Locally compact groups

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Conventions

Our conventions in group theory and topology are mostly standard. The neutral element of a group G is denoted by e or, if the group is abelian and written additively, by 0. For subsets $X, Y \subseteq G$ we put

$$XY = \{xy \mid x \in X \text{ and } y \in Y\}, \quad X^{-1} = \{x^{-1} \mid x \in X\}$$

and for an integer $k \geq 1$ we put

$$X^{\cdot k} = \{x_1 x_2 \cdots x_k \mid x_1, \dots, x_k \in X\}.$$

We call X *symmetric* if $X = X^{-1}$. The *centralizer* of X is denoted by

$$\mathrm{Cen}_G(X) = \{g \in G \mid gx = xg \text{ holds for all } x \in X\}.$$

The *center* of a group is written as $\mathrm{Cen}(G) = \mathrm{Cen}_G(G)$. Our convention for *commutators* is that

$$[a, b] = aba^{-1}b^{-1}.$$

Most of the time, we will consider *left actions.* Such a left action of a group G on a set X will be written as

$$G \times X \longrightarrow X, \quad (g, x) \longmapsto gx.$$

The *stabilizer* of a point $x \in X$ will be denoted by

$$G_x = \{g \in G \mid gx = x\}.$$

For a subset $A \subseteq X$ we put $GA = \{ga \mid g \in G \text{ and } a \in A\}$.

A subset V of a topological space X is called a *neighborhood* of a point $x \in X$ if there is an open set U with $x \in U \subseteq V$. It is our convention is that all compact or locally compact spaces are assumed to be Hausdorff.

We consider 0 to be a natural number. The set of natural numbers is thus

$$\mathbb{N} = \{0, 1, 2, 3, \dots\},$$

and we denote the set of all positive natural numbers by

$$\mathbb{N}_1 = \{1, 2, 3, \ldots\}.$$

The difference of two sets X, Y is written as

$$X - Y = \{x \in X \mid x \notin Y\}$$

and the *symmetric difference* is written as

$$X \triangle Y = (X - Y) \cup (Y - X) = (X \cup Y) - (X \cap Y).$$

The *power set* of a set X is denoted as

$$\mathcal{P}(X) = \{Y \mid Y \subseteq X\}.$$

We use the *axiom of choice* without further ado.

1 Topological Groups

This chapter contains basic results about the point-set topology of topological groups. Whenever we use the Hausdorff condition, this will be mentioned explicitly. However, we do assume that locally compact spaces and compact spaces are Hausdorff.

Definition 1.1. A *topological group* $(G, \cdot, \mathcal{T})$ consists of a group $(G, \cdot)$ and a topology $\mathcal{T}$ on G for which the multiplication map

$$\begin{aligned} G \times G &\longrightarrow G \\ (g, h) &\longmapsto gh \end{aligned}$$

and the inversion map

$$\begin{aligned} G &\longrightarrow G \\ g &\longmapsto g^{-1} \end{aligned}$$

are continuous. We can combine this into one condition by considering the map

$$\begin{aligned} \kappa : G \times G &\longrightarrow G \\ (g, h) &\longmapsto g^{-1}h. \end{aligned}$$

If G is a topological group, then κ is continuous. Conversely, if κ is continuous, then the maps $g \longmapsto g^{-1} = \kappa(g, e)$ and $(g, h) \longmapsto gh = \kappa(\kappa(g, e), h)$ are also continuous.

Suppose that G is a topological group. For every $a \in G$, the *right translation map*

$$\rho_a(g) = ga,$$

the *left translation map*

$$\lambda_a(g) = ag$$

and the *conjugation map*

$$\gamma_a(g) = aga^{-1}$$

are homeomorphisms of G onto itself, with inverses $\lambda_{a^{-1}}$, $\rho_{a^{-1}}$ and $\gamma_{a^{-1}}$, respectively. In particular, the homeomorphism group of G acts transitively on G. It follows that every neighborhood W of a group element $g \in G$ can be written as $W = gU = Vg$, where $U = \lambda_{g^{-1}}(W)$ and $V = \rho_{g^{-1}}(W)$ are neighborhoods of the identity. In what follows, we will mostly write G for a topological group, without mentioning the topology $\mathcal{T}$ explicitly. A neighborhood of the identity element e will be called an *identity neighborhood* for short.

Definition 1.2. We define a *morphism* $f : G \longrightarrow K$ between topological groups G, K to be a continuous group homomorphism.

Example 1.3. The following are examples of topological groups and morphisms.

(a) The additive and the multiplicative groups of the fields $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, and the p-adic fields $\mathbb{Q}_p$, endowed with their usual field topologies, are examples of topological groups. The exponential maps $\exp : \mathbb{R} \longrightarrow \mathbb{R}^*$ and $\exp : \mathbb{C} \longrightarrow \mathbb{C}^*$ are morphisms.

(b) The *circle group* $\mathrm{U}(1) = \{z \in \mathbb{C} \mid |z| = 1\} \subseteq \mathbb{C}^*$ is another example of a topological group. The map $\mathbb{R} \longrightarrow \mathrm{U}(1)$ that maps t to $\exp(2\pi \boldsymbol{i} t) = \cos(2\pi t) + \boldsymbol{i}\sin(2\pi t)$ is a morphism.

(c) Every morphism $f : \mathbb{R} \longrightarrow \mathbb{R}$ is of the form $f(t) = rt$, for a unique real r. This follows from the fact that $\mathbb{Q}$ is dense in $\mathbb{R}$, and that an additive homomorphism $f : \mathbb{Q} \longrightarrow \mathbb{R}$ is determined uniquely determined by the element $r = f(1)$, since $\mathbb{Q}$ is uniquely divisible.

(d) Every morphism $f : \mathrm{U}(1) \longrightarrow \mathrm{U}(1)$ is of the form $f(z) = z^m$, for a unique integer $m \in \mathbb{Z}$.

(e) As a vector space over $\mathbb{Q}$, the group $(\mathbb{R}, +)$ has dimension $2^{\aleph_0}$. Hence the abelian group $\mathbb{R}$ has $2^{2^{\aleph_0}}$ additive endomorphisms, almost all of which are not continuous.

(f) Let H denote the additive group of the reals, endowed with the discrete topology. Then H is a locally compact group and the identity map $\mathrm{id} : H \longrightarrow \mathbb{R}$ is a continuous bijective morphism whose inverse is not continuous.

(g) Let F be a field and let $\mathrm{GL}_n F$ denote the group of invertible $n \times n$-matrices over F. For an $n \times n$-matrix g, let $g^\#$ denote the matrix with entries $(g^\#)_{i,j} = (-1)^{i+j}\det(g'(j,i))$, where $g'(j,i)$ is the $(n-1)\times(n-1)$-matrix obtained by removing column i and row j from the matrix g. Then $gg^\# = g^\# g = \det(g)\mathbb{1}$. Hence if F is a topological field, then $\kappa(g,h) = g^{-1}h = \frac{1}{\det(g)} g^\# h$ depends continuously on g and h, and therefore $\mathrm{GL}_n F$ is a topological group. In particular, the matrix groups $\mathrm{GL}_n \mathbb{Q}$, $\mathrm{GL}_n \mathbb{R}$, $\mathrm{GL}_n \mathbb{C}$, and $\mathrm{GL}_n \mathbb{Q}_p$ are topological groups.

(h) Every group G, endowed either with the discrete or with the trivial nondiscrete topology, is a topological group.

Proposition 1.4. *Suppose that $(G_i)_{i \in I}$ is a family of topological groups. Then the product $G = \prod_{i \in I} G_i$, endowed with the product topology, is again a topological group. For each j,* *the projection map* $\mathrm{pr}_j : G \longrightarrow G_j$ *is an open morphism. If* H *is a topological group and if there are morphisms* $f_j : H \longrightarrow G_j$*, for every* $j \in J$*, then there is a unique morphism* $f : H \longrightarrow G$ *such that* $\mathrm{pr}_j \circ f = f_j$ *holds for all* $j \in J$*.*

Proof. We have to show that the map $\kappa : G \times G \longrightarrow G$ that maps (g, h) to $g^{-1}h$ is continuous. Let $\kappa_j : G_j \times G_j \longrightarrow G_j$ denote the corresponding maps, which are by assumption continuous. Then we have for each j a continuous map $\mathrm{pr}_j \circ \kappa = \kappa_j \circ (\mathrm{pr}_j \times \mathrm{pr}_j)$,

$$\begin{array}{ccc} G \times G & \xrightarrow{\kappa} & G \\ {\scriptstyle \mathrm{pr}_j \times \mathrm{pr}_j}\downarrow & & \downarrow{\scriptstyle \mathrm{pr}_j} \\ G_j \times G_j & \xrightarrow{\kappa_j} & G_j. \end{array}$$

By the universal property of the product topology on G, this implies that κ is continuous.

The remaining claims follow, since these maps on the one hand are continuous (and open) as claimed by the properties of the product topology, and on the other hand are group homomorphisms. □

The following local criterion for morphisms is often useful.

Lemma 1.5. *Let* G, K *be topological groups and let* $f : G \longrightarrow K$ *be a (not necessarily continuous) group homomorphism. Then the following are equivalent.*

(i) *The homomorphism* f *is continuous and hence a morphism of topological groups.*

(ii) *The homomorphism* f *is continuous at one point* $a \in G$*, i.e. for every neighborhood* W *of* $f(a)$*, there exists a neighborhood* V *of* a *such that* $f(V) \subseteq W$*.*

Proof. It is clear that (i) $\Rightarrow$ (ii), because a continuous map is continuous at every point. Suppose that (ii) holds and that $U \subseteq K$ is open. If $g \in f^{-1}(U)$, then $f(a) = f(ag^{-1}g) \in f(ag^{-1})U$. Hence there exists a neighborhood V of a with $f(V) \subseteq f(ag^{-1})U$. Then $ga^{-1}V$ is a neighborhood of g, with $f(ga^{-1}V) \subseteq U$. Hence $f^{-1}(U)$ is open. □

Below we will look into separation properties for topological groups more closely. At this point we just record the following.

Lemma 1.6. *A topological group* G *is Hausdorff if and only if some singleton* $\{a\} \subseteq G$ *is closed.*

Proof. Suppose that $\{a\} \subseteq G$ is closed. The preimage of $\{a\}$ under the continuous map $(g, h) \longmapsto g^{-1}ha$ is the diagonal $\{(g, g) \mid g \in G\} \subseteq G \times G$, which is therefore closed in $G \times G$. Thus G is Hausdorff. Conversely, every singleton set in a Hausdorff space is closed. □

Subgroups

Now we study subgroups of topological groups.

Proposition 1.7. *Let H be a subgroup of a topological group G. Then H is a topological group with respect to the subspace topology. Moreover, the closure $\overline{H}$ is also a subgroup of G. If H is normal in G, then $\overline{H}$ is also normal.*

Proof. It is clear from the definition that a subgroup of a topological group is again a topological group. Let $H \subseteq G$ be a subgroup. The continuity of the map κ from Definition 1.1 ensures that

$$\kappa(\overline{H} \times \overline{H}) = \kappa(\overline{H \times H}) \subseteq \overline{\kappa(H \times H)} = \overline{H}.$$

Thus $\overline{H}$ is a subgroup. Suppose in addition that $H \trianglelefteq G$ is normal. For $a, g \in G$ we put $\gamma_a(g) = aga^{-1}$. Since the conjugation map $\gamma_a : G \longrightarrow G$ is continuous, we have

$$\gamma_a(\overline{H}) \subseteq \overline{\gamma_a(H)} = \overline{H}.$$

This shows that $\overline{H}$ is normal in G. □

Lemma 1.8. *Let G be a topological group and let $A \subseteq G$ be a closed subset. Then the normalizer of A,*

$$\mathrm{Nor}_G(A) = \{g \in G \mid \gamma_g(A) = A\},$$

is a closed subgroup.

Proof. For $a \in A$ let $c_a(g) = gag^{-1}$. Then $c_a : G \longrightarrow G$ is continuous and hence $c_a^{-1}(A) = \{g \in G \mid gag^{-1} \in A\}$ is closed. Therefore

$$S = \bigcap \{c_a^{-1}(A) \mid a \in A\} = \{g \in G \mid \gamma_g(A) \subseteq A\}$$

is a closed semigroup in G, and $\mathrm{Nor}_G(A) = S \cap S^{-1}$ is closed as well. □

Lemma 1.9. *Let G be a Hausdorff topological group, and let $X \subseteq G$ be any subset. Then the centralizer*

$$\mathrm{Cen}_G(X) = \{g \in G \mid [g, X] = e\}$$

is closed. In particular, the center of G is closed.

Proof. Given $x \in X$, the map $g \longrightarrow [g, x] = gxg^{-1}x^{-1}$ is continuous. Therefore $\mathrm{Cen}_G(x) = \{g \in G \mid [g, x] = e\}$ is closed, provided that $\{e\} \subseteq G$ is closed. Then $\mathrm{Cen}_G(X) = \bigcap \{\mathrm{Cen}_G(x) \mid x \in X\}$ is closed as well. □

Lemma 1.10. *Let G be a Hausdorff topological group. If $A \subseteq G$ is an abelian subgroup, then $\overline{A}$ is an abelian subgroup.*

Proof. The commutator map $(g, h) \longrightarrow [g, h]$ is constant on $A \times A$ and hence also constant on the closure $\overline{A \times A} = \overline{A} \times \overline{A}$. □

Lemma 1.11. *Let G be a topological group and suppose that $U \subseteq G$ is an open subset. If $X \subseteq G$ is any subset, then UX and XU are open subsets. In particular, the multiplication map $m : G \times G \longrightarrow G$, $(g, h) \longmapsto gh$ and the map $\kappa : G \times G \longrightarrow G$, $(g, h) \longmapsto g^{-1}h$ are open.*

Proof. For each $x \in X$, the sets $Ux = \rho_x(U)$ and $xU = \lambda_x(U)$ are open. Hence $UX = \bigcup\{Ux \mid x \in X\}$ and $XU = \bigcup\{xU \mid x \in X\}$ are open as well. □

Proposition 1.12. *Let G be a topological group and let $H \subseteq G$ be a subgroup.*
(i) *The subgroup H is open if and only if it contains a nonempty open set.*
(ii) *If H is open, then H is also closed.*
(iii) *The subgroup H is closed if and only if there exists an open set $U \subseteq G$ such that $U \cap H$ is nonempty and closed in U.*

Proof. For (i), suppose that H contains the nonempty open set U. Then $H = UH$ is open by Lemma 1.11. Conversely, if H is open then it contains the nonempty open set H. For (ii), suppose that $H \subseteq G$ is open. Then $G - H = \bigcup\{aH \mid a \in G - H\}$ is also open and therefore H is closed. For (iii), suppose that $U \cap H$ is nonempty and closed in the open set U. Then $U \cap H$ is also closed in the smaller set $U \cap \overline{H} \subseteq U$. Upon replacing G by $\overline{H}$, we may thus assume in addition that H is dense in the ambient group G, and we have to show that $H = G$. The set $U - H = U - (U \cap H)$ is open in U and hence open in G. On the other hand, H is dense in G. Therefore $U - H = \varnothing$ and thus $U \subseteq H$. By (i) and (ii), H is closed in G, whence $H = G$. Conversely, if H is closed, then H is closed in the open set G. □

Corollary 1.13. *Let G be a topological group and let $V \subseteq G$ be a neighborhood of some element $g \in G$. Then V generates an open subgroup of G.*

Corollary 1.14. *Suppose that G is a Hausdorff topological group and that $H \subseteq G$ is a subgroup. If H is locally compact in the subspace topology, then H is closed. In particular, every discrete subgroup of G is closed.*

Proof. Let $C \subseteq H$ be a compact set which is an identity neighborhood in the topological group H. Then there exists an open identity neighborhood U in G such that $U \cap H \subseteq C$. Since C is compact, C is closed in G and hence $U \cap H = U \cap C$ is closed in U and nonempty. Now we may apply Proposition 1.12(iii). □

The product AB of closed subsets A, B in a topological group G need not be closed. An example is the additive group of the reals $(\mathbb{R}, +)$, with the closed subgroups $A = \mathbb{Z}$ and $B = \sqrt{2}\mathbb{Z}$. Then $A + B$ is a countable dense subgroup of $\mathbb{R}$ which is not closed. However, we have the following.

