~~start~~ Solidity is an adjective for condensed abelian groups that is analogous to "non-Archimedean completeness". (1)

To define it we're going to declare a family of free solid abelian groups & then use them to generate all other solid abelian groups.

$$\mathbb{Z}[S]^{\blacksquare} := \varprojlim \mathbb{Z}[S_i] \quad \text{for } S = \varprojlim S_i.$$

as $\mathrm{ProF} = \mathrm{Pro}-\mathrm{Object}(\mathrm{Fin\,Sets})$ & so $\mathbb{Z}[S]^{\blacksquare}$ is indep. of how we present S as an inverse limit of finite sets as any two such ways of presenting S are equivalent as formal projective systems. ($S = \varprojlim_i S_i$, a co-filtered limit of object in Fin(Sets))

(For $\mathcal{F} \in Sh(\mathrm{ProF})$, the nat. arrow $\varinjlim_n \mathbb{Z}[\mathcal{F}]_{\leq n} \to \mathbb{Z}[\mathcal{F}]$ is iso).

From there we have $\mathbb{Z}[S] = \varinjlim_n \mathbb{Z}[S]_{\leq n}$.

$\mathbb{Z}[S] \to \varprojlim_i \mathbb{Z}[S_i]$ is one-one.

Prop^n 9.4.2 :- Nat. map $\mathbb{Z}[S] \longrightarrow \varprojlim_i \mathbb{Z}[S_i]$ is one-one. It's image is precisely the condensed abelian group

$$\bigcup_n \varprojlim_i \mathbb{Z}[S_i]_{\leq n}.$$

→ Requires some results about pre-sheaves etc.

Eg:- Consider $S = \mathbb{N} \cup \{\infty\} \Rightarrow S \in \mathrm{ProF}$. So S is obtained as ~~the~~ $S = \varprojlim S_i$ where $S_i = \{0, 1, \ldots, i-1, \infty\}$ where

$$S \to S_i : \begin{cases} \{0, 1, \ldots, i-1\} \xmapsto{id} \{0, 1, \ldots, i-1\} \\ \text{otheres} \longmapsto \{\infty\}. \end{cases}$$

$$\therefore \mathbb{Z}[S]^{\blacksquare} / ([\infty] = 0) = \varprojlim \mathbb{Z}[S_i] / ([\infty] = 0)$$

$$= \varprojlim \mathbb{Z}[x] / (x^n) = \mathbb{Z}[[x]].$$

Now, let's define Solid (Ab) as a full sub-cat. of Cond (Ab)

Def^n :- Let $M \in \mathrm{Cond}(Ab)$. $\exists$ a nat. restriction map for any $S \in ED$

$$\mathrm{Hom}_{\mathrm{Cond}(Ab)}(\mathbb{Z}[S]^{\blacksquare}, M) \longrightarrow \mathrm{Mor}_{\mathrm{Cond}(Ab)}(S, M)$$

Say M is "solid" if this restriction map is an iso $\forall S \in ED$.

Rmk :- ~~Later we~~ M solid $\Rightarrow$ the map is iso $\forall S \in \mathrm{Prof}$ as well ($ED \subset \mathrm{Prof}$).

Prop^n 9.1.6 :- $M \in \mathrm{Cond}(Ab)$ solid $\Leftrightarrow$ $\forall S \in \mathrm{Prof}$ the nat. map

$$\underline{\mathrm{Hom}}_{\mathrm{Cond}(Ab)}(\mathbb{Z}[S]^{\blacksquare}, M) \longrightarrow \underline{\mathrm{Hom}}_{\mathrm{Cond}(Ab)}(\mathbb{Z}[S], M)$$

is an iso.

