Let $h \in V$ and $x \in F$. Then $h(\partial M)$ is disjoint from $g(L) \supseteq g(C_x)$. On the other hand, $h(x) \in g(C_x)$. The point $h(x)$ is in the interior of $h(M)$. Being connected, $g(C_x)$ is thus contained in the interior of $h(M)$. Thus

$$K \subseteq g(L) \subseteq h(M)$$

and therefore $h(X - M) \subseteq W$. Since $h(A \cap M) \subseteq W$, we have $h(A) \subseteq W$. This proves the claim. □

We recall that a topological space is called *locally connected* if every point has arbitrarily small open connected neighborhoods. An example of a locally compact space that is not locally connected, but where every point has a compact connected neighborhood, is the set

$$X = \overline{\{(x, \sin(1/x)) \mid x > 0\}} \subseteq \mathbb{R}^2.$$

Corollary 2.17. *Let X be a locally compact and locally connected space. Then* $\mathrm{Homeo}(X)$, *endowed with the compact-open topology, is a topological group.*

The following example shows that for the homeomorphism group of a locally compact and totally disconnected space, the compact-open topology may differ from the Arens topology

Example 2.18. Let $C = \{0,1\}^{\mathbb{N}_1}$ denote the Cantor set consisting of of all $\{0,1\}$-valued sequences, endowed with the topology of pointwise convergence. We put $\mathbf{0} = (0,0\ldots)$ and $\mathbf{1} = (1,1,\ldots)$. Let

$$U_n = \{c \in C \mid c_1 = \cdots = c_n = 0\} \quad \text{and} \quad V_n = \{c \in C \mid c_1 = \cdots = c_n = 1\}.$$

These sets are closed and open in C. The sets U_n form an neighborhood basis of $\mathbf{0}$ and the sets V_n form a neighborhood basis of $\mathbf{1}$.

We define a sequence of homeomorphisms h_n of C, for $n \geq 1$, as follows.

$$\begin{aligned} h_n((\underbrace{0,\ldots,0}_{n},0,x_{n+2},\ldots)) &= (\underbrace{0,\ldots,0}_{n},x_{n+2},\ldots) \\ h_n((\underbrace{0,\ldots,0}_{n},1,x_{n+2},\ldots)) &= (\underbrace{1,\ldots,1}_{n},1,x_{n+2},\ldots) \\ h_n((\underbrace{1,\ldots,1}_{n},x_{n+1},\ldots)) &= (\underbrace{1,\ldots,1}_{n},0,x_{n+1},\ldots) \\ h_n(c) &= c \text{ else.} \end{aligned}$$

Thus

$$h_n(U_{n+1}) = U_n, \quad h_n(U_n - U_{n+1}) = V_{n+1}, \quad h_n(V_n) = V_n - V_{n+1}, \\ \text{and} \quad h_n|_{C-(U_n\cup V_n)} = \mathrm{id}_{C-(U_n\cup V_n)}.$$

We note that each h_n fixes $\mathbf{0}$ and we put $X = C - \{\mathbf{0}\}$. We claim that $\lim_n h_n|_X = \mathrm{id}_X$, but that $(h_n|_X)^{-1}$ does not converge to id_X in the compact-open topology of X.

For the first claim, let $K \subseteq X$ be compact and let $W \subseteq X$ be open, with $\mathrm{id}_X \in \langle K; W\rangle$. Thus $K \subseteq W$. Since $K \subseteq C$ is compact and does not contain $\mathbf{0}$, there is $m \geq 1$ such that $U_m \cap K = \varnothing$. Suppose that $\mathbf{1} \in X - K$. Then there exists $n \geq 0$ such that $V_n \cap K = \varnothing$. For $\ell \geq m, n$ we have thus $h_\ell|_K = \mathrm{id}_K$, and thus $h_\ell|_X \in \langle K; W\rangle$. Suppose now that $\mathbf{1} \in K$. Then there exists $n \geq 1$ such that $V_n \subseteq W$. For $\ell \geq m, n$ we have thus $h_\ell|_{K-V_\ell} = \mathrm{id}_{K-V_\ell}$ and $h_\ell(V_\ell) \subseteq V_\ell \subseteq W$, whence $h_\ell|_X \in \langle K; W\rangle$. This shows that $\lim_n h_n|_X = \mathrm{id}_X$.

