"Cohomology of condensed LCA groups"

My condensed gps talk. 1

13.6.23

Cf. Master's thesis of Dagur Ásgeirsson ⊛ Internal hom

For condensed abelian gps M, N, we define the <u>tensor product</u> $M \otimes N$ as sheafification of $ED \to Ab$, $S \mapsto M(S) \otimes N(S)$.

<u>Recall</u> adjoint functors $\mathbb{Z}[-] : CondSet \rightleftarrows CondAb : \text{forget}$.

For $T \in CondSet$, $\mathbb{Z}[T]$ is flat (even $\mathbb{Z}[T(S)]$ free) with respect to tensor product. Moreover $\mathbb{Z}[T_1 \times T_2] = \mathbb{Z}[T_1] \otimes \mathbb{Z}[T_2]$.

As $- \otimes A$ commutes with colimits in ~~Ab~~ Ab (and sheafification is a left adjoint), so does $- \otimes M$ in CondAb. Thus we get adjunction

$$- \otimes M : CondAb \rightleftarrows CondAb : \underline{Hom}(M, -)$$

That is, by definition

$$Hom_{CondAb}(N \otimes M, P) \simeq Hom_{CondAb}(N, \underline{Hom}(M, P)).$$

<u>Recall</u> (Yoneda) If $A \in \mathcal{C}$, $G : \mathcal{C}^{op} \to Set$, then $Nat(Hom(-, A), G) \cong G(A)$.

Thus
$$\begin{aligned} \underline{Hom}(M,N)(S) &= Hom_{CondSet}(\underline{S}, \underline{Hom}(M,N)) && \text{(Yoneda)} \\ &= Hom_{CondAb}(\mathbb{Z}[\underline{S}], \underline{Hom}(M,N)) && \text{(adj.)} \\ &= Hom_{CondAb}(\mathbb{Z}[\underline{S}] \otimes M, N) && \text{(adj.)} \end{aligned}$$

⊛ Internal hom: if A, B are abelian topological gps, then so is $Hom(A, B)$ (with compact-open top)

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Proposition 4.2 Let A and B be Hausdorff topological abelian gps and assume that A is compactly generated. Then there is a natural iso$^{\cong}$ of condensed abelian gps
$$\underline{\mathrm{Hom}}(\underline{A}, \underline{B}) \cong \underline{\mathrm{Hom}(A,B)}$$
(where $\mathrm{Hom}(A,B)$ has$^{\text{as usual}}$ compact-open topology).

Pf We construct $\eta : \underline{\mathrm{Hom}}(\underline{A}, \underline{B}) \to \underline{\mathrm{Hom}(A,B)}$.

For $S \in \mathrm{Prof}$, we have by previous computation
$$\underline{\mathrm{Hom}}(\underline{A}, \underline{B})(S) = \mathrm{Hom}_{\mathrm{Cond\,Ab}}(\underline{A} \otimes \mathbb{Z}[\underline{S}], \underline{B}).$$

A map $f : \underline{A} \otimes \mathbb{Z}[\underline{S}] \to \underline{B}$ gives, by evaluating at $*$, a map $f : A \otimes \mathbb{Z}[S] \to B$. We define
$$\eta(f) : S \to \mathrm{Hom}(A,B) : s \mapsto (a \mapsto f(a \otimes s)).$$

Check: $\eta(f)$ is continuous (so it's really in $\underline{\mathrm{Hom}(A,B)}(S)$).
// also need $\eta(-)$ is cts?? no! sheaf of abstract abelian gps //

Injectivity is immediate.
Surjectivity: a map $S \to \mathrm{Hom}(A,B)$ corresponds to a map $\mathbb{Z}[\underline{A}] \otimes \mathbb{Z}[\underline{S}] \to \underline{B}$.
We need to show it factors through $\underline{A} \otimes \mathbb{Z}[\underline{S}]$.