Pf :- Immediate from def^n of solid & adjunction iso :-

$$\underline{\mathrm{Hom}}_{\mathrm{Cond}(Ab)}(\mathbb{Z}[S]^{\blacksquare}, M) \longrightarrow \underline{\mathrm{Hom}}_{\mathrm{Cond}(Ab)}(\mathbb{Z}[S], M) \quad \blacksquare$$

Def^n :- A complex $C_\bullet \in \mathcal{D}(\mathrm{Cond}(Ab))$ is derived solid (or just "solid" if its understood we're using a complex) if $\forall S \in \mathrm{Prof}$, the natural map

$$R\underline{\mathrm{Hom}}(\mathbb{Z}[S]^{\blacksquare}, C_\bullet) \to R\underline{\mathrm{Hom}}(\mathbb{Z}[S], C) \text{ is an iso.}$$

Rmk :- ~~A~~ we can replace $\underline{\mathrm{Hom}}$ above.

Prop^n 9.1.9 :- If $X \in \mathrm{Cond}(Ab)$ is such that the one-term complex $X \in \mathcal{D}(\mathrm{Cond}(Ab))$ is derived solid, then X is solid. (No Proof to be given).

Pf :- We've natural map:

$$R\underline{\mathrm{Hom}}(\mathbb{Z}[S]^{\blacksquare}, X) \to R\underline{\mathrm{Hom}}(\mathbb{Z}[S], X) \quad \forall S \in ED.$$

is iso in derived cat.

$\mathbb{Z}[S]$ projective $\Rightarrow$ $R\underline{\mathrm{Hom}}(\mathbb{Z}[S], X) = \underline{\mathrm{Hom}}(\mathbb{Z}[S], X)$

Take 0-th coho of two iso complexes $\Rightarrow$ $\underline{\mathrm{Hom}}(\mathbb{Z}[S]^{\blacksquare}, X) = \underline{\mathrm{Hom}}(\mathbb{Z}[S], X)$.

③

" Solidification as measures" ;—

Interpretation of $\mathbb{Z}[S]$ as a space of measures ;—

We claim :— $\exists$ a nat. iso

$$\mathbb{Z}[S]^{■} \cong \underline{\mathrm{Hom}}_{\mathrm{Cond}(Ab)}(C(S,\mathbb{Z});\mathbb{Z}) \quad (9.2.1).$$

vvI
Here $C(S,\mathbb{Z})$ is viewed as Haus. abelian topo. grp with discrete topo. Why discrete ?
Well if S finite $\Rightarrow$ $C(S,\mathbb{Z})$ have discrete topo.

Generally, discrete topo is chosen so that the following holds :—

Lemma 9.2.2 :— The nat map $\phi : \varinjlim_i C(S_i,\mathbb{Z}) \to C(S,\mathbb{Z})$ is iso of topo groups.

Pf :— Check ϕ is ~~to~~ group map.

$\because$ Each $C(S_i,\mathbb{Z})$ discrete $\forall i$ $\Rightarrow$ colim $\varinjlim_i C(S_i,\mathbb{Z})$ is also discrete (by def^n of the topo. on colim).

As $C(S,\mathbb{Z})$ is discrete $\Rightarrow$ To show ϕ is iso of topo grps, enough ts that ϕ is a set bij^n.

Check :— ϕ is one - one (by def^n.)

Pf of onto :— Equivalently we can show that any C^0-map $f : S \to \mathbb{Z}$ factors via some S_i. $S \in \mathrm{Prof}$ $\Rightarrow$ S cpt & so $f(S) \subset \mathbb{Z}$ finite. But then $f : S \to f(S)$ is a morphism of profinite sets, & hence it must factor via some S_i (recall the equivalence b/w ~~cat. of pro-finite~~ Prof & Pro-objects (FinSets) & use def^n of Homsets in the later category).

Pf of 9.2.1:– First assume S is finite, at end we'll use (9.2.1.) $\forall S \in \mathrm{Prof}$.
a limit to prove

S finite $\Rightarrow$ $\mathbb{Z}[S]^{■} = \mathbb{Z}[\underline{S}] = \underline{\mathbb{Z}[S]}$ (Prop^n 5.7.5 in Barz)

(I guess it was discu. in last talk with $\eta^{pre} : \mathbb{Z}[S] \to \mathbb{Z}[S]^{pre}$ & $\underline{S}(E) = C(E, S)$] $\mathbb{Z}[S](T) = C(T, \mathbb{Z}[S])$)