On the other hand, $\mathbf{1} \in h_n(U_n)$ and therefore the sequence $(h_n^{-1}(\mathbf{1}))_{n\in\mathbb{N}_1}$ does not converge to $\mathbf{1}$. Since evaluation at $\mathbf{1}$ is continuous, this shows that the sequence $(h_n^{-1}|_X)_{n\in\mathbb{N}_1}$ does not converge to id_X in the compact-open topology.

Proper Actions

Before we study proper actions, we recall a few facts about proper maps. Proper maps are closely related to compactness by Kuratowski's Theorem.

Theorem 2.19 (Kuratowski). *Let X be a Hausdorff space. The following are equivalent.*

(i) *The space X is compact.*
(ii) *For every Hausdorff space Z, the map $\mathrm{pr}_2 : X \times Z \longrightarrow Z$ is closed.*

Proof. Suppose that X is compact and that $A \subseteq X \times Z$ is closed. We claim that $\mathrm{pr}_2(A)$ is closed. Let $z \in Z - \mathrm{pr}_2(A)$. Thus $(X \times \{z\}) \cap A = \varnothing$. By Wallace's Lemma 1.21, z has a neighborhood $U \subseteq Z$ such that $(X \times U) \cap A = \varnothing$. Since $U \cap \mathrm{pr}_2(A) = \varnothing$, the set A is open.

Suppose that X is not compact. Then there exists a nonempty collection $\mathcal{K}$ of closed subsets of X with the finite intersection property, with $\bigcap \mathcal{K} = \varnothing$. We put $Z = X \cup \{\infty\}$, where $\infty \notin X$, and we define a topology on Z as follows. By definition, a subset $U \subseteq Z$ is open if either $U \subseteq X$, or if there is a nonempty finite subset $\mathcal{F} \subseteq \mathcal{K}$ such that $\bigcap \mathcal{F} \subseteq U$. It is readily verified that this is a topology. If $x, y \in X$ are distinct points, then $\{x\}$ and $\{y\}$ are disjoint neighborhoods of x and y, respectively. Also, there exists $K \in \mathcal{K}$ with $x \notin K$, since $\bigcap \mathcal{K} = \varnothing$. Then $\{\infty\} \cup K$ and $\{x\}$ are disjoint neighborhoods of ∞ and x, respectively. Hence Z is a Hausdorff space. Every neighborhood of ∞ intersects X nontrivially, because $\bigcap \mathcal{F} \neq \varnothing$ for every nonempty finite subset $\mathcal{F} \subseteq \mathcal{K}$. Let $D = \{(x, x) \mid x \in X\} \subseteq X \times Z$. Then $\mathrm{pr}_2(D) = X \subseteq Z$. We claim that $\mathrm{pr}_2(\overline{D})$ is not closed. Otherwise we would have $(x, \infty) \in \overline{D}$, for some $x \in X$. But there is $K \in \mathcal{K}$ with $x \notin K$, and thus $(X - K) \times (\{\infty\} \cup K)$ is a neighborhood of (x, ∞) which is disjoint from D. Hence $\mathrm{pr}_2 : X \times Z \longrightarrow Z$ is not a closed map. $\square$

Definition 2.20. Let

$$f : X \longrightarrow Y$$

be a continuous map between Hausdorff spaces. We call f *proper* if for every Hausdorff space Z, the map

$$f \times \mathrm{id}_Z : X \times Z \longrightarrow Y \times Z$$

is closed.

The following are immediate consequences of the definition and Kuratowski's Theorem.