Lemma 1.15. *Let G be a Hausdorff topological group, and let $A, B \subseteq G$ be closed subsets. If either A or B is compact, then $AB \subseteq G$ is closed.*

The proof uses Wallace's Lemma.

Lemma 1.16 (Wallace). *Let $X_1, \dots X_k$ be Hausdorff spaces containing compact sets $A_j \subseteq X_j$, for $j = 1, \dots, k$. If $W \subseteq X_1 \times \cdots \times X_k$ is an open set containing $A_1 \times \cdots \times A_k$, then there exist open sets U_j with $A_j \subseteq U_j \subseteq X_j$, for $j = 1, \dots, k$, such that*

$$A_1 \times \cdots \times A_k \subseteq U_1 \times \cdots \times U_k \subseteq W.$$

Proof. There is nothing to show for $k = 1$. Suppose that $k = 2$. We put $A = A_1$ and $B = A_2$ and we fix $a \in A$. For every point $b \in B$, we choose an open neighborhood $U_b \times V_b$ of (a, b) such that $U_b \times V_b \subseteq W$. Since $\{a\} \times B$ is compact, finitely many such neighborhoods $U_{b_1} \times V_{b_1}, \dots, U_{b_m} \times V_{b_m}$ cover $\{a\} \times B$. We put $U_a = U_{b_1} \cap \cdots \cap U_{b_m}$ and $V_a = V_{b_1} \cup \cdots \cup V_{b_m}$. Then $\{a\} \times B \subseteq U_a \times V_a \subseteq W$. Now we let $a \in A$ vary. Since A is compact, finitely many such strips $U_{a_1} \times V_{a_1}, \dots, U_{a_n} \times V_{a_n}$ cover $A \times B$. We put $U = U_{a_1} \cup \cdots \cup U_{a_n}$ and $V = V_{a_1} \cap \cdots \cap V_{a_n}$. Then $A \times B \subseteq U \times V \subseteq W$ and the claim is proved for $k = 2$.

For $k \geq 3$ we apply the previous argument to $A = A_1$ and $B = A_2 \times \cdots \times A_k$, and we obtain open sets $U \subseteq X_1$ and $V \subseteq X_2 \times \cdots \times X_k$ with $A \times B \subseteq U \times V \subseteq W$. By induction, we find now open sets $U_2, \dots U_k$ such that $A_2 \times \cdots \times A_k \subseteq U_2 \times \cdots U_k \subseteq V$. Therefore $A_1 \times \cdots \times A_k \subseteq U \times U_2 \times \cdots U_k \subseteq W$. ☐

Proof of Lemma 1.15. Suppose that A is compact and B is closed, and that $g \in G - AB$. We have to show that $G - AB$ contains a neighborhood of g. By assumption, $A^{-1}g \cap B = \varnothing$. If we put $\kappa(g, h) = g^{-1}h$, then $\kappa(A \times \{g\}) \subseteq G - B$. By Wallace's Lemma 1.16 there exists an open neighborhood V of g such that $\kappa(A \times V) \subseteq G - B$, i.e. $A^{-1}V \cap B = \varnothing$. Hence $V \cap AB = \varnothing$ and the claim follows. The case where B is compact and A is closed follows by taking inverses. ☐

Quotients

Suppose that H is a subgroup of a topological group G. We endow the set G/H of left cosets with the quotient topology with respect to the natural map

$$p : G \longrightarrow G/H, \qquad g \longmapsto gH.$$

Thus a subset of G/H is open if and only if its preimage is open. The next result is elementary, but important.

Proposition 1.17. *Let G be a topological group and let H be a subgroup. Then the quotient map*

$$p : G \longrightarrow G/H$$

is open. The quotient G/H is Hausdorff if and only if H is closed in G, and it is discrete if and only if H is open in G.

Proof. Suppose that $U \subseteq G$ is an open set. Then $p^{-1}(p(U)) = UH$ is open by Lemma 1.11, hence $p(U)$ is open by the definition of the quotient topology.

If G/H is Hausdorff, then $\{H\} \subseteq G/H$ is closed, hence $H = p^{-1}(\{H\}) \subseteq G$ is closed as well. Conversely, suppose that $H \subseteq G$ is closed. The map $p \times p : G \times G \longrightarrow G/H \times G/H$ is open, because p is open and because a cartesian product of two open maps is again open. The open set $W = \{(x, y) \in G \times G \mid x^{-1}y \in G - H\}$ maps under $p \times p$ onto the complement of the diagonal in $G/H \times G/H$. Hence the diagonal $\{(gH, gH) \mid g \in G\}$ is closed in $G/H \times G/H$, and therefore G/H is Hausdorff.

If H is open, then every coset gH is open and hence G/H is discrete. Conversely, if G/H is discrete, then H, being the preimage of the open singleton $\{H\} \subseteq G/H$, is open. ☐

If G is Hausdorff and if $K \subseteq G$ is compact then more can be said, see Corollary 1.28 below.

Corollary 1.18. *Suppose that G is a Hausdorff topological group which is compact and that $H \subseteq G$ is a closed subgroup. Then H is open if and only if H has finite index in G.*

Proof. A discrete topological space is compact if and only if it is finite. ☐

Corresponding remarks apply to the set $H\backslash G$ of right cosets, by taking inverses.

Proposition 1.19. *Let G be a topological group. If $N \trianglelefteq G$ is a normal subgroup, then the factor group G/N is a topological group with respect to the quotient topology on G/N. The quotient map $p : G \longrightarrow G/N$ is an open morphism. The factor group G/N is Hausdorff if and only if N is closed. In particular, $G/\overline{N}$ is a Hausdorff topological group.*

Proof. We put $\bar{\kappa}(gN, hN) = g^{-1}hN$ and $p(g) = gN$. Then the diagram

$$\begin{array}{ccc} G \times G & \xrightarrow{\kappa} & G \\ \downarrow{\scriptstyle p\times p} & & \downarrow{\scriptstyle p} \\ G/N \times G/N & \xrightarrow{\bar{\kappa}} & G/N \end{array}$$

commutes, and $p \circ \kappa$ is continuous. Since p is open, $p \times p$ is also open and hence a quotient map. It follows from the universal property of quotient maps that $\bar{\kappa}$ is continuous, and therefore G/N is a topological group. The remaining claims follow from Propositions 1.7 and 1.17. ☐

The next result is the Homomorphism Theorem for topological groups.

Lemma 1.20. *Let $f : G \longrightarrow K$ be a morphism of topological groups, and put $N = \ker(f)$. Then f factors through the open morphism $p : G \longrightarrow G/N$ via a unique morphism $\bar{f}$,*

$$\begin{array}{ccc} G & \xrightarrow{f} & K. \\ {\scriptstyle p}\downarrow & \nearrow{\scriptstyle \bar{f}} & \\ G/N & & \end{array}$$

If f is open, then $\bar{f}$ is also open.

Proof. The group homomorphism $\bar{f}$ exists uniquely by the Homomorphism Theorem for groups. Since p is a quotient map, $\bar{f}$ is continuous and thus a morphism of topological groups. If f is open and if $W \subseteq G/N$ is an open set, then $f(p^{-1}(W)) = \bar{f}(W)$ is open as well. ☐

Corollary 1.21. *Suppose that G, K are topological groups and that K is Hausdorff. If $f : G \longrightarrow K$ is a morphism of topological groups, then f factors through the open morphism $p : G \longrightarrow G/\overline{\{e\}}$,*

$$\begin{array}{ccc} G & \xrightarrow{f} & K. \\ {\scriptstyle p}\downarrow & \nearrow{\scriptstyle \bar{f}} & \\ G/\overline{\{e\}} & & \end{array}$$

Connected Components

Definition 1.22. Let x be a point in a topological space X. The *connected component* of x is the union of all connected subsets of X containing x. This union is closed and connected. We call a topological space X *totally disconnected* if the only connected nonempty subsets of X are the singletons.

The connected component the identity element of a topological group G will be denoted by G°, and we call G° the *identity component* of G. Since the homeomorphism group of G acts transitively on G, the group G is totally disconnected if and only if $G^\circ = \{e\}$. We note that a totally disconnected group is automatically Hausdorff.

Proposition 1.23. *Let G be a topological group. Then the identity component G° is a closed normal subgroup, and G/G° is a totally disconnected Hausdorff topological group.*

Proof. We put $\kappa(g,h) = g^{-1}h$ and we note that a continuous image of a connected set is connected. Since $G^\circ \times G^\circ$ is connected and contains the identity element, $\kappa(G^\circ \times G^\circ) \subseteq G^\circ$. This shows that G° is a subgroup. By the remark above, G° is closed. For every $a \in G$, the set $\gamma_a(G^\circ) = aG^\circ a^{-1}$ is connected and contains the identity, whence $aG^\circ a^{-1} \subseteq G^\circ$. This shows that G° is a closed normal subgroup.

It remains to show that G/G° is totally disconnected. We consider the canonical morphism $p : G \longrightarrow G/G^\circ$ and we put $H = (G/G^\circ)^\circ$ and $N = p^{-1}(H)$. Then N is a closed normal subgroup of G containing G°. We claim that $N = G^\circ$. If we have proved this claim, then $H = \{G^\circ\}$ and thus G/G° is totally disconnected. The restriction-corestriction map $p|_N^H : N \longrightarrow H$ is open, hence H carries the quotient topology with respect to $p|_N^H : N \longrightarrow H$. Suppose that $V \subseteq N$ is closed and open in N and contains the identity. Since G° is connected and contains e, we have $vG^\circ \subseteq V$ for all $v \in V$. Hence $V = p^{-1}(p(V))$, and therefore $p(V)$ is closed and open in H. But H is connected, whence $H = p(V)$ and thus $V = N$. It follows that N is connected, whence $N = G^\circ$. □

Corollary 1.24. *Let $f : G \longrightarrow K$ be a morphism of topological groups. If K is totally disconnected, then f factors through the open morphism $p : G \longrightarrow G/G^\circ$,*

$$\begin{array}{ccc} G & \xrightarrow{\ f\ } & K. \\ {\scriptstyle p}\downarrow & \nearrow_{\bar f} & \\ G/G^\circ & & \end{array}$$

Proof. Since $f(G^\circ) \subseteq K$ is connected, G° is contained in the kernel of f. □

Orbit Spaces of Compact Transformation Groups

We recall that compact spaces are by definition Hausdorff.

Definition 1.25. A *compact transformation group* consists of a continuous action

$$K \times X \longrightarrow X$$

of a compact group K on a Hausdorff space X. We denote the K-orbit of $x \in X$ by

$$K(x) = \{gx \mid g \in K\} \subseteq X.$$

We endow the *orbit space*

$$K \backslash X = \{K(x) \mid x \in X\}$$

with the quotient topology with respect to the map

$$q : X \longrightarrow K\backslash X$$

that maps x to its orbit $K(x)$.

Example 1.26. Let G be a Hausdorff topological group and suppose that $K \subseteq G$ is a compact subgroup.

(i) The group K acts by left multiplication on G, and the orbit space is the set of right cosets $K\backslash G = \{Kg \mid g \in g\}$.

(ii) If $H \subseteq G$ is a closed subgroup, then K acts by left multiplication on G/H. The orbit space $K\backslash(G/H)$ can be identified with the set of double cosets

$$K\backslash G/H = \{KgH \mid g \in G\}$$

via the map $K(gH) = \{kgH \mid k \in K\} \longmapsto KgH$. The topology on $K\backslash G/H$ can be either viewed as the quotient topology with respect to the canonical map $q : G/H \longrightarrow K\backslash G/H$ or as the quotient topology with respect to the canonical map $r : G \longrightarrow K\backslash G/H$. This makes no difference, because a composition of quotient maps is again a quotient map.

(iii) The group K also acts on G from the left by conjugation. The orbit space for this action is the space of K-conjugacy classes.

We recall that a continuous map between Hausdorff spaces is called *proper* if the preimage of every compact set is compact.

Proposition 1.27. *Suppose that $K \times X \longrightarrow X$ is a compact transformation group. Then we have the following.*

(i) *The orbit space $K\backslash X$ is Hausdorff.*

(ii) *The map $q : X \longrightarrow K\backslash X$ is open, closed and proper.*

In particular, $K\backslash X$ is compact (locally compact) if and only if X is compact (locally compact).

Proof. We first show that q is open and closed. If $U \subseteq X$ is open, then $q^{-1}(q(U)) = KU = \bigcup\{gU \mid g \in K\}$ is open, hence q is an open map. Suppose that $A \subseteq X$ is closed. We have to show that $q^{-1}(q(A)) = KA \subseteq X$ is closed. For $z \in X - KA$ we have $K(z) \cap KA = \varnothing$. By Wallace's Lemma 1.16 there exits an open neighborhood U of z such that $KU \cap KA = \varnothing$. Thus $X - KA$ is open, and hence KA is closed.

Next we show that $K\backslash X$ is Hausdorff. Suppose that $x, y \in X$ are points with $q(x) \neq q(y)$. Since the orbits $K(x), K(y) \subseteq X$ are compact and disjoint, there exist disjoint open sets $U, V \subseteq X$ with $K(x) \subseteq U$ and $K(y) \subseteq V$. In particular, $K(y) \cap \overline{U} = \varnothing$. Now we use that q is open and closed. The point $q(x)$ is in the open set $q(U)$, and the point $q(y)$ is not in the closed set $q(\overline{U}) = \overline{q(U)}$. Therefore $K\backslash X$ is Hausdorff.

