Talk: Condensed cohomology

Sources: Scholze's script + Dagur Ásgeirsson's master thesis

We have seen a way to send CHaus to Cond Sets by $X \to \underline{X}$ with good categorial properties. Now we want ~~construct~~ to see that we can detect some of the topological invariants on the condensed sets.

Main focus: how to compute the condensed cohomology groups

~~[illegible]~~ $\mathcal{A}$ is always an abelian category

Def. 1: A cochain complex A° in $\mathcal{A}$ is given by:

- a collection $(A_n)_{n\in\mathbb{Z}}$
- morphisms $d^n: A^n \to A^{n+1}$ $\forall n$ (differentials) with $d\circ d = 0$

A morphism $f: A^\circ \to B^\circ$ is given by morphisms $(f_n: A^n \to B^n)_{n\in\mathbb{Z}}$ which ~~[illegible]~~ commutes with d. The category is called $\mathrm{CoCh}(\mathcal{A})$

A homotopy from f to g with $f, g: A^\circ \to B^\circ$ is given by morphisms $(h_n: A^n \to B^{n-1})_{n\in\mathbb{Z}}$ with $f - g = dh + hd$ (omitting the indices). The category with the objects $\mathrm{CoCh}(\mathcal{A})$ and ~~morphism~~ homotopy classes as morphisms is called $K(\mathcal{A})$.

The i-th cohomology is given by $H^i(A^\circ) = \frac{\ker d^i}{\mathrm{im}\, d^{i-1}}$ and if $H^i(f)$ is isomorphic $\forall i$ f is called quasi-isomorphic.

Def. 2: $N \in K(\mathcal{A})$ is called acyclic if $H^i(N) = 0$ $\forall i$.

- $P \in K(\mathcal{A})$ is K-projective if $\mathrm{Hom}^\circ(P, N)$ is acyclic $\forall N$ acyclic
- ~~$P \in K(\mathcal{A})$ is~~ A K-projective resolution of $M \in K(\mathcal{A})$ is a quasi-isom. $P \to M$ in $K(\mathcal{A})$ where P is K-projective.

Rmk: Every object in Cond Ab has a K-proj. resolution

Simplicial Hypercovers

Def 3: Δ is the category with objects $[n] = \{0, .., n\}$ and morphisms order preserving maps. For a category $\mathcal{C}$ we define a simplicial object of $\mathcal{C}$ as a functor $\Delta^{op} \to \mathcal{C}$. The category is denoted as $\mathrm{Simp}(\mathcal{C})$. Further $\Delta_{\leq n}$ is the full subcategory of Δ with objects $[m]$, $m \leq n$ and $\mathrm{Simp}_n(\mathcal{C})$ are called n-truncated simplicial objects of $\mathcal{C}$.

Prop 4: A simplicial object of $\mathcal{C}$ is fully defined by:

- a collection $(X_n)_n \in \mathcal{C}$ $\forall n \in \mathbb{N}$.
- for $0 \leq i \leq n$ a morphism $d_i : X_j \to X_{j-1}$ (faces) and $s_i : X_i \to X_{i+1}$ satisfying $d_i d_j = d_{j-1} d_i$, $s_i s_j = s_{j+1} s_i$ and $d_i s_j = \begin{cases} s_{j-1} d_i & \text{for } i<j \\ id & \text{for } i=j, i=j+1 \\ s_j d_{i-1} & \text{for } i>j+1 \end{cases}$

The restriction $sk_n : Simp(\mathcal{C}) \to Simp_n(\mathcal{C})$ is called skeleton functor and its right adjoint $cosk_n$ coskeleton functor ($cosk_n : Simp_n(\mathcal{C}) \to Simp(\mathcal{C})$)

Rmk: $cosk_n$ exists in simplicial sets $\forall n$.

Def. 5: Let $\mathcal{C}$ be a category and X an object of $\mathcal{C}$. Then we get the category of semi-representables over X $(SR(\mathcal{C}, X))$

- objects: families $\{X_i \to X\}_{i \in I}$ of morphisms
- morphisms: $\{X_i \to X\}_{i \in I} \longrightarrow \{Y_j \to X\}_{j \in J}$ given by a map of sets $\alpha : I \to J$ and for each $i \in I$ a morphism $\varphi_i : X_i \to Y_{\alpha(i)}$ in $\mathcal{C}$ such that

$$\begin{tikzcd} X_i \arrow[r] \arrow[dr] & Y_{\alpha(i)} \arrow[d] \\ & X \end{tikzcd}$$
commutes.