S finite $\Rightarrow$ $C(S, \mathbb{Z}) = \prod_S \mathbb{Z} = \mathbb{Z}[S]$ as "abelian grps" & so

$$\mathbb{Z}[S]^{■} = \underline{C(S, \mathbb{Z})}$$

But again using finiteness of S, $\exists$ an iso of abelian grps

$$C(S, \mathbb{Z}) \cong \mathrm{Hom}(C(S, \mathbb{Z}), \mathbb{Z})$$

$\left(\because \mathrm{Hom}\left(\prod_S \mathbb{Z}, \mathbb{Z}\right) \cong \prod_S \mathbb{Z}\right.$ as index set is finite$\left.\right)$

and the above induces an iso of Cond. abelian groups:

$$\mathbb{Z}[S]^{■} \approx \underline{\mathrm{Hom}(C(S, \mathbb{Z}), \mathbb{Z})}$$

Thm 5.8.4 $\Rightarrow$ $\mathbb{Z}[S]^{■} \cong \underline{\mathrm{Hom}}(C(S, \mathbb{Z}), \mathbb{Z})$ whenever S finite.

General

For $S \in \mathrm{Prof}$, we've

$$\mathbb{Z}[S]^{■} = \varprojlim_i \mathbb{Z}[S_i] \approx \varprojlim \underline{\mathrm{Hom}}(C(S_i, \mathbb{Z}), \mathbb{Z})$$

Also $\underline{\mathrm{Hom}}$ & $\varprojlim$ commute ie for $\{X_i\}$, $Y \in \mathrm{Cond}(\mathrm{Ab})$,

$$\varprojlim_i \underline{\mathrm{Hom}}(X_i, Y) = \underline{\mathrm{Hom}}(\varinjlim_i X_i, Y)$$

(refer Lemma 9.2.3, Barz)

~~Note~~ (crux is tensor product is left adjoint & hence commutes with colimits).

Thus $\mathbb{Z}[S]^{■} \approx \underline{\mathrm{Hom}}(\varinjlim_i C(S_i, \mathbb{Z}), \mathbb{Z})$.

Already seen: $\varinjlim_i C(S_i, \mathbb{Z}) = C(S, \mathbb{Z})$: true ~~for~~ at level of topo abelian grps.

To show for level of cond. ab. grps :-

Lemma 9.2.4: $\{A_i\}$ is a directed system of discrete topo abelian grps, then $A = \varinjlim_i A_i$ is a discrete topo abelian grp. Also in Cond(Ab)

$$\varinjlim_i \underline{A_i} = \underline{A}$$

Corollary: $S \in \text{Prof} \Rightarrow \mathbb{Z}[S]^{\blacksquare} \simeq \underline{Hom}(C(S, \mathbb{Z}), \mathbb{Z})$

vvI "viewing $C(S, \mathbb{Z})$ as discrete topo abelian group.

Def^n: If $S \in \text{Prof}$, define

$M(S, \mathbb{Z}) := Hom(C(S, \mathbb{Z}), \mathbb{Z})$ as the grp of $\mathbb{Z}$-valued measures on S.

Prop^n 9.2.7: If $S \in ED$ & $A \in$ solid, then $\forall \mu \in M(S, \mathbb{Z})$ one can & $\forall$ ~~f~~ $f : S \to A$ (a morphism of condensed sets) define $\int f\mu \in A(*)$. So, we can integrate.

Pf: $f \rightsquigarrow f : \mathbb{Z}[S] \to A$ extending uniquely to $\tilde{f} : \mathbb{Z}[S]^{\blacksquare} \to A$ as A is solid. & $\tilde{f}(*)$ gives an intg. map.