Now we show that q is proper. Suppose that $B \subseteq K\backslash X$ is compact. Then $A = q^{-1}(B)$ is a closed set which is a union of K-orbits. Let $\boldsymbol{U}$ be an open covering of A. For every $a \in A$ we find a finite subset $\boldsymbol{U}_a \subseteq \boldsymbol{U}$ such that $K(a) \subseteq \bigcup \boldsymbol{U}_a$, because $K(a) \subseteq A$ is compact. By Wallace's Lemma 1.16 we find an open neighborhood V_a of a such that $KV_a \subseteq \bigcup \boldsymbol{U}_a$. Since $B \subseteq \bigcup\{q(V_a) \mid a \in A\}$, there exists finitely many points $a_1, \ldots, a_n \in A$ such that $B \subseteq q(V_{a_1}) \cup \cdots \cup q(V_{a_n})$. It follows that

$$A \subseteq KV_{a_1} \cup \cdots \cup KV_{a_n} \subseteq \bigcup \boldsymbol{U}_{a_1} \cup \cdots \cup \bigcup \boldsymbol{U}_{a_n}.$$

Hence A is compact.

Since q is proper, X is compact if and only if $K\backslash X$ is compact. If X is locally compact and if $U \subseteq X$ is an open neighborhood of $x \in X$ with compact closure, then $q(\overline{U})$ is a compact neighborhood of $q(x)$ because q is open. Hence $K\backslash X$ is locally compact. If $K\backslash X$ is locally compact and if $V \subseteq K\backslash X$ is an open neighborhood of $q(x)$ with compact closure, then $q^{-1}(\overline{V})$ is a compact neighborhood of x because q is proper. ☐

Corollary 1.28. *Suppose that G is a Hausdorff topological group and that $K \subseteq G$ is a compact subgroup. Then the map $q : G \longrightarrow K\backslash G$, $g \longmapsto Kg$ is continuous, open, closed, and proper.*

The same result holds of course for the map $p : G \longrightarrow G/K$.

Corollary 1.29. *Let G be a Hausdorff topological group, with a closed subgroup H. If two of the three spaces G, H, G/H are compact, then all three spaces are compact.*

Lemma 1.30. *Suppose that $K \times X \longrightarrow X$ is a compact transformation group and that $x \in X$ is a fixed point of K. Then every neighborhood V of x contains a K-invariant neighborhood U of x.*

Proof. Suppose that x is a fixed point of the K-action and that V is a neighborhood of x. By Wallace's Lemma 1.16 there exists an open neighborhood U of x such that $KU \subseteq V$. ☐

Corollary 1.31. *Suppose that G is a Hausdorff topological group and that $K \subseteq G$ is a compact subgroup. Then there are arbitrarily small identity neighborhoods $V \subseteq G$ which are invariant under conjugation by elements of K.*

Proof. The compact group K acts as a compact transformation group on G via conjugation, and e is a fixed point for this action. ☐

Recall that a Hausdorff space X is called a *Tychonoff space*, or *completely regular*, or a $T_{3\frac{1}{2}}$*-space* if for every $x \in X$ and every neighborhood V of x there exists a continuous map

$$\varphi : X \longrightarrow [0,1]$$

with $\varphi(x) = 0$ and $\varphi(y) = 1$ for all $y \in X - V$.

Proposition 1.32. *Suppose that $K \times X \longrightarrow X$ is a compact transformation group. If X is a Tychonoff space, then $K\backslash X$ is a Tychonoff space as well.*

We first need a little observation.

Lemma 1.33. *Suppose that X is a set, that $u, v : X \longrightarrow \mathbb{R}$ are two functions which are bounded from below, and that $\varepsilon > 0$. If $|u(x) - v(x)| \leq \varepsilon$ holds for all $x \in X$, then $|\inf u(X) - \inf v(X)| \leq \varepsilon$.*

Proof. For every $n \geq 1$ there exists a point $x_n \in X$ such that $u(x_n) - \inf u(X) \leq 2^{-n}$. Then $v(x_n) \leq \inf u(X) + \varepsilon + 2^{-n}$ and thus $\inf v(X) \leq \inf u(X) + \varepsilon$. □

Proof of Proposition 1.32. Suppose that X is Tychonoff, that $x \in X$, and that $V \subseteq K\backslash X$ is a neighborhood of $q(x)$. There exists a neighborhood U of x with $q(U) \subseteq V$. Since X is a Tychonoff space, there exists a continuous map $\psi : X \longrightarrow [0,1]$ with $\psi(x) = 0$ and $\psi(y) = 1$ for all $y \in X - U$. We put $\tilde{\psi}(z) = \min \psi(K(z))$. Then $\tilde{\psi}$ is constant on the K-orbits and hence $\tilde{\psi}$ descends to a map $\varphi : K\backslash X \longrightarrow [0,1]$ via $\varphi(K(z)) = \tilde{\psi}(z) = \min \psi(K(z))$. We have $\varphi(q(x)) = 0$ and $\varphi(q(z)) = 1$ if $q(z) \notin V$. It remains to show that φ is continuous. For this, it suffices to show that $\tilde{\psi}$ is continuous because $K\backslash X$ carries the quotient topology with respect to the map q,

$$\begin{array}{ccc} X & \xrightarrow{\tilde{\psi}} & [0,1] \\ {\scriptstyle q}\downarrow & \nearrow_{\varphi} & \\ K\backslash X. & & \end{array}$$

Given $z \in X$, we may consider the continuous map $h : K \times X \longrightarrow [0,1]$ with $h(g,y) = |\psi(gz) - \psi(gy)|$. Since $h(g,z) = 0$ holds for all $g \in K$, there exists by Wallace's Lemma 1.16 for every $\varepsilon > 0$ a neighborhood W_ε of z such that $h(g,y) \leq \varepsilon$ holds for all $(g,y) \in K \times W_\varepsilon$. It follows from Lemma 1.33 that $|\inf \psi(K(z)) - \inf \psi(K(y))| \leq \varepsilon$ holds for all $y \in W_\varepsilon$, and this shows that $\tilde{\psi}$ is continuous at z. □

Metrizability of Topological Groups and the Tychonoff Property

In this section we prove several important results about the metrizability and about separation properties for topological groups and their coset spaces. We first review length functions and invariant pseudometrics on groups.

Definition 1.34. A *pseudometric* d on a set X is a map $d : X \times X \longrightarrow \mathbb{R}$ satisfying

$$d(x,x) = 0, \quad 0 \le d(x,y) = d(y,x), \quad d(x,z) \le d(x,y) + d(y,z),$$

for all $x, y, z \in X$. If $d(x,y) = 0$ implies that $x = y$, then d is called a *metric*.

A metric or pseudometric d on a group G is called *left invariant* if the left translation map $\lambda_a : G \longrightarrow G$ is an isometry of the metric or pseudometric space (G,d), for every $a \in G$. In other words, we require for a left invariant metric that $d(x,y) = d(ax,ay)$ holds for all $a, x, y \in G$. Similarly, a pseudometric d is called *right invariant* if $d(xa, ya) = d(x,y)$ holds for all $x, y, a \in G$. If d is a left invariant pseudometric, then $d'(x,y) = d(x^{-1}, y^{-1})$ is a right invariant pseudometric, and vice versa.

Every pseudometric d on a set X induces a topology $\mathcal{T}_d$ on X, such that $U \subseteq X$ is in $\mathcal{T}_d$ if and only if for every $x \in U$ there exists some $\varepsilon > 0$ such that $\{y \in X \mid d(x,y) \le \varepsilon\} \subseteq U$. This topology $\mathcal{T}_d$ is Hausdorff if and only if the pseudometric d is a metric. The following facts are elementary.

Lemma 1.35. *Suppose that d, d' are pseudometrics on a set X, and that $d \le d'$.*

(i) *The metric d is continuous with respect to $\mathcal{T}_{d'}$ and thus $\mathcal{T}_d \subseteq \mathcal{T}_{d'}$.*

(ii) *If d' is continuous with respect to $\mathcal{T}_d$, then $\mathcal{T}_d = \mathcal{T}_{d'}$.*

(iii) *If $\mathcal{T}$ is a topology on X, then $\mathcal{T} = \mathcal{T}_d$ holds if and only if d is continuous with respect to $\mathcal{T}$ and $\mathcal{T} \subseteq \mathcal{T}_d$.*

Definition 1.36. A *length function* ℓ on a group is a map

$$\ell : G \longrightarrow \mathbb{R}_{\ge 0}$$

with the properties

$$\ell(e) = 0, \quad \ell(g) = \ell(g^{-1}), \quad \ell(gh) \le \ell(g) + \ell(h),$$

for all $g, h \in G$. Then

$$|\ell(gh) - \ell(g)|, |\ell(hg) - \ell(g)| \le \ell(h)$$

holds for all $g, h \in G$. Moreover, the set

$$K = \{g \in G \mid \ell(g) = 0\}$$

is a subgroup of G. For $k \in K$ and $g \in G$ we have $\ell(kg) \leq \ell(g) = \ell(k^{-1}kg) \leq \ell(kg)$ and therefore

$$\ell(kg) = \ell(g)$$

holds for all $k \in H$ and $g \in G$. In particular, $\ell(\gamma_k(g)) = \ell(g)$ holds for all $k \in K$ and $g \in G$. If ℓ is a length function, then

$$d(g,h) = \ell(g^{-1}h)$$

is a left invariant pseudometric on G and $d'(g,h) = \ell(gh^{-1})$ is a right invariant pseudometric on G. Conversely, if d is a left or right invariant pseudometric on a group G, then $\ell(g) = d(e,g)$ is a length function.

Proposition 1.37. *Suppose that a left invariant metric d metrizes the topological group G. Then the metric*

$$d_c(x,y) = d(x,y) + d(x^{-1}, y^{-1})$$

also metrizes G. Moreover, the metric completion $\widehat{G}$ of (G, d_c) is a topological group that contains G as a dense subgroup.

Proof. It is clear that d_c is a continuous metric on G. Since $d \leq d_c$, every d-open set is also d_c-open, and thus d_c metrizes G. Let $(\widehat{G}, d_c)$ denote the metric completion of (G, d_c). We wish to extend the map $\kappa : G \times G \longrightarrow G$ that maps (x,y) to $x^{-1}y$ to a map $\widehat{G} \times \widehat{G} \longmapsto \widehat{G}$. The problem is that κ need not be uniformly continuous with respect to d_c. We first prove an auxiliary result.

Claim. Given $x, y \in \widehat{G}$ and $\varepsilon > 0$, there exists $\delta > 0$ such that $d_c(x_1^{-1}y_1, x_2^{-1}y_2) \leq \varepsilon$ holds for all $x_1, x_2, y_1, y_2 \in G$ with $d_c(x_1, x), d_c(x_2, x), d_c(y_1, y), d_c(y_2, y) \leq \delta$.

We choose $x_0, y_0 \in G$ such that $d_c(x_0, x), d_c(y_0, y) \leq \frac{\varepsilon}{10}$. Then we choose $\delta > 0$ with $\delta \leq \frac{\varepsilon}{10}$ in such a way that

$$d(zx_0, x_0), d(zy_0, y_0) \leq \frac{\varepsilon}{10}$$

holds for all $z \in G$ with $d_c(z, e) \leq 2\delta$. We have

$$d(x_1x_2^{-1}, e) = d(x_1^{-1}, x_2^{-1}) \leq d_c(x_1, x_2) \leq d_c(x_1, x) + d_c(x, x_2) \leq 2\delta$$

and therefore

$$\begin{aligned} d(x_1^{-1}y_1, x_2^{-1}y_2) &= d(x_2x_1^{-1}y_1, y_2) \\ &\leq d(x_2x_1^{-1}y_1, y_0) + d(y_0, y_2) \\ &= d(y_1, x_1x_2^{-1}y_0) + d(y_0, y_2) \\ &\leq d(y_1, y_0) + d(y_0, x_1x_2^{-1}y_0) + d(y_0, y_2) \\ &\leq d(y_1, y) + d(y, y_0) + d(y_0, x_1x_2^{-1}y_0) + d(y_0, y) + d(y, y_2) \\ &\leq \delta + \tfrac{\varepsilon}{10} + \tfrac{\varepsilon}{10} + \tfrac{\varepsilon}{10} + \delta \leq \tfrac{\varepsilon}{2}. \end{aligned}$$

In the same way we have $d(y_1^{-1}x_1, y_2^{-1}x_2) \leq \frac{\varepsilon}{2}$ and thus $d_c(x_1^{-1}y_1, x_2^{-1}y_2) \leq \varepsilon$ and the claim follows.

We view the map κ as a subset of $G\times G\times G \subseteq \widehat{G}\times\widehat{G}\times\widehat{G}$, and we let $\hat{\kappa}$ denote its closure in $\widehat{G}\times\widehat{G}\times\widehat{G}$. If $(x_n)_{n\geq 1}$ and $(y_n)_{n\geq 1}$ are sequences in G converging to points $x, y \in \widehat{G}$, then the claim shows that $(x_n^{-1}y_n)_{n\geq 1}$ is a Cauchy sequence in G. Hence $\lim_n(x_n, y_n, x_n^{-1}y_n)$ exits in $\widehat{G}\times\widehat{G}\times\widehat{G}$, and hence $(\mathrm{pr}_1 \times \mathrm{pr}_2)(\hat{\kappa}) = \widehat{G}\times\widehat{G}$.

We claim that $\hat{\kappa}$ is a continuous map. Suppose that $(x, y, z) \in \hat{\kappa}$. Given $\varepsilon > 0$ there exists by the claim $\delta > 0$ such that for all $x', y' \in G$ with $d_c(x', x), d_c(y', y) \leq \delta$ we have $d_c(\kappa(x', y'), z) \leq \varepsilon$. Since $\hat{\kappa}$ is the closure of κ, it follows that $d_c(z', z) \leq \varepsilon$ holds for all $(x', y', z') \in \hat{\kappa}$ with $d_c(x', x), d_c(y', y) \leq \delta$. Hence $\hat{\kappa}$ is a continuous map at (x, y).

Since $\hat{\kappa}$ is continuous, so are the maps $i(x) = \hat{\kappa}(x, e)$ and $m(x, y) = \hat{\kappa}(i(x), y)$. Now $m(m(x, y), z) = m(x, m(y, z))$ holds on the dense subset $G \times G \times G$, and thus everywhere. Hence m is an associative multiplication. Similarly, $m(x, e) = m(e, x) = x$ holds on the dense subset G and hence everywhere, and likewise $m(i(x), x) = e = m(x, i(x)$ holds on G and hence everywhere. Therefore $\widehat{G}$ is a topological group which contains G as a dense subgroup. □

We need also the following result about metrics on quotients.