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For any abelian grp G we have a partial resolution

$$\mathbb{Z}[G \times G] \to \mathbb{Z}[G] \to G \to 0$$
$$[(g_1, g_2)] \mapsto [g_1 + g_2] - [g_1] - [g_2]$$

$\rightsquigarrow$ also for condensed abelian grps, so have

$$\mathbb{Z}[\underline{A} \times \underline{A}] \to \mathbb{Z}[\underline{A}] \to \underline{A} \to 0.$$

Since tensor product is right exact, have exact sequence

$$\mathbb{Z}[\underline{A} \times \underline{A}] \otimes \mathbb{Z}[\underline{S}] \to \mathbb{Z}[\underline{A}] \otimes \mathbb{Z}[\underline{S}] \to \underline{A} \otimes \mathbb{Z}[\underline{S}] \to 0.$$

As $\mathrm{Hom}_{\mathrm{condAb}}(-, \underline{B})$ is left exact, have

$$0 \to \mathrm{Hom}_{\mathrm{condAb}}(\underline{A} \otimes \mathbb{Z}[\underline{S}], \underline{B}) \to \mathrm{Hom}_{\mathrm{condAb}}(\mathbb{Z}[\underline{A}] \otimes \mathbb{Z}[\underline{S}], \underline{B})$$
$$\to \mathrm{Hom}_{\mathrm{condAb}}(\mathbb{Z}[\underline{A} \times \underline{A}] \otimes \mathbb{Z}[\underline{S}], \underline{B}).$$

So: RTP final map is zero ie composition

$$\begin{array}{ccccc} \mathbb{Z}[\underline{A} \times \underline{A}] \otimes \mathbb{Z}[\underline{S}] & \to & \mathbb{Z}[\underline{A}] \otimes \mathbb{Z}[\underline{S}] & \to & \underline{B} \\ \| & & \| & & \\ \mathbb{Z}[\underline{A} \times \underline{A} \times \underline{S}] & & \mathbb{Z}[\underline{A} \times \underline{S}] & & \end{array}$$

is zero.

This corresponds to composition $A \times A \times S \to A \times S \to B$.

But $(a_1, a_2, s) \mapsto (0, s) \mapsto 0$. ✓

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Thm 4.3 Let $A = \underline{\prod_I \mathbb{T}} \in \mathrm{CondAb}$ (I infinite allowed).

(i) For any discrete condensed abelian gp M (i.e. $M = \underline{B}$ for B a discrete abelian gp),

$$R\underline{\mathrm{Hom}}(A, M) = \bigoplus_I M[-1]$$

concentrated in degree 1
same for $H^i_{\mathrm{CondAb}}(A,B) = H^i(R\underline{\mathrm{Hom}}(A,M))$

where the iso$^{\underline{m}}$ is induced by the maps

$$M[-1] = R\underline{\mathrm{Hom}}(\underline{\mathbb{Z}}[1], M) \to R\underline{\mathrm{Hom}}(\underline{\mathbb{R}/\mathbb{Z}}, M) \to R\underline{\mathrm{Hom}}(A, M).$$

where the last map is induced by projection to i^{th} factor $\prod_I \mathbb{T} \to \mathbb{T}$.

(ii) $R\underline{\mathrm{Hom}}(A, \underline{\mathbb{R}}) = 0$.

Rmk Hoffmann-Spitzweck (2007) define $D^b(LCA)$, although LCA is only quasi-abelian. The thm allows one to prove that $D^b(LCA) \to D(\mathrm{CondAb})$ is fully faithful.

Recall (Structure thm) A LCA $\Rightarrow$ $\exists$ LCA A' w/ a compact open subgp s.t. $A \cong \mathbb{R}^n \times A'$.

Proof sketch of (i). Do case $|I| < \infty$, equivalently $|I| = 1$.

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We have a distinguished triangle $(\underline{\mathbb{Z}}, \underline{\mathbb{R}}, \underline{\mathbb{R}/\mathbb{Z}})$

$\Rightarrow$ ~~so~~ also $(\underline{\mathbb{R}}, \underline{\mathbb{R}/\mathbb{Z}}, \underline{\mathbb{Z}}[1])$

$\Rightarrow$ also

$$R\underline{\mathrm{Hom}}(\underline{\mathbb{Z}}[1], M) \to R\underline{\mathrm{Hom}}(\underline{\mathbb{R}/\mathbb{Z}}, M) \to R\underline{\mathrm{Hom}}(\underline{\mathbb{R}}, M)$$

Spectral sequence argument (~~and~~ [illegible] writing $\mathbb{R}^r = \varinjlim [-n,n]^r$ as direct limit of compact spaces)

shows that $R\underline{\mathrm{Hom}}(\underline{\mathbb{R}}, M) = 0$ $\square$.