Prop. 6: If category $\mathcal{C}$ has a fibre product $\Rightarrow$ $SR(\mathcal{C}, X)$ has all finite limits and coskeleton functors exists in Simp $SR(\mathcal{C}, X)$ (can be written as a finite limit here). All finite limits have the form of a finite fiber product.

Def. 7: Let $X \in$ CHaus. A hypercover of X is a simplicial object K of $SR(\mathcal{C}, X)$ such that $K_0 = \{X_i \to X\}_{i \in I}$ jointly surjective ($\coprod_{i \in I} X_i \to X$ surj.) and $\forall n \geq 0$ the map $K_{n+1} \to (cosk_n\, sk_n\, K)_{n+1}$ gives covering family. Means is $K_{n+1} = \{X_i \to X\}_{i \in I}$ and $(cosk_n\, sk_n\, K)_{n+1} = \{Y_j \to X\}_{j \in J}$ and the map is given by $\alpha : I \to J$. Then for each $j \in J$ we have $\{X_i \to Y_j\}_{\alpha(i)=j}$ jointly surjective.

Cohomology

Compare a new cohomology theory to the old ones for example singular cohomology, Čech and Sheaf cohomology.

(For the definition of ~~...~~ condensed cohomology we will use right derived functors from the next talk)

Let $S \in$ CHaus. Then $\Gamma(S,-) = \mathrm{Hom}_{\mathrm{cond Ab}}(\mathbb{Z}[\underline{S}],-)$ functor from cond Ab to Ab.

Now we define for $M \in$ Cond Ab and $S \in$ CHaus

$$H^i_{cond}(S,M) = H^i(R\mathrm{Hom}(\mathbb{Z}[\underline{S}],M))$$

(Here I will only tell you how to compute it)

Prop: $M, N \in K(A)$ and $P \to M$ proj. resolution. Then $R\,\mathrm{Hom}(M,N) = \mathrm{Hom}^\bullet(P,N)$

Now we can compute $H^i_{cond}(S,M)$. For that we look at $\mathbb{Z}[\underline{S}]$ and M as objects $\ldots \to 0 \to M \to 0 \to \ldots$.

How to compute $H^i_{cond}(S,M)$?

Construct $P^\bullet \to \mathbb{Z}[\underline{S}]$ projective resolution by using extremely disconnected hypercover $S_\bullet \to S$.

Inductive construction: $S_0 \twoheadrightarrow S$ extremally disconnected set (for example discrete S). If $S_0, \ldots, S_n$ n-truncated simplicial objects of ED. Then S_{n+1} extremely disconnected surjective onto $(\mathrm{cosk}_n(S_0, \ldots, S_n))_{n+1}$. + face maps: composing face maps from $\mathrm{cosk}_n(S_0, \ldots, S_n)$ with surjection (same for s_n)

This gives all the axioms (without proof here) for a hypercover

Now define $P^i = \mathbb{Z}[\underline{S_i}]$ + alternating sum of face maps as differentials (Moore complex). One can find in the sources that this gives a projective resolution.

$\Rightarrow H^i_{cond}(\underline{S}, M)$ is given by cohomology of

$$0 \to \Gamma(S_0, M) \to \Gamma(S_1, M) \to \ldots$$

Comparison of cohomology

In the classic cohomology theories there exists a natural isomorphism $H^i_{\check{C}ech}(S, \mathbb{Z}) \to H^i_{sheaf}(S, \mathbb{Z})$ for S paracompact. If S is also a CW-space the cohomology groups also agree with $H^i_{sing}(S, \mathbb{Z})$.

Now we get the following comparisons for the ~~...~~ condensed cohomology (without proof):

Thm 9 (Dyckhoff '76): Let $S \in$ CHaus and M discrete abelian group. Then $H^i_{sheaf}(S, M) \cong H^i_{cond}(S, \underline{M})$ natural.

$\hat{\mathrm{I}}$ sheafication from constant presheaf

~~Thm 10: For any~~ For not discrete groups M the theory ~~...~~ hasn't the same nice properties.

Thm 10: For any $S \in$ CHaus let $C(S, \mathbb{R})$ cts. real valued functions. Then $H^i_{cond}(S, \underline{\mathbb{R}}) = \begin{cases} C(S, \mathbb{R}) & \text{if } i = 0 \\ 0 & \text{if } i > 0 \end{cases}$