Lemma 1.38. *Let ℓ be a length function on a group G and let $H \subseteq G$ be a subgroup. Then $d(x, y) = \ell(xy^{-1})$ is a right invariant pseudometric on G and*

$$d_H(xH, yH) = \inf \ell(xHy^{-1}) = \inf d(x, yH) = \inf\{d(xh_1, yh_2) \mid h_1, h_2 \in H\}$$

is a pseudometric on G/H. If G is a topological group and if ℓ is continuous, then d_H is continuous on G/H.

Proof. It is clear that d is a right invariant pseudometric and that d_H is symmetric and nonnegative. For $x, y, z \in G$ and $h, h' \in H$ we have

$$d_H(xH, zH) \leq d(x, zh') \leq d(x, yh) + d(yh, zh') = d(x, yh) + d(y, zh'h^{-1}),$$

whence

$$d_H(xH, zH) \leq d_H(xH, yH) + d_H(yH, zH).$$

Thus d_H is a pseudometric on G/H.

Suppose that G is a topological group and that ℓ is continuous. Then d is continuous and the map $d'(x, y) = d_H(xH, yH)$ is a pseudometric on G. Since $0 \leq d' \leq d$, the pseudometric d' is also continuous. The map $p : G \longrightarrow G/H$ is open and therefore the map $p \times p : G \times G \longrightarrow G/H \times G/H$ is also open and hence a quotient map. Therefore d_H is continuous on $G/H \times G/H$. □

Theorem 1.39 (Birkhoff–Kakutani). *Let G be a Hausdorff topological group. Then the following are equivalent.*

(i) *The topology on G is metrizable by a left invariant metric.*
(ii) *The topology on G is metrizable by a right invariant metric.*
(iii) *The topology on G is metrizable.*
(iv) *Some nonempty open subset $U \subseteq G$ is metrizable.*
(v) *The identity element $e \in G$ has a countable neighborhood basis.*

If a group Γ acts as a group of automorphisms on the group G and if e has a countable neighborhood basis consisting of Γ-invariant sets, then the metric d can be chosen to be left invariant and in addition Γ-invariant, i.e.

$$d(gx, gy) = d(x, y) = d(\gamma(x), \gamma(y))$$

holds for all $g, x, y \in G$ and $\gamma \in \Gamma$.

Corollary 1.40. *If a topological group G is metrizable and if $K \subseteq G$ is a compact subgroup, then there exists a left invariant metric d on G which is in addition K-invariant from the right, i.e. $d(x, y) = d(gxk, gyk)$ holds for all $k \in K$ and $x, y, g \in G$. In particular, every metrizable compact group admits a complete metric which is left and right invariant.*

Proof. If G is metrizable, then e has by Corollary 1.31 a countable neighborhood basis consisting of K-invariant neighborhoods. Hence the metric d on G can be chosen to be left invariant and invariant under the conjugation action of K. In a metrizable compact space every metric is complete because every Cauchy sequence has a convergent subsequence. □

The Birkhoff–Kakutani Theorem makes no claims about metric completeness. Indeed, there exist completely metrizable topological groups which admit no left invariant *complete* metric. The proof of Theorem 1.39 relies on the following technical lemma.

Lemma 1.41. *Let G be a topological group. Suppose that $(K_n)_{n\in\mathbb{Z}}$ is a family of symmetric identity neighborhoods with the property that $K_n^{\cdot 3} \subseteq K_{n+1}$ holds for all $n \in \mathbb{Z}$, and with $\langle \bigcup_{n\in\mathbb{Z}} K_n \rangle = G$. Then there exists a continuous length function $\ell : G \longrightarrow \mathbb{R}_{\geq 0}$ such that*

$$\{g \in G \mid \ell(g) < 2^n\} \subseteq K_n \subseteq \{g \in G \mid \ell(g) \leq 2^n\}.$$

holds for all $n \in \mathbb{Z}$. In particular,

$$\bigcap_{n\in\mathbb{Z}} K_n = \{g \in G \mid \ell(g) = 0\}.$$

If a group Γ acts as a group of automorphisms on the group G such that all sets K_n are Γ-invariant, then the length function ℓ can be chosen to be invariant under the Γ-action on G.

Proof. First of all we note that for every $g \in G$ there exist integers $n_1, \ldots, n_k$ with $g \in K_{n_1} \cdots K_{n_k}$, because the union of the K_n generates G and because every K_n is symmetric. For $g \in G$ we put

$$\ell(g) = \inf\{t \geq 0 \mid \text{ there is some } k \geq 1 \text{ and } n_1, \ldots, n_k \in \mathbb{Z} \\ \text{with } t = 2^{n_1} + \cdots + 2^{n_k} \text{ and } g \in K_{n_1} K_{n_2} \cdots K_{n_k}\}.$$

If $g \in K_{m_1} \cdots K_{m_r}$ and $h \in K_{n_1} \cdots K_{n_s}$, then $gh \in K_{m_1} \cdots K_{m_r} K_{n_1} \cdots K_{n_s}$. It follows that ℓ satisfies the triangle inequality. Since each K_n is symmetric, we have $\ell(g) = \ell(g^{-1})$ for all $g \in G$. Finally, $\ell(e) = 0$ since $e \in K_n$ holds for every $n \in \mathbb{Z}$. This shows that ℓ is a length function. Moreover, $\ell(g) \leq 2^n$ if $g \in K_n$. If all sets K_n are Γ-invariant, then ℓ is also Γ-invariant.

Next we show the continuity of ℓ. Let $g \in G$ be any element, and let $\varepsilon > 0$. We choose $n \in \mathbb{Z}$ in such a way that $2^n \leq \varepsilon$. Since $\ell(a) \leq 2^n$ holds for all $a \in K_n$, we have $|\ell(g) - \ell(h)| \leq 2^n \leq \varepsilon$ for all $h \in gK_n$. Hence ℓ is continuous.

Suppose that $\ell(g) < 2^n$. Then there exists $k \geq 1$ and numbers $n_1, \ldots, n_k \in \mathbb{Z}$ with $g \in K_{n_1} \cdots K_{n_k}$ and with $2^{n_1} + \cdots + 2^{n_k} < 2^n$.

Claim. Suppose that $2^{n_1} + \cdots + 2^{n_k} < 2^n$. *Then* $K_{n_1} \cdots K_{n_k} \subseteq K_n$.

Proof of the claim. We note that $n_j < n$ holds for $j = 1, \ldots, k$. Hence $K_{n_1} \cdots K_{n_k} \subseteq K_{n-1} \cdots K_{n-1}$. This proves the claim for $k = 1, 2, 3$. For $k \geq 4$ we proceed by induction on k. Suppose that $k \geq 4$. If $2^{n_1} + \cdots + 2^{n_k} < 2^{n-1}$, then $K_{n_1} \cdots K_{n_{k-1}} \subseteq K_{n-1}$ by the induction hypothesis, and $K_{n_k} \subseteq K_{n-1}$, whence $K_{n_1} \cdots K_{n_k} \subseteq K_{n-1} K_{n-1} \subseteq K_n$. There remains the case where $2^{n-1} \leq 2^{n_1} + \cdots + 2^{n_k} < 2^n$. We choose the smallest $r \in \{1, \ldots, k\}$ with $2^{n-1} \leq 2^{n_1} + \cdots + 2^{n_r}$. Then $2^{n_1} + \cdots + 2^{n_{r-1}} < 2^{n-1}$ and $2^{n_{r+1}} + \cdots + 2^{n_k} < 2^{n-1}$. By the induction hypotheses $K_{n_1} \cdots K_{n_{r-1}} \subseteq K_{n-1}$ and $K_{n_{r+1}} \cdots K_{n_k} \subseteq K_{n-1}$. Thus $K_{n_1} \cdots K_{n_k} \subseteq K_{n-1} K_{n_r} K_{n-1} \subseteq K_{n-1} K_{n-1} K_{n-1} \subseteq K_n$. Note that this is true also for the extremal cases $r = 1$ and $r = k$. Hence the claim is true.

It follows that $\{g \in G \mid \ell(g) < 2^n\} \subseteq K_n \subseteq \{g \in G \mid \ell(g) \leq 2^n\}$, and in particular $\bigcap_{n \in \mathbb{Z}} K_n = \{g \in G \mid \ell(g) = 0\}$. □

Proof of Theorem 1.39. It is clear that (i) $\Leftrightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv). If a nonempty open set $U \subseteq G$ is metrizable, then every left translate of U is metrizable and hence some identity neighborhood is metrizable. In a metrizable space, every point has a countable neighborhood basis. Therefore we have (iv) $\Rightarrow$ (v).

It remains to show that (v) $\Rightarrow$ (i). Let $\{V_n \mid n \geq 1\}$ be a countable neighborhood basis of e. We put $W_n = V_n \cap V_n^{-1}$. If the V_n are all Γ-invariant, then the W_n are Γ-invariant as well. We now put $K_n = G$ for $n \geq 0$, and $K_{-1} = W_1$. We then proceed inductively as follows. Given K_{-n}, we choose $m \geq 1$ in such a way that $W_m^{\cdot 3} \subseteq K_{-n} \cap V_{n+1}$ holds. This is possible because the W_m form a neighborhood basis of e. We put $K_{-(n+1)} = W_m$.

Let ℓ denote a continuous length function as in Lemma 1.41 for the family $(K_n)_{n\in\mathbb{Z}}$. If the V_n are Γ-invariant, then the K_n are also Γ-invariant and hence we may assume in this case that ℓ is Γ-invariant. Then $d(g,h) = \ell(g^{-1}h)$ is a continuous left invariant pseudometric on G. Let $U \subseteq G$ be an open set and suppose that $g \in U$. There exists an integer $n \geq 1$ such that $gV_n \subseteq U$. It follows that the set $\{h \in G \mid d(g,h) < 2^{-n}\}$ is contained in $gK_{-n} \subseteq gV_n \subseteq U$. Therefore the pseudometric d metrizes the topology of G. Since G is Hausdorff, d is a metric. □

The following is another important consequence of Lemma 1.41.

Theorem 1.42. *Suppose that G is a topological group and that $H \subseteq G$ is a closed subgroup. Then G/H is a Tychonoff space. In particular, every Hausdorff topological group is a Tychonoff space.*

Proof. If $H \subseteq G$ is a closed subgroup, then G/H is Hausdorff by Proposition 1.17. Let $p : G \longrightarrow G/H$ denote the canonical projection and suppose that $W \subseteq G/H$ is a neighborhood of the point $x = aH$. Since p is continuous, there exists a symmetric identity neighborhood $K_0 \subseteq G$ with $p(K_0 a) \subseteq W$. We put $K_n = G$ for all $n \geq 1$, and we choose inductively symmetric identity neighborhoods K_n for $n = -1, -2, -3, \ldots$ such that $K_n^{\cdot 3} \subseteq K_{n+1}$. Let ℓ denote a continuous length function for the family $(K_n)_{n\in\mathbb{Z}}$ as in Lemma 1.41. Then $\{g \in G \mid \ell(g) < 2^0\} \subseteq K_0$. Let d_H denote the corresponding continuous pseudometric on G/H, as in Lemma 1.38. Then $\varphi(gH) = \max\{d_H(aH, gH), 1\}$ is a continuous function on G/H, with $\varphi(aH) = 0$. If $gH \notin p(K_0 a)$, then $K_0 a \cap gH = \varnothing$ and therefore $d_H(aH, gH) \geq 1$, whence $\varphi(gH) = 1$. □

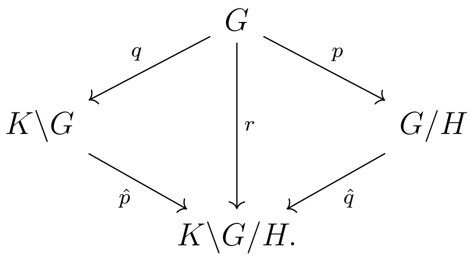
Proposition 1.43. *Suppose that G is a metrizable topological group and that H is a closed subgroup. Then G/H is metrizable.*

Proof. Since G is metrizable, there exists by Theorem 1.39 a right invariant metric d on G that metrizes G. From Lemma 1.38 we see that d_H is a continuous pseudometric on G/H. Suppose that $W \subseteq G/H$ is an open set containing gH. Since p is continuous, there exists $\varepsilon > 0$ such that the set $\{x \in G \mid d(g,x) < \varepsilon\}$ is mapped by $p : G \longrightarrow G/H$ into W. Then $\{xH \mid d_H(gH, xH) < \varepsilon\} \subseteq W$, which shows that d_H determines the topology on G/H. Since G/H is Hausdorff, d_H is a metric. □

We extend the previous result to certain double coset spaces. This will become relevant in the next chapter.

Lemma 1.44. *Let G be a Hausdorff topological group and let ℓ be a continuous length function on G, with its associated right invariant pseudometric $d(x,y) = \ell(xy^{-1})$. Suppose that $H \subseteq G$ is a closed subgroup. Suppose also that $K = \{g \in G \mid \ell(g) = 0\}$ is compact and that the metric $\bar{d}(Kx, Ky) = d(x,y)$ metrizes the quotient topology on $K\backslash G$. Then the pseudometric $\bar{d}_H(KxH, KyH) = d_H(xH, yH)$ metrizes the topology on $K\backslash G/H$.*

Proof. We consider the diagram of quotient maps



Since G/H is Hausdorff and K is compact, $K\backslash G/H$ is Hausdorff by Proposition 1.27. The pseudometric $d_H(xH, yH) = \inf \ell(xHy^{-1})$ is continuous on G/H. For $k \in K$ and $x \in G$ we have $\ell(kx) = \ell(x)$ and therefore

$$d_H(k_1xH, k_2xH) = \inf \ell(k_1xHy^{-1}k_2^{-1}) = \inf \ell(xHy^{-1}) = d_H(xH, yH)$$

holds for all $k_1, k_2 \in K$ and all $x, y \in G$. Hence

$$\bar{d}_H(KxH, KyH) = d_H(xH, yH)$$

is a pseudometric on $K\backslash G/H$. Since $\hat{q} \times \hat{q} : G/H \longrightarrow K\backslash G/H$ is an open map, $\bar{d}_H$ is continuous.

We claim that the continuous pseudometric $\bar{d}_H$ determines the topology on $K\backslash G/H$. Suppose that $W \subseteq K\backslash G/H$ is an open set containing KgH. Since $\bar{d}$ metrizes $K\backslash G$, there exists $\varepsilon > 0$ such that the set $V = \{Kx \mid \bar{d}(Kg, Kx) < \varepsilon\} \subseteq K\backslash G$ is mapped by $\hat{p}$ into W. If $\bar{d}_H(KgH, KyH) < \varepsilon$, then there exist $h \in H$ with $d(g, yh) = \bar{d}(Kg, Kyh) < \varepsilon$ and therefore $\hat{p}(Kyh) = KyH \in \hat{p}(V) \subseteq W$. This shows that the pseudometric $\bar{d}_H$ topologizes $K\backslash G/H$. Since $K\backslash G/H$ is Hausdorff, $\bar{d}_H$ is a metric. ☐

2 | Around the Baire Property

We recall some notions related to the Baire Category Theorem.