Lemma 2.21. *Let $f : X \longrightarrow Y$ be a proper map. Then the following hold.*
(i) *The map f is closed.*
(ii) *If f is injective, then f is a topological embedding.*
(iii) *For every subset $B \subseteq Y$ and $A = f^{-1}(B)$, the restriction-corestriction $f|_A^B : A \longrightarrow B$ is proper.*
(iv) *For every closed set $A \subseteq X$ the restriction-corestriction $f|_A^{f(A)} : A \longrightarrow f(A)$ is proper.*
(v) *If f is constant, then X is compact.*
(vi) *For every compact set $B \subseteq Y$, the preimage of B is compact.*

Proof. Claim (i) follows from the definition by putting $Z = \{0\}$. A closed injective continuous map is a topological embedding, hence (ii) also follows. For (iii) we note that $A \times Z = (f \times \mathrm{id}_Z)(B \times Z)$ and therefore the restriction-corestriction $A \times Z \longrightarrow B \times Z$ is a closed map, provided that $f \times \mathrm{id}_Z$ is a closed map. For (iv) we note that if E is closed in $A \times Z$, then $(f \times \mathrm{id}_Z)(E)$ is closed in $Y \times Z$ and hence closed in $f(A) \times Z$. Claim (v) follows from (iv) and Kuratowski's Theorem 2.19. In the special case that $B = \{b\}$ is a singleton, (vi) follows from (iv) and (v). In general, let $\mathcal{U}$ be an open covering of $f^{-1}(B)$. The preimage of every point $b \in B$ is compact and hence there is a finite subset $\mathcal{U}_b \subseteq \mathcal{U}$ such that $f^{-1}(b) \subseteq \bigcup \mathcal{U}_b$. The set $A_b = f(X - \bigcup \mathcal{U}_b)$ is closed and does not contain b, hence $V_b = Y - A_b$ is an open neighborhood of b. Since B is compact, there exist finitely many points $b_1, \ldots, b_m \in B$ such that $B = \bigcup_{j=1}^m V_{b_j}$. Let $x \in X$. If $x \notin \bigcup \mathcal{U}_{b_j}$, then $f(x) \in f(X - \bigcup \mathcal{U}_{b_j}) = A_{b_j}$. This shows that $f^{-1}(B) \subseteq \bigcup_{j=1}^m \bigcup \mathcal{U}_{b_j}$ and therefore $f^{-1}(B)$ is compact. □

These properties characterize proper maps.

Proposition 2.22. *Let $f : X \longrightarrow Y$ be a continuous map between Hausdorff spaces X, Y. Then the following are equivalent.*

(i) *The map f is proper.*

(ii) *The map f is closed and the preimage of every point $y \in Y$ is compact.*

If Y is locally compact and if every compact set $B \subseteq Y$ has compact preimage, then f is proper and X is locally compact.

Proof. By Lemma 2.21, (i) $\Rightarrow$ (ii). Suppose that (ii) holds. Let Z be a Hausdorff space and let $E \subseteq X \times Z$ be a closed set. We claim that $F = (f \times \mathrm{id}_Z)(E)$ is closed in $Y \times Z$. Let $(y, z) \in (Y \times Z) - F$. The preimage of (y, z) is the compact set $(f \times \mathrm{id}_Z)^{-1}(y, z) = f^{-1}(y) \times \{z\}$, and this set is disjoint from E. By Wallace's Lemma 1.21 there are open sets $U \subseteq X$ and $V \subseteq Z$ with $f^{-1}(y) \times \{z\} \subseteq U \times V \subseteq (X \times Z) - E$. Since f is closed, the set $A = f(X - U)$ is closed and hence $W = Y - A$ is open. Also, $y \in W$ because $f^{-1}(y)$ is disjoint from $X - U$. Since $f^{-1}(W) \subseteq U$, the open set $f^{-1}(W) \times V$ is disjoint from E and hence $W \times V$ an open neighborhood of (y, z) which is disjoint from F.

Suppose that Y is locally compact and that every compact subset $B \subseteq Y$ has compact preimage. Let $x \in X$ and let $V \subseteq Y$ be an open neighborhood of $f(x)$ with compact closure. Then $f^{-1}(\overline{V})$ is an compact neighborhood of x, hence X is locally compact. It remains to show that f is closed. Let $A \subseteq X$ be closed subset with $y \in \overline{f(A)}$. Let U be a compact neighborhood of y. Then $f(A) \cap U = f(A \cap f^{-1}(U)$ is compact and therefore closed. Hence $y \in \overline{f(A)} \cap U = f(A) \cap U$ and therefore $f(A)$ is closed. □

Now we consider proper actions.