Lemma: In Cond(Ab),

$$\underline{Hom}(\bigoplus_i A_i, B) = \prod_i \underline{Hom}(A_i, B)$$ (No pf. to be given)

Pf: $\forall E \in ED$,

$$\underline{\mathrm{Hom}}\Big(\bigoplus_i A_i, B\Big)(E) = \mathrm{Hom}\Big(\mathbb{Z}[E] \otimes \bigoplus_i A_i, B\Big)$$
$$= \mathrm{Hom}\Big(\bigoplus_i (\mathbb{Z}[E] \otimes A_i), B\Big)$$
$$= \prod_i \mathrm{Hom}(\mathbb{Z}[E] \otimes A_i, B)$$
$$= \prod_i \underline{\mathrm{Hom}}(A_i, B)(E).$$

<u>Thm 5.2.5</u>:— $\forall S \in \mathrm{Prof}$, $\exists$ a set I with $|I| \leq 2^{|S|}$ st $\mathbb{Z}[S]^{\blacksquare} \cong \prod_I \mathbb{Z}$ where prod. is taken in the catg of condensed abelian grps.

$\to$ $S \in \mathrm{Prof} \Rightarrow C(S, \mathbb{Z})$ is free abelian

$\therefore$ $\mathbb{Z}[S]^{\blacksquare} \cong \underline{\mathrm{Hom}}(C(S,\mathbb{Z}), \mathbb{Z}) \cong \underline{\mathrm{Hom}}\Big(\bigoplus_I \mathbb{Z}, \mathbb{Z}\Big) = \prod_I \mathbb{Z}$. with ($|I| \leq 2^{|S|}$)

<u>Thm 2.3.1</u>:— $S \in \mathrm{Prof} \Rightarrow$ the cond. ab. grps $\mathbb{Z}[S]^{\blacksquare}$ is both solid & derived solid.

$\to$ ~~$\forall T \in$~~ for any $T \in \mathrm{Prof}$, we've a natural morph
$$R\underline{\mathrm{Hom}}(\mathbb{Z}[T]^{\blacksquare}, \mathbb{Z}[S]^{\blacksquare}) \to R\underline{\mathrm{Hom}}(\mathbb{Z}[T], \mathbb{Z}[S]^{\blacksquare})$$

Thm 5.2.5 $\Rightarrow$ $\mathbb{Z}[S] = \prod_I \mathbb{Z}$ for index set I.

$\therefore$ $R\underline{\mathrm{Hom}}(\mathbb{Z}[T]^{\blacksquare}, \mathbb{Z}[S]^{\blacksquare}) = \bigoplus_I R\underline{\mathrm{Hom}}(\mathbb{Z}[T]^{\blacksquare}, \mathbb{Z})$

& $R\underline{\mathrm{Hom}}(\mathbb{Z}[T], \mathbb{Z}[S]^{\blacksquare}) = \bigoplus_I R\underline{\mathrm{Hom}}(\mathbb{Z}[T], \mathbb{Z})$.

<u>ETS</u>:— The result holds for $S = *$ aka $\mathbb{Z}[S]^{\blacksquare} = \mathbb{Z}$.

The object $R\underline{\mathrm{Hom}}(\mathbb{Z}[T], \mathbb{Z})$ is quite simple.

From our coho ~~conditions~~, calculation,

$$Ext^i(\mathbb{Z}[T], \mathbb{Z}) = H^i(T, \mathbb{Z}) = 0 \quad \forall i > 0$$

$$Ext^0(\mathbb{Z}[S], \mathbb{Z}) = Hom(\mathbb{Z}[T], \mathbb{Z}) = C(T, \mathbb{Z}).$$

$$\therefore RHom(\mathbb{Z}[S], \mathbb{Z}) = C(T, \mathbb{Z})[0]$$

Now, we compute

$$RHom(\mathbb{Z}[T]^{■}, \mathbb{Z}).$$

Firstly, $\mathbb{Z}[T]^{■} = \prod_J \mathbb{Z}$ ~~and~~ & Thm 9.2.5 ⇒

Now thm 8.6.1 ⇒ ($RHom(\prod_J \mathbb{Z}, \mathbb{Z}) = \bigoplus_J \mathbb{Z}[0]$).

$$\Rightarrow RHom(\mathbb{Z}[T]^{■}, \mathbb{Z}) = RHom(\prod_J \mathbb{Z}, \mathbb{Z}) = \bigoplus \mathbb{Z}[0]$$

(From last talk)

But $C(T, \mathbb{Z}) = \bigoplus_J \mathbb{Z}$ thus done ■

Prop^n 9.3.24 Let $\mathcal{K}$ is a cat with

$$Obj(\cancel{\mathcal{R}})(\mathcal{K}) = Mor_{Cond(CAb)}(S \to A) \text{ with } A \in Solid(Ab)$$

Then $S \to \mathbb{Z}[S]^{■}$ is initial in this catg. ie

every $S \to A$ factors via $S \to \mathbb{Z}[S]^{■}$.