Definition 2.1. Let X be a topological space. A subset $N \subseteq X$ is called *nowhere dense* if its closure $\overline{N}$ has empty interior. Equivalently, there exists a dense open set $U \subseteq X$ which is disjoint from N. Any subset of a nowhere dense set is again nowhere dense. A countable union of nowhere dense sets is called a *meager* set (a *set of first category* in the older literature). It follows that subsets of meager sets are meager, and that countable unions of meager sets are again meager. A subset $M \subseteq X$ is meager if and only if there exists a countable family of dense open sets $(U_n)_{n\geq 1}$ in X with $M \cap \bigcap_{n\geq 1} U_n = \varnothing$. A topological space X is called a *Baire space* if for every countable family of dense open sets $(U_n)_{n\geq 0}$, the intersection $\bigcap_{n\geq 0} U_n$ is again dense. This condition can be phrased in several ways.

Lemma 2.2. *Let X be a topological space. Then the following are equivalent.*

(i) *X is a Baire space.*
(ii) *The union of countably many closed subsets with empty interiors has empty interior.*
(iii) *Every nonempty open subset of X is nonmeager.*
(iv) *The complement of every meager set is dense.*

In descriptive set theory, the Polish space $\mathbb{N}^{\mathbb{N}}$ is often called *the Baire space*. We will not use this terminology.

Proof. Suppose that X is a Baire space and that $(A_n)_{n\geq 1}$ is a family of closed sets with empty interiors. Then each of the open sets $U_n = X - A_n$ is dense, and thus $\bigcap_{n\geq 1} U_n$ is also dense. Thus $\bigcup_{n\geq 1} A_n$ has empty interior. Hence (i) implies (ii). Suppose that (ii) holds. Then in particular, every meager set has empty interior. Hence every nonempty open set is nonmeager, and (iii) holds. Suppose that (iii) holds and that $M \subseteq X$ is meager. Then the interior of M is meager and therefore empty. Hence $X - M$ is dense. If (iv) holds and if $(U_n)_{n\geq 1}$ is a family of dense open sets, then $M = X - \bigcap_{n\geq 1} U_n = \bigcap_{n\geq 1}(X - U_n)$ is meager and hence $\bigcap_{n\geq 1} U_n$ is dense. Therefore X is a Baire space. $\square$

In particular, every open subset of a Baire space is again a Baire space in the subspace topology. The class of Baire spaces is, however, not closed under products or passage to closed subsets, and this is one main reason why we consider later the subclass of Čech complete spaces. In particular, we will see that every completely metrizable space and every locally compact space is a Baire space, by Baire's Category Theorem. In any case, we have the following elementary result. We recall that a countable intersection of open sets in a topological space is called a G_δ-set.

Lemma 2.3. *Suppose that X is a Baire space and that $A \subseteq X$ is a dense subspace. If A is a G_δ-set, then A is a Baire space. If $X \neq \varnothing$, then A is not meager.*

Proof. We put $A = \bigcap_{n\geq 1} U_n$, where each $U_n \subseteq X$ is open. Since A is dense, each set U_n is dense. Suppose that $(B_n)_{n\geq 1}$ is a sequence of relatively open dense subsets of A. Then there exist open sets $W_n \subseteq X$ such that $B_n = A \cap W_n$. In particular, each set W_n is dense in X. Hence $\bigcap_{n\geq 1} B_n = \bigcap_{n\geq 1}(U_n \cap W_n) \subseteq A$ is dense in X and therefore dense in A. Hence A is a Baire space. Moreover, $X - A = \bigcup_{n\geq 1}(X - U_n)$ is meager. If A is also meager, then X itself is meager and therefore empty. □

It follows for example that the space of irrational numbers $\mathbb{R} - \mathbb{Q} \subseteq \mathbb{R}$ is a Baire space. On the other hand, $\mathbb{Q}$ is not a Baire space, since $\mathbb{Q}$ is nonempty and meager. The following result is fundamental for the proof of Pettis' Lemma below.

Theorem 2.4 (Banach's Category Theorem). *Let X be a topological space, let $\boldsymbol{U}$ be a collection of open subsets of X and let $A \subseteq X$ be a subset. If for each $U \in \boldsymbol{U}$ the set $A \cap U$ is meager, then $A \cap \bigcup \boldsymbol{U}$ is meager. In particular, an arbitrary union of open meager subsets is again meager.*

Proof. We may assume that $\boldsymbol{U} \neq \varnothing$. Let $\mathcal{C}$ denote the set consisting of all collections $\boldsymbol{W}$ of subsets of X with the following properties. The members of $\boldsymbol{W}$ are open, pairwise disjoint, and each $W \in \boldsymbol{W}$ is contained in some $U \in \boldsymbol{U}$. The set $\mathcal{C}$ is partially ordered by inclusion and $(\mathcal{C}, \subseteq)$ is nonempty and inductive. By Zorn's Lemma, $\mathcal{C}$ contains a maximal element $\boldsymbol{W}$. Put $M = \overline{\bigcup \boldsymbol{U}} - \bigcup \boldsymbol{W}$. Then M is closed and we claim that M has empty interior. If a nonempty open set V is contained in M, then V intersects some $U \in \boldsymbol{U}$ nontrivially. But then $\boldsymbol{W} \cup \{V \cap U\} \in \mathcal{C}$, contradicting the maximality of $\boldsymbol{W}$. Thus M has empty interior. Since M is closed, M is nowhere dense and in particular meager.

For every member W of $\boldsymbol{W}$ the set $A \cap W$ is meager. Hence there exists a countable family of nowhere dense sets $N_{W,n} \subseteq W$, for $n \geq 1$, with $A \cap W = \bigcup_{n\geq 1} N_{W,n}$. We put $N_n = \bigcup\{N_{W,n} \mid W \in \boldsymbol{W}\}$ and we claim that N_n is nowhere dense. If a nonempty open set V is contained in $\overline{N_n}$, then there exists a member $W \in \boldsymbol{W}$ which intersects V nontrivially. Now $\overline{N_n} \cap W \subseteq \overline{N_{W,n}}$, and thus $V \cap W \subseteq \overline{N_{W,n}}$, contradicting the fact that $N_{W,n}$ is nowhere dense. Hence N_n is nowhere dense. It follows that $A \cap \bigcup \boldsymbol{W} = \bigcup_{n\geq 1} N_n$ is meager, and so is $A \cap \bigcup \boldsymbol{U} \subseteq M \cup \bigcup_{n\geq 1} N_n$. □

The Open Mapping Theorem and Pettis' Lemma

There exist topological spaces which are neither meager nor Baire spaces. An example of such a space is the subspace $\mathbb{Q} \cup [0,1] \subseteq \mathbb{R}$. For topological groups this cannot happen.

Proposition 2.5. *Let G be a topological group. Then the following are equivalent.*
(i) *G is a Baire space.*
(ii) *G is not meager.*
(iii) *G contains a non-meager subset.*

Proof. It is clear that (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii), and we show that $\neg$(i) $\Rightarrow \neg$(iii). If G is not a Baire space, then there exists an open nonempty meager subset $U \subseteq G$ by Lemma 2.2. Then $G = \bigcup\{gU \mid g \in G\}$ is also meager by Banach's Category Theorem 2.4, and therefore every subset of G is meager. ☐

There are several open mapping theorems for topological groups. The following is the most basic version. We note that every Hausdorff topological group which is compactly generated is σ-compact.

Theorem 2.6 (The Open Mapping Theorem, I). *Let $f : G \longrightarrow K$ be a morphism of Hausdorff topological groups. If G is σ-compact and if $f(G) \subseteq K$ is not meager, then f is open and K is locally compact.*

In particular, every Hausdorff topological group which is σ-compact and a Baire space is locally compact.

Proof. We assume that $f(G) \subseteq K$ is not meager, and we consider first the case where f is in addition injective. We write $G = \bigcup_{n\in\mathbb{N}} A_n$, with A_n compact. For every $n \in \mathbb{N}$, the restriction-corestriction $f|_{A_n}^{f(A_n)} : A_n \longrightarrow f(A_n)$ is a continuous bijection and hence a homeomorphism. Moreover, each $f(A_n)$ is compact and therefore closed. Since $f(G) = \bigcup_{n\in\mathbb{N}} f(A_n)$ is not meager, there exists by Lemma 2.2 an index $m \in \mathbb{N}$ such that $f(A_m)$ contains a nonempty open set V. It follows from Proposition 1.12 that $f(G) \subseteq K$ is open. Put $U = f^{-1}(V)$. Then $U \subseteq G$ is open, and the restriction-corestriction $f|_U^V : U \longrightarrow V$ is a homeomorphism. It follows from Lemma 1.5 that the corestriction $f|^{f(G)} : G \longrightarrow f(G)$ has a continuous inverse. Hence f is open. Moreover, $f(G)$ is an open locally compact set in K, and thus K is locally compact.

In the general case we put $N = \ker(f)$ and we consider the commutative diagram

$$\begin{array}{ccc} G & \xrightarrow{f} & K. \\ {\scriptstyle p}\downarrow & \nearrow_{\bar f} & \\ G/N & & \end{array}$$

The quotient G/N is σ-compact. Since p is open by Proposition 1.17 and since $\bar{f}$ is open by the previous argument, $f = \bar{f} \circ p$ is open. $\square$

Corollary 2.7 (The Closed Graph Theorem, I). *Let G, K be Hausdorff topological groups and suppose that $f : G \longrightarrow K$ is a group homomorphism. Assume also that G and K are σ-compact, and that G is a Baire space. Then the following are equivalent.*
(i) *f is continuous.*
(ii) *the graph of f is closed in $G \times K$.*

Proof. The graph of a continuous map whose range is a Hausdorff space is closed, hence (i) $\Rightarrow$ (ii). Suppose that (ii) holds. Then $H = \{(g, f(g)) \mid g \in G\}$ is a closed subgroup of $G \times K$ and hence σ-compact. The map $h : H \longrightarrow G$ that maps $(g, f(g))$ to g is continuous and bijective. By Proposition 2.6, h is open and hence the map $g \longmapsto (g, f(g))$ is continuous. $\square$

For two sets A, B we write the *symmetric difference* as

$$A \triangle B = (A - B) \cup (B - A) = (A \cup B) - (A \cap B).$$

Definition 2.8. A subset A of a topological space X is called *Baire measurable* (or *almost open*, or a subset with the *Baire property*) if there exists an open set $U \subseteq X$ such that the symmetric difference $M = U \triangle A$ is meager.

The reader should be warned that a Baire measurable subset is not the same as a subspace which is a Baire space.

Theorem 2.9 (Pettis' Lemma). *Let G be a topological group. If $A \subseteq G$ is Baire measurable and nonmeager, then $A^{-1}A$ is an identity neighborhood.*

Proof. Let $U \subseteq G$ be open such that $M = U \triangle A$ is meager. Then U is not meager and in particular nonempty. We choose $g \in U$ and an open identity neighborhood V with $gVV^{-1} \subseteq U$. We claim that $V \subseteq A^{-1}A$. If $h \in V$, then $g \in U \cap Uh$, whence $U \cap Uh \neq \varnothing$. Thus $(A \cap Ah) \triangle (U \cap Uh) \subseteq (A \triangle U) \cup (A \triangle U)h$ is meager. But then $A \cap Ah \neq \varnothing$, because otherwise $U \cap Uh$ would be a nonempty meager open set in the Baire space G. Hence $V \subseteq A^{-1}A$. $\square$

From Pettis' Lemma we derive several results about the continuity of homomorphisms. For this, the following is a useful property of Baire measurable sets.

Proposition 2.10. *The Baire measurable sets in a topological space X form a σ-algebra which contains all open sets. In particular, every Borel set in X is Baire measurable.*

Proof. Every open set is Baire measurable, and in particular the empty set is among the Baire measurable sets. Suppose that $A \subseteq X$ is Baire measurable. We claim that the complement $B = X - A$ is also Baire measurable. Let $U \subseteq X$ be open such that $U \triangle A$ is meager and put $V = X - \overline{U}$. We note that $M = \overline{U} - U$ is nowhere dense. The symmetric difference of two subsets of a given set is not changed if we replace both sets by their complements. Hence

$$V \triangle B = \overline{U} \triangle A \subseteq (U \triangle A) \cup M$$

is meager, and therefore B is Baire measurable. Suppose that $(A_n)_{n \geq 0}$ is a family of Baire measurable sets. For every A_n there is an open set U_n such that the symmetric difference $M_n = (A_n \cup U_n) - (A_n \cap U_n)$ is meager. We put $A = \bigcup_{n \geq 0} A_n$, $M = \bigcup_{n \geq 0} M_n$ and $U = \bigcup_{n \geq 0} U_n$. Then $A_n - U \subseteq A_n - U_n \subseteq M_n$, whence $A - U \subseteq M$. Likewise, $U - A \subseteq M$, and therefore

$$U \triangle A \subseteq M$$

is meager. This shows that the Baire measurable sets form a σ-algebra. Since every open set is Baire measurable, every Borel set is Baire measurable. ☐

Corollary 2.11. *Suppose that G is a topological group and that $H \subseteq G$ is a subgroup. If H contains a Baire measurable set (eg. a Borel set) which is not meager, then H is open.*

Proof. By Theorem 2.9, $H = H^{-1}H$ is an identity neighborhood and hence open by Proposition 1.12. ☐

Corollary 2.12. *Suppose that G is a topological group which is a Baire space. If $H \subseteq G$ is a dense subgroup which is a G_δ-set, then $H = G$.*

Proof. By Lemma 2.3, H is not meager. Hence H is open by Theorem 2.9 and therefore closed by Proposition 1.12. ☐

We recall that a topological space is called *Lindelöf* if every open covering has a countable subcovering. Examples of Lindelöf spaces are second countable spaces and σ-compact spaces. A not necessarily continuous map between topological spaces is called *Baire measurable* (*Borel measurable*) if the preimage of every open set is Baire measurable (a Borel set). Every continuous map is thus Baire measurable and Borel measurable. The reader should be aware that there are other, non-equivalent notions of Baire measurable functions in the literature.