Definition 2.23. Let G be a Hausdorff topological group and let X be a Hausdorff space. We call an action

$$\alpha : G \times X \longrightarrow X$$

a *topological transformation group* if the map $\alpha : (g, x) \longrightarrow gx$ is continuous. We endow the orbit space $G\backslash X = \{G(x) \mid x \in X\}$ with the quotient topology with respect to the map

$$q : X \longrightarrow G\backslash X, \quad g \longmapsto G(x).$$

We note that q is always open, since $g^{-1}(q(U)) = \bigcup\{g(U) \mid g \in G\}$. If the map

$$\tilde{\alpha} : G \times X \longrightarrow X \times X, \quad (g, x) \longmapsto (gx, x)$$

is proper, then the action α is called *proper*.

Lemma 2.24. *Let $\alpha : G \times X \longrightarrow X$ be a proper action. Then the following hold.*

(i) *For every $x \in X$ the stabilizer G_x is compact.*
(ii) *For every $x \in X$ the map evaluation map $G \longrightarrow X$, $g \longmapsto gx$ is proper and in particular closed.*
(iii) *For every $x \in X$, the induced map $G/G_x \longrightarrow G(x)$, $gG_x \longmapsto gx$ is a homeomorphism.*
(iv) *Every G-orbit $G(x)$ is closed.*
(v) *The orbit space $G\backslash X$ is a Hausdorff space with respect to the quotient topology.*

Proof. The set $G_x \times \{x\}$ is the $\tilde{\alpha}$-preimage of (x, x), hence (i) holds by Lemma 2.21(vi). The $\tilde{\alpha}$-preimage of $X \times \{x\}$ is $G \times \{x\}$, hence (ii) holds by Lemma 2.21(iii). Since a proper map is closed, (iv) follows from (ii). The map $G \longrightarrow G(x)$ is proper by (ii) and by Lemma 2.21(iii) and therefore in particular closed. On the other hand, the map $G \longrightarrow G/G_x$ is a quotient map,

$$\begin{array}{ccc} G & \longrightarrow & G(x) \\ \downarrow & \nearrow & \\ G/G_x. & & \end{array}$$

Hence the map $G/G_x \longrightarrow G(x)$ is a homeomorphism. For (v) we note that the map $q \times q : X \times X \longrightarrow G\backslash X \times G\backslash X$ is open and hence a quotient map. The $q \times q$-preimage of the diagonal D in $G\backslash X \times G\backslash X$ is the closed set $\tilde{\alpha}(G \times X)$. Hence the diagonal D is closed and therefore $G\backslash X$ is Hausdorff. ☐

For locally compact transformation groups there is the following convenient characterization of proper actions.

Theorem 2.25. *Let $\alpha : G \times X \longrightarrow X$ be a topological transformation group. If G is locally compact, then the following are equivalent.*

(i) *The action is proper.*

(ii) *For all $x, y \in X$ there are neighborhoods U and V of x and y, respectively, such that $\{g \in G \mid U \cap g(V) \neq \varnothing\}$ has compact closure in G.*

Proof. First of all we note the following. Let $U, V \subseteq X$ be subsets. Then

$$(gz, z) \in U \times V \iff gz \in U \cap g(V).$$

We put

$$Q = \{(gz, z) \mid g \in G \text{ and } z \in X\} = \tilde{\alpha}(G \times X).$$