Pf A solid $\Rightarrow Mor(S, A) = Hom(\mathbb{Z}[S], A)$

$$= Hom(\mathbb{Z}[S]^{■}, A)$$

■.

Rmk $\forall$ prime p, $\mathbb{Z}_p$ is solid.

Rmk $\mathbb{Z}[[T]] = \prod_{\mathbb{N}} \mathbb{Z}$ solid. (Don't tell).

Thm :

If X is pro-finite, then $C(X, \mathbb{Z})$ is a free $\mathbb{Z}$-module. (Barz, Pg 27)

$\rightarrow$ we're $X = \varprojlim X_i$ for some finite sets X_i.

$\because X \in \text{prof} \Rightarrow X$ cpt & hence any continuous function into $\mathbb{Z}$ has finite image, means it must factor via one of the X_i's. The indicator functions of various pts. in X_i induce continuous functions $X \to \mathbb{Z}$ & it's clear that these are spanning, & it seems plausible that they might be independent, if we take things in stages; if our index set for the inverse limit were well-ordered.

(Bergman) Find a cont. ~~function~~ injection $X \hookrightarrow \prod_I \{0,1\}$

To see such is possible, note if $X = \varprojlim X_i$, then $X \hookrightarrow \prod X_i$, & then we can continuously embed a finite set into a product of copies of $\{0,1\}$

Well order I, identify with ordinal λ; elts$(I) \leftrightarrow \mu$ for $\mu < \lambda$.

For each $\mu < \lambda$, we get an element of I which is $e_\mu : X \xrightarrow{C^0} \{0,1\}$.

Order the "products" $e_{\mu_1} \cdots e_{\mu_r}$ with $\mu_1 > \cdots > \mu_r$ lexicographically, including the empty product for $r = 0$.

Let E be the set of all such products that are not linear combination of smaller products. We show that E is a basis of $C(X, \mathbb{Z})$ using "transfinite induction" on λ. If $\lambda = 0 \Rightarrow$ claim trivial.

(6) If λ is a limit ordinal, then the result is true for all $\mu < \lambda$, then let $S_\mu := \mathrm{Im}(S \hookrightarrow \prod_{\mu' < \mu} \{0,1\})$

Then E_μ be a basis of S_μ, & $E = \bigcup_\mu E_\mu$.

Also, $S = \varinjlim_{\mu < \lambda} S_\mu$. As each E_μ is spanning

we find that E is "spanning" in the limit.

E is independent by construction.

Now let $\lambda = \rho + 1$, a successor ordinal. Let $S' = S_\rho$.

We have closed immersions (due to compactness) $S \hookrightarrow S' \times \{0,1\}$,

so that the projections $S \to S'$ of S is onto.

TE/ $0 \to C(S', \mathbb{Z}) \to C(S, \mathbb{Z})$.

<u>Cokernel of the above ??</u>

Let $T \subset S'$ be $T := \left\{ t \in S' \;\middle|\; \begin{array}{l} (t,0) \in S \text{ and} \\ (t,1) \in S \end{array} \right\}$

<u>Claim</u> $0 \to C(S', \mathbb{Z}) \to C(S, \mathbb{Z}) \to C(T, \mathbb{Z}) \to 0$

exact.

$f \mapsto g$ where $g(t) = f(t,0) - f(t,1)$

Indeed $g = 0 \Leftrightarrow f$ was induced from a map over S' &

that $f \mapsto g$ is onto as for any $g : T \to \mathbb{Z}$, we can define $f(t,0) = g(t)$ & then 0 everywhere else on S.