Theorem 2.13. *Let G, K be topological groups and let $f : G \longrightarrow K$ be a group homomorphism. Assume also that K is Lindelöf and that G is a Baire space. The following are equivalent.*

(i) *f is a morphism of topological groups.*

(ii) *f is Borel measurable.*
(iii) *f is Baire measurable.*
(iv) *The group K has arbitrarily small identity neighborhoods whose preimages are Baire measurable.*

Proof. A closed subset of a Lindelöf space is again Lindelöf. Replacing K by $\overline{f(G)}$, we may thus assume in addition that $f(G)$ is dense in K. It is clear that (i) $\Rightarrow$(ii) $\Rightarrow$(iv) and that (i) $\Rightarrow$(iii) $\Rightarrow$(iv). Suppose that (iv) holds. Let $V \subseteq K$ be an identity neighborhood. We claim that $f^{-1}(V)$ contains an identity neighborhood. We choose an identity neighborhood $U \subseteq K$ such that $U^{-1}U \subseteq V$, and such that $E = f^{-1}(U)$ is Baire measurable. Since K is Lindelöf and $f(G)$ is dense, we find elements $g_1, g_2, g_3, \ldots$ in G such that $K = \bigcup_{n\geq 1} f(g_n)U$. Hence $G = \bigcup_{n\geq 1} g_n E$. Since G is not meager, E cannot be meager. Hence $E^{-1}E$ is an identity neighborhood by Theorem 2.9, and $f(E^{-1}E) \subseteq V$. It follows that f is continuous at the identity element of G. By Lemma 1.5, the map f is continuous and hence a morphism of topological groups. □

Čech Complete Spaces

We introduce another piece of terminology which allows us to unify methods from completely metrizable spaces and locally compact spaces.

Definition 2.14. A Tychonoff space X is called *Čech complete* if there exits a countable family $(\boldsymbol{U}_n)_{n\geq 1}$ of open coverings of X such that the following holds. If $\boldsymbol{F}$ is a family of closed subsets of X which has the finite intersection property, and if for every $n \geq 1$ there exists $F \in \boldsymbol{F}$ and $U \in \boldsymbol{U}_n$ with $F \subseteq U$, then $\bigcap \boldsymbol{F} \neq \varnothing$. In this case $(\boldsymbol{U}_n)_{n\geq 1}$ is called a *complete sequence of open coverings* for X.

It follows right from the definition that a closed subspace of a Čech complete space is again Čech complete.

Lemma 2.15. *Every completely metrizable space is Čech complete.*

Proof. Every metric space is a Tychonoff space. Suppose that (X, d) is a complete metric space. We let $\boldsymbol{U}_n$ denote the collection of all open subsets of diameter at most 2^{-n}, for $n \geq 1$. Suppose that $\boldsymbol{F}$ is a family of closed sets having the finite intersection property, and that for each $n \geq 1$ there exists a member $F_n \in \boldsymbol{F}$ which has diameter at most 2^{-n}. We choose points $x_n \in F_1 \cap F_2 \cap \cdots \cap F_n$. Then $d(x_n, x_{n+1}) \leq 2^{-n}$ and thus $(x_n)_{n\geq 1}$ is a Cauchy sequence in X. We put $x = \lim_n x_n$. Suppose that E is an arbitrary member of $\boldsymbol{F}$. For every $n \geq 1$ we may choose a point $y_n \in E \cap F_n$. Then $d(x_n, y_n) \leq 2^{-n}$ and thus $\lim_n y_n = \lim_n x_n = x$. Therefore $x \in E$, and hence $x \in \bigcap \boldsymbol{F}$. Therefore $(\boldsymbol{U}_n)_{n\geq 1}$ is a complete sequence of open coverings of X. □

Lemma 2.16. *Let K be a compact space and suppose that $X \subseteq K$ is a G_δ-set. Then X is Čech complete. In particular, every locally compact space is Čech complete.*

Proof. Since every compact space is normal, X is a Tychonoff space. Let $U_1, U_2, \ldots \subseteq K$ be open with $X = \bigcap_{n \geq 1} U_n$. We put $\boldsymbol{U}_n = \{X \cap V \mid V \subseteq K \text{ is open and } \overline{V} \subseteq U_n\}$, for $n = 1, 2, \ldots$. Then $\boldsymbol{U}_n$ is an open covering of X, since K is regular. Suppose that $\boldsymbol{E}$ is a collection of relatively closed subsets of Y having the finite intersection property, and that for every n there exists $V_n \in \boldsymbol{U}_n$ and $E_n \in \boldsymbol{E}$ with $E_n \subseteq V_n$. Since K is compact, there exists a point $z \in \bigcap\{\overline{E} \mid E \in \boldsymbol{E}\}$. Since $z \in \overline{E_n} \subseteq U_n$ holds for every n, we have $z \in X$. Hence $(\boldsymbol{U}_n)_{n \geq 1}$ is a complete sequence of open coverings of X. □

Theorem 2.17 (Baire's Category Theorem). *Every Čech complete space is a Baire space. In particular, every completely metrizable space and every locally compact space is a Baire space.*

Proof. Let $(\boldsymbol{U}_n)_{n \geq 1}$ be a complete sequence of open coverings of X. Suppose that $(V_n)_{n \geq 1}$ is a family of open dense subsets of X, and that $W \subseteq X$ is a nonempty open set. We have to show that $W \cap \bigcap_{n \geq 1} V_n \neq \varnothing$.

We put $W_0 = W$, and we choose inductively points x_n and open sets W_n for $n = 1, 2, 3, \ldots$ as follows. We choose a point $x_n \in W_{n-1} \cap V_n$ and an element $U_n \in \boldsymbol{U}_n$ with $x_n \in U_n$. Since X is regular, there is an open neighborhood W_n of x_n such that $\overline{W_n} \subseteq W_{n-1} \cap V_n \cap U_n$. We put $\boldsymbol{F} = \{\overline{W_n} \mid n \geq 1\}$. Then $\bigcap \boldsymbol{F} \neq \varnothing$ because X is Čech complete and thus $W \cap \bigcap_{n \geq 1} V_n \neq \varnothing$. □

In view of Baire's Category Theorem, we investigate Čech complete spaces more closely. First we improve Lemma 2.15 and Lemma 2.16. We recall that every Tychonoff space X embeds into its *Čech-Stone compactification* βX. The Čech-Stone compactification has the following universal property. If $f : X \longrightarrow K$ is a continuous map from a Tychonoff space X to a compact space K, then there is a unique continuous map $\beta f : \beta X \longrightarrow K$ that extends f,

$$\begin{array}{ccc} X & \xrightarrow{\ f\ } & K. \\ \big\downarrow & \nearrow_{\beta f} & \\ \beta X & & \end{array}$$

It follows that X is dense in βX and that β is a functor from the category of Tychonoff spaces and continuous maps to the category of compact spaces and continuous maps.

Proposition 2.18. *Let X be a Tychonoff space. The following are equivalent.*
(i) *X is Čech complete.*
(ii) *Whenever K is a compact space and $\iota : X \longrightarrow K$ is a topological embedding with dense image, the image $\iota(X) \subseteq K$ is a G_δ-set.*

(iii) *There exists a compact space K and a topological embedding $\iota : X \longrightarrow K$ such that the subset $\iota(X) \subseteq K$ is a* G_δ*-set.*

Proof. We show first that (i) $\Rightarrow$ (ii). Let $\iota : X \longrightarrow K$ be a topological embedding and let $(\boldsymbol{U}_n)_{n\geq 1}$ be a complete sequence of open coverings of X. We put

$$W_n = \bigcup\{V \subseteq K \mid V \text{ is open and } \iota^{-1}(V) \in \boldsymbol{U}_n\}.$$

Then $\iota(X) \subseteq W_n$ holds for every $n \geq 1$ and we claim that $\iota(X) = \bigcap_{n\geq 1} W_n$. Suppose that $z \in \bigcap_{n\geq 1} W_n$. We let $\boldsymbol{A}$ denote the collection of all closed neighborhoods of z in K. The set $\boldsymbol{E} = \{A \cap \iota(X) \mid A \in \boldsymbol{A}\}$ has the finite intersection property because z is in the closure of $\iota(X)$. By the definition of W_n, there exists for every $n \geq 1$ an open set $V_n \subseteq K$ containing z with $\iota^{-1}(V_n) \in \boldsymbol{U}_n$. Since K is regular, there exists a closed neighborhood $A_n \in \boldsymbol{A}$ with $z \in A_n \subseteq V_n$. Since $\iota(X)$ is Čech complete, $\bigcap \boldsymbol{E} \neq \varnothing$. On the other hand, $\{z\} = \bigcap \boldsymbol{A} \supseteq \bigcap \boldsymbol{E} \subseteq \iota(X)$. Hence $z \in \iota(X)$. This shows that $\iota(X)$ is a G_δ-set in K.

Since every Tychonoff space X embeds with a dense image into its Čech-Stone compactification βX, we see that (ii) $\Rightarrow$ (iii). In Lemma 2.16 we showed that (iii) $\Rightarrow$ (i). ☐

Corollary 2.19. *If X is a Čech complete space and if $A \subseteq X$ is a* G_δ*-set, then A is Čech complete.*

Now we turn to metric completeness versus Čech completeness. We recall the following classical fact.

Lemma 2.20. *Suppose that X is completely metrizable and that $A \subseteq X$ is a* G_δ*-set. Then A is completely metrizable.*

Proof. Let $U_1, U_2, \ldots \subseteq X$ be open, with $A = \bigcap_{n\geq 1} U_n$ and let d be a complete metric on X. We may assume that $U_n \neq X$ holds for all $n \geq 1$. For each $n \geq 1$ the map $f_n : x \longmapsto d(x, X - U_n)$ is continuous. Hence the map $h : A \longrightarrow X \times \mathbb{R}^{\mathbb{N}_1}$ that maps $x \in A$ to $h(x) = (x, 1/f_1(x), 1/f_2(x), \ldots)$ is continuous as well, with a continuous inverse. Hence h is a topological embedding. The product space $X \times \mathbb{R}^{\mathbb{N}_1}$ is completely metrizable. Therefore it suffices to show that $h(A)$ is closed. If $(x_n)_{n\geq 1}$ is a sequence in A and if $\lim_n h(x_n) = (z, t_1, t_2, \ldots)$, then $\lim_n x_n = z$ and $t_j = \lim_n 1/f_n(z)$, which shows that $f_n(z) \neq 0$ for all n. Thus $z \in A$. ☐

Proposition 2.21. *A metrizable space X is Čech complete if and only if X is completely metrizable.*

Proof. We showed in Lemma 2.15 that every completely metrizable space is Čech complete. Conversely, suppose that (X, d) is metric and Čech complete. Let $\widehat{X}$ denote the metric completion of (X, d), and consider the natural embedding $\iota : X \longrightarrow \beta\widehat{X}$. Then X is dense in $\beta\widehat{X}$ and hence by Proposition 2.18 a G_δ-set in $\beta\widehat{X}$. Hence X is a G_δ-set in $\widehat{X}$, and therefore X is completely metrizable by Lemma 2.20. ☐

Corollary 2.22. *Every locally compact metrizable space is completely metrizable.*

Proposition 2.23. *If X is a completely metrizable space, then a subspace $A \subseteq X$ is completely metrizable if and only if A is a G_δ-set.*

Proof. If $A \subseteq X$ is a G_δ-set, then A is completely metrizable by Lemma 2.20. Conversely, if $A \subseteq X$ is completely metrizable and if we put $Y = \overline{A}$, then A is dense in βY and therefore a G_δ-set in βY by Proposition 2.18. Therefore A is a G_δ-set in Y. On the other hand, the closed set Y is a G_δ-set in X and therefore A is also a G_δ-set in X. ☐

Theorem 2.24. *Every product of Čech complete spaces is a Baire space and every countable product of Čech complete spaces is again Čech complete.*

Proof. First of all we note that a product (of any length) of Tychonoff spaces is again a Tychonoff space. Suppose that $(X_n)_{n\geq 1}$ is a countable sequence of Čech complete spaces. We put $K_n = \beta X_n$ and consider the compact space $K = \prod_{n\geq 1} K_n$. The natural map $\prod_{n\geq 1} X_n \longrightarrow \prod_{n\geq 1} K_n$ is an embedding. It remains to show that its image is a G_δ-set. For each $n \geq 1$ let $U_{1,n}, U_{2,n}, \ldots \subseteq K_n$ be open, with $\bigcap_{k\geq 1} U_{k,n} = X_n$, and put $W_{k,n} = K_1 \times \cdots \times K_{n-1} \times U_{k,n} \times K_{n+1} \times \cdots$. Then $W_{n,k} \subseteq K$ is open and $\bigcap_{k,n\geq 1} W_{k,n} = \prod_{n\geq 1} X_n$. Hence $\prod_{n\geq 1} X_n$ is a G_δ-set in K and thus Čech complete.

Suppose now that $(X_j)_{j\in J}$ is a family of Čech complete spaces. We have to show that the product $X = \prod_{j\in J} X_j$ is a Baire space. Let $U_n \subseteq X$ be open and dense, for $n = 1, 2, \ldots$ and put $A = \bigcap_{n\geq 1} U_n$. We claim that $A \subseteq X$ is dense. For this we show that A intersects every nonempty basic open set W in X. We put $W = \prod_{j\in J} W_j$, where $W_j \subseteq X_j$ is nonempty and open, and $W_j = X_j$ outside a finite nonempty index set $J_0 \subseteq J$. The canonical projection map $p : X \longrightarrow \prod_{j\in J_0} X_j$ is continuous and open and hence the sets $p(U_n)$ are open and dense. Since the finite product $\prod_{j\in J_0} X_j$ is Čech complete and in particular a Baire space, $\bigcap_{n\geq 1} p(U_n)$ is dense. Hence there exist points $x_j \in W_j$, for $j \in J_0$, such that $(x_j)_{j\in J_0} \in p(U_n)$ holds for all $n \geq 1$. We choose arbitrary points $x_j \in X_j$ for all $j \in J - J_0$. Then $(x_j)_{j\in J} \in U_n \cap W$ holds for all n. Hence $W \cap \bigcap_{n\geq 1} U_n \neq \varnothing$. ☐

We recall that a continuous map between Hausdorff spaces is called *proper* if the preimage of every compact set is compact. The following is elementary.

Lemma 2.25. *Suppose that $f : X \longrightarrow Y$ is a continuous map between Hausdorff spaces, that $A \subseteq X$ is a subset and that the restriction $f|_A : A \longrightarrow Y$ is closed and proper. Then $A \subseteq X$ is closed.*

Proof. Suppose that $x \in X - A$. Then $B = A \cap f^{-1}(f(x))$ is compact, because the restriction $f|_A : A \longrightarrow Y$ is proper. Since X is Hausdorff and $B \subseteq X$ is compact, there exists an open set $U \subseteq X$ with $B \subseteq U$ and $x \notin \overline{U}$. The set $f(A - U)$ is closed and does not contain $f(x)$. Hence $f^{-1}(f(A - U))$ is closed in X and does not contain x. In particular, $x \notin \overline{A - U} \subseteq f^{-1}(f(A - U))$. On the other hand, $x \notin \overline{U}$, whence $x \notin \overline{A}$. ☐

The complement of a G_δ-set in a topological space is called an F_σ-set. An F_σ-set is thus a countable union of closed sets.