Suppose that the action is proper. If x and y have different orbits, then (x, y) is not contained in the closed set Q. Hence there are neighborhoods U of x and V of y such that $(U \times V) \cap Q = \varnothing$ and thus $\{g \in G \mid U \cap g(V) \neq \varnothing\} = \varnothing$. If $(x, y) \in Q$, then $x = hy$ for some $h \in G$ and the set

$$P = \{(g, z) \in G \times X \mid (gz, z) = (x, y)\} = \{(g, y) \in G \times X \mid g \in hG_y\}$$

is compact. Let $W \supseteq hG_y$ be open with compact closure. Then $W \times X$ is an open neighborhood of P. Since $A = \tilde{\alpha}((G - W) \times X)$ is closed and does not contain (x, y), there are open neighborhoods U and V of x and y, respectively, with $(U \times V) \cap A = \varnothing$. Therefore $\{(g, z) \in G \times X \mid (gz, z) \in U \times V\} \subseteq W \times X$ and in particular $\{g \in G \mid U \cap g(V) \neq \varnothing\} \subseteq W$.

Suppose now that (ii) holds. The set $\{(g, y) \in G \times X \mid gy = x\}$ is compact for every $(x, y) \in X \times X$. By Proposition 2.22 it remains to show that $\tilde{\alpha}$ is a closed map. Let $E \subseteq G \times X$ be closed and put $F = \tilde{\alpha}(E)$. Suppose that $(x, y) \in \overline{F}$. We choose neighborhoods U of x and V of y such that $\{g \in G \mid U \cap g(V) \neq \varnothing\}$ has compact closure $A \subseteq G$. We claim that $x = ay$ for $(a, y) \in E \cap (A \times \{y\})$. Otherwise we find, by Wallace' Lemma 1.21, a neighborhood $V' \subseteq V$ of y and a neighborhood $U' \subseteq U$ of x such that $U' \cap \{av \mid (a, v) \in E \cap (A \times V')\} = \varnothing$. On the other hand, there is $(g, z) \in E$ with $(gz, z) \in U' \times V'$, and $(g, z) \in A \times V'$. This is a contradiction. Hence $(x, y) \in F$. □

Definition 2.26. Let Γ be a group acting on a Hausdorff space X. If for every $g \in \Gamma$ the map $x \longmapsto gx$ is continuous, then

$$\Gamma \times X \longrightarrow X$$

is a topological transformation group with respect to the discrete topology on Γ. By Theorem 2.25 this action is proper if and only if for all $x, y \in X$ there exist neighborhoods U and V of x and y, respectively, such that the set $\{g \in \Gamma \mid U \cap g(V) \neq \varnothing\}$ is finite. Such an action of a discrete group is called a *properly discontinuous action.*

Elementary Properties of Compact Transformation Groups

Definition 2.27. A topological transformation group $K \times X \longrightarrow X$ is called a *compact transformation group* if the group K is compact.

Proposition 2.28. *Let $K \times X \longrightarrow X$ be a compact transformation group. Then the action is proper. The map $q : X \longrightarrow K\backslash X$ is also proper. In particular, q is open and closed. Moreover, X is locally compact if and only if $K\backslash X$ is locally compact.*

Proof. The action is proper by Theorem 2.25. We claim that q is closed. Let $A \subseteq X$ be a closed subset. We have to show that $q^{-1}(q(A)) = K(A)$ is closed. Suppose that $x \in X - K(A)$. Then $K(x)$ is disjoint from A and by Wallace' Lemma 1.21 there is an open neighborhood U of x such that $K(U) \cap A = \varnothing$. whence $U \cap K(A) = \varnothing$. Thus q is proper by Proposition 2.22. If $K\backslash X$ is locally compact, then X is locally compact by Lemma 2.21 and Proposition 2.22. If $V \subseteq X$ is an open set with compact closure, then $q(V)$ is also open with compact closure and therefore $K\backslash X$ is locally compact if X is locally compact. ☐

Corollary 2.29. *Let G be a Hausdorff topological group and let K be a compact subgroup. Then the map*

$$p : G \longrightarrow G/K$$

is closed, open and proper. Moreover G/K is locally compact if and only if G is locally compact.