To see such a def^n is cont, enough to check $S \setminus (T \times \{0\})$ is open which is obvious.

$\therefore \ \forall n \in \mathbb{Z}\setminus\{0\}$,

$$f^{-1}(n) = g^{-1}(n) \times \{0\} \text{ open by continuity of } f.$$

$$\& \ f^{-1}(0) = \underbrace{\left(S \setminus (T \times \{0\})\right)}_{\text{open}} \cup \underbrace{g^{-1}(0)}_{\text{open}}$$

$$\Downarrow$$

$$f^{-1}(0) \text{ open}.$$

Induction hypothesis $\Rightarrow$ $C(S', \mathbb{Z}) = C(S_p, \mathbb{Z})$ is free, with basis $E_p \subset E$ where there are products not containing e_p in their multiplicand ie ~~product~~ as a term).

Additionally, the products containing e_p must start with e_p ($\because$ we order all terms in a product) where recall here that $e_p : S \to \{0,1\}$ is the map obtained by

$$S \hookrightarrow S' \times \{0,1\} \longrightarrow \{0,1\}$$

But a prod $e_p e_{\mu_{(r-1)}} \cdots e_{\mu_1}$ projects to the map $T \to \mathbb{Z}$ given by $e_{\mu_{(r-1)}} \cdots e_{\mu_1}$.

Hence, Ind^n hypo applied to T $\Rightarrow$ The products project onto a basis of $C(T, \mathbb{Z})$.

$C(T,\mathbb{Z})$ free $\Rightarrow$ exact sequence split hence $C(S,\mathbb{Z})$ free with basis being $E_p \cup \{$terms which start with e_p as a factor$\} = E$ ■

Thm 5.8:

(i) The cate. Solid(Ab) $\subset$ Cond(Ab) is an abelian sub-cate stable under all limits & co-limits & extensions. The objects $\prod_I \mathbb{Z} \in$ Solid(Ab) form a family of compact projective generators for any set I.

And, Solid(Ab) $\subset$ Cond(Ab) admits a rt.-adjoint

$$M \to M^{\blacksquare} : \text{Cond}(Ab) \to \text{Solid}(Ab)$$

that is the ! colimit-preserving extension of $\mathbb{Z}[S] \mapsto \mathbb{Z}[S]^{\blacksquare}$.

(ii) Moreover, $D(\text{Solid}(Ab)) \to D(\text{Cond}(Ab))$ induced by the map Solid(Ab) $\hookrightarrow$ Cond(Ab) is fully faithful & describes its images (which are precisely the objects of $\mathcal{D}(\text{Cond}(Ab))$.

An object $C \in \mathcal{D}(\text{Cond}(Ab))$ is solid $\iff$ $H^i(C) \in$ Solid(Ab) $\forall i \in \mathbb{N}_0$

Let $u \in L^1_{loc}(\Omega)$ & $\alpha \in \mathbb{N}^n$ be a multi-index. Then $v \in L^1_{loc}(\Omega)$ is called the α-th weak derivative of u usually written as $D^\alpha u = v$ if

$$\int_\Omega u D^\alpha \phi = (-1)^{\alpha} \int_\Omega v \cdot \phi \quad \forall \phi \in C_c^\infty(\Omega)$$

$n = 1$, $\Omega = (0,2)$

$$u(x) = \begin{cases} x & 0 \le x \le 1 \\ 1 & 1 \le x \le 2 \end{cases}$$

then

$$v(x) = \begin{cases} 1 & 0 \le x \le 1 \\ 0 & 1 \le x \le 2 \end{cases}$$

wk derivative of u,

Indeed for $\phi \in C_c^\infty((0,2))$, we've

$$\int_0^2 u(x) \frac{\partial \phi}{\partial x} dx = \int_0^1 x \frac{\partial \phi}{\partial x} dx + \int_1^2 \frac{\partial \phi}{\partial x} dx = -\int_0^1 \phi + [x\phi]_0^1 + [\phi]_1^2$$

$$= -\int_0^1 \phi(x) = -\int_0^2 v\phi \quad \text{QED}$$