Corollary 2.26. *Suppose that X, Y are Tychonoff spaces and that $f : X \longrightarrow Y$ is a continuous, surjective, closed, and proper map. Then X is Čech complete if and only if Y is Čech complete.*

Proof. We consider the induced map $\beta f : \beta X \longrightarrow \beta Y$ between the Čech-Stone compactifications. We claim that $\beta f(\beta X - X) = \beta Y - Y$. To show this, we put $Z = (\beta f)^{-1}(Y) \subseteq \beta X$ and we consider the restriction-corestriction $(\beta f)|_Z^Y : Z \longrightarrow Y$. By Lemma 2.25, X is closed in Z. In the other hand, X is dense in Z, whence $Z = X$ and thus $\beta f(\beta X - X) \subseteq \beta Y - Y$. Since Y is dense in βY and $f(X) = Y$, we have $\beta f(\beta X - X) = \beta Y - Y$. Now βf is a closed continuous map and hence $\beta f(\beta X - X)$ is an F_σ-set if and only if $\beta Y - Y$ is an F_σ set. ☐

A Hausdorff space X is called *paracompact* if every open covering of X has a locally finite refinement. Every metric space is paracompact.

Proposition 2.27. *Suppose that $f : X \longrightarrow Y$ is a continuous open surjective map, that X is Čech complete and that Y is paracompact. Then Y is Čech complete.*

Proof. We subdivide the proof into several steps.

Claim. There exists a Tychonoff space Z containing X as a G_δ-subspace and a continuous closed proper map $F : Z \longrightarrow Y$ which extends f.

We consider the map $\beta f : \beta X \longrightarrow \beta Y$ between the Čech-Stone compactifications, and we put $Z = \beta f^{-1}(Y) \supseteq X$. We denote the restriction-corestriction of βf by

$$F : Z \longrightarrow Y.$$

If $B \subseteq Y$ is compact, then $F^{-1}(B) = \beta f^{-1}(B)$ is compact, hence F is proper. If $A \subseteq \beta f(X)$ is closed, then A is compact and $F(A \cap Z) = \beta f(A) \cap Y$ is closed in Y, hence F is closed.

Claim. Suppose that $U \subseteq Z$ is open and that $F(U \cap X) = Y$. Then there exists an open set $U' \subseteq Z$ with $F(U' \cap X) = Y$ and with $\overline{U'} \subseteq U$.

Suppose that $U \subseteq Z$ is open and that $F(X \cap U) = Y$. We choose, for every $y \in Y$, an open set set $V_y \subseteq Z$ as follows. We choose an element $x \in X \cap U$ with $f(x) = y$, and then an open neighborhood $V_y \subseteq U$ of x with $\overline{V_y} \subseteq U$. This is possible because Z is regular. Since $V_y \cap X$ is open in X and since f is open, $U_y = f(V_y \cap X)$ is an open neighborhood of y. Since Y is paracompact, the open cover $\{U_y \mid y \in Y\}$ has a locally finite refinement $\boldsymbol{W}$. For every $W \in \boldsymbol{W}$ we choose $y(W) \in Y$ such that $W \subseteq U_{y(W)}$. We put $U_W = V_{y(W)} \cap F^{-1}(W)$ and we note that $\overline{U_W} \subseteq U$ is open. We also put $U' = \bigcup\{U_W \mid W \in W\}$. If $W \in \boldsymbol{W}$, then $W \subseteq f(V_{y(W)} \cap X)$ and thus $f(U_W \cap X) = W$. Hence $F(U' \cap X) = Y$. Suppose that $z \in Z$ is in the closure of U'. There exists an open neighborhood O of $f(z)$ such that set $\boldsymbol{W}_O = \{W \in \boldsymbol{W} \mid O \cap W \neq \varnothing\}$ is finite. Hence

$$z \in \overline{\bigcup\{U_W \mid W \in \boldsymbol{W}_O\}} = \bigcup\{\overline{U_W} \mid W \in \boldsymbol{W}_O\} \subseteq U.$$

Claim. There exists a closed G_δ*-set* $C \subseteq X$ *such that* $f|_C : C \longrightarrow Y$ *is surjective.*

We write $X = \bigcup_{n\geq 1} U_n$, where $U_n \subseteq Z$ is open. we construct a sequence of open sets Z_n as follows We put $Z_0 = Z$. Given Z_{n-1}, we choose an open set Z_n such that $\overline{Z_n} \subseteq U_n \cap Z_n$, with $F(X \cap Z_N) = Y$. We put $C = \bigcap_{n\geq 0} Z_n = \bigcap_{n\geq 0} \overline{Z_n}$. Thus $C \subseteq X$ is a closed G_δ-set. The set $f^{-1}(y) \subseteq X$ is compact for every y, hence $\bigcap \overline{Z_n} \cap f^{-1}(y) \neq \varnothing$.

The claim of the proposition is true.

The set $C \subseteq X$ is a G_δ-set in the Čech complete space X, and therefore Čech complete. Since $C \subseteq Z$ is closed, $F|_C = f|_C$ is continuous, proper, closed and surjective. Therefore Y is Čech complete. □

Čech Complete Groups

From the results of the previous section we see that the class of Čech complete groups has many favorable properties. It is closed under passage to closed subgroups, to open subgroups and even to G_δ-subgroups, and also closed under passage to countable products. It remains to consider quotients.

Proposition 2.28. *Let* G *be a Hausdorff topological group and* $Y \subseteq G$ *be any* G_δ*-set containing* e *(for example,* $Y = G$*). The following are equivalent.*

(i) *The group* G *is Čech complete.*
(ii) *There is a continuous length function* ℓ *on* G *such that* $K = \{g \in G \mid \ell(g) = 0\}$ *is compact and contained in* Y*, and such that* $K\backslash G$ *is Čech complete and metrizable by the metric* $\bar{d}(Kx, Ky) = \ell(xy^{-1})$*.*
(iii) *There is a compact subgroup* $K \subseteq Y$ *such that* $K\backslash G$ *is Čech complete.*
(iv) *There is a compact subgroup* $K \subseteq G$ *such that* $K\backslash G$ *is Čech complete.*

If G *is* σ*-compact, then the subgroup* K *in (iii) can in addition be chosen to be normal.*

A corresponding result holds of course for the left coset space G/K.

Proof. It is clear that (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv). If (iv) holds, then G is Čech complete by Corollary 2.26 and hence (iv) $\Rightarrow$ (i). It remains to show that (i) $\Rightarrow$ (ii). Suppose that G is Čech complete and that $Y = \bigcap_{n\geq 1} V_n$, where the $V_n \subseteq G$ are open identity neighborhoods.

If G is σ-compact, then there exists an ascending sequence $L_1 \subseteq L_2 \subseteq L_3 \subseteq \cdots$ of compact subsets of G such that $G = \bigcup_{n\geq 1} L_n$.

Let $(\boldsymbol{U}_n)_{n\geq 1}$ be a complete sequence of open coverings. For every n we choose $U_n \in \boldsymbol{U}_n$ with $e \in U_n$. Next we choose inductively closed symmetric identity neighborhoods $K_{-n} \subseteq U_n \cap V_n$, such that $K_{-(n+1)}^{\cdot 3} \subseteq K_{-n}$ holds, for $n = 1, 2, \ldots$.

If $G = \bigcup_{n\geq 1} L_n$ is an ascending union of compact sets, then we choose the K_{-n} in addition in such a way that

$$\text{(c)} \qquad aK_{-(n+1)}a^{-1} \subseteq K_n$$

holds for all $a \in L_n$. This is possible by Wallace's Lemma 1.16.

For $n \geq 0$ we put $K_n = G$. Let $\ell : G \longrightarrow \mathbb{R}$ denote a continuous length function for the the sequence $(K_n)_{n\in\mathbb{Z}}$ as in Lemma 1.41. Put $K = \bigcap_{n\in\mathbb{Z}} K_n = \{g \in G \mid \ell(g) = 0\} \subseteq Y$. If (c) holds, then K is a normal subgroup of G.

We claim that the closed subgroup $K \subseteq G$ is compact. If $\boldsymbol{E}$ is a family of closed subsets of K having the finite intersection property, then $\bigcap \boldsymbol{E} \neq \varnothing$, because each $E \in \boldsymbol{E}$ is contained in some $U_n \in \boldsymbol{U}_n$. Hence K is compact and $K\backslash G$ is Čech complete by Corollary 2.26.

It remains to show the metrizability of $K\backslash G$. We put $d(x, y) = \ell(xy^{-1})$ and we note that $d(x, y) = 0$ holds if and only if $x \in Ky$. Therefore d descends to a metric $\bar{d}$ on $K\backslash G$. Since the map $q : G \longrightarrow K\backslash G$ is open, the map $q \times q : G \times G \longrightarrow K\backslash G \times K\backslash G$ is also open and in particular a quotient map. Hence $\bar{d}$ is a continuous metric on $K\backslash G$.

Suppose that W is an open neighborhood of $Ke \in K\backslash G$ and put $U = q^{-1}(W)$. We claim that there exists $n \geq 1$ with $K_{-n} \subseteq U$. Otherwise, the sets $A_n = K_{-n} - U$ are nonempty, closed and nested, $A_1 \supseteq A_2 \supseteq A_3 \supseteq \cdots$. Then $A = \bigcap_{n\geq 1} A_n \neq \varnothing$, and $A \subseteq K - U = \varnothing$, a contradiction. Hence there exists $n \geq 1$ with $q(K_{-n}) \subseteq W$, and therefore $\{Kg \mid \bar{d}(K, Kg) < 2^{-n}\} \subseteq W$. Therefore $\bar{d}$ metrizes $K\backslash G$. ☐

Corollary 2.29. *If G is a Čech complete topological group G and if there exists an identity neighborhood $V \subseteq G$ which contains no compact subgroup besides $\{e\}$, then G is completely metrizable.*

Lemma 2.30. *Suppose that G is a Čech complete group and that $H \subseteq G$ is a subgroup which is a G_δ-set. Then H is closed.*

Proof. Since the closure $\overline{H} \subseteq G$ is Čech complete, we may in addition assume that H is dense in G. The claim follows now from Corollary 2.12. ☐

Lemma 2.31. *Suppose that G is a Čech complete topological group and that H is a Čech complete dense subgroup. Then $H = G$.*

Proof. The subspace $H \subseteq \beta G$ is dense and therefore a G_δ-set in βG by Proposition 2.18. Hence H is a G_δ-set in G, and therefore closed by Lemma 2.30. □

Proposition 2.32. *Let G be a completely metrizable topological group and let d be a left invariant metric which metrizes G. Then the metric $d_c(x,y) = d(x,y) + d(x^{-1},y^{-1})$ is complete.*

Proof. By Proposition 1.37 the metric completion $\widehat{G}$ of (G,d_c) is a topological group. Since $G \subseteq \widehat{G}$ is dense and Čech complete, G is a G_δ-set in $\widehat{G}$ by Proposition 2.23. Thus $G = \widehat{G}$ by Lemma 2.31. □

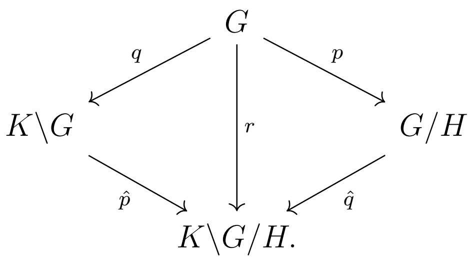
Corollary 2.33. *If G is a completely metrizable topological group, then every left and right invariant metric which metrizes A is complete.*

The result applies in particular to abelian groups, where every left invariant metric is automatically right invariant.

Proof. If d is left and right invariant, then $d_c = 2d$ is complete by Proposition 2.32 hence d is complete as well. □

Theorem 2.34. *If G is a Čech complete group and if H is a closed subgroup, then G/H and H are Čech complete. If G is completely metrizable, then G/H and H are completely metrizable.*

Proof. If G is Čech complete, then the closed subgroup H is also Čech complete. We have to show that G/H is Čech complete. By Proposition 2.28 there exists a continuous length function ℓ on G such that $K = \{g \in G \mid \ell(g) = 0\}$ is compact and $K\backslash G$ is metrizable by the metric $\bar{d}(Kx,Ky) = \ell(xy^{-1})$. We consider the diagram of quotient maps and spaces



Since $p, q, \hat{q}$ are open maps by Proposition 1.17 and Proposition 1.27, $r, \hat{p}$ are open as well. The space $K\backslash G/H$ is metrizable by Lemma 1.44 and in particular paracompact. Since $K\backslash G$ is Čech complete, we conclude from Proposition 2.27 that $K\backslash G/H$ is Čech complete. Then G/H is Čech complete by Corollary 2.26.

Suppose that G is completely metrizable. Then H is also completely metrizable. Moreover, G/H is metrizable by Proposition 1.43 and therefore completely metrizable. □

Polish Groups and the Open Mapping Theorem

A topological space X is called *Polish* if it is separable and completely metrizable. Equivalently, X is second countable and completely metrizable, or metrizable, Čech complete and second countable. A topological group is called Polish if its underlying topology is Polish. For example, the topological group $\mathbb{Z}^{\mathbb{N}}$ is Polish since $\mathbb{Z}$ the discrete group is completely metrizable and second countable.

Lemma 2.35. *A subspace A of a Polish space X is Polish if and only if A is a G_δ-set.*

Proof. Any subspace of a second countable space is second countable. By Proposition 2.23, the subspace $A \subseteq X$ is completely metrizable if and only if it is a G_δ-set. ☐

Proposition 2.36. *Suppose that G is a Polish group and that $H \subseteq G$ is a subgroup. Then H is Polish if and only if H is closed in G. If H is closed, then G/H is also Polish. In particular, G/H is again a Polish group if H is a closed normal subgroup.*

Proof. By Lemma 2.35, a subgroup $H \subseteq G$ is Polish if and only if it is a G_δ-set. In this case, H is closed by Lemma 2.30. The quotient G/H is separable because G is separable and if H is closed, then G/H is completely metrizable by Theorem 2.34. ☐

Our next aim is the Open Mapping Theorem for Polish groups. This requires some preparation.

Lemma 2.37. *Suppose that X is a Polish space. Then there exists a continuous surjective map $\mathbb{N}^{\mathbb{N}} \longrightarrow X$*

Proof. Let d_X be a complete metric on X. On $\mathbb{N}^{\mathbb{N}}$ we use the metric $d(\boldsymbol{a}, \boldsymbol{b}) = 2^{-m}$, where $m = \inf\{k \in \mathbb{N} \mid a_k \neq b_k\}$, with $2^{-\infty} = 0$. Let $Z = \{z_0, z_1, z_2, \ldots\} \subseteq X$ be a countable dense subset. Given $\boldsymbol{a} \in \mathbb{N}^{\mathbb{N}}$, we define a sequence $(s_k^{\boldsymbol{a}})_{k\geq 0}$ as follows. We put $s_0^{\boldsymbol{a}} = z_0$ and

$$s_{k+1}^{\boldsymbol{a}} = \begin{cases} z_{a_k} & \text{if } d_X(z_{a_k}, s_k^{\boldsymbol{a}}) \leq 2^{-k} \\ s_k^{\boldsymbol{a}} & \text{else.} \end{cases}$$

It follows that $d_X(s_k^{\boldsymbol{a}}, x_{k+1}^{\boldsymbol{a}}) \leq 2^k$ and therefore $(s_k^{\boldsymbol{a}})_{k\geq 0}$ is a Cauchy sequence in X. We put $f(\boldsymbol{a}) = \lim_k s_k^{\boldsymbol{a}}$. If $a_k = b_k$ for $k = 0, \ldots, m$, then $s_k^{\boldsymbol{a}} = s_k^{\boldsymbol{b}}$ for $k = 0, \ldots m$ and therefore $d_X(f(\boldsymbol{a}), f(\boldsymbol{b})) \leq 2^{-m+4}$. Hence f is continuous. Given $x \in X$, we define a sequence $\boldsymbol{a}$ by $a_m = \min\{k \in \mathbb{N} \mid d(x, z_k) \leq 2^{-m-1}\}$. Then $f(\boldsymbol{a}) = x$. ☐

We now introduce the Alexandroff–Suslin Operation $\mathcal{A}$.