Proof. The groups K acts from the left on $X = G$ via $(k, x) \longmapsto xk^{-1}$. The orbits of this action are the left cosets gK, hence G/K is the orbit space for this action. The claim follows now from Proposition 2.28. ☐

Lemma 2.30. *Suppose that $K \times X \longrightarrow X$ is a compact transformation group and that $x \in X$ is a fixed point of K. Then every neighborhood V of x contains a K-invariant open neighborhood U of x.*

Proof. Suppose that x is a fixed point of the K-action and that V is a neighborhood of x. By Wallace's Lemma 1.21 there exists an open neighborhood W of x such that $K(W) \subseteq V$. Then $U = K(W)$ is an open and invariant neighborhood of x. ☐

Corollary 2.31. *Suppose that G is a Hausdorff topological group and that $K \subseteq G$ is a compact subgroup. Then there are arbitrarily small open identity neighborhoods $V \subseteq G$ which are invariant under conjugation by elements of K.*

Proof. The compact group K acts as a compact transformation group on G via conjugation, and e is a fixed point for this action. The claim follows thus from Lemma 2.30. ☐

We recall that a Hausdorff space X is called a *Tychonoff space*, or *completely regular*, or a $\mathrm{T}_{3\frac{1}{2}}$*-space* if for every $x \in X$ and every neighborhood V of x there exists a continuous map

$$\varphi : X \longrightarrow [0,1]$$

with $\varphi(x) = 0$ and $\varphi(y) = 1$ for all $y \in X - V$.

Proposition 2.32. *Suppose that $K \times X \longrightarrow X$ is a compact transformation group. If X is a Tychonoff space, then $K\backslash X$ is a Tychonoff space as well.*

We first make a small observation.

Lemma 2.33. *Suppose that X is a set, that $u, v : X \longrightarrow \mathbb{R}$ are two functions which are bounded from below, and that $\varepsilon > 0$. If $|u(x) - v(x)| \leq \varepsilon$ holds for all $x \in X$, then $|\inf u(X) - \inf v(X)| \leq \varepsilon$.*

Proof. For every $n \geq 1$ there exists a point $x_n \in X$ such that $u(x_n) - \inf u(X) \leq 2^{-n}$. Then $v(x_n) \leq \inf u(X) + \varepsilon + 2^{-n}$ and thus $\inf v(X) \leq \inf u(X) + \varepsilon$. ☐

Proof of Proposition 2.32. Suppose that X is a Tychonoff space, that $x \in X$, and that $V \subseteq K\backslash X$ is a neighborhood of $q(x)$. There exists a neighborhood U of x with $q(U) \subseteq V$. Since X is a Tychonoff space, there exists a continuous map $\psi : X \longrightarrow [0,1]$ with $\psi(x) = 0$ and $\psi(y) = 1$ for all $y \in X - U$. We put $\tilde{\psi}(z) = \min \psi(K(z))$. Then $\tilde{\psi}$ is constant on the K-orbits and hence $\tilde{\psi}$ descends to a map $\varphi : K\backslash X \longrightarrow [0,1]$ via $\varphi(K(z)) = \tilde{\psi}(z) = \min \psi(K(z))$. We have $\varphi(q(x)) = 0$ and $\varphi(q(z)) = 1$ if $q(z) \notin V$. It remains to show that φ is continuous. For this, it suffices to show that $\tilde{\psi}$ is continuous because $K\backslash X$ carries the quotient topology with respect to the map q,

$$\begin{array}{ccc} X & \xrightarrow{\ \tilde{\psi}\ } & [0,1] \\ {\scriptstyle q}\big\downarrow & \nearrow_{\varphi} & \\ K\backslash X. & & \end{array}$$

Given $z \in X$, we may consider the continuous map $h : K \times X \longrightarrow [0,1]$ with $h(g,y) = |\psi(gz) - \psi(gy)|$. Since $h(g,z) = 0$ holds for all $g \in K$, there exists by Wallace's Lemma 1.21 for every $\varepsilon > 0$ a neighborhood W_ε of z such that $h(g,y) \leq \varepsilon$ holds for all $(g,y) \in K \times W_\varepsilon$. It follows from Lemma 2.33 that $|\inf \psi(K(z)) - \inf \psi(K(y))| \leq \varepsilon$ holds for all $y \in W_\varepsilon$, and this shows that $\tilde{\psi}$ is continuous at z. ☐