Definition 2.38. Put $\mathcal{S} = \mathbb{N} \cup \mathbb{N}^2 \cup \mathbb{N}^3 \cup \cdots$ and suppose that P is a collection of sets. A *Suslin scheme* is a map $u : \mathcal{S} \longrightarrow P$, $s \longmapsto u_s$. Given an element $\boldsymbol{a} \in \mathbb{N}^{\mathbb{N}}$, we put

$$u_{\boldsymbol{a}} = u_{(a_0)} \cap u_{(a_0,a_1)} \cap u_{(a_0,a_1,a_2)} \cap \cdots$$

and

$$\mathcal{A}(u) = \bigcup \{u_{\boldsymbol{a}} \mid \boldsymbol{a} \in \mathbb{N}^{\mathbb{N}}\} \subseteq \bigcup P.$$

The map $\mathcal{A} : P^{\mathcal{S}} \longrightarrow 2^{\bigcup P}$ is called the Alexandroff–Suslin operation.

Lemma 2.39. *Let X be a Hausdorff space and suppose that $f : \mathbb{N}^{\mathbb{N}} \longrightarrow X$ is a continuous map. Then there exists a Suslin scheme $u : \mathcal{S} \longrightarrow P$, where P denotes the collection of all closed subsets of X, with $\mathcal{A}(u) = f(\mathbb{N}^{\mathbb{N}})$.*

Proof. For $m \geq 1$ and $s \in \mathbb{N}^m$ we put $E_s = \{\boldsymbol{b} \in \mathbb{N}^{\mathbb{N}} \mid (b_0, \ldots, b_{m-1}) = s\}$, and $u_s = \overline{f(E_s)} \subseteq X$. Thus we have set up a Suslin scheme u.

If $x \in \mathcal{A}(u)$, then there exists $\boldsymbol{a} \in \mathbb{N}^{\mathbb{N}}$ with

$$x \in u_{\boldsymbol{a}} = u_{(a_0)} \cap u_{(a_0,a_1)} \cap u_{(a_0,a_1,a_2)} \cap \cdots .$$

Hence

$$x \in \bigcap_{n \geq 0} \overline{f(\{\boldsymbol{b} \mid b_k = a_k \text{ for } k = 0, \ldots, n\})}.$$

Let V be a closed neighborhood of x. For every $n \geq 0$ there exists an element $\boldsymbol{b}_n \in \mathbb{N}^{\mathbb{N}}$ such that $b_{k,n} = a_k$ for $k = 0, \ldots, n$, and with $f(\boldsymbol{b}_n) \in V$. Now $\lim_n \boldsymbol{b}_n = \boldsymbol{a}$ and therefore $\lim_n f(\boldsymbol{b}_n) = f(\boldsymbol{a})$. It follows that $f(\boldsymbol{a}) \in V$, whence $x = f(\boldsymbol{a})$.

Conversely, if $x = f(\boldsymbol{a})$, then $\boldsymbol{a} \in E_{(a_0,\ldots,a_n)}$ for every $n \geq 0$ and thus $f(\boldsymbol{a}) \in u_{\boldsymbol{a}}$. □

Lemma 2.40. *Suppose that X is a topological space and that $A \subseteq X$ is a subset. Then there exists a Baire measurable subset B with $A \subseteq B \subseteq \overline{A}$ such that every Baire measurable subset $Z \subseteq B - A$ is meager.*

Proof. We put $\boldsymbol{U} = \{U \subseteq X \mid U \cap A \text{ is meager}\}$ and we put $W = \bigcup \boldsymbol{U}$. By Banach's Category Theorem 2.4, the set $Y = A \cap W$ is meager. We put

$$B = (X - W) \cup A = (X - W) \cup Y.$$

Then B is Baire measurable and $A \subseteq B$. The set $X - \overline{A}$ is a member of $\boldsymbol{U}$ and thus $X - W \subseteq \overline{A}$, which shows that $B \subseteq \overline{A}$.

Suppose that $Z \subseteq B - A$ is Baire measurable. Then there exists an open set $V \subseteq X$ such that $M = Z \triangle V$ is meager. Since $Z \cap A = \varnothing$, the set $V \cap A$ is meager and therefore $V \subseteq W$. On the other hand, $Z \subseteq X - W$ and therefore $Z \cap V = \varnothing$, whence $Z \subseteq M$. □

Lemma 2.41. *Suppose that X is a topological space. Let P denote the set of all Baire measurable subsets of X and suppose that $u : \mathcal{S} \longrightarrow P$ is a Suslin scheme. Then $\mathcal{A}(u) \subseteq X$ is Baire measurable.*

Proof. We define a Suslin scheme $v : \mathcal{S} \longrightarrow P$ by putting

$$v_{(n_1,\ldots,n_m)} = u_{(n_1)} \cap u_{(n_1,n_2)} \cap \cdots \cap u_{(n_1,\ldots,n_m)}$$

and we note that $v_{\boldsymbol{a}} = u_{\boldsymbol{a}}$ holds for all $\boldsymbol{a} \in \mathbb{N}^{\mathbb{N}}$, whence $\mathcal{A}(u) = \mathcal{A}(v)$. The Suslin scheme v has the additional property that

$$v_{(n_1)} \supseteq v_{(n_1,n_2)} \supseteq \cdots \supseteq v_{(n_1,\ldots,n_m)} \supseteq \cdots .$$

At this stage is is convenient to introduce formally the empty tuple $s = ()$. We put $E_s = \{\boldsymbol{a} \in \mathbb{N}^{\mathbb{N}} \mid (a_0, \ldots, a_{m-1}) = s\}$, with the understanding that $E_{()} = \mathbb{N}^{\mathbb{N}}$, and we put $v_{()} = X$. For $s \in \mathcal{S} \cup \{()\}$ we put

$$A_s = \bigcup \{v_{\boldsymbol{a}} \mid \boldsymbol{a} \in E_s\}.$$

Then $A_{()} = \mathcal{A}(v)$, and $A_s \subseteq v_s$. By Lemma 2.40 there exists a Baire measurable set B with $A_s \subseteq B$ such that every Baire measurable subset $Z \subseteq B - A_s$ is meager. Then the Baire measurable set $B_s = B \cap v_s \supseteq A_s$ has the same property.

For $n \in \mathbb{N}$ and $s = (n_1, \ldots, n_m)$ we put $s.n = (n_1, \ldots, n_m, n)$ and $().n = (n)$. Then

$$A_s = \bigcup_{n \geq 0} A_{s.n}.$$

The set $M_s = B_s - \bigcup_{n \geq 0} B_{s.n}$ is meager, because $A_s = \bigcup_{n \geq 0} A_{s.n} \subseteq \bigcup_{n \geq 0} B_{s.n}$. Hence $M = \bigcup \{M_s \mid s \in \mathcal{S}\}$ is meager as well. We claim that

$$B_{()} - M \subseteq A_{()} = \mathcal{A}(v).$$

Once we have proved this, it follows that $B_{()} - A_{()} \subseteq M$ is meager, and hence that $A_{()}$ is Baire measurable.

Suppose that $x \in B_{()} - M$. Then $x \in B_{()} - M_{()} = \bigcup_{n \geq 0} B_{(n)}$ and thus $x \in B_{(a_0)}$ for some $a_0 \in \mathbb{N}$. Now $x \in B_{(a_0)} - M_{(a_0)} = \bigcup_{n \geq 0} B_{(a_0,n)}$ and therefore $x \in B_{(a_0,a_1)}$ for some $a_1 \in \mathbb{N}$. Continuing inductively we find $\boldsymbol{a} \in \mathbb{N}^{\mathbb{N}}$ such that

$$x \in B_{(a_0)} \cap B_{(a_0,a_1)} \cap B_{(a_0,a_1,a_2)} \cap \cdots \subseteq v_{\boldsymbol{a}} \subseteq \mathcal{A}(v).$$ □

Proposition 2.42. *Suppose that X is a Polish space, that Y is a Hausdorff space and that $f : X \longrightarrow Y$ is a continuous map. Then $f(X) = A \subseteq Y$ is Baire measurable.*

Proof. By Lemma 2.37, there exist a continuous surjective map $g : \mathbb{N}^{\mathbb{N}} \longrightarrow X$. Hence $A = f(g(\mathbb{N}^{\mathbb{N}}))$. By Lemma 2.39 there exists a Suslin scheme $u : \mathcal{S} \longrightarrow P$, where P denotes the set of closed subsets of Y, with $A = \mathcal{A}(u)$. By Lemma 2.41 the set $\mathcal{A}(u)$ is Baire measurable. □

Theorem 2.43 (The Open Mapping Theorem, II). *Suppose that G is a Polish group, that K is a Hausdorff topological group and that $f : G \longrightarrow K$ is a continuous homomorphism. If $f(G)$ is not meager, then f is open.*

Proof. We assume that $f(G)$ is not meager and we put $N = \ker(f)$. Replacing G by the Polish group G/N, we may assume that f is injective. By Proposition 2.42, $f(G)$ is Baire measurable. Since $f(G)$ is not meager, $f(G)$ is open by Pettis' Lemma 2.9. If $U \subseteq G$ is an identity neighborhood, then there exist a closed identity neighborhood $V \subseteq G$ with $V^{-1}V \subseteq U$. Then $E = f(V)$ is Baire measurable by Proposition 2.42. Since G is separable, $f(G)$ is a countable union of translates of E, and therefore E is not meager. Hence $E^{-1}E \subseteq f(U)$ is an identity neighborhood in K. It follows that the inverse of the corestriction $f|^{f(G)} : G \longrightarrow f(G)$ is continuous at the identity, and hence continuous everywhere by Lemma 1.5. Hence f is open. □

Corollary 2.44 (The Closed Graph Theorem, II). *Suppose that G and K are Polish groups and that $f : G \longrightarrow K$ is a group homomorphism. Then the following are equivalent.*
(i) *f is continuous.*
(ii) *The graph of f is closed in $G \times K$.*
(iii) *The graph of f is a G_δ-set in $G \times K$.*

Proof. The graph of a continuous map is closed, hence (i) $\Rightarrow$ (ii). A closed set in a metric space is a G_δ-set, hence (ii) $\Rightarrow$ (iii). Suppose that $H = \{(g, f(g)) \mid g \in G\}$ is a G_δ-set in $G \times K$. Then H is a Polish group by Lemma 2.35. The map $h : H \longrightarrow G$ that maps $(g, f(g))$ to g is continuous and bijective and hence open by Theorem 2.43. Hence the map $g \longmapsto (g, f(g))$ is continuous, and therefore f is continuous. □

Theorem 2.45 (The Open Mapping Theorem, III). *Suppose that $f : G \longrightarrow K$ is a morphism of topological groups, that G is Polish, and that K is metrizable. If for every identity neighborhood $V \subseteq G$, the set $\overline{f(V)}$ has nonempty interior, then f is open.*

Proof. In view of Theorem 2.43 it suffices to prove that $f(G)$ is not meager. Suppose to the contrary that $f(G)$ is meager. Then there exist closed sets A_n with empty interiors, for $n \geq 1$, with $f(G) \subseteq \bigcup_{n \geq 1} A_n$. The sets $B_n = f^{-1}(A_n) \subseteq G$ are closed, and $G = \bigcup_{n \geq 1} B_n$. Since G is a nonempty Baire space, at least one set B_m contains a nonempty open set $U \subseteq B_m$. Let $g \in U$. Then $g^{-1}U$ is an identity neighborhood in G and therefore $W = \overline{f(g^{-1}U)}$ is a closed identity neighborhood in K. But then $\overline{f(U)} = f(g)W \subseteq A_m$ is a neighborhood of $f(g)$, a contradiction. Therefore $f(G)$ is not meager and hence f is open. □

Bibliography

[1] S. Banach, Théorème sur les ensembles de première catégorie, Fundamenta Mathematicae **16** (1930), 395–398.

[2] Y. Cornulier and P. de la Harpe, *Metric geometry of locally compact groups*, EMS Tracts in Mathematics, 25, European Mathematical Society (EMS), Zürich, 2016. MR3561300

[3] A. Deitmar and S. Echterhoff, *Principles of harmonic analysis*, second edition, Universitext, Springer, Cham, 2014. MR3289059

[4] J. Dugundji, *Topology*, Allyn and Bacon, Inc., Boston, MA, 1978. MR0478089

[5] E. Hewitt and K. A. Ross, *Abstract harmonic analysis. Vol. I*, second edition, Grundlehren der Mathematischen Wissenschaften, 115, Springer-Verlag, Berlin, 1979. MR0551496

[6] K. H. Hofmann and S. A. Morris, *The structure of compact groups*, third edition, revised and augmented., De Gruyter Studies in Mathematics, 25, De Gruyter, Berlin, 2013. MR3114697

[7] K. H. Hofmann and S. A. Morris, Open mapping theorems for topological groups, preprint 2009.

[8] I. Kaplansky, *Lie algebras and locally compact groups*, reprint of the 1974 edition, Chicago Lectures in Mathematics, University of Chicago Press, Chicago, IL, 1995. MR1324106

[9] C. Kuratowski, *Topologie. I et II*, reprint of the fourth (Part I) and third (Part II) editions, Éditions Jacques Gabay, Sceaux, 1992. MR1296876

[10] J. C. Oxtoby, *Measure and category*, second edition, Graduate Texts in Mathematics, 2, Springer-Verlag, New York, 1980. MR0584443

[11] B. J. Pettis, On continuity and openness of homomorphisms in topological groups, Ann. of Math. (2) **52** (1950), 293–308. MR0038358

[12] M. Stroppel, *Locally compact groups*, EMS Textbooks in Mathematics, European Mathematical Society (EMS), Zürich, 2006. MR2226087

[13] T. Tao, *Hilbert's fifth problem and related topics*, Graduate Studies in Mathematics, 153, American Mathematical Society, Providence, RI, 2014. MR3237440

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